Routhian reduction for quasi-invariant Lagrangians

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Abstract

In this paper we describe Routhian reduction as a special case of standard symplectic reduction, also called Marsden-Weinstein reduction. We use this correspondence to present a generalization of Routhian reduction for quasi-invariant Lagrangians, i.e. Lagrangians that are invariant up to a total time derivative. We show how functional Routhian reduction can be seen as a particular instance of reduction of a quasi-invariant Lagrangian, and we exhibit a Routhian reduction procedure for the special case of Lagrangians with quasi-cyclic coordinates. As an application we consider the dynamics of a charged particle in a magnetic field.

1 Introduction and outline

In modern geometric approaches to Routhian reduction it is often mentioned that this reduction technique is the Lagrangian analogue of symplectic or Marsden-Weinstein reduction [14] (see for instance the introduction of [3]). This assertion is usually justified by the fact that, roughly speaking, for Routhian reduction one first restricts the system to a fixed level set of the momentum map and then reduces by taking the quotient with respect to the symmetry group. In this paper we show, among other things, that the analogy between Routhian reduction and Marsden-Weinstein reduction holds at a more fundamental level: in fact we will show that Routhian reduction is simply a special instance of general Marsden-Weinstein reduction (from now on referred to as MW-reduction). More specifically, by applying the MW-reduction procedure to the tangent bundle of a manifold, equipped with the symplectic structure induced by the Poincaré-Cartan 2-form associated with a Lagrangian, we will show that the resulting reduced symplectic space is 'tangent bundle-like', and that the reduced symplectic structure is

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again defined by a Poincaré-Cartan form, augmented with a gyroscopic 2-form. Of course, this symplectic description of the reduced system, obtained via the Routh's reduction method, is well-known in the literature. The difference with our approach, however, lies in the fact that we arrive at the reduced symplectic structure following the Marsden-Weinstein method. Until now, the symplectic nature of a Routh-reduced system was obtained either by reducing the variational principle (see [7, 13] and references therein) or by directly reducing the second order vector field describing the given system (see [4]).

The advantage of interpreting Routhian reduction in terms of MW-reduction lies in the fact that we are able to extend the concept of Routhian reduction to quasi-invariant Lagrangian systems, i.e. Lagrangian systems which are invariant up to a total time derivative. Such a generalization lies at hand: it is well known that a quasi-invariant Lagrangian determines a strict invariant energy and a strict invariant symplectic structure on the tangent bundle. On the other hand, the actual reduction of quasi-invariant Lagrangians exploits the full power of MW-reduction and is therefore in our opinion a very interesting application of this reduction procedure. The generalization to quasi-invariant Lagrangians is the main result of this paper.

Lagrangians with a quasi-cyclic coordinate. In the remainder of the introduction, we illustrate some of the concepts used in this paper by means of a simple, but clarifying example: the case of a Lagrangian with a single quasi-cyclic coordinate. This is a generalization of the classical procedure of Routh dealing with Lagrangians with a cyclic coordinate, and will serve as a conceptual introduction for the geometric techniques introduced later on, when we deal with the case of general quasi-invariant Lagrangians in Theorems 7 and 8.

We begin by recalling the classical form of Routh's result on the reduction of Lagrangians with cyclic coordinates (or, stated in a slightly different way, the reduction of Lagrangians which are invariant with respect to an abelian group action). For simplicity, we confine ourselves to the case of one cyclic coordinate. Subsequently, we will illustrate how this theorem can be extended to cover the case of quasi-cyclic coordinates.

Given a Lagrangian $L:\mathbb{R}^{2n}\to\mathbb{R}$ for a system with n degrees of freedom $(q^1,\ldots,q^n)$ for which, say, q^1 is a *cyclic coordinate* (i.e. $\partial L/\partial q^1=0$). The momentum $p_1=\partial L/\partial\dot q^1$ is a first integral of the Euler-Lagrange equations of motion. If $\partial^2L/\partial\dot q^1\partial\dot q^1\neq 0$ holds, there exists a function ψ such that $p_1=\mu$ is equivalent to $\dot q^1=\psi(q^2,\ldots,q^n,\dot q^2,\ldots,\dot q^n)$.

Theorem 1 (Routh reduction [17])**.** *Let $L:\mathbb{R}^{2n}\to\mathbb{R}$ be a regular Lagrangian for a system with n degrees of freedom $(q^1,\ldots,q^n)$. Assume that q^1 is a cyclic coordinate and that $\partial^2L/\partial\dot q^1\partial\dot q^1\neq 0$ so that $\dot q^1$ can be expressed as $\dot q^1=\psi(q^2,\ldots,q^n,\dot q^2,\ldots,\dot q^n)$. Consider the Routhian $R^\mu:\mathbb{R}^{2(n-1)}\to\mathbb{R}$ defined as the function $R^\mu=L-\dot q^1\mu$ where all instances of $\dot q^1$ are replaced by ψ. The Routhian is now interpreted as the Lagrangian for a system with $(n-1)$ degrees of freedom $(q^2,\ldots,q^n)$.*

Any solution $(q^1(t),\ldots,q^n(t))$ of the Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q^i}\right)-\frac{\partial L}{\partial q^i}=0,\ \ i=1,\ldots,n$$

with momentum $p_1=\mu$, projects onto a solution $(q^2(t),\ldots,q^n(t))$ of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial R^\mu}{\partial\dot q^k}\right)-\frac{\partial R^\mu}{\partial q^k}=0,\ \ k=2,\ldots,n.$$

Conversely, any solution of the Euler-Lagrange equations for R^μ can be lifted to a solution of the Euler-Lagrange equations for L with momentum $p_1=\mu$.

The number of degrees of freedom of the system with Lagrangian R^μ is reduced by one, and this technique is called Routh-reduction. We now formulate a generalization of this theorem for a Lagrangian system with a *quasi-cyclic coordinate* q^1, i.e. there exists a function f depending on $(q^1, \ldots, q^n)$ such that

$$\frac{\partial L}{\partial q^1} = \dot{q}^i \frac{\partial f}{\partial q^i}.$$

If q^1 is quasi-cyclic, it is easy to show that there is an associated first integral of the Lagrangian system given by $F := \partial L/\partial \dot{q}^1 - f$. Note that if $\partial^2 L/\partial \dot{q}^1 \partial \dot{q}^1 \neq 0$, we can again solve the equation $F = \mu$, where μ is a constant, to obtain an expression for $\dot{q}^1$ in terms of the remaining variables. In the next theorem we now show how the classical procedure of Routh may be extended to cover the case of a Lagrangian with a quasi-cyclic coordinate. We defer the proof of this theorem to section 5.1.

Theorem 2 (Routh reduction for a quasi-cyclic coordinate)**.** *A regular Lagrangian $L : \mathbb{R}^{2n} \to \mathbb{R}$ for a system with n degrees of freedom $(q^1, \ldots, q^n)$ with a quasi-cyclic coordinate q^1 is Routh-reducible if (i) $\partial^2 L/\partial \dot{q}^1 \partial \dot{q}^1 \neq 0$ and if (ii) there exist $(n-1)$ functions Γ_k independent of q^1 such that*

$$\frac{\partial f}{\partial q^k} = \Gamma_k(q^2, \ldots, q^n) \frac{\partial f}{\partial q^1}, \; k = 2, \ldots, n. \tag{1}$$

For μ a constant, consider the Routhian $R^\mu : \mathbb{R}^{2(n-1)} \to \mathbb{R}$ defined as

$$R^\mu = L - (\mu + f(q))(\dot{q}^1 + \Gamma_i \dot{q}^i),$$

where all instances of $\dot{q}^1$ are replaced by the expression obtained from the equation $\partial L/\partial \dot{q}^1 = \mu + f$. The Routhian is independent of q^1 and can be seen as a Lagrangian for a system with $(n-1)$ degrees of freedom $(q^2, \ldots, q^n)$.

Then, any solution $(q^1(t), \ldots, q^n(t))$ of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \; i = 1, \ldots, n$$

such that $\partial L/\partial \dot{q}^1 - f = \mu$, projects onto a solution $(q^2(t), \ldots, q^n(t))$ of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial R^\mu}{\partial \dot{q}^k}\right) - \frac{\partial R^\mu}{\partial q^k} = 0, \; k = 2, \ldots, n.$$

Conversely, any solution of the Euler-Lagrange equations for R^μ can be lifted to a solution of the Euler-Lagrange equations for L for which $\partial L/\partial \dot{q}^1 - f = \mu$

Readers familiar with methods from differential geometry might recognize that the functions Γ_k determine a *connection* on the configuration space. The condition (ii) from the above theorem can be interpreted geometrically as the existence of a connection for which *df* annihilates the horizontal distribution, or alternatively, such that f is covariantly constant: $Df = 0$ (with Df denoting the restriction of *df* to the horizontal distribution). It turns out that this condition is essential to Routhian reduction in the context of quasi-invariant Lagrangians.

We note that the requirement that *df* annihilates the horizontal distribution implies in this case that there exists an equivalent Lagrangian L' (*i.e.* a Lagrangian that differs from L by a total time derivative) which is strictly invariant so that Routhian reduction in the classical sense can be applied. However, we should warn against dismissing quasi-invariant Routh reduction too hastily since Routh reduction is possible also for quasi-invariant Lagrangians with nontrivial non-equivariance cocycle. We refer to [10] for a general discussion on quasi-invariant Lagrangian systems and in particular the property that the vanishing of this non-equivariance cocycle is a necessary condition for a quasi-invariant Lagrangian to be equivalent to a strict invariant Lagrangian.

To conclude this introduction, we note that the study of Routhian reduction for quasi-invariant Lagrangians was partially inspired on a technique called *functional Routhian reduction* described in [2], where it is used to obtain a control law for a three-dimensional bipedal robot. We will return to this example in section 5.2.

Plan of the paper. In sections 2 and 3 we show that classical Routhian reduction is precisely MW-reduction. We start with the well-known description of MW-reduction in the cotangent bundle framework. Although a description of cotangent bundle reduction may be found in [12], we will elaborate on this and prove the results because this will show useful when considering quasi-invariant Lagrangians. Next, in section 4 we describe MW-reduction for quasi-invariant Lagrangians. In section 5 we conclude with a number of examples.

2 Tangent and cotangent bundle reduction

In this section, we recall some standard results on group actions and principal bundles and we formulate Marsden-Weinstein reduction theorem in its standard form. We then specialize to the reduction of a cotangent bundle with the canonical symplectic form or a tangent bundle with a symplectic form which is obtained through pullback along the Legendre transformation. The material in this section is well-known and more information can be found in [11, 16].

2.1 Momentum maps and symplectic reduction

Notations. Throughout this paper we shall mainly adopt the notations from [3] and [15]. Let M be a manifold on which a group G acts on the right. This action is denoted by $\Psi : M \times G \to M$ and is such that $\Psi_{gh} = \Psi_h \circ \Psi_g$ for all $g, h \in G$, with $\Psi_g :\equiv \Psi(\cdot, g)$. The action Ψ induces a mapping on the Lie-algebra level

$$\varphi : M \times \mathfrak{g} \to TM : (m, \xi) \mapsto \varphi_m(\xi) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Psi(m, \exp \epsilon\xi).$$

The mapping $\mathfrak{g} \to \mathfrak{X}(M)$ associating to a Lie-algebra element ξ the corresponding infinitesimal generator $\xi_M \in \mathfrak{X}(M) : m \to \varphi_m(\xi)$ is a Lie-algebra morphism. The isotropy group $G_m < G$ of an element $m \in M$ is the subgroup of G determined by $\Psi(m, g) = m$. The Lie-algebra of G_m is denoted by $\mathfrak{g}_m$. The orbit $\mathcal{O}_m$ of m is the subset of M consisting of the elements of the form $\Psi(m, g)$ with $g \in G$ arbitrary. Finally, we will sometimes consider the dual to φ_m, i.e. the map $\varphi_m^* : T_m^*M \to \mathfrak{g}^*$. With a slight abuse of notation, the symbol φ^* will also be used to map a 1-form to a $\mathfrak{g}^*$-valued function on M, pointwise defined by $\varphi^*(\alpha)(m) = \varphi_m^*(\alpha(m))$, with α a 1-form and $m \in M$ arbitrary.

We will often assume that the action on a manifold M is free and proper. This guarantees that the space of orbits M/G is a manifold and that the projection $\pi : M \to M/G$ is a principal fibre bundle [8]. We assume that the reader is familiar with the concept of associated bundles of a principal manifold and, in particular, the bundle $\tilde{\mathfrak{g}}$ associated with the Lie-algebra $\mathfrak{g}$ on which the group acts on the left by means of the adjoint action. The adjoint action of G on its Lie-algebra $\mathfrak{g}$ is denoted by Ad_g, and is defined as the differential at the identity of the conjugation mapping. The dual to the adjoint action is called the coadjoint action and is denoted by Ad_g^*, i.e. $Ad_g^*(\mu) \in \mathfrak{g}^*$ for $\mu \in \mathfrak{g}^*$. We denote elements in $\tilde{\mathfrak{g}}$ by $\tilde{\xi}$ and they represent orbits of points in $Q \times \mathfrak{g}$ under the action of G defined by $(q, \xi) \mapsto (qg, Ad_{g^{-1}}\xi)$ with $q \in Q, g \in G$ and $\xi \in \mathfrak{g}$ arbitrary. In this sense we sometimes write $\tilde{\xi} = [q, \xi]_G$.

A principal connection on a manifold M on which G acts freely and properly is an equivariant $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on M such that, in addition, $\mathcal{A}(\xi_M) = \xi$ for all $\xi \in \mathfrak{g}$. The equivariance property is expressed by $\mathcal{A}_{\Psi_g(m)}(T\Psi_g(v_m)) = Ad_{g^{-1}}(\mathcal{A}_m(v_m))$, for any $m \in M, v_m \in T_mM$ and $g \in G$. The kernel of $\mathcal{A}$ determines a G-invariant distribution on M which is called the *horizontal* distribution since it is complementary to the vertical distribution $V\pi = \ker T\pi$, with $\pi : M \to M/G$. In this paper we will consider the dual of the linear map $\mathcal{A}_m : T_mM \to \mathfrak{g}$ which is understood to be a map $\mathcal{A}_m^* : \mathfrak{g}^* \to T_m^*M$. If $\mu \in \mathfrak{g}^*$, then the 1-form $\mathcal{A}^*(\mu) : M \to T^*M$ is defined pointwise by $m \mapsto \mathcal{A}_m^*(\mu)$. Again, with a slight abuse of notation, we sometimes write $\mathcal{A}^*(\mu) = \mathcal{A}_\mu$.

Throughout the paper we encounter products of bundles over the same base manifold B, say $E_1 \to B$ and $E_2 \to B$. The fibred product $E_1 \times_B E_2$ over the base manifold is often denoted simply by $E_1 \times E_2$ and consists of pairs (e_1, e_2) with $e_1 \in E_1$ and $e_2 \in E_2$ such that e_1 and e_2 project onto the same point in B.

Symplectic reduction. Let (M, ω) be a symplectic manifold on which G acts freely on the right, $\Psi : M \times G \to M$. The action Ψ is *canonical* if $\Psi_g^*\omega = \omega$ for all $g \in G$. If the infinitesimal generators ξ_M are globally hamiltonian vector fields, i.e. if there is a function J_ξ for any $\xi \in \mathfrak{g}$ such that $i_{\xi_M}\omega = -dJ_\xi$, then the map $J : M \to \mathfrak{g}^*$, is called a momentum map associated to the action.

Following [1], we define the *non-equivariance cocycle* associated to a momentum map of the canonical action:

$$\sigma : G \to \mathfrak{g}^* : g \mapsto J(mg^{-1}) - Ad_{g^{-1}}^*(J(m)),$$

where m is arbitrary in M. If M is connected this definition is independent of the choice of the point m and determines a $\mathfrak{g}^*$-valued one-cocycle σ in G, i.e. for $g, h \in G$ it satisfies

$$\sigma(gh) = \sigma(g) + Ad_{g^{-1}}^*\sigma(h).$$

If M is not connected we restrict the analysis to a connected component. Therefore, without further mentioning it, we will always assume that the manifolds we are considering are connected. Given another momentum map J' associated to the same action, its non-equivariance cocycle σ' determines the same element as σ in the first $\mathfrak{g}^*$-valued cohomology of G, i.e. $[\sigma] = [\sigma'] \in H^1(G, \mathfrak{g}^*)$. Note that for reasons of conformity, we haven chosen to define σ following [15] for left actions: recall that a right action composed with the group inversion is a left action.

If the moment map is not equivariant one can show (see [16]) that it becomes equivariant with respect to the *affine* action of G on $\mathfrak{g}^*$ determined using the cocycle σ and given by

$$(g, \mu) \mapsto Ad_g^*\mu + \sigma(g^{-1}).$$

Due to the fact that G acts freely on M - this is the only case we consider - any value of J is regular and, therefore, $J^{-1}(\mu)$ will be a submanifold of M for all $\mu \in J(M)$ [15].

Theorem 3 (Marsden-Weinstein reduction)**.** *Let (M, ω) be a symplectic manifold with G acting freely, properly and canonically on M. Let J be a momentum map for this action with non-equivariance cocycle σ. Assume that $\mu \in J(M)$, and denote by G_μ the isotropy of μ under the affine action of G on $\mathfrak{g}^*$. Then (M_μ, ω_μ), with $M_\mu = J^{-1}(\mu)/G_\mu$, is a symplectic manifold such that the 2-form ω_μ is uniquely determined by $i_\mu^*\omega = \pi_\mu^*\omega_\mu$, with $i_\mu : J^{-1}(\mu) \to M$ and $\pi_\mu : J^{-1}(\mu) \to M_\mu = J^{-1}(\mu)/G_\mu$.*

Let H denote a function on M, which is invariant under the action of G. Then, the Hamiltonian vector field X_H is tangent to $J^{-1}(\mu)$ and there exists a Hamiltonian h on M_μ with $\pi_\mu^* h = i_\mu^* H$, such that the restriction of X_H to $J^{-1}(\mu)$ is π_μ-related to X_h.

2.2 Cotangent bundle reduction

Consider now the case of a cotangent bundle T^*Q with its canonical symplectic structure $\omega_Q := d\theta_Q$, where θ_Q is the Cartan 1-form[1]. Let G be a Lie group acting freely and properly on Q from the right. Since a cotangent bundle is a special case of a symplectic manifold, the Marsden-Weinstein theorem obviously applies to T^*Q. However, because of the extra structure present on a cotangent bundle much more can be said in this case than one would expect from the Marsden-Weinstein theorem: see [11, 12].

The group G acts on Q by a right action Ψ and hence also on T^*Q by the cotangent lift of this action: $(g, \alpha) \mapsto T^*\Psi_{g^{-1}}(\alpha)$. The map $J := \varphi^* : T^*Q \to \mathfrak{g}^*$, defined by $\langle J(\alpha_q), \xi \rangle = \langle \alpha_q, \varphi_q(\xi) \rangle$, is a momentum map for this action. One can easily show that J is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$, or in other words, $J \circ T^*\Psi_{g^{-1}} = Ad_g^* \circ J$.

Recall that we assume that the action of G is free and proper so that the quotient Q/G is a manifold. In this case the quotient projection $\pi : Q \to Q/G$ defines a principal fiber bundle with structure group G. We denote the bundle of vertical vectors with respect to the projection π by $V\pi$. The subbundle $V^0\pi$ of T^*Q is defined as the annihilator of $V\pi$.

Fix a principal connection $\mathcal{A}$ on Q and let $\phi_{\mathcal{A}}^\mu$ be the map $J^{-1}(\mu) \to V^0\pi; \alpha_q \mapsto \phi_{\mathcal{A}}^\mu(\alpha_q) := \alpha_q - \mathcal{A}_q^*(\mu)$. This is an equivariant diffeomorphism w.r.t. the standard action of G_μ on $V^0\pi$, and its projection onto the quotient spaces is denoted by $[\phi_\mu^{\mathcal{A}}] : J^{-1}(\mu)/G_\mu \to V^0\pi/G_\mu$. The space $V^0\pi$ can be identified with $T^*(Q/G) \times Q$ and, consequently, the quotient space $V^0\pi/G_\mu$ can be identified with the product bundle $T^*(Q/G) \times Q/G_\mu$. We therefore conclude that the choice of a connection $\mathcal{A}$ allows us to identify $J^{-1}(\mu)/G_\mu$ with the bundle $T^*(Q/G) \times Q/G_\mu$ by means of the diffeomorphism $[\phi_\mu^{\mathcal{A}}]$.

Next, the 1-form $\mathcal{A}_\mu$ (which is also denoted by $\mathcal{A}^*(\mu)$) determines a G_μ-invariant 1-form on Q. It is not hard to show that $d\mathcal{A}_\mu$ is a 2-form on Q, projectable to a 2-form $\mathcal{B}_\mu$ on Q/G_μ. This follows from the invariance under the action of G_μ and the annihilation of fundamental vector fields of the form ξ_Q with ξ in the Lie-algebra $\mathfrak{g}_\mu$ of G_μ. In the following we consider the 2-form on $T^*(Q/G) \times Q/G_\mu$ determined as the sum of

- the pull-back to $T^*(Q/G) \times Q/G_\mu$ of $\omega_{Q/G}$ on $T^*(Q/G)$;
- the pull-back to $T^*(Q/G) \times Q/G_\mu$ of $\mathcal{B}_\mu$ on Q/G_μ.

[1] Let $\alpha \in T^*Q$, then $\theta_Q(\alpha)(X) = \langle \alpha, T\pi_Q(X) \rangle$ for arbitrary $X \in T_\alpha(T^*Q)$.

Let π_1, π_2 and p_μ be the projections $\pi_1 : T^*(Q/G)\times Q/G_\mu \to T^*(Q/G)$, $\pi_2 : T^*(Q/G)\times Q/G_\mu \to Q/G_\mu$ and $p_\mu : Q \to Q/G_\mu$, respectively. We further denote the natural injection $V^0\pi \to T^*Q$ by i_0. The above mentioned 2-form on $T^*(Q/G) \times Q/G_\mu$ equals

$$\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu.$$

Theorem 4 (Cotangent bundle reduction)**.** *Given a free and proper action of G on Q and consider its canonical lift to T^*Q. Let μ be any value of the momentum map, with isotropy subgroup G_μ. By fixing a principal connection $\mathcal{A}$, the symplectic manifold (M_μ, ω_μ) is symplectomorphic to $(T^*(Q/G) \times Q/G_\mu, \pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu)$, with symplectomorphism $[\phi_\mu^\mathcal{A}]$.*

We can summarize this in the diagram presented in Figure 1.

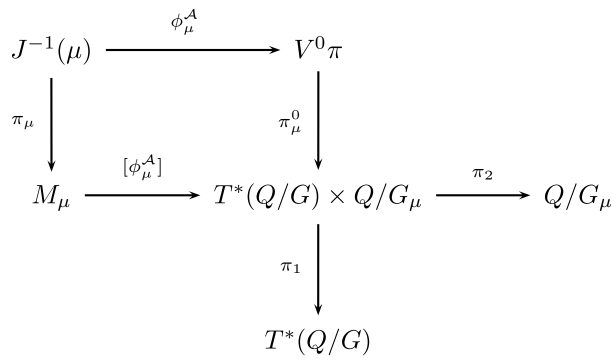

Figure 1: Cotangent bundle reduction.

Although this result is not new and can be found for instance in [11, 12], we include a proof because its method will turn out to be useful later on.

Proof. We know that $[\phi_\mu^\mathcal{A}]$ is a diffeomorphism, and therefore it only remains to show that the symplectic 2-form $\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu$ is pulled back to ω_μ under this map. We use the fact that ω_μ is uniquely determined by $i_\mu^*\omega_Q = \pi_\mu^*\omega_\mu$, with $i_\mu : J^{-1}(\mu) \to T^*Q$ the natural inclusion and $\pi_\mu : J^{-1}(\mu) \to M_\mu$ the projection to the quotient space. Due to the uniqueness property, it is therefore sufficient to show that

$$\pi_\mu^*([\phi_\mu^\mathcal{A}]^*(\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu)) = i_\mu^*\omega_Q. \tag{2}$$

We will slightly reformulate this condition, by using the fact that $i_\mu^*\theta_Q = (\phi_\mu^\mathcal{A})^*\big(i_0^*(\theta_Q + \pi_Q^*\mathcal{A}_\mu)\big)$ and $[\phi_\mu^\mathcal{A}] \circ \pi_\mu = \pi_\mu^0 \circ \phi_\mu^\mathcal{A}$:

$$(\pi_\mu^0)^*(\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu) = di_0^*(\theta_Q + \pi_Q^*\mathcal{A}_\mu) \tag{3}$$

The latter equality follows easily from the properties of the maps involved: we have that (i) $(\pi_2 \circ \pi_\mu^0)^*\mathcal{B}_\mu = (\pi_Q \circ i_0)^* d\mathcal{A}_\mu$ and (ii) $(\pi_1 \circ \pi_\mu^0)^*\theta_{Q/G} = i_0^*\theta_Q$ hold. □

The above description of cotangent bundle reduction can be seen as a special case of the more general result stating that if two symplectomorphic manifolds are both MW-reducible for the same symmetry group and have compatible actions, then the reduced spaces are also symplectomorphic. More specifically, given two symplectic manifolds (P,Ω) and (P',Ω') and a symplectomorphism $f : P \to P'$, i.e. $f^*\Omega' = \Omega$. We assume in addition that both P and P' are equipped with a canonical free and proper action of G. Let $J : P \to \mathfrak{g}^*$ and $J' : P' \to \mathfrak{g}^*$ denote corresponding momentum maps for these actions on P and P' respectively. We say that f is equivariant if $f(pg) = f(p)g$ for arbitrary $p \in P$, $g \in G$. Note that the non-equivariance cocycles for J and J' are equal up to a coboundary. Withouth loss of generality we assume $f^*J' = J$ and that the non-equivariance cocycles coincide. This in turn guarantees that the affine actions on $\mathfrak{g}^*$ coincide and that the isotropy group of an element $\mu \in \mathfrak{g}^*$ coincides for both affine actions. Finally, fix a value $\mu \in \mathfrak{g}^*$ of both J and J'.

Theorem 5. *If f is an equivariant symplectic diffeomorphism $P \to P'$ such that $J' = J \circ f$, then under MW-reduction, the symplectic manifolds (P_μ, Ω_μ) and (P'_μ, Ω'_μ) are symplectically diffeomorphic under the map*

$$[f_\mu] : P_\mu \to P'_\mu; [p]_{G_\mu} \mapsto [f(p)]_{G_\mu}.$$

Proof. This is a straightforward result. Since f is a diffeomorphism for which $J' = J \circ f$, the restriction f_μ of f to $J^{-1}(\mu)$ determines a diffeomorphism from $J^{-1}(\mu)$ to $J'^{-1}(\mu)$. The equivariance implies that f_μ reduces to a diffeomorphism $[f_\mu]$ from $P_\mu = J^{-1}(\mu)/G_\mu$ to $P'_\mu = J'^{-1}(\mu)/G_\mu$. It is our purpose to show that $[f_\mu]^*\Omega'_\mu = \Omega_\mu$ or, since both π_μ and π'_μ are projections, that $\pi^*_\mu\Omega_\mu = f^*_\mu(\pi'^*_\mu\Omega'_\mu)$. The determining property for Ω_μ and Ω'_μ is $\pi^*_\mu\Omega_\mu = i^*_\mu\Omega$ (similarly for Ω'_μ). From diagram chasing we have that $i^*_\mu\Omega = f^*_\mu(i'^*_\mu\Omega')$. Then

$$\pi^*_\mu\Omega_\mu = i^*_\mu\Omega = f^*_\mu(i'^*_\mu\Omega') = f^*_\mu(\pi'^*_\mu\Omega'_\mu) = \pi^*_\mu([f_\mu]^*\Omega'_\mu),$$

since $\pi'_\mu \circ f_\mu = [f_\mu] \circ \pi_\mu$ by definition. This concludes the proof. □

2.3 Tangent bundle reduction

We start by recalling the symplectic formulation of Lagrangian systems on the tangent bundle TQ of a manifold Q, and its relation to the canonical symplectic structure on T^*Q through the Legendre transform. Next, we shall consider Lagrangians invariant under the action of G, and study a general Marsden-Weinstein reduction scheme for such systems.

Definition 1. *A Lagrangian system is a pair (Q,L) where Q is called the configuration manifold and L is a smooth function on TQ. A Lagrangian system (Q,L) is said to be regular if the fibre derivative $\mathbb{F}L : TQ \to T^*Q; v_q \mapsto \mathbb{F}L(v_q)$ is a diffeomorphism. The map $\mathbb{F}L$ is called the Legendre transformation and is defined by*

$$\langle \mathbb{F}L(v_q), w_q\rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(v_q + \epsilon w_q),$$

for arbitrary $v_q, w_q \in T_qQ$.

Definition 2. *Given a free and proper action Ψ of G on Q, then a Lagrangian system (Q,L) is said to be invariant if L is an invariant function for the lifted action $(v_q, g) \mapsto T\Psi_g(v_q)$.*

Given a regular Lagrangian system (Q,L), one can define a symplectic structure on TQ by using the Legendre transform: we denote the 2-form on TQ obtained by pulling back ω_Q under $\mathbb{F}L$, by $\Omega^L_Q = (\mathbb{F}L)^*\omega_Q$. We will only consider regular Lagrangians throughout this paper. The following results are standard.

Theorem 6. *The lifted action $T\Psi$ of G on TQ is a canonical action for the symplectic manifold (TQ, Ω_Q^L). A momentum map is given by $J_L = J \circ \mathbb{F}L : TQ \to \mathfrak{g}^*$, and J_L is equivariant w.r.t. to the coadjoint action on $\mathfrak{g}^*$. Furthermore the Legendre transformation is an equivariant symplectomorphism between the symplectic manifolds (TQ, Ω_Q^L) and (T^*Q, ω_Q).*

The above theorem guarantees that Theorem 5 is applicable. We are now ready to draw the diagram in Figure 2, with $\mu \in \mathfrak{g}^*$.

$$\begin{array}{ccccc}
(TQ, \Omega_Q^L) & \xrightarrow{\mathbb{F}L} & (T^*Q, \omega_Q) & & \\
\text{MW-red} \downarrow & & \text{MW-red} \downarrow & & \\
(J_L^{-1}(\mu)/G_\mu, \Omega_\mu) & \xrightarrow{[\mathbb{F}L_\mu]} & (J^{-1}(\mu)/G_\mu, \omega_\mu) & \xrightarrow{[\phi_\mu^{\mathcal{A}}]} & (T^*(Q/G) \times Q/G_\mu, \pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu)
\end{array}$$

Figure 2: Diagram relating tangent and cotangent reduction

Next, we will show that the manifold $J_L^{-1}(\mu)/G_\mu$ is diffeomorphic to the fibred product $T(Q/G) \times Q/G_\mu$ if L satisfies an additional regularity assumption. Lagrangians satisfying this condition are called *G-regular*. We shall compute the map $[\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu]$ and show that it coincides with a Legendre transform for a function defined on $T(Q/G) \times Q/G_\mu$. This fact will eventually allow us to show that the reduced symplectic spaces are again originating from a Lagrangian system on Q/G_μ. The Lagrangian of this 'reduced' Lagrangian system is precisely the Routhian known from classical Routhian reduction.

We use the fixed connection $\mathcal{A}$ on Q to identify TQ/G with the bundle $T(Q/G) \times \tilde{\mathfrak{g}}$ in the standard way. This identification is obtained as follows: let $[v_q]_G \in TQ/G$ be arbitrary and fix a representative $v_q \in TQ$. The image in $T(Q/G) \times \tilde{\mathfrak{g}}$ of $[v_q]_G$ is defined as the element $(T\pi(v_q), \tilde{\xi})$ with $\pi : Q \to Q/G$ and $\tilde{\xi} = [q, \mathcal{A}(v_q)]_G \in \tilde{\mathfrak{g}}$. This map is invertible and determines a diffeomorphism (see for instance [3]). To define the inverse: let $(v_x, \tilde{\xi})$ be arbitrary in $T(Q/G) \times \tilde{\mathfrak{g}}$, and consider the tangent vector $v_q = (v_x)_q^h + \varphi_q(\xi)$ at $q \in \pi^{-1}(x)$, with $(v_x)_q^h$ the horizontal lift determined by $\mathcal{A}$ and ξ such that $\tilde{\xi} = [q, \xi]_G$. The inverse of $(v_x, \tilde{\xi})$ is the orbit $[v_q]_G \in TQ/G$ (the latter is well defined: one can show that it is independent of the point q, see also [3]).

Completely analogous one can show that TQ/G_μ is diffeomorphic to $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$. Indeed, let $[v_q]_{G_\mu} \in TQ/G_\mu$ be arbitrary and fix a representative $v_q \in TQ$, then the image of $[v_q]_{G_\mu}$ is defined by $(v_x, p_\mu(q), \tilde{\xi})$, with $T\pi(v_q) = v_x$ and $\tilde{\xi} = [q, \mathcal{A}(v_q)]_G$ (recall that $p_\mu : Q \to Q/G_\mu$). The construction of the inverse map uses the previous diffeomorphism and consists of three steps. Let $(v_x, y, \tilde{\xi}) \in T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$ be arbitrary. First, we consider the element $[v_q]_G$ in TQ/G which is the inverse of $(v_x, \tilde{\xi}) \in T(Q/G) \times \tilde{\mathfrak{g}}$. Secondly, we take a representative v_q of $[v_q]_G$ at a point $q \in p_\mu^{-1}(y)$. And finally, we consider $[v_q]_{G_\mu}$. It is not hard to show that this inverse is well-defined (i.e. independent of the chosen representative v_q).

An invariant Lagrangian L determines a function on the quotient TQ/G, and under the identification determined above, a function l on $T(Q/G) \times \tilde{\mathfrak{g}}$. We define the fibre derivative $\mathbb{F}_{\tilde{\xi}} l : T(Q/G) \times \tilde{\mathfrak{g}} \to T(Q/G) \times \tilde{\mathfrak{g}}^*$ by

$$\langle \mathbb{F}_{\tilde{\xi}} l(v_x, \tilde{\xi}), (v_x, \tilde{\eta}) \rangle := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} l(v_x, \tilde{\xi} + \epsilon \tilde{\eta}) \,.$$

Definition 3. *An invariant Lagrangian L is said to be G-regular if the map $\mathbb{F}_{\tilde{\xi}}l : T(Q/G) \times \tilde{\mathfrak{g}} \to T(Q/G) \times \tilde{\mathfrak{g}}^*$ is a diffeomorphism.*

We remark here that according to the previous definition, G-regularity depends on the chosen connection $\mathcal{A}$. However, we mention here that G-regularity can alternatively be defined as a condition on L directly. We refer the reader to [9] for a detailed discussion on G-regularity.

A momentum value μ determines in the quotient spaces a mapping $\tilde{\mu} : Q/G_\mu \to \tilde{\mathfrak{g}}^*$ as follows: let $y \in Q/G_\mu$ be arbitrary

$$\langle \tilde{\mu}(y), \tilde{\xi} \rangle = \langle \mu, \xi \rangle,$$

with ξ the unique representative of $\tilde{\xi} = [q, \xi]_G$ at a point $q \in p_\mu^{-1}(y)$. Recall that p_μ denotes the projection $p_\mu : Q \to Q/G_\mu$. Due to the identification $TQ/G_\mu \cong T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$ the manifold $J_L^{-1}(\mu)/G_\mu$ is a subset of $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$. In the following lemma we characterize this subset in terms of $\tilde{\mu}$ and $\mathbb{F}_{\tilde{\xi}}l$.

Lemma 1. *There is a one-to-one correspondence between $J_L^{-1}(\mu)/G_\mu$ and the subset of $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$ determined as the set of points $(v_x, y, \tilde{\xi})$ that satisfy the condition $\mathbb{F}_{\tilde{\xi}}l(v_x, \tilde{\xi}) = (v_x, \tilde{\mu}(y))$.*

Proof. Consider a point $[v_q]_{G_\mu}$ in $J_L^{-1}(\mu)/G_\mu$ and let v_q be a representative. Then, by definition, $J_L(v_q) = \mu$, and since L is invariant, we have $L(v_q) = l(v_x, \tilde{\xi})$, with $(v_x, \tilde{\xi})$ the element in $TQ/G \cong T(Q/G) \times \tilde{\mathfrak{g}}$ corresponding to $[v_q]_G$. Using the definition of the momentum map J_L, we obtain

$$\langle J_L(v_q), \eta \rangle = \frac{d}{d\epsilon}\bigg|_0 L(v_q + \epsilon\varphi_q(\eta)) = \frac{d}{d\epsilon}\bigg|_0 l(v_x, \tilde{\xi} + \epsilon\tilde{\eta}) = \langle \mathbb{F}_{\tilde{\xi}}l(v_x, \tilde{\xi}), (v_x, \tilde{\eta}) \rangle,$$

with $\tilde{\eta} = [q, \eta]_G$. □

Lemma 2. *Let L be G-regular invariant Lagrangian. Then there is a diffeomorphism between $J_L^{-1}(\mu)/G_\mu$ and $T(Q/G) \times Q/G_\mu$.*

Proof. We define a map $J_L^{-1}(\mu)/G_\mu \to T(Q/G) \times Q/G_\mu$ and its inverse. Let $[v_q]_{G_\mu} \in J_L^{-1}(\mu)/G_\mu$, with $v_q \in J_L^{-1}(\mu)$ a representative at q. We again use the fixed connection $\mathcal{A}$ on Q, and we introduce the maps:

$$p_1([v_q]_{G_\mu}) := T\pi(v_q), \qquad p_2([v_q]_{G_\mu}) := p_\mu(q), \qquad p_3([v_q]_{G_\mu}) := [q, \mathcal{A}(v_q)]_G,$$

with $\pi : Q \to Q/G$ and $p_\mu : Q \to Q/G_\mu$. These maps $p_{1,2,3}$ are simply the restrictions to $J_L^{-1}(\mu)/G_\mu$ of the projections on the first, second and third factors in the product $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$. It is easily verified that $(p_1, p_2) : J_L^{-1}(\mu)/G_\mu \to T(Q/G) \times Q/G_\mu$ is smooth.

We now define the inverse map ψ_μ of (p_1, p_2). Let $(v_x, y) \in T(Q/G) \times Q/G_\mu$ be arbitrary and define the element $\tilde{\xi} \in \tilde{\mathfrak{g}}_x$ such that $(v_x, \tilde{\xi}) = (\mathbb{F}_{\tilde{\xi}}l)^{-1}(v_x, \tilde{\mu}(y))$ (here we use the condition that L is G-regular). Now consider the tangent vector $v_q = (v_x)_q^h + \varphi_q(\xi)$, where ξ is such that $\tilde{\xi} = [q, \xi]_G$ and $q \in p_\mu^{-1}(y)$. By construction we have on the one hand that $J_L(v_q) = \mu$ and on the other hand $(p_1, p_2)([v_q]_{G_\mu}) = (v_x, y)$. □

Combined with Figure 2, we can now draw the diagram in Figure 3 below.

There is an interesting local criterium for a Lagrangian L to be G-regular. We say that L is *locally G-regular* if given a point $[v_q]_{G_\mu}$ in $J_L^{-1}(\mu)/G_\mu$, there is a neighborhood U of $[v_q]_{G_\mu}$ such that the restriction $(p_1, p_2)|_U$ is a diffeomorphism from U to its image $(p_1, p_2)(U)$.

$$
\begin{array}{ccc}
(TQ, \Omega_Q^L) & \xrightarrow{\quad \mathbb{F}L \quad} & (T^*Q, \omega_Q) \\
\text{MW-red} \Big\downarrow & & \Big\downarrow \text{MW-red} \\
(T(Q/G) \times Q/G_\mu, \tilde{\Omega}_\mu) & \xrightarrow{[\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu} & (T^*(Q/G) \times Q/G_\mu, \pi_1^* \omega_{Q/G} + \pi_2^* \mathcal{B}_\mu)
\end{array}
$$

Figure 3: Diagram relating tangent and cotangent reduction for G-regular Lagrangians

Lemma 3. *An invariant Lagrangian L is locally G-regular if one of the following two equivalent conditions hold:*

1. $T(J_L^{-1}(\mu)) \oplus V_{J_L^{-1}(\mu)}\varphi = T_{J_L^{-1}(\mu)}(TQ)$, *with $V\varphi \subset V\tau_Q \subset T(TQ)$ defined as the set of tangent vectors of the form $(\varphi_q(\xi))^v_{v_q} = \xi_Q^v(v_q)$, $\xi \in \mathfrak{g}$ arbitrary, where ξ_Q is the fundamental vector field of the action on Q corresponding to ξ, and $\cdot^v$ denotes the vertical lift $TQ \times TQ \to T(TQ)$.*
2. *The 'vertical' Hessian of l, defined as*
$$D^2 l(v_x, \tilde{\xi})(\tilde{\eta}, \tilde{\eta}') := \frac{\partial^2 l}{\partial\epsilon\partial\epsilon'}(v_x, \tilde{\xi} + \epsilon\tilde{\eta} + \epsilon'\tilde{\eta}')|_{\epsilon=\epsilon'=0}$$
for $v_x \in T(Q/G)$ and $\tilde{\xi}, \tilde{\eta}, \tilde{\eta}' \in \tilde{\mathfrak{g}}_x$, is invertible.

Proof. Note that for all $v_q \in J_L^{-1}(\mu)$,

$$\dim T_{v_q}(J_L^{-1}(\mu)) + \dim V\varphi(v_q) = (\dim TQ - \dim \mathfrak{g}) + \dim \mathfrak{g} = \dim T_{v_q}(TQ).$$

The direct-sum decomposition in (1) is therefore equivalent to the statement that $T(J_L^{-1}(\mu)) \cap V_{J_L^{-1}(\mu)}\varphi = 0$. We will now prove that this is equivalent to the vertical Hessian of l being invertible.

Assume that the intersection $T(J_L^{-1}(\mu)) \cap V\varphi$ contains a non-zero element. Such an element is necessarily of the form $(\xi_Q)^v(v_q)$, where $\xi \in \mathfrak{g}$ and $\xi \neq 0$. Expressing the fact that this element is contained in $T(J_L^{-1}(\mu))$ implies that for every $\eta \in \mathfrak{g}$, $\langle TJ_L((\xi_Q)^v(v_q)), \eta\rangle = 0$. This can be made more explicit as follows:

$$\begin{aligned}
\langle TJ_L((\xi_Q)^v(v_q)), \eta\rangle &= \frac{d}{ds} J_L(v_q + s\varphi_q(\xi))(\eta)\Big|_{s=0} \\
&= \frac{d}{ds} \mathbb{F}_{\tilde{\xi}} l(v_x, \tilde{\zeta} + s\tilde{\xi})(\tilde{\eta})\Big|_{s=0} \\
&= D^2 l(v_x, \tilde{\zeta})(\tilde{\xi}, \tilde{\eta}),
\end{aligned}$$

where we have decomposed v_q in its vertical and horizontal parts as $v_q = (v_x)_q^h + \varphi_q(\zeta)$. Since this holds for every $\eta \in \mathfrak{g}$, we conclude that $(\xi_Q)^v(v_q)$ is contained in the intersection $T(J_L^{-1}(\mu)) \cap V_{J_L^{-1}(\mu)}\varphi$ if and only if the associated section $\tilde{\xi}$ is in the null space of $D^2 l(v_x, \tilde{\zeta})$. Hence, the two statements in Lemma 3 are equivalent.

If $D^2 l(v_x, \tilde{\zeta})$ is invertible, then via the implicit function theorem, the reduced Legendre transformation is locally invertible. The method of proof of the previous Lemma 2 can be used to show that, locally, (p_1, p_2) is invertible. □

Note in passing that if the given L is a mechanical Lagrangian, i.e. it is of type kinetic minus potential, then L is G-regular if the locked inertia tensor, defined by the restriction of the kinetic energy metric to the fundamental vector fields (see e.g. [13]), is non-degenerate. In our language, the 'reduced' locked inertia tensor coincides with $D^2 l_x : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*$ in the following sense:

$$D^2 l_x(\tilde{\xi}, \tilde{\eta}) = J(q)(\xi, \eta),$$

with $\tilde{\xi} = [q, \xi]_G$ and $\tilde{\eta} = [q, \eta]_G$ arbitrary.

3 Routhian reduction

In this section, we make a start with Routhian reduction. We consider a Lagrangian $L : TQ \to \mathbb{R}$ which is invariant under the action of a Lie group G and as before we consider a connection $\mathcal{A}$ in the bundle $\pi : Q \to Q/G$. Furthermore, let $\mu \in \mathfrak{g}^*$ be a fixed momentum value, and define the function R^μ as $R^\mu = L - \mathcal{A}_\mu$ (recall that $\mathcal{A}_\mu : TQ \to \mathbb{R}$ is the connection 1-form contracted with $\mu \in \mathfrak{g}^*$). By definition R^μ is G_μ-invariant and in particular, its restriction to $J_L^{-1}(\mu)$ is reducible to a function $[R^\mu]$ on the quotient $J_L^{-1}(\mu)/G_\mu$. In turn, we denote the function on $T(Q/G) \times Q/G_\mu$ corresponding to $[R^\mu]$ by $\mathcal{R}^\mu$, i.e.$\mathcal{R}^\mu = \psi_\mu^*[R^\mu]$. The function $\mathcal{R}^\mu$ is called the *Routhian.*

We begin by reconsidering some aspects from the reduction theory of tangent bundles, which we relate to the geometry of the Routhian. Recall from the diagram in Figure 3 that we may write the symplectic 2-form $\tilde{\Omega}_\mu$ obtained from MW-reduction as

$$\left([\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu\right)^* \left(\pi_1^* \omega_{Q/G} + \pi_2^* \mathcal{B}_\mu\right).$$

Lemma 4. *The map $[\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu$ is the fibre derivative of the Routhian $\mathcal{R}^\mu$, i.e. for $(v_x, y), (w_x, y) \in T(Q/G) \times Q/G_\mu$ arbitrary*

$$\left\langle \left([\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu\right)(v_x, y), (w_x, y) \right\rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{R}^\mu(v_x + \epsilon w_x, y) =: \langle \mathbb{F}\mathcal{R}^\mu(v_x, y), (w_x, y) \rangle.$$

Proof. Fix elements $(v_x, y) \in T(Q/G) \times Q/G_\mu$ and fix a $v_q \in J_L^{-1}(\mu)$ that projects onto $\psi_\mu(v_x, y)$. By definition of the maps involved, we have

$$\left([\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu\right)(v_x, y) = (\pi_\mu^0 \circ \phi_\mu^{\mathcal{A}})(\mathbb{F}L(v_q)) = \pi_\mu^0(\mathbb{F}L(v_q) - \mathcal{A}_\mu(q)).$$

Fix a curve $\epsilon \mapsto \zeta(\epsilon)$ in $J_L^{-1}(\mu)$ that projects onto the curve $\epsilon \mapsto \psi_\mu(v_x + \epsilon w_x, y)$ in $J_L^{-1}(\mu)/G_\mu$ and such that $\zeta(0) = v_q$ and $\dot{\zeta}(0)$ is vertical to the projection $\tau_Q \circ i_\mu : J_L^{-1}(\mu) \to Q$. The existence of such a curve is best shown using Lemma 3 and some coordinate computations. For that purpose, fix a bundle adapted coordinate chart on $Q \to Q/G$, and let (x^i, g^a) denote the coordinate functions with $i = 1, \dots, \dim Q/G$ and $a = 1, \dots, \dim G$. From Lemma 3, where it was shown that $TJ_L^{-1}(\mu)$ is transversal to $V\varphi$, we deduce that (x^i, v^i, g^a) are (local) coordinate functions for $J_L^{-1}(\mu)$, with (x^i, v^i) a standard coordinate chart on $T(Q/G)$ associated to (x^i) on Q/G. In this coordinate chart we put $v_q = (x_0^i, v_0^i, g_0^a)$ and $w_x = (x_0^i, w_0^i)$, and we define the curve $\zeta(\epsilon)$ to be the curve $\epsilon \mapsto (x_0^i, v_0^i + \epsilon w_0^i, g_0^a)$. Then the tangent to ζ at $\epsilon = 0$ is the vertical lift of some $w_q \in T_q Q$ with $T\pi(w_q) = w_x$.

Finally, from the definition of $\mathcal{R}^\mu$,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{R}^\mu(v_x + \epsilon w_x, y) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (L(\zeta(\epsilon)) - \mathcal{A}_\mu(\zeta(\epsilon))) = \langle \mathbb{F}L(v_q) - \mathcal{A}_\mu(q), w_q \rangle.$$

Since $\mathbb{F}L(v_q) - \mathcal{A}_\mu(q) \in V^0\pi$, the right-hand side of this equation can be rewritten as a contraction with (w_x, y):

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{R}^\mu(v_x + \epsilon w_x, y) = \langle \pi^0_\mu(\mathbb{F}L(v_q) - \mathcal{A}_\mu(q)), (w_x, y)\rangle.$$

This concludes the proof. □

The above lemma allows us to compute the reduced symplectic 2-form on the manifold $T(Q/G) \times Q/G_\mu$.

$$(\mathbb{F}\mathcal{R}^\mu)^*\big(\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu\big) = (\mathbb{F}\mathcal{R}^\mu)^*\big(\pi_1^*\omega_{Q/G}\big) + \overline{\pi}_2^*\mathcal{B}_\mu,$$

with $\overline{\pi}_2 : T(Q/G) \times Q/G_\mu \to Q/G_\mu$. In order to complete the symplectic reduction we now study the energy-function (this is the Hamiltonian function for the Euler-Lagrange equations). Recall that the energy E_L corresponding with the Lagrangian system (Q, L) is the function on TQ defined by $E_L(v_q) = \langle \mathbb{F}L(v_q), v_q\rangle - L(v_q)$, for $v_q \in TQ$ arbitrary. The energy for the Routhian $\mathcal{R}^\mu$ is defined by

$$E_{\mathcal{R}^\mu}(v_x, y) = \langle \mathbb{F}\mathcal{R}^\mu(v_x, y), (v_x, y)\rangle - \mathcal{R}^\mu(v_x, y),$$

with $(v_x, y) \in T(Q/G) \times Q/G_\mu$ arbitrary.

Lemma 5. *The energy $E_{\mathcal{R}^\mu}$ is the reduced Hamiltonian, i.e. it satisfies:*

$$((p_1, p_2) \circ \pi_\mu)^* E_{\mathcal{R}^\mu} = i_\mu^* E_L,$$

with $\pi_\mu : J_L^{-1}(\mu) \to J_L^{-1}(\mu)/G_\mu$ and $i_\mu : J_L^{-1}(\mu) \to TQ$.

Proof. Let $v_q \in J_L^{-1}(\mu)$, such that $((p_1, p_2) \circ \pi_\mu)(v_q) = (v_x, y)$. Then

$$\begin{aligned} i_\mu^* E_L(v_q) &= \langle \mathbb{F}L(v_q), v_q\rangle - L(v_q) \\ &= \langle \phi_\mu^{\mathcal{A}}(\mathbb{F}L_\mu(v_q)) + \mathcal{A}_q^*(\mu), v_q\rangle - L(v_q) \\ &= \big\langle \big([\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu\big)(v_x, y), (v_x, y)\big\rangle - \mathcal{R}^\mu(v_x, y). \end{aligned}$$

Using Lemma 4 this concludes the proof. □

We end this section with some additional definitions in order to interpret the MW-reduced system as a Lagrangian system (we also refer to [9]). For that purpose consider a manifold M fibred over N with projection $\kappa : M \to N$. Roughly said, a Lagrangian L with configuration space M is said to be *intrinsically constrained* if it does not depend on the velocities of the fibre coordinates of $\kappa : M \to N$. This is made more precise in the following definition.

Definition 4. *A Lagrangian system (M, L) on a fibred manifold $\kappa : M \to N$ is intrinsically constrained if L is the pull-back of a function L' on $T_M N = TN \times_N M$ along the projection $TM \to T_M N$.*

For notational simplicity we will identify L with L'. If we fix a coordinate neighborhood (x^i, y^a) on M adapted to the fibration, we can write the Euler-Lagrange equations for this system. The fact that the Lagrangian is intrinsically constrained is locally expressed by the fact that $L(x, \dot{x}, y)$ is independent of $\dot{y}$, and the Euler-Lagrange equations then read:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0,\ i = 1, \ldots, n \text{ and } \frac{\partial L}{\partial y^a} = 0,\ a = 1, \ldots, k.$$

The latter k equations determine constraints on the system. We now wish to write these equations as Hamiltonian equations w.r.t. a presymplectic 2-form on $T_M N$. For that purpose, we associate to the Lagrangian $L : T_M N \to \mathbb{R}$ a *Legendre transform* $\mathbb{F}L : T_M N \to T_M^* N$. The definition is given by, for $(v_n, m), (w_n, m) \in T_M N$ arbitrary

$$\langle \mathbb{F}L(v_n, m), (w_n, m)\rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(v_n + \epsilon w_n, m),$$

In coordinates $\mathbb{F}L(x^i, \dot{x}^i, y^a)$ simply reads $(x^i, \partial L/\partial \dot{x}^i, y^a)$. Finally, if we write the projection $T_M^* N \to T^* N; (\alpha_n, m) \to \alpha_n$ by κ_1, then it is not hard to show that the pull-back to $T_M N$ of the canonical symplectic form ω_N under the map $\kappa_1 \circ \mathbb{F}L : T_M N \to T^* N$ determines a presymplectic 2-form, locally equal to

$$d\left(\frac{\partial L}{\partial \dot{x}^i}\right) \wedge dx^i.$$

We define the energy as the function

$$E_L : T_M N \to \mathbb{R}; (v_n, m) \to \langle \mathbb{F}L(v_n, m), (v_n, m)\rangle - L,$$

and the solutions $m(t)$ to the Euler-Lagrange equations solve the equation

$$\left.(i_{\dot\gamma}(\kappa_1 \circ \mathbb{F}L)^* \omega_N = -dE_L)\right|_\gamma$$

with $\gamma(t) = (\dot{n}(t), m(t))$ and $n(t) = \kappa(m(t))$ (see also [5, 6]).

If the original intrinsically constrained Lagrangian system (M, L) is non-conservative with a *gyroscopic force term*, i.e. a 2-form β on M is given and the force term is the function $TM \to T^*M; v_m \mapsto -i_{v_m}\beta_m$, then the Euler-Lagrange equations of motion are Hamiltonian w.r.t (pre)-symplectic form $(\kappa_1 \circ \mathbb{F}L)^*\omega_N + \kappa_2^*\beta$ and with Hamiltonian E_L:

$$\left.(i_{\dot\gamma}\left((\kappa_1 \circ \mathbb{F}L)^*\omega_N + \kappa_2^*\beta\right) = -dE_L)\right|_\gamma.$$

Here κ_2 denotes the projection to the second factor in $T_M N$, i.e. $\kappa_2 : T_M N \to M$. In the case of Routhian reduction, the reduced space is of this type: the total space corresponds to Q/G_μ and the base space N to Q/G.

Theorem 7. *Given a G-invariant, G-regular Lagrangian L defined on the configuration space Q. Then the MW-reduction of the symplectic manifold (Q, Ω_L) for a momentum value $J_L = \mu$ is the symplectic manifold*

$$\big(T(Q/G) \times Q/G_\mu, (\mathbb{F}\mathcal{R}^\mu)^*\big(\pi_1^*\omega_{Q/G}\big) + \overline{\pi}_2^*\mathcal{B}_\mu\big).$$

The reduced Hamiltonian of E_L is the energy $E_{\mathcal{R}^\mu}$. The equations of motion for this Hamiltonian vector field are precisely the Euler-Lagrange equations of motion for an intrinsically constrained Lagrangian system on $Q/G_\mu \to Q/G$ with Lagrangian $\mathcal{R}^\mu$ and gyroscopic force term determined by the 2-form $\mathcal{B}_\mu$ on Q/G_μ.

It is remarkable that the 2-form $\mathcal{B}_\mu$ is such that the presymplectic 2-form $(\mathbb{F}\mathcal{R}^\mu)^*\big(\pi_1^*\omega_{Q/G}\big) + \overline{\pi}_2^*\mathcal{B}_\mu\big)$ is symplectic. A next step in Routhian reduction would be to identify $\mathcal{B}_\mu$ as a 2-from which is built up o.a. out of the curvature of $\mathcal{A}$ and a nondegenerate part on the fibres of $Q/G_\mu \to Q/G$. Since this is not the scope of this paper, we refer the reader to [9, 13].

4 Quasi-invariant Lagrangians

In this section we study a possible generalization of the Routhian reduction procedure to quasi-invariant Lagrangians. We refer the reader to [10] and references therein for further details on quasi-invariant Lagrangians. We assume throughout this section that Q is a connected manifold, which ensures that given a function f for which $df = 0$ implies that f is constant.

4.1 Quasi-invariance and cocycles

We begin by defining what it means for a Lagrangian to be quasi-invariant under a group action. We then show that the transformation behaviour of a quasi-invariant Lagrangian induces a certain cocycle on the space of 1-forms, and we study the properties of this cocycle.

Definition 5. *A Lagrangian system (Q, L) is quasi-invariant if the Lagrangian satisfies*

$$(T\Psi_g)^* L(v_q) = L(v_q) + \langle v_q, dF_g(q)\rangle,$$

with v_q arbitrary and for some function $F : G \times Q \to \mathbb{R}$. We denote a quasi-invariant Lagrangian system as a triple (Q, L, F).

Clearly, the function F is not arbitrary: from the fact that Ψ defines a right action it follows that $(T\Psi_{gh})^* L = \big((T\Psi_g)^* \circ (T\Psi_h)^*\big)L$ and one can see that $dF : G \to \mathcal{X}^*(Q)$ should define a group 1-cocycle with values in the G-module of 1-forms on Q, i.e. for $g_1, g_2 \in G$ arbitrary

$$\Psi_{g_1}^* dF_{g_2} - dF_{g_1 g_2} + dF_{g_1} = 0.$$

Consider the map $f : \mathfrak{g} \times Q \to \mathbb{R}$ defined by

$$f(\xi, q) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(\exp \epsilon\xi, q).$$

Clearly, f is linear in its first argument, and thus determines a map $Q \to \mathfrak{g}^*$ which is denoted by the same symbol. We now define a 1-cocycle with values in $\mathfrak{g}^*$.

Lemma 6. *The map*

$$\sigma_F : G \to \mathfrak{g}^* : g \mapsto Ad^*_{g^{-1}} f(q) - \big(\Psi^*_{g^{-1}} f\big)(q) + Ad^*_{g^{-1}}\big(\varphi^*_q\big(dF_{g^{-1}}(q)\big)\big).$$

does not depend on the chosen point q and determines a group 1-cocycle with values in $\mathfrak{g}^*$.

Proof. We first show that the differential of

$$q \mapsto f_{Ad_g\xi}(q) - \Psi_g^* f_\xi(q) + \langle (Ad_g\xi)_Q(q), dF_g(q)\rangle$$

vanishes for arbitrary $\xi \in \mathfrak{g}$. This implies that the above definition of σ_F does not depend on the chosen point q.

We start from the cocycle property of the map $g \mapsto dF_g$, i.e. we have $\Psi_{g_1}^* dF_{g_2} - dF_{g_1 g_2} + dF_{g_1} = 0$. Let $g_1 = g$ and $g_2 = \exp \epsilon\xi$, and take the derivative at $\epsilon = 0$, then

$$\Psi_g^* df_\xi - \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} dF_{g \exp \epsilon\xi} = 0.$$

To compute the second term we again use the cocycle property with $g_1 = g(\exp \epsilon\xi)g^{-1}, g_2 = g$, i.e. $dF_{g \exp \epsilon\xi} = dF_{(\exp \epsilon Ad_g \xi)g} = \Psi^*_{\exp \epsilon Ad_g \xi} dF_g + dF_{\exp \epsilon Ad_g \xi}$. The derivative with respect to ϵ at 0 equals

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} dF_{g \exp \epsilon\xi}(q) = d\left(\langle\langle (Ad_g\xi)_Q, dF_g\rangle\rangle\right)(q) + df_{Ad_g\xi}.$$

We conclude that the map $\langle \sigma_F(g), \xi\rangle = f_{Ad_{g^{-1}}\xi}(q) - \Psi^*_{g^{-1}} f_\xi(q) + \langle\langle (Ad_{g^{-1}}\xi)_Q(q), dF_{g^{-1}}(q)\rangle\rangle$ is independent of q and therefore well-defined. From straightforward computations it follows that it is a group 1-cocycle with values in $\mathfrak{g}^*$: for g_1, g_2 arbitrary,

$$Ad^*_{g_1^{-1}} \sigma_F(g_2) - \sigma_F(g_1 g_2) + \sigma_F(g_1) = 0.$$

This concludes the proof. □

This 1-cocycle induces a $\mathfrak{g}^*$-valued 1-cocycle on the Lie-algebra, given by

$$\xi \mapsto -ad^*_\xi f + \xi_Q(f) - \varphi^*(df_\xi);$$

and hence also a real valued 2-cocycle $\Sigma_f(\xi, \eta) = \xi_Q(f_\eta) - \eta_Q(f_\xi) - f_{[\xi,\eta]}$. This is the cocycle used in the infinitesimal version of quasi-invariant Lagrangians discussed in for instance [10]. If only an infinitesimal action is given, i.e. a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(Q); \xi \mapsto \xi_Q$; or by complete lifting, an infinitesimal action on TQ, then the above definition of 1-cocycle σ_F corresponds infinitesimally to Σ_f. It is often easier to compute Σ_f instead of σ_F in examples, see section 5.

4.2 The momentum map

As mentioned in the introduction, Noether's theorem is applicable to quasi-invariant Lagrangians as well: for each Lagrangian that is quasi-invariant under a group action, there exists a momentum map which is conserved. In this section, we study the properties of this momentum map, with a view towards performing symplectic reduction later on.

We begin by investigating the equivariance of the Legendre transformation.

Lemma 7. *Let (Q, L, F) denote a quasi-invariant system. Then, for $g \in G$ arbitrary, the Legendre map $\mathbb{F}L$ transforms as*

$$\mathbb{F}L(T\Psi_g(v_q)) = T^*\Psi_{g^{-1}}\left(\mathbb{F}L(v_q) + dF_g(q)\right) = T^*\Psi_{g^{-1}}\left(\mathbb{F}L(v_q)\right) - dF_{g^{-1}}(qg).$$

Proof. To show this equality, fix an element $w_{qg} \in TQ$, and let $w_q = T\Psi_{g^{-1}}(w_{qg})$. Then, by definition of the fibre derivative,

$$\begin{aligned}\langle w_{qg}, \mathbb{F}L(T\Psi_g(v_q))\rangle &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(T\Psi_g(v_q) + \epsilon w_{qg}) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(L(v_q + \epsilon w_q) + \langle v_q + \epsilon w_q, dF_g(q)\rangle\right) \\ &= \langle w_{qg}, T^*\Psi_{g^{-1}}\left(\mathbb{F}L(v_q) + dF_g(q)\right)\rangle.\end{aligned}$$

From $\Psi^*_g(dF_{g^{-1}}) = -dF_g$ (let $g_1 = g, g_2 = g^{-1}$ in the cocycle identity for dF) we have the property that $T^*\Psi_{g^{-1}}(dF_g(q)) = -dF_{g^{-1}}(qg)$ for $q \in Q$ and $g \in G$ arbitrary. This concludes the proof. □

The above lemma justifies the next definition.

Definition 6. *Let (Q, L, F) denote a quasi-invariant Lagrangian system. Then we define a right action Ψ_{aff} on T^*Q as follows. For $\alpha_q \in T^*Q$ arbitrary, we put:*

$$\Psi_{\mathrm{aff},g}(\alpha_q) = T^*\Psi_{g^{-1}}\left(\alpha_q + dF_g(q)\right) = T^*\Psi_{g^{-1}}\left(\alpha_q\right) - dF_{g^{-1}}(qg).$$

We say that Ψ_{aff} is the affine action on T^*Q associated to the 1-cocycle dF.

We should check that the affine action is well defined. For that purpose, we need to verify that for g_1, g_2 arbitrary

$$T^*\Psi_{(g_1g_2)^{-1}}\big(\alpha_q + dF_{g_1g_2}(q)\big) = T^*\Psi_{g_2^{-1}}\Big(T^*\Psi_{g_1^{-1}}\big(\alpha_q + dF_{g_1}(q)\big) + dF_{g_2}(qg_1)\Big).$$

This is a straightforward consequence from the fact that dF is a group 1-cocycle.

Lemma 8. *Let (Q, L, F) denote a quasi-invariant Lagrangian system. Then,*

1. *the lifted action $T\Psi$ is a canonical action for the symplectic structure (TQ, Ω_Q^L);*
2. *the map $J_L^f = \varphi^* \circ \mathbb{F}L - \tau_Q^* f : TQ \to \mathfrak{g}^*$ is a momentum map with non-equivariance cocycle σ_F and the energy E_L is an invariant function on TQ;*
3. *the affine action Ψ_{aff} is a canonical action for the symplectic structure (T^*Q, ω_Q); the map $J^f = \varphi^* - \pi_Q^* f$ is a momentum map with non-equivariance cocycle σ_F;*
4. *$\mathbb{F}L$ is a symplectomorphism between (TQ, Ω_Q^L) and (T^*Q, ω_Q), and is equivariant w.r.t. to the lifted action on TQ and the affine action on T^*Q associated to dF.*

Proof. The affine action Ψ_{aff} on T^*Q acts by symplectic transformations, i.e. from local computations it follows that

$$(\Psi_{\mathrm{aff},g})^*\theta_Q = \theta_Q + \pi_Q^* dF_g.$$

Together with Lemma 7, i.e. $\mathbb{F}L \circ T\Psi_g = \Psi_{\mathrm{aff},g} \circ \mathbb{F}L$, assertions (1) and (4) follow:

$$(T\Psi_g)^*\Omega_Q^L = d(\mathbb{F}L \circ T\Psi_g)^*\theta_Q = \mathbb{F}L^* d\Psi_{\mathrm{aff},g}^*\theta_Q = \Omega_L.$$

The latter equality holds since θ_Q is invariant under the affine action up to an exact form.

To show that J_L^f is a momentum map we use an argument involving coordinate expressions. Let $(q^i), i = 1, \ldots, \dim Q$ denote coordinate functions on Q, and let $(q^i, \dot{q}^i)$ be the associated coordinate system on TQ. Then it is not hard to show that

$$\xi_{TQ}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial f_\xi}{\partial q^i} - \frac{\partial \xi_Q^j}{\partial q^i}\frac{\partial L}{\partial \dot{q}^j},$$

holds, with $j = 1, \ldots, \dim Q$ and ξ_Q^j the coordinate expression of ξ_Q: $\xi_Q = \xi_Q^j \partial_j$. From some tedious computations it follows that

$$i_{\xi_{TQ}}\Omega_Q^L = -dJ_\xi^f$$

for $\xi \in \mathfrak{g}$ arbitrary. We now compute the non-equivariance cocycle of J^f_L. Fix any $\xi \in \mathfrak{g}$ and $v_q \in T_qQ$, then

$$\begin{aligned}\langle J^f_L(T\Psi_g(v_q)), \xi\rangle &= \langle \mathbb{F}L(T\Psi_g(v_q)), \varphi_{qg}(\xi)\rangle - f_\xi(qg)\\ &= \big\langle T^*\Psi_{g^{-1}}\big(\mathbb{F}L(v_q) + dF_g(q)\big), T\Psi_g\big(\varphi_q(Ad_g\xi)\big)\big\rangle - f_\xi(qg)\\ &= \langle \mathbb{F}L(v_q), \varphi_q(Ad_g\xi)\rangle - f_{Ad_g\xi}(q) + \big(f_{Ad_g\xi}(q) - f_\xi(qg) + \langle dF_g(q), \varphi_q(Ad_g\xi)\rangle\big)\\ &= \langle Ad^*_g J^f_L(v_q), \xi\rangle + \langle \sigma_F(g^{-1}), \xi\rangle.\end{aligned}$$

Finally, the fact that the energy is invariant easily follows from Lemma 7, and from this we conclude that (2) holds.

Since $\mathbb{F}L$ is a symplectic diffeomorphism and since $J^f \circ \mathbb{F}L = J^f_L$, we conclude that J^f is a momentum map with cocycle σ_F. This proves (3). □

The above lemma ensures that the equivariance conditions for Theorem 5 are satisfied. In that case we can study the MW-reduction and the structure of the corresponding quotient spaces. If these quotient spaces are 'tangent and cotangent bundle like' we shall say that the MW-reduction is a Routhian reduction procedure.

Following Theorem 5 we have that the reduced Legendre transformation $[\mathbb{F}L_\mu]$ is a symplectic diffeomorphism relating the symplectic structures on $(J^f_L)^{-1}(\mu)/G_\mu$ and $(J^f)^{-1}(\mu)/G_\mu$. The subgroup G_μ is the isotropy subgroup of the affine action of G on $\mathfrak{g}^*$ corresponding to the 1-cocycle σ_F. We now study the structure of the reduced manifolds $(J^f_L)^{-1}(\mu)/G_\mu$ and $(J^f)^{-1}(\mu)/G_\mu$, and their respective symplectic 2-forms.

Let $\mathcal{A}$ be a principal connection with horizontal projection operator $TQ \to TQ : v_q \mapsto v^h_q := v_q - \varphi_q(\mathcal{A}_q(v_q))$. Similarly, we can restrict a covector α_q to horizontal tangent vectors: $T^*Q \mapsto T^*Q : \alpha_q \mapsto \alpha^h_q$, with $\langle v_q, \alpha^h_q\rangle = \langle v^h_q, \alpha_q\rangle$. Note that $\alpha^h_q = \alpha_q - (\mathcal{A}^*_q \circ \varphi^*_q)(\alpha_q)$. The covariant exterior derivative (see [8]) of a function λ on Q is denoted by $D\lambda$ and is defined pointwise as $D\lambda_q = d\lambda^h_q$. We first study the symplectic structure of $(J^f)^{-1}(\mu)/G_\mu$. Similar to the invariant situation, we contract the connection 1-form on the Lie-algebra level with $\mu + f$ to obtain a 1-form $\mathcal{A}^f_\mu = q \mapsto \langle \mu + f(q), \mathcal{A}_q\rangle$ on Q.

Lemma 9. *Consider a quasi-invariant Lagrangian system (Q, L, F), for which there exists a principal connection $\mathcal{A}$ such that $DF_g = 0$, for arbitrary $g \in G$. Then,*

1. *the 2-form $d\mathcal{A}^f_\mu$ is invariant under the action of G_μ on Q and is projectable to a 2-form on Q/G_μ denoted by $\mathcal{B}^f_\mu$;*
2. *there exists a symplectic diffeomorphism*

$$[\phi^{\mathcal{A},f}_\mu] : ((J^f)^{-1}(\mu)/G_\mu, \omega_\mu) \to (T^*(Q/G) \times Q/G_\mu, \pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}^f_\mu).$$

Proof. For the proof of both statements we rely on the following identities, for $g \in G_\mu$ and $q \in Q$:

$$\begin{aligned}&\mathcal{A}^*_{qg} = T^*\Psi_{g^{-1}} \circ \mathcal{A}^*_q \circ Ad^*_{g^{-1}}\\ &\mu = Ad^*_{g^{-1}}\mu + \sigma_F(g)\\ &Ad^*_{g^{-1}} f(qg) = f(q) - (Ad^*_{g^{-1}} \circ \varphi^*_{qg})(dF_{g^{-1}}(qg)) + \sigma_F(g)\\ &Ad^*_{g^{-1}} \circ \varphi^*_{qg} = \varphi^*_q \circ T^*\Psi_g.\end{aligned}$$

1. The first statement is proven if we can show that $\mathcal{A}^f_\mu$ is invariant under G_μ up to an exact 1-form. Thus consider any element $q \in Q$ and $g \in G_\mu$, then

$$\begin{aligned}(\Psi_g^*\mathcal{A}^f_\mu)(q) &= \langle \mu + f(qg), Ad_{g^{-1}} \cdot \mathcal{A}_q\rangle \\ &= \Big\langle (\mu - \sigma_F(g)) + \big(f(q) - (Ad^*_{g^{-1}} \circ \varphi^*_{qg})(dF_{g^{-1}}(qg)) + \sigma_F(g)\big), \mathcal{A}_q \Big\rangle \\ &= (\mathcal{A}^f_\mu)(q) + dF_g(q).\end{aligned}$$

The latter equality holds because $dF^h_g(q) = DF_g(q) = 0$. To show that the 2-form is projectable, we prove in addition that $i_{\xi_Q} d\mathcal{A}^f_\mu = 0$. This follows on the one hand from $\mathcal{L}_{\xi_Q}\mathcal{A}^f_\mu = df_\xi$ which is obtained using the previous equation with $g = \exp \epsilon\xi$, and on the other hand from $\mathcal{L}_{\xi_Q} = i_{\xi_Q} d + di_{\xi_Q}$:

$$i_{\xi_Q} d\mathcal{A}^f_\mu = \mathcal{L}_{\xi_Q}\mathcal{A}^f_\mu - df_\xi = 0.$$

2. Similar to the case of an invariant Lagrangian system we relate $(J^f)^{-1}(\mu)$ with $V^0\pi$ by means of the connection: $\phi^{\mathcal{A},f}_\mu : (J^f)^{-1}(\mu) \to V^0\pi; \alpha_q \mapsto \alpha_q - \mathcal{A}^*_q(\mu + f(q))$. The next step is to study the affine action of G_μ on $(J^f)^{-1}(\mu)$ through this diffeomorphism. Let $g \in G_\mu$, and $\alpha_q \in (J^f)^{-1}(\mu)$, then

$$\begin{aligned}\phi^{\mathcal{A},f}_\mu\big(T^*\Psi_{g^{-1}}(\alpha_q + dF_g(q))\big) &= T^*\Psi_{g^{-1}}(\alpha_q + dF_g(q)) - \mathcal{A}^*_{qg}(\mu + f(qg)) \\ &= T^*\Psi_{g^{-1}}\big(\alpha_q - \mathcal{A}^*_q(\mu + f(q)\big) + dF^h_g(qg),\end{aligned}$$

We conclude that $\phi^{\mathcal{A},f}_\mu$ is equivariant w.r.t the affine action on $(J^f)^{-1}(\mu)$ and the standard lifted action on T^*Q restricted to $V^0\pi$ if the condition $DF_g = 0$ holds. The reduced map is denoted by $[\phi^{\mathcal{A},f}_\mu]$ and maps $(J^f)^{-1}(\mu)/G_\mu$ to $T^*(Q/G) \times Q/G_\mu$. The fact that it is a symplectic map the follows from analogous arguments as in the invariant case. □

4.3 The reduced phase space

We are now ready to take the final step towards a Routhian reduction procedure for quasi-invariant Lagrangians. It concerns the realization of $(J^f_L)^{-1}(\mu)/G_\mu$ as a tangent space $T(Q/G) \times Q/G_\mu$. We therefore reintroduce G-regular quasi-invariant Lagrangians. It should be clear that the definitions here are also valid in the strict invariant case. Let $R^\mu = L - \mathcal{A}^f_\mu$ denote the 'Routhian' as a function on TQ. We first show that it is G_μ-invariant. For that purpose let $g \in G_\mu$ and $v_q \in T_qQ$, then

$$\begin{aligned}R^\mu(T\Psi_g(v_q)) &= L(v_q) + \langle dF_g(q), v_q\rangle - \langle(\Psi^*_g\mathcal{A}^f_\mu)(q), v_q\rangle \\ &= L(v_q) - \langle \mathcal{A}^f_\mu(q), v_q\rangle = R^\mu(v_q).\end{aligned}$$

We know from the strict invariant case that TQ/G_μ can be identified with $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$. Let us denote $\mathfrak{R}^\mu$ denote the function on the latter space obtained from projecting R^μ. We now define the fibre derivative $\mathbb{F}_{\tilde\xi}\mathfrak{R}^\mu$ of $\mathfrak{R}^\mu$ w.r.t the $\tilde{\mathfrak{g}}$-fibre:

$$\langle \mathbb{F}_{\tilde\xi}\mathfrak{R}^\mu(v_x, y, \tilde\xi), (v_x, y, \tilde\eta)\rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathfrak{R}^\mu(v_x, y, \tilde\xi + \epsilon\tilde\eta),$$

with $(v_x, y, \tilde\xi) \in T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$ and $\tilde\eta \in \tilde{\mathfrak{g}}_x$ arbitrary.

Definition 7. *Let (Q, L, F) denote a quasi-regular Lagrangian system. We say that the system is G-regular if the function $\mathbb{F}_{\tilde\xi}\mathfrak{R}^\mu : T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}} \to T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}}^*$ is a diffeomorphism.*

It is not so hard to show that there is a one-to-one identification with $(J_L^f)^{-1}(\mu)/G_\mu$ and the set of points $(v_x, y, \tilde{\xi})$ in $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$ for which $\mathbb{F}_{\tilde{\xi}}\Re^\mu(v_x, y, \tilde{\xi}) = (v_x, y, 0)$. We consider the map $(p_1, p_2) : (J_L^f)^{-1}(\mu)/G_\mu \to T(Q/G) \times Q/G_\mu$ taking a point $[v_q]_G$ to the first two factors of the corresponding point $(T\pi(v_q), p_\mu(q), [q, \mathcal{A}(v_q)]_G)$ in the fibred product $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$.

Lemma 10. *If (Q, L, F) is a G-regular quasi-invariant Lagrangian system, then the mapping $(p_1, p_2) : (J_L^f)^{-1}(\mu)/G_\mu \to T(Q/G) \times Q/G_\mu$ is a diffeomorphism with inverse ψ_μ.*

The proof is completely analogous to the proof of Lemma 2: the inverse of (v_x, y) is defined as the point in $(J_L^f)^{-1}(\mu)/G_\mu$ that corresponds to $(\mathbb{F}_{\tilde{\xi}}\Re^\mu)^{-1}(v_x, y, 0)$ in $T(Q/G) \times Q/G_\mu \times \tilde{\mathfrak{g}}$. Let $[R^\mu]$ denote the quotient of the restriction of R^μ to $(J_L^f)^{-1}(\mu)$. Similar to the previous case we define $\mathcal{R}^\mu$ to be function on $T(Q/G) \times Q/G_\mu$ such that $(p_1, p_2)^*(\mathcal{R}^\mu) = [R^\mu]$. Note that $\mathcal{R}^\mu$ could also be obtained by $\mathcal{R}^\mu(v_x, y) = \Re^\mu(v_x, y, \tilde{\xi})$, with $(v_x, y, \tilde{\xi}) = (\mathbb{F}_{\tilde{\xi}}\Re^\mu)^{-1}(v_x, y, 0)$.

Lemma 11. *Let (Q, L, F) denote a G-regular quasi-invariant Lagrangian system and let $\mathcal{A}$ be a principal connection such that $DF_g = 0$. Let μ denote a value of J_L^f. Then*

1. *the map $[\phi_\mu^{\mathcal{A},f}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu$ is the fibre derivative of $\mathcal{R}^\mu$;*
2. *the energy of $\mathcal{R}^\mu$ is the MW-reduced hamiltonian of the energy E_L on the symplectic manifold (Q, Ω_Q^L).*

The proof is again completely similar to the proof of Lemma's 4 and 5. We conclude that the MW-reduction of a G-regular quasi-invariant Lagrangian L is again a 'Lagrangian' system on the manifold $T(Q/G) \times Q/G_\mu$, with Lagrangian $\mathcal{R}^\mu$: the symplectic structure is of the form $(\mathbb{F}\mathcal{R}^\mu)^*(\pi_1^*\omega_{Q/G}) + \overline{\pi}_2^*\mathcal{B}_\mu^f$.

Theorem 8. *Let (Q, L, F) denote a G-regular quasi-invariant Lagrangian system and let $\mathcal{A}$ be a principal connection such that $DF_g = 0$. Let μ denote a value of J_L^f. Then the MW-reduction of the symplectic manifold (Q, Ω_L) for the regular momentum value μ is the symplectic manifold*

$$\big(T(Q/G) \times Q/G_\mu, (\mathbb{F}\mathcal{R}^\mu)^*\big(\pi_1^*\omega_{Q/G}\big) + \overline{\pi}_2^*\mathcal{B}_\mu^f\big).$$

The reduced Hamiltonian is the energy $E_{\mathcal{R}^\mu}$. The equations of motion for this Hamiltonian vector field are precisely the Euler-Lagrange equations of motion for an intrinsically constrained Lagrangian system on $Q/G_\mu \to Q/G$ with Lagrangian $\mathcal{R}^\mu$ and gyroscopic force term associated to the 2-form $\mathcal{B}_\mu^f$ on Q/G_μ.

5 Examples

5.1 Quasi-cyclic coordinates

We continue here the description started in the introduction of a Lagrangian L with a single quasi-cyclic coordinate. Recall that if $(q^1, \dots, q^n)$ are coordinates on $Q = \mathbb{R}^n$ and $L(q^i, \dot{q}^i)$ is a Lagrangian, then we say that q^1 is quasi-cyclic if there exists a function $f(q^1, \dots, q^n)$ such that

$$\frac{\partial L}{\partial q^1} = \dot{q}^i \frac{\partial f}{\partial q^i}.$$

The group $G = \mathbb{R}$ acts on $\mathbb{R}^n$ by translation in q^1. Since $\mathfrak{g} \equiv \mathbb{R}$, a principal connection $\mathcal{A}$ here becomes an ordinary G-invariant 1-form on $\mathbb{R}^n$. The infinitesimal version Σ_f of the definition of the cocycle σ_F is identically zero, and we can conclude that also σ_F vanishes. Since the group is abelian, we have that $G_\mu = G$. The quotient space is $T(Q/G)$ and Q/G is labeled by the configuration space coordinates $(q^2, \ldots, q^n)$.

The condition that the system should be G-regular is locally expressed by $\partial^2 L/\partial\dot{q}^1\partial\dot{q}^1 \neq 0$ and, secondly, the condition that there exists a (principal) connection $\mathcal{A}$ such that $Df = 0$ (i.e. df restricted to the horizontal distribution should vanish) boils down to the condition that there should exist functions Γ_k, $k = 2, \ldots, n$, independent of q^1, for which

$$\frac{\partial f}{\partial q^k} = \Gamma_k(q^2, \ldots, q^n)\frac{\partial f}{\partial q^1},\ k = 2, \ldots, n\,.$$

This is precisely the condition (1) from the introduction (cf. Theorem 2). The connection $\mathcal{A}$ then reads $\mathcal{A} = dq^1 + \Gamma_k dq^k$, with summation over $k = 2, \ldots, n$. Note that $Df = 0$ implies that the connection has vanishing curvature (the horizontal distribution is involutive because it is annihilated by an exact 1-form). Assume now that both of the above conditions hold and keep the value of the momentum $\mu = \partial L/\partial\dot{q}^1 - f$ fixed. We solve this relation for $\dot{q}^1$ by writing $\dot{q}^1 = \psi(q^k, \dot{q}^k)$, with $k = 2, \ldots, n$. The Routhian then is the function

$$R^\mu(q^k, \dot{q}^k) = L - (\mu + f)(\dot{q}^1 + \Gamma_k\dot{q}^k),$$

where all instances of $\dot{q}^1$ on the right hand side have been replaced by the function ψ. It now remains to compute the 2-form $\mathcal{B}_\mu^f$ which is the projection of $d[(\mu+f)(dq^1+\Gamma_k dq^k)]$. After some straightforward computations in which the condition $df^h = 0$ is used, we obtain

$$\mathcal{B}_\mu^f = \frac{1}{2}(\mu + f)\left(\frac{\partial\Gamma_k}{\partial q^s} - \frac{\partial\Gamma_s}{\partial q^k}\right) dq^k \wedge dq^s.$$

The latter is identically zero since the connection has zero curvature due to $Df = 0$. This also follows from the following

$$\begin{aligned}\frac{\partial\Gamma_k}{\partial q^s} &= \frac{1}{\partial f/\partial q^1}\frac{\partial^2 f}{\partial q^k\partial q^s} - \frac{1}{(\partial f/\partial q^1)^2}\frac{\partial f}{\partial q^k}\frac{\partial^2 f}{\partial q^s\partial q^1}\\ &= \frac{1}{\partial f/\partial q^1}\frac{\partial^2 f}{\partial q^k\partial q^s} - \frac{1}{\partial f/\partial q^1}\Gamma_k\Gamma_s\frac{\partial^2 f}{\partial q^1\partial q^1} = \frac{\partial\Gamma_s}{\partial q^k}.\end{aligned}$$

We conclude that the Routhian reduction for Lagrangian systems with a single quasi-cyclic coordinate is the Lagrangian system on the reduced space with Lagrangian the Routhian $L - \mathcal{A}_\mu^f$. This concludes the proof of Theorem 2.

5.2 Functional Routhian reduction

Our motivation for studying Routh-reduction for quasi-invariant Lagrangians was inspired from the reduction technique called *functional Routhian reduction* used in [2]. We will argue here that functional Routhian reduction can be seen as Routhian reduction for a quasi-invariant Lagrangian. Consider a Lagrangian L of type kinetic minus potential energy define on a configuration space (locally) $(q^1, \ldots, q^{n-1}, q^n)$. The coordinate q^n was denoted in [2] by ϕ and the coordinates q^k for $k = 1, \ldots, n-1$ by θ^k. The Lagrangian L is of the form

$$L = \frac{1}{2}\left(M_{ij}(\theta)\dot{q}^i\dot{q}^j\right) - W(\theta, \dot{\theta}, \phi) - V(\theta, \phi),$$

with $M_{ij}(\theta)$ mass-inertia functions depending only on θ^k and $W = (\lambda(\phi)/M_{nn}(\theta))M_{nk}(\theta)\dot{\theta}^k$ and $V = V_{fct}(\theta) - \frac{1}{2}\lambda(\phi)^2/M_{nn}(\theta)$.

It should be immediately clear that ϕ is not a cyclic coordinate, nor a quasi-invariant cyclic coordinate. We will however define a 'momentum map' J_L^λ associated to the would-be cyclic coordinate ϕ:

$$J_L^\lambda(\theta, \dot{\theta}, \phi) = \partial_{\dot{\phi}} L(\theta, \phi, \dot{\theta}, \dot{\phi}) - \lambda(\phi) = M_{kn}(\theta)\dot{\theta}^k + M_{nn}(\theta)\dot{\phi} - \lambda(\phi).$$

Note that, since λ only depends on ϕ we may use the standard connection $\mathcal{A} = d\phi$ when working in a local coordinate system. The Lagrangian L transforms as a quasi-invariant Lagrangian when restricted to the level set $J_L^\lambda = 0$:

$$\left.\frac{\partial L}{\partial \phi}\right|_{J_L^\lambda = 0} = \lambda'(\phi)\dot{\phi}.$$

Strictly speaking this example is not described in the theory outlined above. We hope however that it is clear to the reader that is an even more general type of Routh-reduction for quasi-invariant Lagrangians that is valid only on a specific level set of the momentum map. The correspondence between both techniques is also seen from the fact that in [2] the authors define the *functional Routhian* L_{fct} as the function

$$L_{fct}(\theta, \dot{\theta}) = \left(L(q^i, \dot{q}^i) - \lambda(\phi)\dot{\phi}\right)_{J_L^\lambda = 0}.$$

This is precisely the function $\mathcal{R}^\mu$, with $\mu = 0$ in our analysis of quasi-invariant Lagrangians. Note that all regularity conditions are satisfied and especially the horizontal condition $dF_g^h = 0$ is satisfied since λ is independent of θ.

5.3 Charged particle in a constant magnetic field

In [10] the example of a charged particle in a constant magnetic field B is studied. The Lagrangian for this system is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + eB(\dot{x}y - \dot{y}x)$. The coordinates x and y are quasi-cyclic, and from this we may write that $f(x, y) = (-eBy, eBx) \in \mathbb{R}^2 \cong \mathfrak{g}^*$. The 2-cocycle Σ_f is not vanishing and proportional to eB. The (infinitesimal) affine action on $\mathfrak{g}^*$ is completely determined by this 2-cocycle Σ_f and due to the abelian nature of the group, the Lie-algebra of isotropy subgroup $G_{(\mu_1,\mu_2)}$ is trivial since it is spanned by the kernel of Σ_f. In turn $G_{(\mu_1,\mu_2)} = \{e\}$. The conserved momenta read: $m\dot{x} + 2eBy = \mu_1$ and $m\dot{y} - 2eBx = \mu_2$. Therefore the quotient space is $\mathbb{R}^2$. From the structure of the momenta equations it is immediately seen that the system is G-regular. Further the standard connection 1-form $\mathcal{A} = (dx, dy)^T$, with trivial horizontal distribution implies that $Df = df^h = 0$. Therefore all conditions are met, and the Routhian is then a function on $\mathbb{R}^2$ depending only on x, y:

$$R^\mu = \frac{-1}{2m}\left((\mu_1 - 2eBy)^2 + (\mu_2 + 2eBx)^2\right).$$

The symplectic 2-form $\mathcal{B}_\mu^f$ is precisely $2eBdx\wedge dy$. The Routhian reduced equations of motion the read: $i_{(\dot{x},\dot{y})}\mathcal{B}_\mu^f = dR^\mu$, or simply the momenta equations $m\dot{x} + 2eBy = \mu_1$ and $m\dot{y} - 2eBx = \mu_2$.

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