On the generalized Helmholtz conditions for Lagrangian systems with dissipative forces

M. Crampin, T. Mestdag and W. Sarlet

Department of Mathematical Physics and Astronomy, Ghent University
Krijgslaan 281, B-9000 Ghent, Belgium

July 7, 2009

Abstract. In two recent papers necessary and sufficient conditions for a given system of second-order ordinary differential equations to be of Lagrangian form with additional dissipative forces were derived. We point out that these conditions are not independent and prove a stronger result accordingly.

Keywords. Lagrangian systems, dissipative forces, inverse problem, Helmholtz conditions.

MSC (2000). 70H03, 70F17, 49N45

1 Introduction

The Helmholtz conditions, for the purposes of this paper, are the necessary and sufficient conditions for a given system of second-order ordinary differential equations $f_a(\ddot{q}, \dot{q}, q, t) = 0$ to be of Euler-Lagrange type, that is, for there to exist a Lagrangian $\Lambda(\dot{q}, q, t)$ such that

$$f_a = \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}^a}\right) - \frac{\partial \Lambda}{\partial q^a}. \tag{1}$$

Here q^a are the generalized coordinates (collectively abbreviated to q), $\dot{q}^a$ the corresponding generalized velocities, and so on. We shall state the conditions shortly. The Lagrangian is supposed to be of first order, by which, to be explicit, we mean the following, throughout the paper: a function is of first order if it depends on $\dot{q}$, q and t alone, that is, if it is independent of $\ddot{q}$ and higher-order derivative (or more properly jet) coordinates.

In two recent papers the problem of finding analogous necessary and sufficient conditions for a given set of functions $f_a(\ddot{q}, \dot{q}, q, t)$ to take the more general form

$$f_a = \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}^a}\right) - \frac{\partial \Lambda}{\partial q^a} + \frac{\partial D}{\partial \dot{q}^a} \tag{2}$$

for first-order functions Λ (a Lagrangian) and D (a dissipation function) has been discussed. We shall say that in this case the equations $f_a = 0$ are of Lagrangian form with dissipative forces of gradient type. The problem is solved, in the sense that a set of necessary and sufficient conditions is given, in terms of standard coordinates, in the fairly recent paper [3]. In a very recent paper [2] a version of the conditions expressed in terms of quasi-velocities, or as the authors call them nonholonomic velocities, is obtained. We shall quote the conditions from [3] explicitly in Section 2. These conditions are described as generalized Helmholtz conditions to distinguish them from the Helmholtz conditions discussed in our opening paragraph, which may be called the classical Helmholtz conditions; these must of course comprise a special case of the generalized conditions.

The main purpose of the present paper is to point out that the generalized Helmholtz conditions as stated in [3] are not independent: in fact two of them are redundant, in that they can be derived from the remaining ones. This we show in Section 2 below. We use the same formalism as [3]. However, since the version of the generalized conditions obtained in [2] is equivalent to that in [3], it is clearly the case that the same redundancy is present there as well. By taking advantage of the improvement in the formulation of the generalized Helmholtz conditions that we achieve, we are able to give a shorter and more elegant proof of their sufficiency than the one to be found in [3].

There are in fact several interesting questions raised by the two papers [2, 3], only one of which will be dealt with here. In the third and final section of our paper we give an outline of these additional points of interest, which will receive a fuller airing elsewhere.

We employ the Einstein summation convention throughout.

We end this introduction with a brief summary of the results about the classical Helmholtz conditions that we shall need.

The classical Helmholtz conditions, which are the necessary and sufficient conditions for the functions $f_a(\ddot{q}, \dot{q}, q, t)$ to be the Euler-Lagrange expressions of some first-order Lagrangian function, as given in equation (1), are that the f_a should satisfy

$$\frac{\partial f_a}{\partial \ddot{q}^b} = \frac{\partial f_b}{\partial \ddot{q}^a} \tag{3}$$

$$\frac{\partial f_a}{\partial \dot{q}^b} + \frac{\partial f_b}{\partial \dot{q}^a} = 2\frac{d}{dt}\left(\frac{\partial f_b}{\partial \ddot{q}^a}\right) \tag{4}$$

$$\frac{\partial f_a}{\partial q^b} - \frac{\partial f_b}{\partial q^a} = \tfrac{1}{2}\frac{d}{dt}\left(\frac{\partial f_a}{\partial \dot{q}^b} - \frac{\partial f_b}{\partial \dot{q}^a}\right). \tag{5}$$

It is a consequence of these (and not an extra condition, as stated in [3]) that

$$\frac{\partial^2 f_a}{\partial \ddot{q}^b \partial \ddot{q}^c} = 0$$

(this follows from the vanishing of the coefficient of $\ddot{q}^c$ in condition (4)). Thus we may write $f_a = g_{ab}\ddot{q}^b + h_a$, the coefficients being of first order, with $g_{ab} = g_{ba}$ as a result of

2

condition (3). The Helmholtz conditions can be re-expressed in terms of g_{ab} (assumed to be symmetric) and h_a, when they reduce to the following three conditions:

$$\frac{\partial g_{ab}}{\partial \dot{q}^c} - \frac{\partial g_{ac}}{\partial \dot{q}^b} = 0 \tag{6}$$

$$\frac{\partial h_a}{\partial \dot{q}^b} + \frac{\partial h_b}{\partial \dot{q}^a} = 2\frac{\bar{d}}{dt}(g_{ab}) \tag{7}$$

$$2\left(\frac{\partial h_a}{\partial q^b} - \frac{\partial h_b}{\partial q^a}\right) = \frac{\bar{d}}{dt}\left(\frac{\partial h_a}{\partial \dot{q}^b} - \frac{\partial h_b}{\partial \dot{q}^a}\right), \tag{8}$$

where

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \dot{q}^c\frac{\partial}{\partial q^c}.$$

That is to say, the f_a are the Euler-Lagrange expressions of some first-order Lagrangian if and only if $f_a = g_{ab}\ddot{q}^b + h_a$ for some first-order functions g_{ab} and h_a such that $g_{ab} = g_{ba}$ and (6)–(8) hold. This reformulation can be found in the book by Santilli [5], for example.

2 The generalized Helmholtz conditions

We next turn to the analysis of the generalized Helmholtz conditions obtained in [3] as the necessary and sufficient conditions for a system of equations $f_a = 0$ to be of Lagrangian form with dissipative forces of gradient type, as given in equation (2). Following the notation of [3] we set

$$r_{ab} = \frac{\partial f_a}{\partial q^b} - \frac{\partial f_b}{\partial q^a} + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial f_b}{\partial \dot{q}^a} - \frac{\partial f_a}{\partial \dot{q}^b}\right)$$

$$s_{ab} = \frac{1}{2}\left(\frac{\partial f_a}{\partial \dot{q}^b} + \frac{\partial f_b}{\partial \dot{q}^a}\right) - \frac{d}{dt}\left(\frac{\partial f_b}{\partial \ddot{q}^a}\right).$$

The generalized Helmholtz conditions as given in [3] are that r_{ab} and s_{ab} are of first order, and that in addition

$$\frac{\partial f_a}{\partial \ddot{q}^b} = \frac{\partial f_b}{\partial \ddot{q}^a} \tag{9}$$

$$\frac{\partial s_{ab}}{\partial \dot{q}^c} = \frac{\partial s_{ac}}{\partial \dot{q}^b} \tag{10}$$

$$\frac{\partial r_{ab}}{\partial \dot{q}^c} = \frac{\partial s_{ac}}{\partial q^b} - \frac{\partial s_{bc}}{\partial q^a} \tag{11}$$

$$0 = \frac{\partial r_{ab}}{\partial q^c} + \frac{\partial r_{bc}}{\partial q^a} + \frac{\partial r_{ca}}{\partial q^b}. \tag{12}$$

Our main concern will be with analysing conditions (10)–(12), which correspond to (2.3e), (2.3f) and (2.3g) of [3]; we shall show that conditions (10) and (12) are redundant, being consequences of the remaining conditions.

3

Our first aim is to understand exactly what it means for r_{ab} and s_{ab} to be of first order, bearing in mind condition (9) above.

From the vanishing of the coefficient of $\dddot{q}^c$ in s_{ab} we have

$$\frac{\partial^2 f_a}{\partial \ddot{q}^b \partial \ddot{q}^c} = 0.$$

As before we may write $f_a = g_{ab}\ddot{q}^b + h_a$, the coefficients being of first order and g_{ab} symmetric. The coefficient of $\dddot{q}^c$ in r_{ab} is

$$\tfrac{1}{2}\frac{\partial}{\partial \ddot{q}^c}\left(\frac{\partial g_{bd}}{\partial \dot{q}^a}\ddot{q}^d + \frac{\partial h_b}{\partial \dot{q}^a} - \frac{\partial g_{ad}}{\partial \dot{q}^b}\ddot{q}^d - \frac{\partial h_a}{\partial \dot{q}^b}\right),$$

whence

$$\frac{\partial g_{bc}}{\partial \dot{q}^a} = \frac{\partial g_{ac}}{\partial \dot{q}^b}.$$

The coefficient of $\ddot{q}^c$ in s_{ab}, namely

$$\frac{\partial g_{ac}}{\partial \dot{q}^b} + \frac{\partial g_{bc}}{\partial \dot{q}^a} - 2\frac{\partial g_{ab}}{\partial \dot{q}^c},$$

vanishes as a consequence. The coefficient of $\ddot{q}^c$ in r_{ab} is

$$\frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^a} - \tfrac{1}{2}\left(\frac{\partial^2 h_a}{\partial \dot{q}^b \partial \dot{q}^c} - \frac{\partial^2 h_b}{\partial \dot{q}^a \partial \dot{q}^c}\right),$$

an expression which for later convenience we write as ρ_{abc}; we must of course have $\rho_{abc} = 0$. The remaining terms in r_{ab} and s_{ab} are all of first order, and we have

$$r_{ab} = \frac{\partial h_a}{\partial q^b} - \frac{\partial h_b}{\partial q^a} - \tfrac{1}{2}\frac{\bar{d}}{dt}\left(\frac{\partial h_a}{\partial \dot{q}^b} - \frac{\partial h_b}{\partial \dot{q}^a}\right)$$

$$s_{ab} = \tfrac{1}{2}\left(\frac{\partial h_a}{\partial \dot{q}^b} + \frac{\partial h_b}{\partial \dot{q}^a}\right) - \frac{\bar{d}}{dt}(g_{ab});$$

compare with (7) and (8), and also with equations (2.16b) and (2.17c) of [3].

The redundancy of condition (10) is a consequence of the vanishing of ρ_{abc}, as we now show. We have

$$\frac{\partial s_{ac}}{\partial \dot{q}^b} = \tfrac{1}{2}\left(\frac{\partial^2 h_a}{\partial \dot{q}^b \partial \dot{q}^c} + \frac{\partial^2 h_c}{\partial \dot{q}^a \partial \dot{q}^b}\right) - \frac{\partial}{\partial \dot{q}^b}\left(\frac{\bar{d}}{dt}(g_{ac})\right)$$

$$= \tfrac{1}{2}\left(\frac{\partial^2 h_a}{\partial \dot{q}^b \partial \dot{q}^c} + \frac{\partial^2 h_c}{\partial \dot{q}^a \partial \dot{q}^b}\right) - \frac{\bar{d}}{dt}\left(\frac{\partial g_{ac}}{\partial \dot{q}^b}\right) - \frac{\partial g_{ac}}{\partial q^b},$$

using the commutation relation

$$\left[\frac{\partial}{\partial \dot{q}^a}, \frac{\bar{d}}{dt}\right] = \frac{\partial}{\partial q^a}. \tag{13}$$

4

It follows that

$$\frac{\partial s_{ac}}{\partial \dot{q}^b} - \frac{\partial s_{bc}}{\partial \dot{q}^a} = -\rho_{abc} = 0.$$

That is to say, condition (10) holds automatically as a consequence of the first-order property. Furthermore, $\rho_{abc} = 0$ is equivalent to equation (2.17b) of [3]; in other words, the redundancy of (10) is actually implicit in [3], though not apparently recognized there.

Before proceeding to consider condition (12) we turn aside to make some remarks about the classical Helmholtz conditions.

The calculations just carried out are essentially the same as those which lead to the version of the classical Helmholtz conditions given in equations (6)–(8) at the end of the introduction. It is easy to see that in that case $r_{ab} = s_{ab} = 0$ are necessary conditions. This observation, together with the part of the argument concerning the vanishing of the coefficients of $\dddot{q}$ and $\ddot{q}$, leads to the following conditions:

$$\frac{\partial g_{ac}}{\partial \dot{q}^b} + \frac{\partial g_{bc}}{\partial \dot{q}^a} = 2\frac{\partial g_{ab}}{\partial \dot{q}^c} \tag{14}$$

$$\frac{\partial h_a}{\partial \dot{q}^b} + \frac{\partial h_b}{\partial \dot{q}^a} = 2\frac{\bar{d}}{dt}(g_{ab}) \tag{15}$$

$$\frac{\partial g_{ab}}{\partial \dot{q}^c} - \frac{\partial g_{ac}}{\partial \dot{q}^b} = 0 \tag{16}$$

$$\frac{\partial^2 h_a}{\partial \dot{q}^b \partial \dot{q}^c} - \frac{\partial^2 h_b}{\partial \dot{q}^a \partial \dot{q}^c} = 2\left(\frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^a}\right) \tag{17}$$

$$2\left(\frac{\partial h_a}{\partial q^b} - \frac{\partial h_b}{\partial q^a}\right) = \frac{\bar{d}}{dt}\left(\frac{\partial h_a}{\partial \dot{q}^b} - \frac{\partial h_b}{\partial \dot{q}^a}\right). \tag{18}$$

These are the conditions quoted in Remark 3 of Section 1 of [3]. However, it is now evident that two of them are redundant. Clearly condition (14) (which is the vanishing of the coefficient of $\ddot{q}^c$ in s_{ab}) follows from condition (16) and the symmetry of g_{ab}. Condition (17) is the condition $\rho_{abc} = 0$. The second part of the argument above, that leading to the relation $\rho_{abc} = \partial s_{bc}/\partial \dot{q}^a - \partial s_{ac}/\partial \dot{q}^b$, shows that in the classical case condition (17) follows from the other conditions. When these two redundant conditions are removed we obtain the classical Helmholtz conditions in the form given at the end of the introduction.

These results in the classical case are actually very well known, though not apparently to the authors of [3], and have been known for a long time: they are to be found, for example, in Santilli's book of 1978 [5] (which is in fact referred to in [3]).

For the sake of clarity we should point out a difference between the two cases: in the classical case condition (17) is completely redundant; in the generalized case it is not redundant, but occurs twice in the formulation of the conditions in [3], once in the requirement that r_{ab} should be of first order and once as the condition $\partial s_{ab}/\partial \dot{q}^c = \partial s_{bc}/\partial \dot{q}^a$.

We now return to the generalized conditions, and prove that condition (12) follows from condition (11). It will be convenient to write condition (12) as

$$\sum_{a,b,c} \frac{\partial r_{ab}}{\partial q^c} = 0,$$

where $\sum_{a,b,c}$ stands for the cyclic sum over a, b and c, here and below. As a preliminary remark, note that if k_{abc} is symmetric in b and c (say) then $\sum_{a,b,c} k_{abc} = \sum_{a,b,c} k_{bac}$. Now

$$\frac{\partial r_{ab}}{\partial q^c} = \frac{\partial^2 h_a}{\partial q^b \partial q^c} - \frac{\partial^2 h_b}{\partial q^a \partial q^c} - \frac{1}{2} \frac{\bar{d}}{dt} \left(\frac{\partial^2 h_a}{\partial q^c \partial \dot{q}^b} - \frac{\partial^2 h_b}{\partial q^c \partial \dot{q}^a} \right),$$

and so by the preliminary remark

$$\sum_{a,b,c} \frac{\partial r_{ab}}{\partial q^c} = -\frac{1}{2} \frac{\bar{d}}{dt} \left(\sum_{a,b,c} \left(\frac{\partial^2 h_a}{\partial q^c \partial \dot{q}^b} - \frac{\partial^2 h_b}{\partial q^c \partial \dot{q}^a} \right) \right)$$

$$= -\frac{1}{2} \frac{\bar{d}}{dt} \left(\sum_{a,b,c} \left(\frac{\partial^2 h_a}{\partial q^c \partial \dot{q}^b} - \frac{\partial^2 h_a}{\partial q^b \partial \dot{q}^c} \right) \right).$$

On the other hand, using the commutation relation (13) and the fact that $\partial/\partial q^a$ and $\bar{d}/dt$ commute it is easy to see that condition (11) leads to

$$\frac{1}{2} \sum_{a,b,c} \left(\frac{\partial^2 h_a}{\partial q^c \partial \dot{q}^b} - \frac{\partial^2 h_a}{\partial q^b \partial \dot{q}^c} \right) = -\frac{\bar{d}}{dt} (\rho_{abc}) = 0.$$

We therefore reach the following proposition, which is stronger than the corresponding result in [3].

Proposition. *The necessary and sufficient conditions for the equations $f_a(\ddot{q}, \dot{q}, q, t) = 0$ to be of Lagrangian form with dissipative forces of gradient type as in (2) are that the functions r_{ab} and s_{ab} are of first order, that*

$$\frac{\partial f_a}{\partial \ddot{q}^b} = \frac{\partial f_b}{\partial \ddot{q}^a},$$

and that

$$\frac{\partial r_{ab}}{\partial \dot{q}^c} = \frac{\partial s_{ac}}{\partial q^b} - \frac{\partial s_{bc}}{\partial q^a}. \tag{19}$$

Just as in the classical case we can give an equivalent formulation of these conditions in terms of g_{ab} and h_a. Bearing in mind that r_{ab} and s_{ab} being of first order are essential hypotheses, we find that the following conditions are equivalent to those given in the

proposition above: $f_a = g_{ab}\ddot{q}^b + h_a$ with g_{ab} symmetric, where g_{ab}, h_a are of first order and further satisfy

$$\frac{\partial g_{ab}}{\partial \dot{q}^c} - \frac{\partial g_{ac}}{\partial \dot{q}^b} = 0$$

$$\frac{\partial^2 h_a}{\partial \dot{q}^b \partial \dot{q}^c} - \frac{\partial^2 h_b}{\partial \dot{q}^a \partial \dot{q}^c} = 2\left(\frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^a}\right)$$

$$\sum_{a,b,c}\left(\frac{\partial^2 h_a}{\partial q^b \partial \dot{q}^c} - \frac{\partial^2 h_a}{\partial q^c \partial \dot{q}^b}\right) = 0.$$

The first of these is one of the classical conditions. The second is the condition $\rho_{abc} = 0$, which holds in the classical case as we have shown. The third is just condition (19) above expressed in terms of g_{ab} and h_a (or as it turns out, in terms of h_a alone), and $r_{ab} = s_{ab} = 0$ in the classical case. It is evident therefore that the conditions above are indeed a generalization of those for the classical case.

We end this section by giving an alternative proof of the sufficiency of the generalized Helmholtz conditions, based on this formulation of them, which is shorter and in our view more elegant than the proof in [3] (necessity is an easy if tedious calculation).

We note first that if g_{ab} is symmetric and satisfies

$$\frac{\partial g_{ab}}{\partial \dot{q}^c} = \frac{\partial g_{ac}}{\partial \dot{q}^b}$$

then

$$g_{ab} = \frac{\partial^2 K}{\partial \dot{q}^a \partial \dot{q}^b}$$

for some function $K = K(\dot{q}, q, t)$ (a well-known result, which also appears in [3]). Of course K is not determined by this relation; in fact if $\Lambda = K + P_a \dot{q}^a + Q$, where P_a and Q are any functions of q and t, then Λ has the same Hessian as K (the same g_{ab}, in other words). Our aim is to choose P_a and Q so that the given equations are of Lagrangian form with dissipative forces of gradient type as in (2), with Lagrangian Λ, assuming that the generalized Helmholtz conditions above hold. In fact we won't need to consider Q because it can be absorbed into D: if Λ is a Lagrangian and D a dissipation function for some functions f_a, so are $\Lambda + Q$ and $D + \dot{q}^a \partial Q/\partial q^a$. We shall therefore take $Q = 0$ below.

Let E_a be the Euler-Lagrange expressions of K. Then $E_a = g_{ab}\ddot{q}^b + k_a$ for some first-order k_a, by construction, so $f_a - E_a = h_a - k_a = \kappa_a$ say, where κ_a is also of first order. Moreover, f_a satisfies the generalized Helmholtz conditions by assumption, and E_a does so by construction (it satisfies the classical conditions after all), whence κ_a satisfies

$$\frac{\partial^2 \kappa_a}{\partial \dot{q}^b \partial \dot{q}^c} - \frac{\partial^2 \kappa_b}{\partial \dot{q}^a \partial \dot{q}^c} = 0 \tag{20}$$

$$\sum_{a,b,c}\left(\frac{\partial^2 \kappa_a}{\partial q^b \partial \dot{q}^c} - \frac{\partial^2 \kappa_a}{\partial q^c \partial \dot{q}^b}\right) = 0. \tag{21}$$

7

Let us set
$$\frac{\partial \kappa_a}{\partial \dot{q}^b} - \frac{\partial \kappa_b}{\partial \dot{q}^a} = R_{ab}.$$

Then by (20) R_{ab} is independent of $\dot{q}$, and by (21)
$$\sum_{a,b,c} \frac{\partial R_{ab}}{\partial q^c} = 0.$$

There are therefore functions $P_a(q, t)$ such that
$$R_{ab} = 2\left(\frac{\partial P_a}{\partial q^b} - \frac{\partial P_b}{\partial q^a} \right) = \frac{\partial \kappa_a}{\partial \dot{q}^b} - \frac{\partial \kappa_b}{\partial \dot{q}^a},$$

which is to say that if we set
$$\pi_{ab} = \frac{\partial \kappa_a}{\partial \dot{q}^b} - \left(\frac{\partial P_a}{\partial q^b} - \frac{\partial P_b}{\partial q^a} \right)$$

then $\pi_{ba} = \pi_{ab}$. Moreover,
$$\frac{\partial \pi_{ab}}{\partial \dot{q}^c} = \frac{\partial^2 \kappa_a}{\partial \dot{q}^b \partial \dot{q}^c} = \frac{\partial \pi_{ac}}{\partial \dot{q}^b}.$$

It follows (just as is the case for g_{ab}) that there is a first-order function D' such that
$$\pi_{ab} = \frac{\partial^2 D'}{\partial \dot{q}^a \partial \dot{q}^b},$$

from which we obtain
$$\kappa_a = \left(\frac{\partial P_a}{\partial q^b} - \frac{\partial P_b}{\partial q^a} \right) \dot{q}^b + \frac{\partial D'}{\partial \dot{q}^a} + S_a$$

where S_a is independent of $\dot{q}$. Now take
$$\Lambda = K + P_a \dot{q}^a.$$

Denoting the Euler-Lagrange expressions of K by E_a as before, the Euler-Lagrange expressions for Λ are
$$E_a + \left(\frac{\partial P_a}{\partial q^b} - \frac{\partial P_b}{\partial q^a} \right) \dot{q}^b + \frac{\partial P_a}{\partial t} = E_a + \kappa_a - \frac{\partial D'}{\partial \dot{q}^a} - S_a + \frac{\partial P_a}{\partial t}.$$

Thus, putting
$$D = D' + \left(S_a - \frac{\partial P_a}{\partial t} \right) \dot{q}^a,$$

we get
$$f_a = \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}^a} \right) - \frac{\partial \Lambda}{\partial q^a} + \frac{\partial D}{\partial \dot{q}^a}$$

as required.

This method of proof works equally well in the classical case. The proof is constructive, in the same sense that the one in [3] is, in either case. It is particularly well adapted to the familiar situation in which g_{ab} is independent of $\dot{q}$, when one can take the kinetic energy $\frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b$ for K.

8

3   Concluding remarks

We wish to make four remarks in conclusion, and to announce that these remarks are the basis for further work and results concerning the issues raised here, which will be the subject of a future second paper.

The first remark concerns the nature of conditions (10)–(12) on the derivatives of r_{ab} and s_{ab}, as originally expressed in [3] (that is, ignoring the question of dependence). In particular, bearing in mind the fact that r_{ab} is skew in its indices, the condition

$$\sum_{a,b,c} \frac{\partial r_{ab}}{\partial q^c} = 0$$

is suggestive: if perchance the r_{ab} were functions of the q alone this would have a natural interpretation in terms of the exterior calculus, being the condition for the 2-form $r_{ab}dq^a \wedge dq^b$ to be closed, that is, to satisfy $d(r_{ab}dq^a \wedge dq^b) = 0$. This point is made, in somewhat different terms, in [3] (and we appealed to the same general result in our proof of sufficiency of the generalized Helmholtz conditions in Section 2). The authors of [3] go on to say, however, that the condition above 'can be interpreted as the vanishing curvature of a symplectic space', which seems to us not to be entirely convincing. In fact it is possible to interpret the three conditions (10)–(12) collectively as signifying the vanishing of a certain exterior derivative of a certain 2-form on the space of coordinates $t, q, \dot{q}, \ddot{q}, \ldots$, a 2-form whose coefficients involve both r_{ab} and s_{ab}. This interpretation really arises from seeing the problem in the context of the so-called variational bicomplex (see [6] for a recent review).

Secondly, we contend that the problem we are dealing with should really be regarded as one about (second-order) dynamical systems. The point is that a dynamical system may be represented as a system of differential equations in many different coordinate formulations; the question of real interest is whether there is *some* representation of it which takes the form of an Euler-Lagrange system with dissipation, not just whether a given representation of it takes that form. Of course this point applies equally, mutatis mutandis, to the case in which there is no dissipation. Now the Helmholtz conditions as discussed in [2, 3], in both the classical and the generalized versions, suffer from the disadvantage that they are conditions for a *given* system of differential equations to be of Euler-Lagrange type. There is, however, an alternative approach to the problem which does deal with dynamical systems rather than equations, at least in the case in which the system can be expressed in normal form $\ddot{q}^a = F^a(\dot{q}, q, t)$. In this approach one asks (in the absence of dissipation) for conditions for the existence of a so-called multiplier, a non-singular matrix with elements g_{ab}, such that $g_{ab}(\ddot{q}^b - F^b)$ takes the Euler-Lagrange form (so that in particular when the conditions are satisfied g_{ab} will be the Hessian of the Lagrangian with respect to the velocity variables). The basic idea is to put $h_a = -g_{ab}F^b$ in the conditions at the end of the introduction, and regard the results as a system of partial differential equations equations for g_{ab} with F^a known. The seminal paper in this approach is Douglas's of 1941 [1], which analyses in great detail the case of two

degrees of freedom. For a recent review of developments since then see Sections 5 and 6 of [4] and references therein. One can in fact also formulate conditions on a multiplier for a second-order dynamical system, expressible in normal form, to be representable as equations of Lagrangian form with dissipative forces of gradient type; these generalize the known results for representation in Lagrangian form without dissipation in an interesting way.

The new ingredient in [2], by comparison with [3], is the expression of the generalized Helmholtz conditions in terms of quasi-velocities. As presented in the paper this is quite a long drawn out procedure, because in effect the conditions are rederived from scratch. Our third remark is that in principle this should be unnecessary: a truly satisfactory formulation of the conditions should be tensorial, in the sense of being independent of a choice of coordinates (and of course quasi-velocities are just a certain type of velocity coordinates). The approach described in the previous paragraph leads to conditions which have this desirable property.

Fourthly and finally, there is the question of whether generalized Helmholtz conditions can be derived for other kinds of "generalized force" terms than $\partial D/\partial \dot{q}^a$. One important case is that in which such a term is of gyroscopic type. We have obtained such conditions in this case, again using the approach discussed in our second remark above.

These points will be discussed in full detail in a forthcoming paper.

References

[1] J. Douglas, Solution of the inverse problem of the calculus of variations, Trans. Amer. Math. Soc. **50** (1941) 71–128

[2] U. Jungnickel, G. Kielau, P. Maisser and A. Müller, A generalization of the Helmholtz conditions for the existence of a first-order Lagrangian using nonholonomic velocities, Z. Angew. Math. Mech. **89** (2009) 44–53

[3] G. Kielau and P. Maisser, A generalization of the Helmholtz conditions for the existence of a first-order Lagrangian, Z. Angew. Math. Mech. **86** (2006) 722–735

[4] O. Krupková and G. E. Prince, Second order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations. In "Handbook of Global Analysis" ed. Krupka and Saunders (Elsevier 2008) 837–904

[5] R. M. Santilli, "Foundations of Theoretical Mechanics I. The Inverse Problem in Newtonian Mechanics" (Spinger-Verlag 1978)

[6] R. Vitolo, Variational sequences. In "Handbook of Global Analysis" ed. Krupka and Saunders (Elsevier 2008) 1115–1163