The inverse problem for invariant Lagrangians on a Lie group

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Abstract. We discuss the problem of the existence of a regular invariant Lagrangian for a given system of invariant second-order ordinary differential equations on a Lie group G, using approaches based on the Helmholtz conditions. Although we deal with the problem directly on TG, our main result relies on a reduction of the system on TG to a system on the Lie algebra of G. We conclude with some illustrative examples.

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1 Introduction

The inverse problem of the calculus of variations consists in finding conditions for the existence of a regular Lagrangian for a given set of second-order ordinary differential equations on a manifold, $\ddot{q}^i = f^i(q,\dot{q})$, so that the given equations are equivalent to the Euler-Lagrange equations of the Lagrangian. In order for a Lagrangian $L(q,\dot{q})$ to exist we must be able to find $g_{ij}(q,\dot{q})$, so-called multipliers, such that

$$g_{ij}(\ddot{q}^j - f^j) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}.$$

It is shown for example in [9, 21] that the multipliers must satisfy

$$\begin{aligned}
&\det(g_{ij}) \neq 0, \qquad g_{ji} = g_{ij},\\
&\frac{d}{dt}(g_{ij}) + \tfrac{1}{2}\frac{\partial f^k}{\partial \dot{q}^j} g_{ik} + \tfrac{1}{2}\frac{\partial f^k}{\partial \dot{q}^i} g_{kj} = 0,\\
&g_{ik}\left(\frac{d}{dt}\left(\frac{\partial f^k}{\partial \dot{q}^j}\right) - 2\frac{\partial f^k}{\partial q^j} - \tfrac{1}{2}\frac{\partial f^l}{\partial \dot{q}^j}\frac{\partial f^k}{\partial \dot{q}^l}\right) = g_{jk}\left(\frac{d}{dt}\left(\frac{\partial f^k}{\partial \dot{q}^i}\right) - 2\frac{\partial f^k}{\partial q^i} - \tfrac{1}{2}\frac{\partial f^l}{\partial \dot{q}^i}\frac{\partial f^k}{\partial \dot{q}^l}\right),\\
&\frac{\partial g_{ij}}{\partial \dot{q}^k} = \frac{\partial g_{ik}}{\partial \dot{q}^j};
\end{aligned}$$

and conversely, if one can find functions satisfying these conditions then the equations $\ddot{q}^i = f^i$ are derivable from a Lagrangian. These conditions are generally referred to as the Helmholtz conditions. The solution (g_{ij}) is the Hessian of the sought-for Lagrangian with respect to the velocity variables, and $\det(g_{ij}) \neq 0$ is the condition for the Lagrangian to be regular. We refer to the recent survey [12] and the monograph [2] for comments on the history of the problem, milestones in the literature and accounts of the different paths that have been followed in the past.

We will focus here on the case where the manifold is a Lie group. An immediate example is the one where the second-order system is the geodesic spray of the canonical connection on the group: this connection is specified in terms of left-invariant vector fields X and Y by $\nabla_X Y = \frac{1}{2}[X, Y]$. The inverse problem for this specific type of second-order system has been solved explicitly for almost all Lie groups up to dimension 6 by Thompson and his collaborators (see [10, 20, 23] and the references therein): in each case the authors were able to decide if a Lagrangian exists or not, and to provide a Lagrangian in the affirmative cases.

The second-order equations of the canonical connection are invariant under left translations. Surprisingly, if a Lagrangian exists, it is not necessarily invariant. The main goal of this paper is to solve a type of inverse problem which is on the one hand broader than that discussed in [23] etc. in that it deals with any invariant system of second-order ordinary differential equations on a Lie group, but on the other hand more restricted in that the Lagrangian, if it exists, is required to be invariant also. That is to say, we will deal with the following rather natural problem: given an invariant second-order system on a Lie group G, when does there exist a regular Lagrangian for it that is also invariant under G? We call this the invariant inverse problem. The invariant inverse problem for the specific case of the geodesic spray of the canonical connection has been studied in [18]. In the current paper, by contrast, we will deal with the general invariant inverse problem.

It is unfortunately not straightforward to adapt the solution of the inverse problem by the Helmholtz conditions to the invariant inverse problem. Clearly, if there is an invariant Lagrangian then the corresponding multiplier matrix (its Hessian) must itself be invariant (in an appropriate sense). The difficulty is this: one may find a multiplier which satisfies the Helmholtz conditions and is invariant; one is then guaranteed that there is a Lagrangian, but not that the Lagrangian is invariant. Roughly speaking, to obtain the Lagrangian one has to integrate the multiplier, and invariance may be lost as a result. In fact extra conditions, of a cohomological nature, must be satisfied. The occurrence of such cohomological conditions was discussed already nearly twenty years ago, in a similar but more limited context, by Marmo and Morandi [13]. We will present a version of their result, which amounts in fact to a small generalization of it, in Theorem 1. The conditions also appear in [1], using the rather different framework of the variational bicomplex.

It is however possible to adopt quite a different approach from these authors, by taking advantage of invariance to carry out a reduction of the problem, which turns out to simplify it in concept, and to make the solution considerably more useful in applications. We next explain this alternative approach in a little more detail.

Because of the invariance of our problem, G will be a symmetry group of the second-order system. It follows that the space of interest is effectively the Lie algebra $\mathfrak{g}$ of G rather than the whole manifold TG, and we can first perform a reduction. The dynamical vector field Γ corresponding to the system of differential equations, namely

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \dot{q}^i} \in \mathfrak{X}(TG),$$

reduces to a vector field γ on $\mathfrak{g}$ given in terms of Cartesian coordinates (w^i) (so that the w^i are the components of $w \in \mathfrak{g}$ with respect to some chosen basis of $\mathfrak{g}$) by

$$\gamma = \gamma^i \frac{\partial}{\partial w^i}.$$

On the other hand, if $L \in C^\infty(TG)$ is a regular invariant Lagrangian then its restriction to $\mathfrak{g} = T_eG$ is a function (also called a Lagrangian) $l \in C^\infty(\mathfrak{g})$. We will take optimal advantage of the following observation (which is proved in [14] for example, though we will give a different derivation below): finding a solution $g(t) \in G$ of the Euler-Lagrange equations of L is equivalent to finding a solution $w(t) \in \mathfrak{g}$ of the so-called Euler-Poincaré equations

$$\frac{d}{dt}\left(\frac{\partial l}{\partial w}\right) = \mathrm{ad}^*_w \frac{\partial l}{\partial w},$$

(where ad^* is the adjoint action of $\mathfrak{g}$ on its dual), or equivalently if C^k_{ij} are the structure constants of $\mathfrak{g}$ corresponding to the basis used to define the coordinates,

$$\frac{d}{dt}\left(\frac{\partial l}{\partial w^j}\right) = C^k_{ij}\frac{\partial l}{\partial w^k} w^i. \tag{1}$$

To obtain the corresponding solution $g(t)$ of the Euler-Lagrange equations we need to solve in addition the equation $g(t)^{-1}\dot{g}(t) = w(t)$.

The invariant inverse problem on a Lie group G has therefore the following equivalent reduced version: if Γ is invariant, when does there exist a Lagrangian $l \in C^\infty(\mathfrak{g})$ such that its Euler-Poincaré equations (1) are equivalent to the equations $\dot{w}^i = \gamma^i$ for the reduced vector field γ on $\mathfrak{g}$? As we will show in Theorem 2 below, if such a Lagrangian l exists for γ, the original vector field Γ will be the Euler-Lagrange field for some invariant Lagrangian L. The advantage of such an approach is that the Lagrangian being sought is simply a function of the coordinates w^i on the Lie algebra $\mathfrak{g} = T_eG$, rather than a function of the coordinates $(q^i, \dot{q}^i)$ on TG satisfying invariance conditions. The solution to this existence problem will be given in part by a set of reduced Helmholtz conditions for γ, involving a multiplier matrix (k_{ij}) which, in the end, is the Hessian of the function l we want to find. In addition, cohomological conditions will again make their appearance here. It turns out, as we will establish in Theorem 4 below, that one of the functions of the reduced Helmholtz conditions is to ensure that certain cochains are cocycles, and determine cohomology classes in the cohomology of $\mathfrak{g}$. What is not resolved by the Helmholtz conditions is whether these cocycles can be made into coboundaries; that they can is the additional requirement for the existence of a Lagrangian.

The two approaches, leading respectively to Theorem 1 and Theorem 4, involve classes in the cohomology of $\mathfrak{g}$ which, while differently derived, are the same. Nevertheless there is a subtle difference between the two forms of the inverse problem, which it is worth pointing out. The procedure described in Theorem 1 and [13] associates with a certain set of Lagrangians a pair of cohomology classes, whose vanishing is the condition for there to be a Lagrangian in the set which is invariant. The procedure described in Theorem 4 in effect associates with a certain set of invariant functions a pair of cohomology classes, whose vanishing is the condition for there to be an invariant function in the set which is a Lagrangian.

The geometrical framework that we will use is based on a reformulation of the Euler-Lagrange equations and of the Helmholtz conditions in terms of a suitable adapted frame. The requisite background material is given in Section 2. The solution of the invariant inverse problem using invariant multipliers in the Helmholtz conditions on TG is presented in Theorem 1 in Section 3.

The Euler-Poincaré equations are derived in Section 4, and the reduced Helmholtz conditions in Section 5. Section 6 is devoted to the proof of Theorem 4, which is the solution of the invariant inverse problem using the reduced Helmholtz conditions, and is the main result of the paper. Next, we investigate the geometric structure of the reduced space. In Section 7 we show that Equation (1) is a particular example of a so-called Lagrangian system on a Lie algebroid, where the Lie algebroid at hand is related in a natural way to the Lie algebra $\mathfrak{g}$ of the Lie group G. We make the link between the current set-up and Martínez's approach [15] to Lagrangian systems on Lie algebroids. This will result in a coordinate-independent reformulation of the reduced Helmholtz conditions and of the cohomology conditions. The paper ends with some examples and some suggestions for future work.

Although the paper focusses entirely on left-invariant Lagrangians, it can easily be adjusted to the right-invariant case.

2 Calculus along the tangent bundle projection

One can find in the literature several reformulations of the Helmholtz conditions that are independent of the choice of coordinates on the manifold M: see for example [6, 17]. We will follow closely the one given in [7, 16], which is based on a calculus of tensor fields along the tangent bundle projection $\tau : TM \to M$. By a vector field along τ we mean a section of the pullback bundle $\tau^*TM \to TM$, and likewise for tensor fields. A section of $\tau^*TM \to TM$ can be interpreted as a map $X : TM \to TM$ with the property that $\tau \circ X = \tau$, and can be expressed in terms of local coordinates as

$$X = X^i(q, \dot{q})\frac{\partial}{\partial q^i} \in \mathfrak{X}(\tau).$$

There is a 1-1 correspondence between vector fields along τ and vertical vector fields on TM. This correspondence is made explicit by the so-called vertical lift $X^{\rm V}$ of X, given by

$$X^{\rm V} = X^i\frac{\partial}{\partial \dot{q}^i}.$$

Any vector field on M induces a vector field along τ in an obvious way. If $X = X^i(q)\partial/\partial q^i$ is a vector field on M, its complete lift $X^{\rm C}$ is the following vector field on TM:

$$X^{\rm C} = X^i\frac{\partial}{\partial q^i} + \frac{\partial X^i}{\partial q^j}\dot{q}^j\frac{\partial}{\partial \dot{q}^i}.$$

Here are some convenient formulae for the brackets of complete and vertical lifts:

$$[X^{\rm C}, Y^{\rm C}] = [X, Y]^{\rm C}, \qquad [X^{\rm C}, Y^{\rm V}] = [X, Y]^{\rm V} \qquad \text{and} \qquad [X^{\rm V}, Y^{\rm V}] = 0.$$

Here X and Y are vector fields on M throughout.

The vertical and complete lifts require no additional machinery for their definitions. However, if we have a second-order differential equation field, or dynamical vector field, Γ at our disposal, say

$$\Gamma = \dot{q}^i\frac{\partial}{\partial q^i} + f^i\frac{\partial}{\partial \dot{q}^i}$$

(representing the second-order equations $\ddot{q}^i = f^i$), we can use it to define the so-called horizontal lift of a vector field along τ. The horizontal lift X^{H} of $X \in \mathfrak{X}(\tau)$ is

$$X^{\mathrm{H}} = X^i \left(\frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial \dot{q}^j} \right), \qquad \Gamma_i^j = -\tfrac{1}{2} \frac{\partial f^j}{\partial \dot{q}^i}.$$

Any vector field Z on TM can be decomposed into a horizontal and vertical component: $Z = X^{\mathrm{H}} + Y^{\mathrm{V}}$, for $X, Y \in \mathfrak{X}(\tau)$. In case X is induced by a vector field on M, the three lifts are related as follows:

$$X^{\mathrm{H}} = \tfrac{1}{2}(X^{\mathrm{C}} - [\Gamma, X^{\mathrm{V}}]).$$

Another useful fact, which it is easy to establish by a coordinate calculation, is that $[\Gamma, X^{\mathrm{C}}]$ is always vertical.

The Lie brackets of the dynamics Γ with horizontal and vertical vector fields define important objects for the calculus along τ. It can be shown that the horizontal and vertical components of these brackets take the form

$$[\Gamma, X^{\mathrm{V}}] = -X^{\mathrm{H}} + (\nabla X)^{\mathrm{V}} \qquad \text{and} \qquad [\Gamma, X^{\mathrm{H}}] = (\nabla X)^{\mathrm{H}} + (\Phi(X))^{\mathrm{V}}.$$

The operator Φ is a type (1,1) tensor field along τ and is called the Jacobi endomorphism. The other operator, ∇, acts as a derivative on $\mathfrak{X}(\tau)$, in the sense that for $f \in C^\infty(TM)$ and $X \in \mathfrak{X}(\tau)$, $\nabla(fX) = f\nabla X + \Gamma(f)X$. It is therefore called the dynamical covariant derivative. Finally we will need the vertical derivative $\mathrm{D}^{\mathrm{V}}_X$ associated with any $X \in \mathfrak{X}(\tau)$. This acts on vector fields along τ, but is completely determined by its actions on vector fields Y on M and on functions f on TM by the formulae $\mathrm{D}^{\mathrm{V}}_X Y = 0$ and $\mathrm{D}^{\mathrm{V}}_X f = X^{\mathrm{V}}(f)$.

In the framework of the calculus along the tangent bundle projection the multiplier matrix is regarded as an operator $g : \mathfrak{X}(\tau) \times \mathfrak{X}(\tau) \to C^\infty(TM)$, that is as a type (0,2) tensor field along τ, with local expression $g = g_{ij}(q, \dot{q}) dq^i \otimes dq^j$. The actions of both the dynamical covariant derivative and the vertical derivative can easily be extended to (0,2) tensor fields along τ: by definition, for $X, Y, Z \in \mathfrak{X}(\tau)$

$$\nabla g(X, Y) = \Gamma(g(X, Y)) - g(\nabla X, Y) - g(X, \nabla Y)$$

and

$$\mathrm{D}^{\mathrm{V}}_X g(Y, Z) = X^{\mathrm{V}}(g(Y, Z)) - g(\mathrm{D}^{\mathrm{V}}_X Y, Z) - g(Y, \mathrm{D}^{\mathrm{V}}_X Z).$$

The inverse problem can now be rephrased as the search for a type (0,2) tensor field g along τ which is non-singular and satisfies for all $X, Y, Z \in \mathfrak{X}(\tau)$ the conditions

$$\begin{aligned}
&g(X, Y) = g(Y, X),\\
&\nabla g = 0,\\
&g(\Phi(X), Y) = g(X, \Phi(Y)),\\
&\mathrm{D}^{\mathrm{V}}_X g(Y, Z) = \mathrm{D}^{\mathrm{V}}_Y g(X, Z).
\end{aligned}$$

These are the Helmholtz conditions in coordinate-independent form.

It will also be desirable to have a coordinate-independent version of the Euler-Lagrange equations. It is easy to see that the Euler-Lagrange field Γ of a regular Lagrangian L is uniquely determined by the fact that it is a second-order differential equation field and satisfies

$$\Gamma(X^{\mathrm{V}}(L)) - X^{\mathrm{C}}(L) = 0$$

for every vector field X on M. In particular, if $\{X_i\}$ is a basis of vector fields on M then $\{X_i^{\mathrm{C}}, X_i^{\mathrm{V}}\}$ is an induced basis for vector fields on TM, and the following set of equations is equivalent to the Euler-Lagrange equations:

$$\Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) = 0.$$

We now consider the effect of a diffeomorphism of M on the Euler-Lagrange equations. Let φ be a diffeomorphism of M and $T\varphi$ the induced diffeomorphism of TM. For any $X \in \mathfrak{X}(M)$, $T(T\varphi)(X^{\mathrm{C}}) = (T\varphi X)^{\mathrm{C}}$ and $T(T\varphi)(X^{\mathrm{V}}) = (T\varphi X)^{\mathrm{V}}$ (these are of course the counterparts of the bracket relations quoted earlier). Moreover, if $\Gamma \in \mathfrak{X}(TM)$ is a second-order differential equation field so is $T(T\varphi)\Gamma$ (this is the counterpart of the fact, stated earlier, that $[\Gamma, Z^{\mathrm{C}}]$ is always vertical).

Let L be a regular Lagrangian with Euler-Lagrange field Γ. Then $T\varphi^*L$ is a regular Lagrangian; we claim that its Euler-Lagrange field is $T(T\varphi)^{-1}\Gamma$. The proof goes as follows. For any function f, vector field X and diffeomorphism φ, $X(\varphi^* f) = \varphi^*((T\varphi X)(f))$. We know that Γ is uniquely determined by the Euler-Lagrange equations $\Gamma(X^{\mathrm{V}}(L)) = X^{\mathrm{C}}(L)$ for all $X \in \mathfrak{X}(M)$. Now

$$\begin{aligned} X^{\mathrm{C}}(T\varphi^*L) &= T\varphi^*\left((T\varphi X)^{\mathrm{C}}(L)\right) = T\varphi^*\left(\Gamma((T\varphi X)^{\mathrm{V}}(L))\right) \\ &= T(T\varphi)^{-1}\Gamma\left(T\varphi^*((T\varphi X)^{\mathrm{V}}(L)\right) = T(T\varphi)^{-1}\Gamma\left(X^{\mathrm{V}}(T\varphi^*L)\right). \end{aligned}$$

If L is regular and $T\varphi^*L = L$ then $T(T\varphi)\Gamma = \Gamma$. But although the Lagrangian uniquely determines the Euler-Lagrange equations, it is not in general true that the Euler-Lagrange equations uniquely determine the Lagrangian, so if $T(T\varphi)\Gamma = \Gamma$ all we can conclude is that $T\varphi^*L$ is a Lagrangian for Γ; if different from L it may be called an alternative Lagrangian. That genuinely alternative Lagrangians (Lagrangians not differing by a total derivative) can exist even in the most familiar circumstances is well-known: the free particle is the most obvious example, and lest that look too suspiciously special we could mention also motion in a spherically symmetric potential in Euclidean 3-space [11].

3 The invariant inverse problem

For the remainder of the paper the configuration manifold M will be a connected Lie group G. We will use λ_g and ρ_g to denote left and right multiplication by $g \in G$. Both maps can be extended to actions $T\lambda_g$ and $T\rho_g$ of G on TG.

We assume that we have a left-invariant second-order differential equation field Γ on TG: thus $T(T\lambda_g)\Gamma = \Gamma$ for all $g \in G$. The question under discussion is whether Γ admits an invariant regular Lagrangian, that is, whether there is a function L on TG whose Hessian with respect to velocity coordinates is non-singular and which satisfies $T\lambda_g^*L = L$ for all $g \in G$, such that Γ is the Euler-Lagrange field of L. We can conclude from the analysis at the end of the last section that the Euler-Lagrange field of an invariant regular Lagrangian is invariant. But if we start with an invariant second-order differential equation field on the other hand, and it admits a regular Lagrangian, then all we can conclude is that its left translates are alternative Lagrangians, possibly different.

We now begin to develop the machinery we need for a deeper study of the problem.

By left-translating a basis $\{E_i\}$ of the Lie algebra $\mathfrak{g}$ of G we obtain a left-invariant basis $\{\hat{E}_i\}$ of $\mathfrak{X}(G)$. Similarly, $\{\tilde{E}_i\}$ will denote the right-invariant basis of $\mathfrak{X}(G)$ obtained via right translation. These bases are related by

$$\hat{E}_i(g) = A_i^j(g)\tilde{E}_j(g), \tag{2}$$

where $(A_i^j(g))$ is the matrix representation of ad_g; in particular $A_i^j(e) = \delta_i^j$ (where e is the identity of G). We will identify the Lie algebra with the left-invariant vector fields: then $[\hat{E}_i, \hat{E}_j] = C_{ij}^k\hat{E}_k$ where the C_{ij}^k are the structure constants of $\mathfrak{g}$, and $[\tilde{E}_i, \tilde{E}_j] = -C_{ij}^k\tilde{E}_k$. (This is the convention in [14], for example.)

In the following, a vector v_g in T_gG will have coordinates (w^i) with respect to $\{\hat{E}_i\}$, so that $v_g = w^i\hat{E}_i(g)$. Then (w^i) are exactly the coordinates of the Lie algebra element $w = T\lambda_{g^{-1}}v_g$ with respect to the basis $\{E_i\}$ of $\mathfrak{g}$.

The following property is true for any action of a connected Lie group on a manifold: a tensor field is invariant under an action if and only if its Lie derivative by every fundamental vector field vanishes. When the manifold is a Lie group and the action is left multiplication, the fundamental vector fields are the right-invariant vector fields, for which $\{\tilde{E}_i\}$ is a basis. A function f on G is left-invariant if and only if $\tilde{E}_i(f) = 0$ for all i, and a vector field X on G is left-invariant if and only if $[\tilde{E}_i, X] = 0$. In particular, for the left-invariant $\hat{E}_j$, $[\tilde{E}_i, \hat{E}_j] = 0$. In view of the bracket relations in the two bases it follows that

$$\tilde{E}_i(A_j^k) + A_j^l C_{li}^k = 0 \qquad \text{and} \qquad A_i^k A_j^l C_{kl}^m = A_n^m C_{ij}^n. \tag{3}$$

The Lagrangian L and the dynamical vector field Γ both live on the tangent manifold TG. To characterize their invariance we need to know the infinitesimal generators of the induced action $T\lambda_g$ of G on TG. Given that the flow of a complete lift is tangent to the flow of the underlying vector field, it is easy to see that the infinitesimal generators of $T\lambda_g$ are exactly the complete lifts $\{\tilde{E}_i^{\mathrm{C}}\}$ of the infinitesimal generators of the action λ_g of G on G. So a function $F \in C^\infty(TG)$ is left-invariant if and only if $\tilde{E}_i^{\mathrm{C}}(F) = 0$, and a vector field $Z \in \mathfrak{X}(TG)$ is left-invariant if and only if $[\tilde{E}_i^{\mathrm{C}}, Z] = 0$. Note that $\hat{E}_i^{\mathrm{C}}$ and $\hat{E}_i^{\mathrm{V}}$ are invariant vector fields, by virtue of the bracket relations for complete and vertical lifts given earlier. The functions w^i are also invariant; they are linear fibre coordinates on TG, and satisfy $\hat{E}_j^{\mathrm{V}}(w^i) = \delta_j^i$.

The following observations will be important. First, if a function f satisfies $\hat{E}_i^{\mathrm{V}}(f) = 0$ for all i the f is (the pull-back to TG of) a function on G. Second, if $\hat{E}_i^{\mathrm{V}}(f) = f_i$ is a function on G for all i then $f - f_iw^i$ is a function on G.

Recall that we interpret the Hessian of a Lagrangian L as a type (0,2) tensor field g along the tangent bundle projection $\tau : TG \to G$. If L is invariant then the coefficients $K_{ij} = \hat{E}_i^{\mathrm{V}}\hat{E}_j^{\mathrm{V}}(L) = g(\hat{E}_i, \hat{E}_j)$ will also be invariant functions. Now when we use the Helmholtz condition approach to the inverse problem, if we are interested only in invariant Lagrangians we will certainly need to add to the Helmholtz conditions the extra condition that the multiplier g should be invariant. As we pointed out earlier, if we start from an invariant second-order field, it is often possible to find non-invariant Lagrangians. Examples of this behaviour can be found in the papers [10, 20, 23] for the case of the canonical connection. The reason is that in these examples the extra condition about the invariance of the multiplier is usually not imposed on the problem. However, as we pointed out before and will shortly explain in more detail, the invariance of the multiplier, while necessary for the existence of an invariant Lagrangian, is not sufficient.

The invariance of g can be defined in a coordinate-independent way as follows. We first define a vector field X along τ to be left-invariant if its vertical lift X^{V} is left-invariant. Since $\{\hat{E}_i\}$ is a basis for $\mathfrak{X}(G)$, it serves also as a basis for vector fields along τ. Then a vector field along τ, $X = \Xi^i \hat{E}_i$, is invariant if $\tilde{E}_j^{\mathrm{C}}(\Xi^i) = 0$, or if its coefficients $\Xi^i \in C^\infty(TG)$ are invariant functions. We will say that a type (0,2) tensor field g along τ is invariant if $g(X, Y)$ is an invariant function for all invariant vector fields X and Y along τ. It is easy to verify that this holds if and only if the coefficients of g with respect to $\{\hat{E}_i\}$ are invariant.

We now state and prove a theorem which shows what requirements in addition to the Helmholtz conditions and the invariance of the multiplier are necessary and sufficient for the existence of an invariant Lagrangian. Let us call a type (0,2) tensor field g along τ which satisfies the Helmholtz conditions for an invariant second-order differential equation field Γ and is invariant an invariant multiplier for Γ.

Theorem 1. *An invariant multiplier for an invariant second-order differential equation field Γ determines a cohomology class in $H^1(\mathfrak{g})$ and one in $H^2(\mathfrak{g})$. The field Γ is derivable from an invariant Lagrangian if and only if the corresponding cohomology classes vanish.*

Proof. Suppose that g is an invariant multiplier. We set $K_{ij} = g(\hat{E}_i, \hat{E}_j)$. By the very fact that we have a solution of the Helmholtz conditions we know that there is a regular Lagrangian for Γ, say L, such that $K_{ij} = \hat{E}_i^{\mathrm{V}} \hat{E}_j^{\mathrm{V}}(L)$. Now L need not be invariant; but from the invariance of the K_{ij} we have

$$0 = \tilde{E}_k^{\mathrm{C}}(K_{ij}) = \hat{E}_i^{\mathrm{V}} \hat{E}_j^{\mathrm{V}}(\tilde{E}_k^{\mathrm{C}}(L)) = 0,$$

whence $\tilde{E}_k^{\mathrm{C}}(L) = a_{kl} w^l + b_k$ for certain functions a_{kl} and b_k on G. Since L is known to be a Lagrangian and Γ is invariant,

$$\begin{aligned} 0 &= \tilde{E}_i^{\mathrm{C}}\left(\Gamma(\hat{E}_j^{\mathrm{V}}(L)) - \hat{E}_j^{\mathrm{C}}(L)\right) = \Gamma(a_{ij}) - \hat{E}_j^{\mathrm{C}}(a_{ik} w^k + b_i) \\ &= w^k \left(\hat{E}_k(a_{ij}) - \hat{E}_j(a_{ik}) - a_{il} C^l_{jk}\right) - \hat{E}_j(b_i). \end{aligned}$$

We can set to zero the coefficient of w^k and the remaining term separately (both are functions on G). From the second we see that b_i is constant. From the first,

$$\hat{E}_k(a_{ij}) - \hat{E}_j(a_{ik}) - a_{il} C^l_{jk} = 0.$$

Let ϑ^i be the 1-forms on G dual to the $\hat{E}_i$ (so that $\vartheta = \vartheta^i E_i$ is the Maurer-Cartan form, not that it matters); then for each i the 1-form $a_{ij}\vartheta^j$ is closed, from which it follows that $a_{ij} = \hat{E}_j(f_i)$ for some functions f_i on G. We have

$$\tilde{E}_i^{\mathrm{C}}(L) = w^j \hat{E}_j(f_i) + b_i; \tag{4}$$

note that this is of the form total derivative plus constant. Next,

$$0 = \tilde{E}_i^{\mathrm{C}} \tilde{E}_j^{\mathrm{C}}(L) - \tilde{E}_j^{\mathrm{C}} \tilde{E}_i^{\mathrm{C}}(L) + C^k_{ij} \tilde{E}_k^{\mathrm{C}}(L) = w^k \hat{E}_k \left(\tilde{E}_i(f_j) - \tilde{E}_j(f_i) + C^l_{ij} f_l\right) + C^k_{ij} b_k,$$

from which it follows that

$$\tilde{E}_i(f_j) - \tilde{E}_j(f_i) + C^l_{ij} f_l = \alpha_{ij} \tag{5}$$

is constant, and $C^k_{ij} b_k = 0$. Now we can regard the b_i as the coefficients, with respect to the basis of $\mathfrak{g}^*$ dual to the basis of $\mathfrak{g}$ with which we are working, of a linear map $b : \mathfrak{g} \to \mathbf{R}$, so that $b(\xi) = b_i \xi^i$. Similarly, the α_{ij} are the coefficients of an alternating bilinear map $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$, so that $\alpha(\xi,\eta) = \alpha_{ij}\xi^i\eta^j$. We now show that, viewed from the perspective of the cohomology of $\mathfrak{g}$ with values in $\mathbf{R}$, b and α are cocycles; that is, they satisfy the cocycle conditions

$$b(\{\xi,\eta\}) = 0 \qquad \text{and} \qquad \alpha(\xi,\{\eta,\zeta\}) + \mu(\eta,\{\zeta,\xi\}) + \mu(\zeta,\{\xi,\eta\}) = 0$$

($\{\cdot,\cdot\}$ is the Lie algebra bracket); or in terms of the structure constants,

$$b_k C^k_{ij} = 0 \qquad \text{and} \qquad \alpha_{il}C^l_{jk} + \alpha_{jl}C^l_{ki} + \alpha_{kl}C^l_{ij} = 0.$$

Indeed, we have just seen that $b_k C^k_{ij} = 0$. Operating with $\tilde{E}_k$ again on Equation (5) and taking the cyclic sum we see that α_{ij} is a cocycle too. Moreover, f_i is determined only up to the addition of a constant; and the addition of a constant leaves b unchanged and changes α by a coboundary.

If α_{ij} and b_i are both cohomologous to zero, then $b_i = 0$, and by choice of additive constants we can assume that $\tilde{E}_i(f_j) - \tilde{E}_j(f_i) + C^l_{ij}f_l = 0$. But then $f_i = \tilde{E}_i(f)$ for some function f on G. But then $L - w^j\hat{E}_j(f) = L - \dot{f}$ is invariant, and of course has Γ as its Euler-Lagrange field and has the same Hessian as L. □

In [13] the authors restrict their attention to Lagrangians satisfying just $\tilde{E}^{\mathrm{C}}_i(L) = w^j\hat{E}_j(f_i)$, that is, to Lagrangians which change only by addition of a total derivative under the action of G; they call such Lagrangians quasi-invariant, and appeal to physics to justify this choice. From a purely mathematical point of view such a restriction is unnecessary, and the more general situation is easily analysed, as we have seen. One possible interpretation of the significance of the element of $H^1(\mathfrak{g})$ is this: it is not difficult to see that $b_i = -\tilde{E}^{\mathrm{C}}_i(\mathcal{E})$, where $\mathcal{E}$ is the energy of L; so $b_i = 0$ is the condition for the energy to be invariant (even though L itself might not be). We will have more to say about the significance of b_i later.

4 The Euler-Poincaré equations

We now turn to the reduction of Γ to the Lie algebra $\mathfrak{g}$.

Left-invariant functions on TG are in 1-1 correspondence with functions on the Lie algebra: on the one hand restriction of any function on TG to T_eG determines a function on $T_eG = \mathfrak{g}$; on the other hand, any function on $\mathfrak{g} = T_eG$ can be extended to a left-invariant function on the whole of TG by requiring it to be constant along each orbit of the action. From now on we will use the following convention: capital letters such as F stand for left-invariant functions, vector fields, etc. on TG; the corresponding small letters such as f stand for their restrictions to $T_eG = \mathfrak{g}$.

A vector field $Z = \Xi^j\hat{E}^{\mathrm{C}}_j + F^j\hat{E}^{\mathrm{V}}_j \in \mathfrak{X}(TG)$ is left-invariant if and only if $[\tilde{E}^{\mathrm{C}}_i, Z] = 0$, that is, if and only if $\tilde{E}^{\mathrm{C}}_i(\Xi^j) = 0$ and $\tilde{E}^{\mathrm{C}}_i(F^j) = 0$. Thus Z is invariant if and only if its components Ξ^j and F^j are all invariant functions. We can therefore identify them with functions ξ^j and f^j on the Lie algebra. Note that $f^j\hat{E}^{\mathrm{V}}_j|_e$ can be identified with a vector field on T_eG, since it is vertical; the same is not true for $\xi^j\hat{E}^{\mathrm{C}}_j|_e$, however: it is defined on T_eG, but as a vector field it is transverse to it.

A set $\{\xi^j\}$ of $n = \dim\mathfrak{g}$ functions on $\mathfrak{g}$ can be interpreted in two equivalent ways. First, the elements of the set could be viewed as the coefficients of a $C^\infty(\mathfrak{g},\mathfrak{g})$-map, namely the map $\xi : w \mapsto \xi^i(w)E_i$. A second interpretation is to view them as the components of a vector field

$\bar{\xi}$ on $\mathfrak{g}$, where $\bar{\xi} = \xi^j \partial/\partial w^j$. This equivalence of interpretations is a manifestation of the fact that the vector bundles $T\mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ are isomorphic, so there is a 1-1 correspondence between their sections.

The two sets $\{\xi^j\}$ and $\{f^j\}$ together define a section of the vector bundle $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$, or equivalently the bundle $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We will adopt the following convention: an invariant vector field $Z = \Xi^j \hat{E}_j^{\rm C} + F^j \hat{E}_j^{\rm V} \in \mathfrak{X}(TG)$ reduces to the section $z = (\xi, f)$ of $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$ where the first element $\xi = \xi^j E_j$ is interpreted as a $C^\infty(\mathfrak{g}, \mathfrak{g})$-map and the second $f = f^j \partial/\partial w^j$ is a vector field on $\mathfrak{g}$. In particular, for an invariant second-order field

$$\Gamma = w^i \hat{E}_j^{\rm C} + \Gamma^j \hat{E}_j^{\rm V} \in \mathfrak{X}(TG)$$

the first invariance condition, $\tilde{E}_i^{\rm C}(w^j) = 0$, is trivially satisfied, so the only condition is $\tilde{E}_i^{\rm C}(\Gamma^j) = 0$. Let Δ be the identity map in $C^\infty(\mathfrak{g}, \mathfrak{g})$; then Γ reduces to the section (Δ, γ) of $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$, where

$$\gamma = \gamma^i \frac{\partial}{\partial w^i} \in \mathfrak{X}(\mathfrak{g})$$

will be often called the reduced vector field on $\mathfrak{g}$.

Let $L \in C^\infty(TG)$ be a left-invariant regular Lagrangian with Euler-Lagrange field Γ. We have shown in Section 2 that this second-order differential equation field can be characterized by the equations

$$\Gamma(\hat{E}_i^{\rm V}(L)) - \hat{E}_i^{\rm C}(L) = 0. \tag{6}$$

We have also shown that if L is left-invariant then so also is Γ. We now compute its reduced vector field γ on $\mathfrak{g}$.

The Euler-Lagrange equations (6) are of the form

$$w^k \hat{E}_k^{\rm C} \hat{E}_i^{\rm V}(L) + \Gamma^k \hat{E}_k^{\rm V} \hat{E}_i^{\rm V}(L) - \hat{E}_i^{\rm C}(L) = 0.$$

With the help of (3), the relations between the complete and vertical lifts of elements in the two bases is given by

$$\hat{E}_i^{\rm C} = A_i^j \tilde{E}_j^{\rm C} + w^k C_{ki}^j \hat{E}_j^{\rm V} \qquad \text{and} \qquad \hat{E}_i^{\rm V} = A_i^j \tilde{E}_j^{\rm V}. \tag{7}$$

As a consequence the first term in the Euler-Lagrange equations vanishes:

$$w^k \hat{E}_k^{\rm C} \hat{E}_i^{\rm V}(L) = w^k A_k^j \tilde{E}_j^{\rm C} \hat{E}_i^{\rm V}(L) = w^k A_k^j \hat{E}_i^{\rm V} \tilde{E}_j^{\rm C}(L) + w^k A_k^j [\tilde{E}_j^{\rm C}, \hat{E}_i^{\rm V}](L) = 0.$$

On the other hand, for the last term we get

$$\hat{E}_i^{\rm C}(L) = w^k C_{ki}^j \hat{E}_j^{\rm V}(L).$$

The Euler-Lagrange equations, adapted to the frame $\{\hat{E}_i\}$, are therefore

$$\Gamma^k \hat{E}_k^{\rm V} \hat{E}_i^{\rm V}(L) = w^k C_{ki}^j \hat{E}_j^{\rm V}(L).$$

Notice that when L is globally defined and smooth, and regular, Γ must vanish when $w^k = 0$, that is, on the zero section of TG. So a necessary condition for an invariant second-order differential equation Γ to be derivable from a global regular invariant Lagrangian (or even one smooth and regular in a neighbourhood of the zero section) is that Γ should vanish on the zero section. In fact this is just the requirement that $b_i = 0$ in Theorem 1, since

$$b_i = \tilde{E}_i^{\rm C}(L)|_{w^k=0} = \Gamma_{w^k=0}(\tilde{E}_i^{\rm V}(L)). \tag{8}$$

Let $l \in C^\infty(\mathfrak{g})$ be the restriction of the left-invariant Lagrangian $L \in C^\infty(TG)$ to the Lie algebra. Then the restriction of $\hat{E}_k^{\mathrm{V}}(L)$ to $\mathfrak{g}$ is $\partial l/\partial w^k$, and so on. The defining relation for the reduced vector field $\gamma \in \mathfrak{X}(\mathfrak{g})$ of Γ is therefore

$$\gamma\Big(\frac{\partial l}{\partial w^l}\Big) = C^j_{ml} w^m \frac{\partial l}{\partial w^j}. \tag{9}$$

These are the so-called Euler-Poincaré equations [14].

Evidently if l is globally defined, smooth and regular on $\mathfrak{g}$ then γ must vanish at the origin (this is the counterpart of the property of Γ noted above). So for a vector field γ on $\mathfrak{g}$ to be derivable via the Euler-Poincaré equations from a smooth and regular (reduced) Lagrangian it is necessary that $\gamma(0) = 0$. We will come back to this point later.

The Euler-Poincaré equations should be interpreted as differential equations with solution $w(t)$ in the Lie algebra. We have chosen the coordinates (w^i) in such a way that they are not only the coordinates for $w = w^i E_i$ in $\mathfrak{g}$, but also the fibre coordinates of any translate $v_g = T\lambda_g w \in T_g G$. To find the solution $(g(t), \dot{g}(t)) \in TG$ of the Euler-Lagrange equations that corresponds to $w(t)$, one simply needs to integrate the equation $g^{-1}(t)\dot{g}(t) = w(t)$.

So far as the inverse problem is concerned, we can use the foregoing analysis to reduce the problem to one on $\mathfrak{g}$, as set out in the following theorem.

Theorem 2. *Let Γ be an invariant second-order differential equation field on a Lie group G, and γ the corresponding reduced vector field on $\mathfrak{g}$. Then Γ admits a regular invariant Lagrangian L on TG if and only if γ admits a regular Lagrangian on $\mathfrak{g}$, in the sense that there is a smooth function l whose Hessian is non-singular, such that γ is the vector field uniquely determined by the Euler-Poincaré equations of l.*

Proof. Clearly, if L is a regular invariant Lagrangian for Γ, its restriction l to $\mathfrak{g}$ is a regular Lagrangian for γ. Conversely, suppose that l is a regular Lagrangian for γ on $\mathfrak{g}$, and let L be the unique invariant function on TG which agrees with l on $T_eG = \mathfrak{g}$. Consider the functions

$$\varphi_i = \Gamma^k \hat{E}_k^{\mathrm{V}} \hat{E}_i^{\mathrm{V}}(L) - w^k C^j_{ki} \hat{E}_j^{\mathrm{V}}(L),$$

where $\Gamma = w^k \hat{E}_k^{\mathrm{C}} + \Gamma^k \hat{E}_k^{\mathrm{V}}$. We showed earlier that Γ^k is invariant, and so is w^k. Since $\tilde{E}_i^{\mathrm{C}}$ commutes with $\hat{E}_j^{\mathrm{V}}$, both $\hat{E}_j^{\mathrm{V}}(L)$ and $\hat{E}_k^{\mathrm{V}}\hat{E}_i^{\mathrm{V}}(L)$ are invariant. So φ_i is invariant. But the restriction of φ_i to $\mathfrak{g}$ vanishes, by the Euler-Poincaré equations; so φ_i vanishes everywhere on TG. But as we showed earlier, the vanishing of φ_i is equivalent to the Euler-Lagrange equations for L. Moreover, L is regular since l is. Thus L is a regular invariant Lagrangian and Γ is its Euler-Lagrange field. $\square$

5 The reduced Helmholtz conditions

In this section we will show that in the case of an invariant Lagrangian, not only the Euler-Lagrange equations, but also the Helmholtz conditions can be restated as conditions at the level of the Lie algebra.

Recall that we interpret the Hessian of a Lagrangian as a type (0,2) tensor field g along the tangent bundle projection $\tau : TG \to G$. Due to the invariance of the Lagrangian the coefficients

$K_{ij} = \hat{E}_i^{\mathrm{V}} \hat{E}_j^{\mathrm{V}}(L) = g(\hat{E}_i, \hat{E}_j)$ will also be invariant functions. In what follows we will denote the restrictions of these functions to $\mathfrak{g}$ by k_{ij}.

Let us now evaluate the Helmholtz conditions, which we have stated in a coordinate free way in the second section, in the basis $\{\hat{E}_i\}$. The first conditions are simply

$$\det(K_{ij}) \neq 0, \qquad K_{ij} = K_{ji}. \tag{10}$$

The Jacobi endomorphism and the dynamical derivative are determined by the horizontal structure on TG. Since $[\Gamma, \hat{E}_i^{\mathrm{V}}] = -\hat{E}_i^{\mathrm{C}} + (w^j C_{ji}^k - \hat{E}_i^{\mathrm{V}}(\Gamma^k))\hat{E}_k^{\mathrm{V}}$, it is easy to see that the horizontal lift of $\hat{E}_i$ is

$$\hat{E}_i^{\mathrm{H}} = \hat{E}_i^{\mathrm{C}} + \tfrac{1}{2}\left(-w^j C_{ji}^k + \hat{E}_i^{\mathrm{V}}(\Gamma^k)\right)\hat{E}_k^{\mathrm{V}} = \hat{E}_i^{\mathrm{C}} - \Lambda_i^k \hat{E}_k^{\mathrm{V}}$$

say. Now both $\tilde{E}_i^{\mathrm{C}}(w^j) = 0$ and $\tilde{E}_i^{\mathrm{C}}(\Gamma^j) = 0$, so all $\tilde{E}_i^{\mathrm{H}}$ are invariant. From now on

$$\lambda_i^k = -\tfrac{1}{2}\left(\frac{\partial \gamma^k}{\partial w^i} - w^j C_{ji}^k\right)$$

denotes the restriction of the invariant function Λ_i^k to $\mathfrak{g}$. It is easy to see that the horizontal lift of an invariant vector field along τ is invariant, and vice versa.

We next consider the dynamical covariant derivative ∇. We have $[\Gamma, \hat{E}_i^{\mathrm{V}}] = -\hat{E}_i^{\mathrm{H}} + (\nabla \hat{E}_i)^{\mathrm{V}}$. Now both Γ and $\hat{E}_i^{\mathrm{V}}$ are invariant, so by the Jacobi identity $[\Gamma, \hat{E}_i^{\mathrm{V}}]$ must be invariant also. Since the horizontal part of the bracket, $-\hat{E}_i^{\mathrm{H}}$, is invariant, $(\nabla \hat{E}_i)^{\mathrm{V}}$ and therefore $(\nabla \hat{E}_i)$ must be invariant in turn. In general, if $X = X^i \hat{E}_i \in \mathfrak{X}(\tau)$ is invariant, then $\nabla X = X^i \nabla \hat{E}_i + \Gamma(X^i)\hat{E}_i$ is also invariant. We may summarize this result by saying that ∇ itself is invariant. Furthermore, the coefficients of ∇ with respect to the invariant basis are invariant functions, which can be reduced to functions on $\mathfrak{g}$. In fact we can calculate $[\Gamma, \hat{E}_i^{\mathrm{V}}]$ explicitly, obtaining $[\Gamma, \hat{E}_i^{\mathrm{V}}] = -\hat{E}_i^{\mathrm{H}} + \Lambda_i^k \hat{E}_k^{\mathrm{V}}$, so that

$$\nabla \hat{E}_i = \tfrac{1}{2}(w^j C_{ji}^k - \hat{E}_i^{\mathrm{V}}(\Gamma^k))\hat{E}_k = \Lambda_i^k \hat{E}_k,$$

and the coefficients are just the functions Λ_i^k which we know already to be invariant.

Given that $\tilde{E}_k^{\mathrm{C}}(K_{ij}) = 0$ the Helmholtz condition $\nabla g = 0$, when evaluated on the pair $(\hat{E}_i, \hat{E}_j)$, gives

$$\Gamma^k \hat{E}_k^{\mathrm{V}}(K_{ij}) - K_{kj}\Lambda_i^k - K_{ik}\Lambda_j^k = 0. \tag{11}$$

The components of the Jacobi endomorphism with respect to the current basis can be calculated from $[\Gamma, \hat{E}_j^{\mathrm{H}}]$. One finds that

$$\begin{aligned} \Phi(\hat{E}_j) \;=\; & \Big(\tfrac{1}{2}\Gamma^i \hat{E}_i^{\mathrm{V}} \hat{E}_j^{\mathrm{V}}(\Gamma^l) + \tfrac{1}{2}\Gamma^i C_{ij}^l - \tfrac{1}{4}\hat{E}_j^{\mathrm{V}}(\Gamma^i)\hat{E}_i^{\mathrm{V}}(\Gamma^l) \\ & - \tfrac{3}{4}C_{ij}^k w^i \hat{E}_k^{\mathrm{V}}(\Gamma^l) + \tfrac{1}{4}w^i C_{ik}^l \hat{E}_j^{\mathrm{V}}(\Gamma^k) - \tfrac{1}{4}w^m w^n C_{mj}^k C_{nk}^l\Big)\hat{E}_l = \Phi_j^l \hat{E}_l. \end{aligned}$$

Again, the coefficients Φ_j^l are invariant functions, and restrict to functions on $\mathfrak{g}$ given by

$$\phi_j^l = \tfrac{1}{2}\gamma^i \frac{\partial^2 \gamma^l}{\partial w^i \partial w^j} + \tfrac{1}{2}\gamma^i C_{ij}^l - \tfrac{1}{4}\frac{\partial \gamma^i}{\partial w^j}\frac{\partial \gamma^l}{\partial w^i} - \tfrac{3}{4}C_{ij}^k w^i \frac{\partial \gamma^l}{\partial w^k} + \tfrac{1}{4}w^i C_{ik}^l \frac{\partial \gamma^k}{\partial w^j} - \tfrac{1}{4}w^m w^n C_{mj}^k C_{nk}^l.$$

This somewhat uncouth-looking formula can be civilized by expressing it in terms of the quantities

$$\psi_j^i = \tfrac{1}{2}\left(\frac{\partial \gamma^i}{\partial w^j} + C_{kj}^i w^k\right),$$

when it becomes

$$\phi^l_j = \gamma(\psi^l_j) - w^k C^i_{kj}\psi^l_i + w^k C^l_{ki}\psi^i_j - \psi^k_j\psi^l_k.$$

Again, for any invariant X, $\Phi(X)$ is an invariant vector field along τ. The Helmholtz condition involving the Jacobi endomorphism is simply

$$K_{ij}\Phi^i_k = K_{ik}\Phi^i_j. \tag{12}$$

Finally, the $\mathrm{D^V}$-condition is

$$\hat{E}^{\mathrm{V}}_l(K_{ij}) = \hat{E}^{\mathrm{V}}_i(K_{lj}). \tag{13}$$

The conditions (10), (11), (12) and (13) are all invariant; it is therefore enough to find a solution $k_{ij} \in C^\infty(\mathfrak{g})$ of the restriction of these conditions to $\mathfrak{g} = T_eG$, which may be called the reduced Helmholtz conditions. The solution of the full conditions on TG can then be found by left translating the solution on $\mathfrak{g}$.

For any $\gamma = \gamma^i\partial/\partial w^i \in \mathfrak{X}(\mathfrak{g})$, we call a matrix (k_{ij}) of functions on $\mathfrak{g}$ a multiplier matrix for γ if it satisfies the reduced Helmholtz conditions

$$\det(k_{ij}) \neq 0, \qquad k_{ij} = k_{ji},$$

$$\gamma^k\frac{\partial k_{ij}}{\partial w^k} - k_{kj}\lambda^k_i - k_{ik}\lambda^k_j = 0,$$

$$k_{ij}\phi^i_k = k_{ik}\phi^i_j,$$

$$\frac{\partial k_{ij}}{\partial w^l} = \frac{\partial k_{lj}}{\partial w^i}.$$

We have shown

Theorem 3. *Suppose given an invariant second-order differential equation field Γ, with reduced vector field γ. Then there is an invariant multiplier matrix (K_{ij}) for Γ on TG if and only there is a multiplier matrix (k_{ij}) for γ on $\mathfrak{g}$.*

6 The reduced inverse problem

Theorem 2 shows that the problem of finding an invariant regular Lagrangian for an invariant second-order differential equation field on TG can be reduced to that of finding a regular Lagrangian for the reduced vector field on $\mathfrak{g}$. From Theorem 3 we can infer that the existence of a multiplier matrix, that is, a solution of the reduced Helmholtz conditions, for the reduced vector field on $\mathfrak{g}$ is a necessary condition for it to admit a Lagrangian. However, as we know, the relationship between Helmholtz conditions and Lagrangians in the invariant inverse problem is a little more complicated than is the case for the ordinary inverse problem. While the existence of a multiplier matrix on $\mathfrak{g}$ is sufficient to guarantee the existence of an invariant multiplier matrix on TG, the existence of an invariant multiplier matrix on TG is not sufficient to guarantee the existence of an invariant Lagrangian on TG. The following theorem supplies in effect the extra conditions, working now entirely in terms of reduced quantities on $\mathfrak{g}$.

Theorem 4. *A multiplier matrix for $\gamma \in \mathfrak{X}(\mathfrak{g})$ determines a cohomology class in $H^1(\mathfrak{g})$ and one in $H^2(\mathfrak{g})$. The vector field γ is derivable from a Lagrangian if and only if the corresponding cohomology classes vanish.*

Proof. Suppose the functions k_{ij} on $\mathfrak{g}$ satisfy the reduced Helmholtz conditions, so that (k_{ij}) is a multiplier matrix. From the last of the Helmholtz conditions,

$$\frac{\partial k_{ik}}{\partial w^j} = \frac{\partial k_{ij}}{\partial w^k},$$

and the assumed symmetry of k_{ij} in its indices, it follows that there is a function l on $\mathfrak{g}$ such that

$$k_{ij} = \frac{\partial^2 l}{\partial w^i \partial w^j};$$

l is determined up to the addition of a term linear in the w^k (and the addition of a constant, but this we can ignore). Then

$$\frac{\partial}{\partial w^i}\left(\gamma\left(\frac{\partial l}{\partial w^j}\right) - C^l_{kj} w^k \frac{\partial l}{\partial w^l}\right) = \gamma^k \frac{\partial k_{ij}}{\partial w^k} + \frac{\partial \gamma^k}{\partial w^i} k_{jk} - C^k_{ij}\frac{\partial l}{\partial w^k} - C^l_{kj} w^k k_{il}.$$

Let us denote the term in brackets on the left-hand side (whose vanishing is the Euler-Poincaré equations) by V_j. Then the Helmholtz condition

$$\gamma^k \frac{\partial k_{ij}}{\partial w^k} - k_{kj}\lambda^k_i - k_{ik}\lambda^k_j = 0$$

is equivalent to

$$\frac{\partial V_i}{\partial w^j} + \frac{\partial V_j}{\partial w^i} = 0.$$

It follows that

$$\frac{\partial^2 V_i}{\partial w^j \partial w^k} = -\frac{\partial^2 V_j}{\partial w^i \partial w^k} = \frac{\partial^2 V_k}{\partial w^i \partial w^j} = -\frac{\partial^2 V_i}{\partial w^k \partial w^j},$$

whence

$$\frac{\partial^2 V_i}{\partial w^j \partial w^k} = 0.$$

But this says that there are constants μ_{ij} and ν_i, the μ_{ij} being skew in their indices, such that

$$V_i = \gamma^k \frac{\partial^2 l}{\partial w^i \partial w^k} - C^l_{ki} w^k \frac{\partial l}{\partial w^l} = \mu_{ji} w^j + \nu_i. \tag{14}$$

As before, we can regard the ν_i as the coefficients of a linear map $\nu : \mathfrak{g} \to \mathbf{R}$, and the μ_{ij} as the coefficients of an alternating bilinear map $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$. We now show that ν and μ satisfy the cocycle conditions $\nu_l C^l_{ij} = 0$ and $\mu_{il} C^l_{jk} + \mu_{jl} C^l_{ki} + \mu_{kl} C^l_{ij} = 0$. In fact these conditions hold by virtue of the Helmholtz conditions, especially the condition $k_{il}\phi^l_j = k_{jl}\phi^l_i$.

Now μ_{ij} is half of the skew part of

$$\frac{\partial V_j}{\partial w^i} = \gamma^k \frac{\partial k_{ij}}{\partial w^k} + \frac{\partial \gamma^k}{\partial w^i} k_{jk} - C^k_{ij}\frac{\partial l}{\partial w^k} - C^l_{kj} w^k k_{il},$$

so that

$$\mu_{ij} = \tfrac{1}{2}\left(\frac{\partial \gamma^l}{\partial w^i} + C^l_{ki} w^k\right) k_{jl} - \tfrac{1}{2}\left(\frac{\partial \gamma^l}{\partial w^j} + C^l_{kj} w^k\right) k_{il} - C^k_{ij}\frac{\partial l}{\partial w^k}.$$

Earlier, we set

$$\tfrac{1}{2}\left(\frac{\partial \gamma^l}{\partial w^i} + C^l_{ki} w^k\right) = \psi^l_i;$$

we now put

$$\chi_{ij} = \psi_i^l k_{jl} - \psi_j^l k_{il} = \mu_{ij} + C_{ij}^k \frac{\partial l}{\partial w^k}.$$

We now consider $\phi_i^l k_{jl} - \phi_j^l k_{il}$, where (as we showed earlier)

$$\phi_i^l = \gamma(\psi_i^l) - w^k C_{ki}^j \psi_j^l + w^k C_{kj}^l \psi_i^j - \psi_i^k \psi_k^l.$$

We look first at the terms in $\phi_i^l k_{jl} - \phi_j^l k_{il}$ which involve $\gamma(\psi_i^l)$: these are

$$\gamma(\psi_i^l)k_{jl} - \gamma(\psi_j^l)k_{il} = \gamma(\chi_{ij}) - \psi_i^l \gamma(k_{jl}) + \psi_j^l \gamma(k_{il}).$$

We substitute for the $\gamma(k_{il})$ terms from the appropriate Helmholtz condition, and find in the end that

$$\phi_i^l k_{jl} - \phi_j^l k_{il} = \gamma(\chi_{ij}) + w^k C_{ki}^l \chi_{jl} - w^k C_{kj}^l \chi_{il} = 0.$$

Now since μ_{ij} is constant,

$$\gamma(\chi_{ij}) = C_{ij}^l \gamma\left(\frac{\partial l}{\partial w^l}\right) = C_{ij}^l \nu_l + C_{ij}^l \left(\mu_{kl} + C_{kl}^m \frac{\partial l}{\partial w^m}\right) w^k,$$

so that

$$\begin{aligned}
0 &= \gamma(\chi_{ij}) + (\chi_{il} C_{jk}^l + \chi_{jl} C_{ki}^l) w^k \\
&= C_{ij}^l \nu_l + C_{ij}^l \left(\mu_{kl} + C_{kl}^m \frac{\partial l}{\partial w^m}\right) w^k + (\chi_{il} C_{jk}^l + \chi_{jl} C_{ki}^l) w^k \\
&= C_{ij}^l \nu_l + (\mu_{kl} C_{ij}^l + \mu_{il} C_{jk}^l + \mu_{jl} C_{ki}^l) w^k + (C_{ij}^l C_{kl}^m + C_{jk}^l C_{il}^m + C_{ki}^l C_{jl}^m) w^k \frac{\partial l}{\partial w^m} \\
&= C_{ij}^l \nu_l + (\mu_{il} C_{jk}^l + \mu_{jl} C_{ki}^l + \mu_{kl} C_{ij}^l) w^k.
\end{aligned}$$

Since this expression is affine in w^k with constant coefficients, these coefficients must vanish. Therefore both $C_{ij}^l \nu_l = 0$ and $\mu_{il} C_{jk}^l + \mu_{jl} C_{ki}^l + \mu_{kl} C_{ij}^l = 0$, as required.

If we change l to $l' = l + \theta_k w^k$, the corresponding change in the cocycles is from (ν, μ) to (ν', μ') where $\nu'_i = \nu_i$ and $\mu'_{ij} = \mu_{ij} - \theta_k C_{ij}^k$, or $\nu' = \nu$ and $\mu'(\xi, \eta) = \mu(\xi, \eta) - \theta(\{\xi, \eta\})$. That is, both components of (ν, ν') and (μ, μ') belong to the same cohomology class, respectively. If the cohomology classes of ν and μ vanish then we can find θ such that $l' = l + \theta_k w^k$ is a Lagrangian. □

By setting $w^i = 0$ in Equation (14) we see that

$$\nu_i = \gamma^k(0) \frac{\partial^2 l}{\partial w^i \partial w^k}(0).$$

But as we pointed out earlier, it is a necessary condition for γ to be derivable from a Lagrangian l on $\mathfrak{g}$ that $\gamma(0) = 0$. The significance of the vanishing of ν as a condition for γ to be derivable from a Lagrangian is clear.

We have derived two sets of conditions for the existence of an invariant Lagrangian, each involving a pair of cohomology classes. One would hope that the two pairs of cohomology classes are the same. This is in fact the case, as we now show.

First we show that b_i and ν_i are the same constants. From Equation (8) we have

$$b_i = \Gamma_{w^k=0}(\tilde{E}_i^{\mathrm{V}}(L)) = (\Gamma^j \hat{E}_j^{\mathrm{V}} \tilde{E}_i^{\mathrm{V}}(L))|_{w^k=0}.$$

Since b_i is constant it is enough to evaluate the right-hand side at e; here the distinction between $\tilde{E}_i^{\mathrm{V}}$ and $\hat{E}_i^{\mathrm{V}}$ disappears, and we obtain

$$b_i = \gamma^k(0) \frac{\partial^2 l}{\partial w^i \partial w^k}(0) = \nu_i.$$

To find the relationship between α_{ij} and μ_{ij} it turns out to be convenient to work entirely in terms of the right-invariant fields $\tilde{E}_i$; in the end we will evaluate everything at e, using the constancy of the α_{ij}, and again we can take advantage of the fact that at e the distinction between $\tilde{E}_i$ and $\hat{E}_i$ disappears. The relations between the complete and vertical lifts of $\tilde{E}_i$ and $\hat{E}_i$ given in Equation (7) now come into play. In particular, $\Gamma = w^j A_i^j \tilde{E}_i^{\mathrm{C}} + \Gamma^j A_i^j \tilde{E}_i^{\mathrm{V}}$; we set $A_j^i w^j = v^i$, $A_j^i \Gamma^j = \tilde{\Gamma}^i$.

It follows from Equation (4) that $\hat{E}_i(f_j) = \hat{E}_i^{\mathrm{V}}(\tilde{E}_j^{\mathrm{C}}(L))$, whence

$$\tilde{E}_i(f_j) = \tilde{E}_i^{\mathrm{V}}(\tilde{E}_j^{\mathrm{C}}(L)) = \tilde{E}_i^{\mathrm{V}}(\Gamma(\tilde{E}_j^{\mathrm{V}}(L))).$$

Using the expression above for Γ, and the evident fact that $\tilde{E}_i^{\mathrm{V}}(v^j) = \delta_i^j$, we find that

$$\begin{aligned}
&\tilde{E}_i^{\mathrm{C}}(\tilde{E}_j^{\mathrm{V}}(L)) + v^k \tilde{E}_i^{\mathrm{V}}(\tilde{E}_k^{\mathrm{C}}(\tilde{E}_j^{\mathrm{V}}(L))) \\
&\quad + \tilde{E}_i^{\mathrm{V}}(\tilde{\Gamma}^k)\tilde{E}_k^{\mathrm{V}}(\tilde{E}_j^{\mathrm{V}}(L)) + \tilde{\Gamma}^k \tilde{E}_k^{\mathrm{V}}(\tilde{E}_i^{\mathrm{V}}(\tilde{E}_j^{\mathrm{V}}(L))) - \tilde{E}_i^{\mathrm{V}}(\tilde{E}_j^{\mathrm{C}}(L)) = 0,
\end{aligned}$$

or (remembering that $[\tilde{E}_i, \tilde{E}_j] = -C_{ij}^k \tilde{E}_k$)

$$\begin{aligned}
&\tilde{E}_i^{\mathrm{V}}(\tilde{E}_j^{\mathrm{C}}(L)) - \tilde{E}_j^{\mathrm{V}}(\tilde{E}_i^{\mathrm{C}}(L)) + C_{ij}^k \tilde{E}_k^{\mathrm{V}}(L) \\
&\quad = \Gamma(\tilde{E}_i^{\mathrm{V}}(\tilde{E}_j^{\mathrm{V}}(L)) - C_{ik}^l v^k \tilde{E}_l^{\mathrm{V}}(\tilde{E}_j^{\mathrm{V}}(L)) + \tilde{E}_i^{\mathrm{V}}(\tilde{\Gamma}^k)\tilde{E}_k^{\mathrm{V}}(\tilde{E}_j^{\mathrm{V}}(L)).
\end{aligned}$$

On taking the skew part we find that

$$\begin{aligned}
&\tilde{E}_i^{\mathrm{V}}(\tilde{E}_j^{\mathrm{C}}(L)) - \tilde{E}_j^{\mathrm{V}}(\tilde{E}_i^{\mathrm{C}}(L)) + C_{ij}^k \tilde{E}_k^{\mathrm{V}}(L) \\
&\quad = \tfrac{1}{2}\left(\tilde{E}_i^{\mathrm{V}}(\tilde{\Gamma}^k) + C_{li}^k v^l\right)\tilde{E}_k^{\mathrm{V}}(\tilde{E}_j^{\mathrm{V}}(L)) - \tfrac{1}{2}\left(\tilde{E}_j^{\mathrm{V}}(\tilde{\Gamma}^k) + C_{lj}^l v^l\right)\tilde{E}_k^{\mathrm{V}}(\tilde{E}_i^{\mathrm{V}}(L)).
\end{aligned}$$

The left-hand side is $\alpha_{ij} + C_{ij}^k(\tilde{E}_k^{\mathrm{V}}(L) - f_k)$. At e, the right-hand side is

$$\tfrac{1}{2}\left(\frac{\partial \gamma^k}{\partial w^i} + C_{li}^k w^l\right)k_{jk} - \tfrac{1}{2}\left(\frac{\partial \gamma^k}{\partial w^j} + C_{lj}^k w^l\right)k_{ik} = \psi_i^k k_{jk} - \psi_j^k k_{ik} = \chi_{ij}.$$

Thus

$$\alpha_{ij} = \chi_{ij} - C_{ij}^k(\tilde{E}_k^{\mathrm{V}}(L) - f_k)|_e.$$

Now let l be the restriction of L to T_eG. Of course L is not assumed to be invariant, so this differs from the association between l and L given earlier; nevertheless, it is true that

$$\frac{\partial^2 l}{\partial w^i \partial w^j} = k_{ij},$$

where (k_{ij}) satisfies the reduced Helmholtz conditions. So we can write $\alpha_{ij} = \mu_{ij} + C_{ij}^k f_k(e)$. It is apparent that α_{ij} and μ_{ij} define the same cohomology class (they differ by a coboundary).

7 The Lie algebroid

Our policy while working on TG in earlier sections was to write everything in terms of G-invariant quantities, that is, quantities determined by their values on $T_eG = \mathfrak{g}$. This paves the way towards expressing the whole theory in terms of $\mathfrak{g}$, or more accurately in terms of a vector bundle over $\mathfrak{g}$, namely $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$. We can identify invariant vector fields on TG, via their restrictions to T_eG, with sections of $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$, as we pointed out earlier. The bracket of two invariant vector fields remains invariant, and so the bracket of vector fields on TG determines a bracket of sections of $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$. This is evidently $\mathbf{R}$-bilinear and skew, and it satisfies the Jacobi identity by construction. We will now obtain an explicit formula for this bracket, and deduce that it is a Lie algebroid bracket, i.e. a bracket with the above properties that satisfies an appropriate Leibniz rule when sections are being multiplied with functions on the base manifold.

Let $\xi^i \hat{E}_i^{\mathrm{C}} + X^i \hat{E}_i^{\mathrm{V}}$ and $\eta^i \hat{E}_i^{\mathrm{C}} + Y^i \hat{E}_i^{\mathrm{V}}$ be two invariant vector fields, so that $\tilde{E}_j^{\mathrm{C}}(\xi^i) = \tilde{E}_j^{\mathrm{C}}(X^i) = \tilde{E}_j^{\mathrm{C}}(\eta^i) = \tilde{E}_j^{\mathrm{C}}(Y^i) = 0$. These invariance conditions, when expressed in terms of the vector fields of the invariant basis, become for example $\hat{E}_j^{\mathrm{C}}(\xi^i) = w^k C_{kj}^l \hat{E}_l^{\mathrm{V}}(\xi^i)$, using Equation (7). Thus

$$[\xi^i \hat{E}_i^{\mathrm{C}}, \eta^j \hat{E}_j^{\mathrm{C}}] = \Big(\xi^i \eta^j C_{ij}^k + \xi^i w^j C_{ji}^l \hat{E}_l^{\mathrm{V}}(\eta^k) - \eta^i w^j C_{ji}^l \hat{E}_l^{\mathrm{V}}(\xi^k)\Big) \hat{E}_k^{\mathrm{C}},$$

while

$$[\xi^i \hat{E}_i^{\mathrm{C}}, Y^j \hat{E}_j^{\mathrm{V}}] = -Y^j \hat{E}_j^{\mathrm{V}}(\xi^k) \hat{E}_k^{\mathrm{C}} + \Big(\xi^i Y^j C_{ij}^k + \xi^i w^j C_{ji}^l \hat{E}_l^{\mathrm{V}}(Y^k)\Big) \hat{E}_k^{\mathrm{V}}.$$

The bracket may be written as follows. We identify ξ, η with $\mathfrak{g}$-valued functions on $\mathfrak{g}$, X, Y with vector fields on $\mathfrak{g}$; $\bar{\xi}$ is the vector field corresponding to ξ. We think of $w^k C_{kj}^i$ as the components of a type (1,1) tensor field on $\mathfrak{g}$ which we denote by $\mathcal{A}$: thus

$$\mathcal{A} = w^k C_{kj}^i \frac{\partial}{\partial w^i} \otimes dw^j.$$

The Lie algebra bracket $\{\cdot,\cdot\}$ extends naturally to an algebraic bracket on $\mathfrak{g}$-valued functions on $\mathfrak{g}$, so that $\{\xi, \eta\} = \xi^j \eta^k C_{jk}^i E_i$. Then

$$\begin{aligned} [\![(\xi, X), (\eta, Y)]\!] \;=\; & \big(\{\xi, \eta\} + \mathcal{A}(\bar{\xi})(\eta) - \mathcal{A}(\bar{\eta})(\xi) + X(\eta) - Y(\xi), \\ & [\mathcal{A}(\bar{\xi}), Y] - [\mathcal{A}(\bar{\eta}), X] + \mathcal{A}(\overline{Y(\xi)}) - \mathcal{A}(\overline{X(\eta)}) + [X, Y]\Big). \end{aligned}$$

For any function f on $\mathfrak{g}$ we have $[\![(\xi, X), f(\eta, Y)]\!] = f[\![(\xi, X), (\eta, Y)]\!] + \rho(\xi, X)(f)(\eta, Y)$, as required, where the so-called anchor of the Lie algebroid is given by

$$\rho(\xi, X) = \mathcal{A}(\bar{\xi}) + X \in \mathfrak{X}(\mathfrak{g}).$$

Thus the bracket $[\![\cdot,\cdot]\!]$ does indeed define a Lie algebroid structure on $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$.

We denote by e_i the section $(E_i, 0)$ of $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$, and W_i the section $(0, \partial/\partial w^i)$; then $\{e_i, W_i\}$ is a basis of sections, and we have

$$[\![e_i, e_j]\!] = C_{ij}^k e_k, \quad [\![e_i, W_j]\!] = C_{ij}^k W_k, \quad [\![W_i, W_j]\!] = 0.$$

We denote by δ the induced exterior derivative operator on sections of exterior powers of the dual of the algebroid, and by $\{e^i, W^i\}$ the basis dual to $\{e_i, W_i\}$. Then for any function f on $\mathfrak{g}$,

$$\delta f = \frac{\partial f}{\partial w^i}(w^k C_{kj}^i e^j + W^i),$$

while

$$\delta e^i = -\tfrac{1}{2}C^i_{jk}e^j \wedge e^k, \quad \delta W^i = -C^i_{jk}e^j \wedge W^k.$$

Using these formulae we can express the Euler-Poincaré equations in terms of the Lie algebroid structure, as follows. The vertical endomorphism S on the Lie algebroid is just $S(\xi, X) = (0, \bar{\xi})$. The invariant Lagrangian is represented by a function l on $\mathfrak{g}$. We define, in analogy to the usual case, a Cartan form θ and an energy function $\mathcal{E}$ by

$$\theta = S(\delta l), \quad \mathcal{E} = \langle \bar{\Delta}, \delta l \rangle - l,$$

where $\Delta = w^i e_i$ and $\bar{\Delta} = w^i W_i$. As we will show by a direct calculation, provided that l is regular (in the sense that its Hessian is non-singular) the equation

$$i_\Gamma \delta\theta = -\delta\mathcal{E}$$

determines a unique section Γ, which is of second-order differential equation type, so that it takes the form $w^i e_i + \gamma^i W_i$, and the γ^i satisfy the Euler-Poincaré equations for l. In fact

$$\begin{aligned}
\theta &= \frac{\partial l}{\partial w^i} e^i \\
\delta\theta &= \left(\frac{\partial^2 l}{\partial w^j \partial w^l} w^k C^l_{ki} - \tfrac{1}{2}\frac{\partial l}{\partial w^k} C^k_{ij} \right) e^i \wedge e^j - \frac{\partial^2 l}{\partial w^i \partial w^j} e^i \wedge W^j \\
\mathcal{E} &= w^i \frac{\partial l}{\partial w^i} - l \\
\delta\mathcal{E} &= w^l \frac{\partial^2 l}{\partial w^i \partial w^l} (w^k C^i_{kj} e^j + W^i).
\end{aligned}$$

Let us write $\Gamma = \xi^i e_i + f^i W_i$. Then the vanishing of the W_i component of $i_\Gamma \delta\theta + \delta\mathcal{E}$ gives

$$(-\xi^j + w^j) \frac{\partial^2 l}{\partial w^i \partial w^j} = 0,$$

whence $\xi^i = w^i$ when l is regular. When this result is inserted in $i_\Gamma \delta\theta + \delta\mathcal{E}$ several terms cancel, and the remaining terms in the e_i component reduce to

$$\gamma^j \frac{\partial^2 l}{\partial w^i \partial w^j} - w^k C^j_{ki} \frac{\partial l}{\partial w^j},$$

as required.

The above derivation of the Euler-Poincaré equations was inspired by Martínez' framework [15] for Lagrangian systems on a Lie algebroid. This framework is based on the so-called prolongation algebroid of the underlying Lie algebroid. It should be remarked that although the underlying algebroid of the current system is just the Lie algebra $\mathfrak{g}$, the algebroid we have defined in this section does not coincide with the prolongation algebroid of the Lie algebra. The prolongation algebroid can most easily be defined as follows. Observe that both components of a section of $\mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g}$ have a natural bracket structure. For the first, we have the natural extension of the Lie algebra bracket to $C^\infty(\mathfrak{g}, \mathfrak{g})$-functions, and for the second component we have the Lie bracket of vector fields. The easiest way to combine these two into one Lie bracket structure is as follows:

$$[\![(\xi, X), (\eta, Y)]\!]^1 = (\{\xi, \eta\} + X(\eta) - Y(\xi), [X, Y]). \tag{15}$$

It is easy to check that this is a Lie algebroid whose anchor map $\rho^1 : \mathfrak{g} \times T\mathfrak{g} \to T\mathfrak{g}$ is simply the projection on the second component. In our basis

$$[\![e_i, e_j]\!]^1 = C_{ij}^k e_k, \qquad [\![e_i, W_j]\!]^1 = 0, \qquad [\![W_i, W_j]\!]^1 = 0.$$

A short calculation reveals that indeed the expression $i_\Gamma \delta^1 \theta^1 = -\delta^1 \mathcal{E}$ leads again to the Euler-Poincaré equations. It is, however, easy to guess the relationship between the two algebroid structures: the section map $(\xi, X) \mapsto (\xi, \mathcal{A}(\bar{\xi}) + X)$ is an isomorphism of the first of the Lie algebroids with the second. The more complicated structure of the first Lie algebroid (or of that presentation of the common Lie algebroid if one regards isomorphic algebroids as identical in principle) arises from our desire to work always with invariant objects on TG and objects on $\mathfrak{g}$ derived from them by restriction.

We conclude this section by showing that the restrictions of the horizontal lift, the Jacobi endomorphism and the dynamical derivative to $\mathfrak{g}$ have a direct interpretation in the first Lie algebroid.

At the level of the Lie algebra, $\mathfrak{g}$-valued functions on $\mathfrak{g}$ (sections of $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$) play the role that the vector fields along the tangent bundle projection played at the level of TG. That is, there is a well-defined vertical lift of such a function $\xi = \xi^i(w) E_i$ to the section $\xi^{\mathrm{V}} = \xi^i W_i$ of the Lie algebroid. Moreover, one can easily verify that each vector field γ on $\mathfrak{g}$ defines a horizontal lift

$$\xi^{\mathrm{H}} = \xi^i (e_i - \lambda_i^j W_j)$$

to sections of the Lie algebroid, or equivalently, a splitting of the short exact sequence

$$0 \to \{0\} \times T\mathfrak{g} \to \mathfrak{g} \times T\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g} \to 0$$

of vector bundles over $\mathfrak{g}$. Observe that the restriction to $\mathfrak{g}$ of the horizontal lift of an invariant vector field along τ is in fact the horizontal lift of the restriction to $\mathfrak{g}$ of that vector field along τ.

The functions ψ_i^j we have introduced in previous sections have a nice interpretation in the algebroid set-up. The projection by the anchor map of the horizontal algebroid section $E_i^{\mathrm{H}} = e_i - \lambda_i^j W_j$ to a vector field on $\mathfrak{g}$ is exactly

$$\rho(E_i^{\mathrm{H}}) = (w^k C_{ki}^l - \lambda_i^l) \frac{\partial}{\partial w^l} = \psi_i^l \frac{\partial}{\partial w^l}.$$

As before, we can define a Jacobi endomorphism and a dynamical derivative by considering the horizontal and vertical parts of the brackets of the algebroid section $\Gamma = w^i e_i + \gamma^i W_i$:

$$[\![\Gamma, \eta^{\mathrm{V}}]\!] = -\eta^{\mathrm{H}} + (\nabla\eta)^{\mathrm{V}}, \qquad [\![\Gamma, \eta^{\mathrm{H}}]\!] = (\nabla\eta)^{\mathrm{H}} + (\Phi(\eta))^{\mathrm{V}},$$

where ∇ acts like a derivative in the sense that for $f \in C^\infty(\mathfrak{g})$, $\nabla(f\eta) = f\nabla\eta + \gamma(f)\eta$, and Φ is tensorial with coefficients ϕ_j^i as before. We have $\nabla E_i = \lambda_i^k E_k$, and in fact in both cases the operators are nothing but the original operators restricted to $\mathfrak{g} = T_e G$.

We define the vertical derivative for $\xi \in C^\infty(\mathfrak{g}, \mathfrak{g})$ as the map $\mathrm{D}_\xi^{\mathrm{V}} : C^\infty(\mathfrak{g}, \mathfrak{g}) \to C^\infty(\mathfrak{g}, \mathfrak{g})$, determined by $\mathrm{D}_\xi^{\mathrm{V}} f = \bar{\xi}(f)$ for $f \in C^\infty(\mathfrak{g})$, $\mathrm{D}_\xi^{\mathrm{V}} \eta = 0$ for a vector η of $\mathfrak{g}$ (in other words a constant element of $C^\infty(\mathfrak{g}, \mathfrak{g})$), and the obvious Leibniz rule for multiplication by functions.

We can now restate the reduced Helmholtz conditions in a coordinate-free form. The multiplier (k_{ij}) is a matrix of functions on $\mathfrak{g}$, or equivalently a map $k : C^\infty(\mathfrak{g},\mathfrak{g}) \times C^\infty(\mathfrak{g},\mathfrak{g}) \to C^\infty(\mathfrak{g})$, which satisfies the conditions

$$\begin{aligned}
&\det k \neq 0, \quad k(\xi,\eta) = k(\eta,\xi),\\
&\nabla k = 0,\\
&k(\Phi(\eta),\zeta) = k(\eta,\Phi(\zeta)),\\
&\mathrm{D}^{\mathrm{V}}_\xi k(\eta,\zeta) = \mathrm{D}^{\mathrm{V}}_\eta k(\xi,\zeta).
\end{aligned}$$

for all $\xi,\eta,\zeta \in C^\infty(\mathfrak{g},\mathfrak{g})$.

Suppose that the Hessian of $l \in C^\infty(\mathfrak{g})$ is k. Then if $\theta = (\partial l/\partial w^i)e^i$ as before, we can define a 2-form $\delta^{\mathrm{H}}\theta$ on the algebroid by requiring it to vanish whenever one of its arguments is a vertical section (i.e. it is semi-basic) and by setting

$$\delta^{\mathrm{H}}\theta(\xi^{\mathrm{H}},\eta^{\mathrm{H}}) = \rho(\xi^{\mathrm{H}})(\theta(\eta^{\mathrm{H}})) - \rho(\eta^{\mathrm{H}})(\theta(\xi^{\mathrm{H}})) - \theta([\![\xi^{\mathrm{H}},\eta^{\mathrm{H}}]\!]).$$

Then $\delta^{\mathrm{H}}\theta = \frac{1}{2}\mu_{ij}e^i\wedge e^j$, where

$$\begin{aligned}
\mu_{ij} &= \rho(E_i^{\mathrm{H}})(\theta(E_j^{\mathrm{H}})) - \rho(E_j^{\mathrm{H}})(\theta(E_i^{\mathrm{H}})) - \theta([\![E_i^{\mathrm{H}},E_j^{\mathrm{H}}]\!])\\
&= \psi_i^l k_{lj} - \psi_j^l k_{li} - C^k_{ij}\frac{\partial l}{\partial w^k}.
\end{aligned}$$

These coefficients are exactly those we have encountered in the previous section. Recall from the proof of Theorem 4 that the reduced Helmholtz conditions ensure that the μ_{ij} are constants and that they form a cocycle. A necessary condition for a Lagrangian to exist is that μ_{ij} is a coboundary. We can now re-express this statement in terms of the Lie algebroid. For a 2-form $\mu = \delta^{\mathrm{H}}\theta$ with constant coefficients we get that $\delta\mu = -\frac{1}{2}\mu_{ij}C^i_{lk}e^l\wedge e^k\wedge e^j$. Therefore, δ-closure of the 2-form μ amounts to the cocycle condition. On the other hand, the condition that the μ_{ij} are of the form $\alpha_k C^k_{ij}$ for some α_k is equivalent to μ being exact. Similarly, the reduced Helmholtz conditions ensure that the semi-basic 1-form $\nu = i_\Gamma\delta\theta + \delta\mathcal{E} - i_\Gamma\mu$ has constant coefficients ν_i that form a cocycle, or that $\delta\nu = 0$. Theorem 4 states that ν should vanish for a Lagrangian to exist.

8 Examples and applications

The method of reduced Helmholtz conditions really comes into its own when one has to deal with any specific problem. In practice — certainly, if the following examples are representative — the cohomological conditions do not play much of a role. Where there is no invariant Lagrangian this is because the reduced Helmholtz conditions fail, often at the level of regularity. Where one is able to find a solution of the Helmholtz conditions one is usually able to integrate it by hand, and check directly for which integration constants the Euler-Poincaré equations are equivalent to the equations associated with the vector field γ.

To save space, in the following examples we will write k_{ijl} for $\partial k_{ij}/\partial w^l$, and we will implicitly assume that the conditions $k_{ij} = k_{ji}$, $k_{ijl} = k_{ilj}$ and so on are satisfied.

8.1 The canonical connection on a Lie group

The canonical connection on a Lie group is defined as a covariant derivative operator by $\nabla_X Y = \frac{1}{2}[X, Y]$, where X and Y are any two left-invariant vector fields on G. As we mentioned in the Introduction, the invariant inverse problem for the canonical connection has been studied by Muzsnay in [18]; however, he uses methods different from ours.

The connection coefficients of the canonical connection with respect to the left-invariant basis $\{\hat{E}_i\}$ of $\mathfrak{X}(G)$ are just $\frac{1}{2}C^i_{jk}$. So the coefficients Γ^i of the corresponding second-order differential equation field (the geodesic spray) are in this case $\Gamma^i = \frac{1}{2}C^i_{jk}w^j w^k = 0$. The reduced equations are therefore simply $\dot{w}^i = 0$. In fact the geodesics through the identity of G are just the 1-parameter subgroups.

If a left-invariant Lagrangian L exists, then

$$C^k_{ij}\hat{E}^{\mathrm{V}}_k(L)w^i = 0 \qquad \text{or} \qquad C^k_{ij}\frac{\partial l}{\partial w^k}w^i = 0. \tag{16}$$

In view of relation (7), L must also be right-invariant, i.e. $\hat{E}^{\mathrm{C}}_j(L) = 0$, and thus bi-invariant. At the level of the Lie algebra, this means that $l \in C^\infty(\mathfrak{g})$ will be ad-invariant. This observation is in fact Proposition 2 in [18]. Thus a Lagrangian is a function which is constant on the adjoint orbits in $\mathfrak{g}$ and whose Hessian is non-singular.

We will use our methods to investigate the invariant inverse problem for the canonical connection. The first observation is that in this case the reduced Helmholtz condition $k_{il}\phi^l_j = k_{jl}\phi^l_i$ is a consequence of the other conditions. Since $\gamma^i = 0$,

$$\gamma^k\frac{\partial k_{ij}}{\partial w^k} - k_{kj}\lambda^k_i - k_{ik}\lambda^k_j = \tfrac{1}{2}w^l(k_{kj}C^k_{li} + k_{ik}C^k_{lj}) = 0. \tag{17}$$

On the other hand, $\phi^l_j = -\frac{1}{4}w^m w^n C^k_{mj}C^l_{nk}$. But

$$w^m w^n C^k_{mj}C^l_{nk}k_{il} - w^m w^n C^k_{mi}C^l_{nk}k_{jl} = -w^m w^n C^k_{mj}C^l_{ni}k_{kl} + w^m w^n C^k_{mi}C^l_{nj}k_{kl} = 0,$$

so the condition $k_{il}\phi^l_j = k_{jl}\phi^l_i$ holds by virtue of condition (17) and the symmetry of k_{kl}.

A second general remark concerns the cohomological conditions. In this case Equation (14) reads

$$\mu_{ji}w^j + \nu_i = -C^l_{ki}w^k\frac{\partial l}{\partial w^l},$$

from which immediately $\nu_i = 0$. Moreover

$$\mu_{ij} = -C^k_{ij}\frac{\partial l}{\partial w^k}(0),$$

so μ_{ij} is a coboundary. Thus the cohomological conditions are automatically satisfied for the canonical connection. Any function l whose Hessian satisfies the reduced Helmholtz conditions and is such that $C^k_{ij}\partial l/\partial w^k(0) = 0$ will be a Lagrangian; in particular, if l satisfies the reduced Helmholtz conditions and we set

$$l' = l - w^k\frac{\partial l}{\partial w^k}(0)$$

then l' will be a Lagrangian. So the inverse problem for the canonical connection reduces essentially to the analysis of condition (17), in the form $w^l(k_{kj}C^k_{li}+k_{ik}C^k_{lj}) = 0$, and the condition

$k_{ijk} = k_{ikj}$. Where there is no Lagrangian this will often become apparent by the fact that there is no non-singular (k_{ij}) satisfying the first of these conditions.

We will examine two specific situations, one in which there is no Lagrangian, one in which there is one.

The first case is that of the Heisenberg algebra, which is a 3-dimensional algebra with the single non-trivial bracket relation $\{E_1, E_3\} = E_2$. Condition (17) amounts simply to

$$k_{12}w^3 = k_{22}w^3 = k_{22}w^1 = k_{32}w^1 = -k_{12}w^1 + k_{32}w^3 = 0.$$

Evidently $k_{12} = k_{22} = k_{32} = 0$, and there is no non-singular 3×3 matrix (k_{ij}) satisfying the Helmholtz conditions.

For our second example we take the 4-dimensional Lie algebra with bracket relations

$$\{E_2, E_3\} = E_1, \qquad \{E_2, E_4\} = E_2, \qquad \{E_3, E_4\} = -E_3$$

(this is the algebra $A_{4,8}$ in the classification of Patera et al. [19]). Condition (17) says in this case that the matrix

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix} \begin{bmatrix} 0 & w^3 & -w^2 & 0 \\ 0 & w^4 & 0 & -w^2 \\ 0 & 0 & -w^4 & w^3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

must be skew-symmetric. This leads to the following 7 independent equations for the 10 unknowns k_{ij} (with $i \leq j$):

$$\begin{aligned} k_{11}w^3 + k_{12}w^4 &= 0 = k_{11}w^2 + k_{13}w^4, \\ k_{12}w^3 + k_{22}w^4 &= 0 = k_{13}w^2 + k_{33}w^4, \\ k_{24}w^2 + k_{34}w^3 &= 0, \\ k_{22}w^2 - k_{24}w^4 &= (k_{14} + k_{23})w^3 \\ k_{33}w^3 - k_{34}w^4 &= (k_{14} + k_{23})w^2. \end{aligned}$$

Evidently k_{44} is unconstrained by these equations. It turns out that $k_{24} = k_{34} = 0$. The remaining unknowns can conveniently be expressed in terms of k_{11} and k_{23}. If for convenience we set $k_{11} = (w^4)^2 F$ (for $w^4 \neq 0$), $k_{23} = G$ and $k_{44} = H$ then (k_{ij}) is

$$\begin{bmatrix} (w^4)^2 F & -w^3 w^4 F & -w^2 w^4 F & w^2 w^3 F - G \\ -w^3 w^4 F & (w^3)^2 F & G & 0 \\ -w^2 w^4 F & G & (w^2)^2 F & 0 \\ w^2 w^3 F - G & 0 & 0 & H \end{bmatrix}.$$

We next look at the conditions $k_{ijk} = k_{ikj}$. From $k_{124} = k_{241} = 0$ we find that

$$w^3 \frac{\partial (w^4 F)}{\partial w^4} = 0.$$

From $k_{224} = k_{242} = 0$ we obtain

$$(w^3)^2 \frac{\partial F}{\partial w^4} = 0.$$

It follows that $F = 0$, except possibly where $w^3 = 0$ or $w^4 = 0$. Thus $k_{11} = 0$, except possibly where $w^3 = 0$ or $w^4 = 0$; but then by continuity $k_{11} = 0$ everywhere; and similarly for the other coefficients involving F. We are left with

$$\begin{bmatrix} 0 & 0 & 0 & -G \\ 0 & 0 & G & 0 \\ 0 & G & 0 & 0 \\ -G & 0 & 0 & H \end{bmatrix}.$$

This is evidently non-singular provided that G is non-zero, whatever H may be. Continuing to analyse the consequences of the condition $k_{ijk} = k_{ikj}$ we find that G must be constant and H must be a function of w^4 alone. This gives as potential Lagrangians

$$l(w^1, w^2, w^3, w^4) = \lambda(w^2w^3 - w^1w^4) + \alpha_1 w^1 + \alpha_2 w^2 + \alpha_3 w^3 + h(w^4)$$

where λ and the α s are constants with λ non-zero, and h is an arbitrary smooth function of its argument. According to the general remarks made earlier, l will in fact be a Lagrangian if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (h doesn't play a role here because $C^4_{ij} = 0$). It is easy to check this directly. In fact for the potential Lagrangian above it is easy to see by direct calculation that all ν_i and almost all μ_{ij} vanish, except that $\mu_{23} = -\alpha_1$, $\mu_{24} = -\alpha_2$ and $\mu_{34} = \alpha_3$ (and their skew counterparts). We have shown that there will exist a Lagrangian $l' = l + \theta_k w^k$ whose Euler-Poincaré equations are exactly the equations associated to γ if we can find θ_k such that $\mu_{ij} = \theta_k C^k_{ij}$. One easily verifies that this condition is only satisfied for $\theta_k = -\alpha_k$. The sought-for Lagrangian l' is therefore the one above where one sets $\alpha_k = 0$.

It is interesting to note that the most general Lagrangian in this case is not just a quadratic form.

The method used by Muzsnay in [18] deals directly with the equation

$$w^j C^k_{ij} \frac{\partial l}{\partial w^k} = 0$$

as a set of partial differential equations for l. In effect, Muzsnay derives an integrability condition for this equation by differentiating it, to obtain

$$C^k_{ij} \frac{\partial l}{\partial w^k} + w^l C^k_{il} \frac{\partial^2 l}{\partial w^j \partial w^k} = 0.$$

The part of this equation symmetric in i and j is our Helmholtz condition (17), the skew part states that the cocycle μ_{ij} must vanish. The examples we have considered above are two of the many examples dealt with in [18]. Muzsnay's results are of course broadly the same as ours; however, in the second case though he shows that a Lagrangian exists he does not indicate how to find one, whereas we have obtained the most general one. As Muzsnay points out, the example is also treated in [10]. By using only the unreduced Helmholtz conditions on TG, the authors of [10] look for a (not necessarily invariant) Lagrangian L for the canonical geodesic flow on any 4 dimensional Lie group. Although they are not able to give an expression of the most general Lagrangian, they observe in the case of the Lie algebra $\mathrm{A}_{4,8}$ that the quadratic part of the Lagrangian above (written in terms of invariant forms on G in their set-up) generates the flow of the canonical connection. They also notice that the quadratic part is a bi-invariant metric (as the theory predicts).

8.2 The Bloch-Iserles equations

These equations appear in e.g. [3, 4]. The space of interest is Sym(n), the linear space of symmetric $n \times n$ matrices. The equation is

$$\dot{w} = [w^2, N], \tag{18}$$

where $w \in \mathrm{Sym}(n)$, N is a skew-symmetric $n \times n$ matrix, and the right-hand side is the commutator of matrices. With the help of N one can give Sym(n) the structure of a Lie algebra, the Lie algebra bracket being

$$\{w_1, w_2\} = w_1 N w_2 - w_2 N w_1, \qquad w_1, w_2 \in \mathrm{Sym}(n).$$

Can we find a Lagrangian $l \in C^\infty(\mathrm{Sym}(n))$ for which Equation (18) is of Euler-Poincaré type with respect to the above Lie algebra? The answer is in fact given in [4]: a corresponding Lagrangian is

$$l(w) = \tfrac{1}{2}\,\mathrm{tr}(w^2). \tag{19}$$

We will show that the reduced Helmholtz conditions, applied to the current Lie algebra and dynamical system, lead to the correct Lagrangian.

To make things more accessible we will consider only the case $n = 2$. For a basis of the Lie algebra we take the matrices

$$E_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad E_z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Further, without loss of generality we can take N to be

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The non-vanishing Lie algebra brackets are then $\{E_x, E_y\} = 2E_z$, $\{E_x, E_z\} = E_y$ and $\{E_y, E_z\} = 2E_z$. An arbitrary element of the Lie algebra is of the form

$$w = xE_x + yE_y + zE_z = \begin{bmatrix} x & y \\ y & z \end{bmatrix},$$

and Equation (18) is

$$\begin{bmatrix} \dot{x} & \dot{y} \\ \dot{y} & \dot{z} \end{bmatrix} = \begin{bmatrix} -2y(x+z) & x^2 - z^2 \\ x^2 - z^2 & 2y(x+z) \end{bmatrix}.$$

We use now the notation of the Lie algebroid formulation of the Helmholtz conditions from Section 7. For Φ we find

$$\begin{aligned} \Phi(E_x) &= (-3y^2 + \tfrac{1}{2}z^2)E_x + (\tfrac{3}{2}xy - 2yz)E_y + (4y^2 - \tfrac{1}{2}xz)E_z, \\ \Phi(E_y) &= (3xy - 4yz)E_x + (4xz - \tfrac{3}{2}x^2 - \tfrac{3}{2}z^2)E_y + (3yz - 4xy)E_z, \\ \Phi(E_z) &= (4y^2 - \tfrac{1}{2}xz)E_x + (\tfrac{3}{2}yz - 2xy)E_y + (-3y^2 + \tfrac{1}{2}x^2)E_z, \end{aligned}$$

and for ∇

$$\begin{aligned} \nabla E_x &= -(x + \tfrac{1}{2}z)E_y - yE_z, \\ \nabla E_y &= (2x + z)E_x - (x + 2z)E_z, \\ \nabla E_z &= yE_x + (z + \tfrac{1}{2}x)E_y. \end{aligned}$$

The ∇-equations in this case (taking the symmetry of k_{ij} into account) are

$$\begin{aligned}
&\gamma(k_{xx}) + (2x+z)k_{xy} + 2yk_{xz} = 0,\\
&\gamma(k_{xy}) + (x+\tfrac{1}{2}z)k_{yy} + yk_{yz} - (2x+z)k_{xx} + (x-2z)k_{xz} = 0,\\
&\gamma(k_{xz}) + (x+\tfrac{1}{2}z)k_{yz} + yk_{zz} - yk_{xx} - (z+\tfrac{1}{2}x)k_{xy} = 0,\\
&\gamma(k_{yy}) - 2(2x+z)k_{xy} + 2(x+2z)k_{yz} = 0,\\
&\gamma(k_{yz}) - (2x+z)k_{xz} + (x+2z)k_{zz} - yk_{xy} - (z+\tfrac{1}{2}x)k_{yy} = 0,\\
&\gamma(k_{zz}) - 2yk_{xz} - (2z+x)k_{yz} = 0.
\end{aligned} \tag{20}$$

The Φ-equations are

$$\begin{aligned}
&k_{xx}(3yx-4yz) + k_{xy}(-\tfrac{3}{2}x^2+4xz-\tfrac{3}{2}z^2) + k_{xz}(-4yx+3yz)\\
&\quad = k_{xy}(-3y^2+\tfrac{1}{2}y^2) + k_{yy}(\tfrac{3}{2}yx-2yz) + k_{yz}(4y^2-\tfrac{1}{2}xz),\\
&k_{xx}(4y^2-\tfrac{1}{2}xz) + k_{xy}(-2yx+\tfrac{3}{2}yz) + k_{xz}(-3y^2+\tfrac{1}{2}x^2)\\
&\quad = k_{xz}(-3y^2+\tfrac{1}{2}z^2) + k_{yz}(\tfrac{3}{2}xy-2yz) + k_{zz}(4y^2-\tfrac{1}{2}xz),\\
&k_{xy}(4y^2-\tfrac{1}{2}xz) + k_{yy}(-2xy+\tfrac{3}{2}yz) + k_{yz}(-3y^2+\tfrac{1}{2}x^2)\\
&\quad = k_{xz}(3xy-4yz) + k_{yz}(-\tfrac{3}{2}x^2+4xz-\tfrac{3}{2}z^2) + k_{zz}(-4xy+3yz).
\end{aligned} \tag{21}$$

We will first try to find a solution of (20) in which all k_{ij} are constants. In that case, adding one half times the (y,y)-equation to the (x,x)- and (z,z)-equations gives $k_{xy}=0$ from which also $k_{xz}=0$ and $k_{yz}=0$. Then the (x,z)-equation gives $k_{xx}=k_{zz}$ and the (y,z)-equation gives $k_{zz}=\frac{1}{2}k_{yy}$. So the solutions of (20) with constant coefficients are of the form

$$k = c\begin{bmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

It is easy to see that a multiplier k of this form also satisfies Equations (21). The Hessian of the function

$$l(x,y,z) = \tfrac{1}{2}(x^2+2y^2+z^2)$$

takes the above form; this is exactly the Lagrangian (19).

An expression for the most general solution of Equations (20) and (21) is beyond the scope of the current paper. However, instead of looking for constant solutions k_{ij} as above, we could use an additional symmetry assumption. For example, it seems natural to require that $k_{xx}=k_{zz}$ (but that they are not necessarily constants). A tedious calculation reveals that in that case the only possible solution of the reduced Helmholtz conditions is again the multiplier in (22) with constant coefficients.

8.3 An illustrative example on the Lie group of the affine line

There are only two distinct Lie algebras of dimension 2. In this example we will use the Lie group of the affine line (the Euclidean group). An element of this group is an affine map $\mathbf{R}\to\mathbf{R} : t\mapsto \exp(q_1)t+q_2$ and can be represented by the matrix

$$\begin{bmatrix} \exp(q_1) & q_2\\ 0 & 1 \end{bmatrix}.$$

The corresponding Lie algebra is given by the set of matrices of the form

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}.$$

A basis for this algebra is

$$E_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for which $\{E_x, E_y\} = E_y$. Let $A = aE_x + bE_y$ be a constant vector in the Lie algebra. We will determine whether there exists a regular Lagrangian for the dynamical system

$$\dot{w} = \{w, \{w, A\}\},$$

or, in the above basis,

$$\dot{x} = 0, \qquad \dot{y} = x(bx - ay).$$

For this system

$$\begin{aligned} \Phi(E_x) &= \tfrac{1}{4}(a-1)^2 xyE_y, & \nabla E_x &= (-bx + \tfrac{1}{2}(a-1)y)E_y, \\ \Phi(E_y) &= -\tfrac{1}{4}(a-1)^2 x^2 E_y, & \nabla E_y &= \tfrac{1}{2}(a+1)xE_y. \end{aligned}$$

The ∇-equations are therefore

$$\begin{aligned} &x(bx - ay)k_{xxy} - 2(-xb + \tfrac{1}{2}(a-1)y)k_{xy} = 0, \\ &x(bx - ay)k_{xyy} - \tfrac{1}{2}(a+1)xk_{xy} - (-bx + \tfrac{1}{2}(a-1)y)k_{yy} = 0, \\ &x(bx - ay)k_{yyy} - (a+1)xk_{yy} = 0, \end{aligned}$$

and the only Φ-equation is

$$-(a-1)^2 x^2 k_{xy} = (a-1)^2 xyk_{yy}.$$

If we differentiate the Φ-equation with respect to x and y we obtain two more equations for the k_{ijk}:

$$\begin{aligned} -(a-1)^2(x^2 k_{xxy} + 2xk_{xy}) &= (a-1)^2(xyk_{xyy} + yk_{yy}), \\ -(a-1)x^2 k_{xyy} &= (a-1)^2(xyk_{yyy} + xk_{yy}). \end{aligned}$$

The component k_{xx} of the Hessian and its derivative k_{xxx} are absent from these equations, and they will also not show up in any derived equation; there will therefore always remain freedom of choice for the x-derivative of k_{xx}. We get 6 homogeneous linear equations in the 5 unknowns k_{xy}, k_{yy}, k_{xxy}, k_{xyy} and k_{yyy}. If the rank of this system is less than 5 the system will have a non-zero solution. When $a = 1$, the rank is clearly 3. It can easily be verified that in all other cases the rank is 4.

For reasons of clarity, we will deal first with the case where $a = 1$.

1. The case where $a = 1$. In this case the Φ-equation is identically satisfied. The ∇-equations are now

$$\begin{aligned} &x(bx - y)k_{xxy} + 2bxk_{xy} = 0, \\ &x(bx - y)k_{xyy} - xk_{xy} + bxk_{yy} = 0, \\ &x(bx - y)k_{yyy} - 2xk_{yy} = 0. \end{aligned}$$

From the first and the last of these equations, we find that

$$k_{xy} = \frac{f_1(y)}{(bx-y)^2} \qquad \text{and} \qquad k_{yy} = \frac{f_2(x)}{(bx-y)^2}$$

respectively, as long as $x \neq 0$ and $bx - y \neq 0$. By substituting this result in the second equation and by interpreting k_{xyy} once as $\partial k_{yy}/\partial x$ and once as $\partial k_{xy}/\partial y$, we get a system of ODE's from which we can determine $f_1(y)$ and $f_2(x)$. They are

$$f_1(y) = -b\alpha_2 - \alpha_1 y \qquad \text{and} \qquad f_2(x) = \alpha_1 x + \alpha_2.$$

The solution of the ∇-equations is therefore of the form

$$k = \begin{bmatrix} \frac{b^2\alpha_2 - b\alpha_1(bx-y)}{2(bx-y)^2} + f(x) & -\frac{b\alpha_2+\alpha_1 y}{(bx-y)^2} \\ -\frac{b\alpha_2+\alpha_1 y}{(bx-y)^2} & \frac{\alpha_1 x+\alpha_2}{(bx-y)^2} \end{bmatrix}, \qquad \det(k) = \frac{f(x)(\alpha_1 x + \alpha_2) - \alpha_1^2}{(bx-y)^2}.$$

This matrix is, however, not defined on the whole of $\mathbf{R}^2$. By continuity it exists on $x = 0$ but it is not defined on $bx - y = 0$. So, there is no regular multiplier on $\mathbf{R}^2$.

This is not the end of the story, however. The dynamical equations are now $\dot{x} = 0$ and $\dot{y} = x(bx - y)$. Notice that the lines $x = 0$ and $bx - y = 0$ are both invariant under the flow. They divide the space $\mathbf{R}^2$ into regions, each invariant under the flow of the dynamical system. The matrix above is well-defined on the invariant region with $bx - y \neq 0$. It will be a multiplier provided its determinant is not zero, that is, provided $f(x)(\alpha_1 x + \alpha_2) - \alpha_1^2 \neq 0$. The function $l(x, y) = -\alpha_2 \ln|bx - y| - \alpha_1 x \ln|bx - y| + \alpha_3 y + h(x)$, with $h''(x) = f(x)$, has the above matrix as its Hessian. However, for l to give the required Euler-Poincaré equations, α_2 and α_3 must vanish. A non-degenerate Lagrangian on $bx - y \neq 0$ is therefore

$$l(x, y) = -\alpha_1 x \ln|bx - y| + h(x),$$

where α_1 is a non-vanishing constant, and $h(x)$ is an arbitrary function which is not of the form $\alpha_1(x \ln|x| - x) + \alpha_4 x + \alpha_5$ for any constants α_4 and α_5.

In the following cases it will happen that there is no regular Lagrangian defined on the whole of $\mathbf{R}^2$, but it may be possible to find Lagrangians for subsets of $\mathbf{R}^2$ invariant under the dynamical flow.

2. The case where $a \neq 1$. In this case, the Φ-equations come into play. As before, we can search first for the most general class of solutions of the ∇-equations, and then restrict to only those that also satisfy the Φ-equation. Notice that e.g. the last of the ∇-equations leads to a further division of this case in subcases. We have

$$k_{yy} = \begin{cases} f_2(x)(bx - ay)^{-\frac{1}{a}-1}, & a \neq 0, \\ f_2(x)\exp\left(\dfrac{y}{bx}\right), & a = 0,\ b \neq 0, \\ 0, & a = 0,\ b = 0. \end{cases}$$

We will only summarize the results.

2A. The case where $a \neq 0$. There is a regular Lagrangian of the form

$$l(x, y) = \frac{\alpha_1}{1-a}|ay - bx|^{1-\frac{1}{a}}|x|^{\frac{1}{a}} + h(x),$$

where α_1 is a non-zero constant and $h(x)$ is an arbitrary, but non-affine function. The lines $x = 0$ and $ay - bx = 0$ are invariant under the flow.

2B. The case where $a = 0$ and $b \neq 0$. There is a regular Lagrangian of the form

$$l(x, y) = \alpha_1 b^2 x \exp\left(\frac{y}{bx}\right) + h(x),$$

where α_1 is a non-zero constant and $h(x)$ is an arbitrary but non-affine function. The line $x = 0$ is invariant under the flow.

2C. The case where $a = 0$ and $b = 0$. This is a degenerate case, there is no regular multiplier.

Having decided in all cases whether a Lagrangian l on the Lie algebra $\mathfrak{g}$ exists or not, it is instructive to give an expression for the corresponding Lagrangians L at the level of the Lie group G. If (q_1, q_2) are coordinates on the Lie group, then a left-invariant basis of vector fields is given by

$$\hat{E}_x = \frac{\partial}{\partial q_1}, \qquad \hat{E}_y = \exp(q_1)\frac{\partial}{\partial q_2}.$$

Fibre coordinates $(w^i) = (x, y)$ with respect to this basis and $(\dot{q}_1, \dot{q}_2)$ with respect to the coordinate field basis are related as $x = \dot{q}_1, y = \exp(-q_1)\dot{q}_2$. A right-invariant basis is

$$\tilde{E}_x = \frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}, \qquad \tilde{E}_y = \frac{\partial}{\partial q_2}.$$

The complete and vertical lifts of the left-invariant basis fields are

$$\hat{E}_x^{\mathrm{C}} = \frac{\partial}{\partial q_1}, \qquad \hat{E}_y^{\mathrm{C}} = \exp(q_1)\left(\frac{\partial}{\partial q_2} + \dot{q}_1\frac{\partial}{\partial \dot{q}_2}\right), \qquad \hat{E}_x^{\mathrm{V}} = \frac{\partial}{\partial \dot{q}_1}, \qquad \hat{E}_y^{\mathrm{V}} = \exp(q_1)\frac{\partial}{\partial \dot{q}_2}.$$

We can now rewrite a second-order field Γ in any of the following forms

$$\begin{aligned}
\Gamma &= \dot{q}_1\frac{\partial}{\partial q_1} + \dot{q}_2\frac{\partial}{\partial q_2} + f_1\frac{\partial}{\partial \dot{q}_1} + f_2\frac{\partial}{\partial \dot{q}_2} \\
&= \dot{q}_1\frac{\partial}{\partial q_1} + \exp(-q_1)\dot{q}_2\left(\exp(q_1)\left(\frac{\partial}{\partial q_2} + \dot{q}_1\frac{\partial}{\partial \dot{q}_2}\right)\right) \\
&\quad + f_1\frac{\partial}{\partial \dot{q}_1} + (\exp(-q_1)(f_2 - \dot{q}_1\dot{q}_2))\exp(q_1)\frac{\partial}{\partial \dot{q}_2} \\
&= x\hat{E}_x^{\mathrm{C}} + y\hat{E}_y^{\mathrm{C}} + \Gamma_x\hat{E}_x^{\mathrm{V}} + \Gamma_y\hat{E}_y^{\mathrm{V}}.
\end{aligned}$$

In the example under consideration, $\Gamma_x = 0$ and $\Gamma_y = x(bx - ay)$, so

$$f_1 = 0, \qquad f_2 = (1 - a)\dot{q}_1\dot{q}_2 + \exp(q_1)b\dot{q}_1^2).$$

Let's look, for example, at case 2B ($a = 0$), where we have stated above that there exist a regular Lagrangian on the Lie algebra of the form $l(x, y) = \alpha_1 b^2 x \exp(y/bx) + h(x)$ ($\alpha_1 \neq 0$, h non-affine). By using left translations we can extend this to a Lagrangian on the whole of TG:

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \alpha_1 b^2 \dot{q}_1 \exp\left(\frac{\exp(-q_1)\dot{q}_2}{b\dot{q}_1}\right) + h(\dot{q}_1).$$

Obviously, this Lagrangian is invariant:

$$\tilde{E}_1^{\mathrm{C}}(L) = \frac{\partial L}{\partial q_1} + q_2\frac{\partial L}{\partial q_2} + \dot{q}_2\frac{\partial L}{\partial \dot{q}_2} = 0 \quad \text{and} \quad \tilde{E}_2^{\mathrm{C}}(L) = \frac{\partial L}{\partial q_2} = 0.$$

A short calculation shows that the Euler-Lagrange equations for the above Lagrangian do indeed return the differential equations $\ddot{q}_1 = 0$ and $\ddot{q}_2 = \dot{q}_1\dot{q}_2 + b\exp(q_1)\dot{q}_1^2$, as they should.

Our analysis reveals only whether there is an invariant Lagrangian. In the case 2C ($a = b = 0$) where no such Lagrangian exists there could still be a (necessarily non-invariant) Lagrangian for the second-order system $\ddot{q}_1 = 0$, $\ddot{q}_2 = \dot{q}_1\dot{q}_2$ on the two-dimensional Lie group. In [9] Douglas gave a more-or-less complete classification of the inverse problem for two-dimensional systems. A modern geometric approach to Douglas's classification can be found in [22]. A meticulous analysis using the methods described there shows that a regular Lagrangian must exist, even in the case 2C where we concluded that there is no invariant Lagrangian. In more detail, our case 1 belongs to Douglas's case I, and our cases 2A, 2B and 2C to his case IIa1.

Observe that if $a = b = 0$ we are back in the example of the canonical connection. According to [23], the most general Lagrangian for the case 2C, subject to the regularity condition, is given by

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \dot{q}^1\theta(q_1, q_2, z) + \psi(\dot{q}^1), \qquad z = \dot{q}_2/\dot{q}_1,$$

where ψ is an arbitrary function and θ is a solution of the PDE

$$z\theta_{zz} + z\theta_{zq_2} + \theta_{q_1 z} - \theta_{q_2} = 0$$

(subscripts denote derivatives, as usual). For example, the function

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \tfrac{1}{2}\dot{q}_1^2 + \exp(-q_1)\frac{\dot{q}_2^2}{2\dot{q}_1}$$

is a Lagrangian for the system in 2C. It is clearly not invariant since

$$\tilde{E}_1^{\rm C}(L) = \exp(-q_1)\frac{\dot{q}_2^2}{2\dot{q}_1}.$$

In fact, there does not exist a function θ for which the Lagrangian is invariant and regular. The relations $\tilde{E}_1^{\rm C}(L) = 0$ and $\tilde{E}_2^{\rm C}(L) = 0$ imply that $\theta_{q_1} + z\theta_z = 0$ and $\theta_{q_2} = 0$, respectively. By taking the z-derivative of the first relation and by applying the second in the defining relation of θ, we can conclude that also $\theta_z = 0$ and $\theta_{q_1} = 0$. But then θ is a constant and the Lagrangian is clearly degenerate.

9 Outlook

We discuss briefly two possible extensions of the current framework. First of all, let M be a manifold with a given symmetry group G. One can then set up an inverse problem for G-invariant Lagrangians on M. In that case, it has been shown in [5] that the Euler-Lagrange equations reduce to the so-called *Lagrange*-Poincaré equations. On the other hand, the technique of adapted frames can be easily extended to manifolds with a symmetry group; a description of the reduced equations for arbitrary second-order equations can be found in [8]. So the question would be when these reduced equations are of Lagrange-Poincaré form.

The second extension can be situated at the level of Lie algebroids. In this paper, we have discussed the inverse problem for Lagrangians on a Lie algebra $\mathfrak{g}$. The original inverse problem deals with Lagrangians on TM. Both $\mathfrak{g}$ and TM are the two simplest cases of a Lie algebroid. So it seems natural to study an inverse problem for arbitrary Lie algebroids (the corresponding Lagrangian equations were given in e.g. [15]). The situation in the previous paragraph then coincides with the case that the Lie algebroid is TM/G, the so-called Atiyah algebroid.

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