

The Hamilton-Pontryagin Principle and Multi-Dirac Structures for Classical Field Theories

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Abstract

We introduce a variational principle for field theories, referred to as the Hamilton-Pontryagin principle, and we show that the resulting field equations are the Euler-Lagrange equations in implicit form. Secondly, we introduce multi-Dirac structures as a graded analog of standard Dirac structures, and we show that the graph of a multisymplectic form determines a multi-Dirac structure. We then discuss the role of multi-Dirac structures in field theory by showing that the implicit Euler-Lagrange equations for fields obtained from the Hamilton-Pontryagin principle can be described intrinsically using multi-Dirac structures. Lastly, we show a number of illustrative examples, including time-dependent mechanics, nonlinear scalar fields, Maxwell's equations, and elastostatics.

1 Introduction

In this paper, we describe classical field theories by means of a new variational principle, referred to as the *Hamilton-Pontryagin variational principle* and we show that the resulting field equations can be described in terms of *multi-Dirac structures*, a field theoretic extension of the concept of Dirac structures (Courant [1990]). To set the stage for the paper, we begin with reviewing the Hamilton-Pontryagin principle and Dirac structures in the context of mechanical systems.

The Hamilton-Pontryagin Variational Principle in Mechanics. Let $L(q^i, v^i)$ be a Lagrangian. The *Hamilton-Pontryagin principle*, originally coined by Livens [1919] and rediscovered by Schwinger, DeRaad, Milton, and Tsai [1998] and Gitman and Tyutin [1990] among others, is a variational principle in which the position coordinates q^i and the velocity coordinates v^i are treated independently. The relation

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$\dot{q}^i = v^i$ is imposed as a constraint by means of a Lagrange multiplier p_i , leading to an action functional of the form

$$\begin{aligned} S(q, v, p) &= \int_{t_0}^{t_1} (L(q^i(t), v^i(t)) + \langle p_i(t), \dot{q}^i(t) - v^i(t) \rangle) dt \\ &= \int_{t_0}^{t_1} (\langle p_i(t), \dot{q}^i(t) \rangle - E(q^i(t), v^i(t), p_i(t))) dt, \end{aligned} \tag{1.1}$$

where $E(q^i, v^i, p_i) := \langle p_i, v^i \rangle - L(q^i, v^i)$ is the generalized energy associated to the Lagrangian. By taking arbitrary variations with respect to q^i , v^i and p_i , we obtain the Euler-Lagrange equations in implicit form:

$$p_i = \frac{\partial L}{\partial v^i}, \quad \dot{p}_i = \frac{\partial L}{\partial q^i}, \quad \text{and} \quad \dot{q}^i = v^i. \tag{1.2}$$

In the context of variational principles, it was shown by [Yoshimura and Marsden \[2006b\]](#) that the Hamilton-Pontryagin principle and the associated implicit equations of motion can be described in an intrinsic way by means of Dirac structures. The discrete Hamilton-Pontryagin principle was developed in [Bou-Rabee and Marsden \[2009\]](#); [Leok and Ohsawa \[2011\]](#) for the design of accurate variational integrators.

Dirac Structures and Lagrange-Dirac Systems in Mechanics. The notion of Dirac structures was originally developed by [Courant and Weinstein \[1988\]](#); [Courant \[1990\]](#) and [Dorfman \[1993\]](#) as a unified concept of (pre-)symplectic and Poisson structures, inspired by Dirac’s theory of constraints that was established by [Dirac \[1950\]](#) for constrained systems including constraints due to degenerate Lagrangians.

It was soon realized that Dirac structures play an important role in mechanics. In particular, it was shown that interconnected systems, such as LC circuits and nonholonomic systems can be effectively formulated in the context of implicit Hamiltonian systems ([Courant \[1990\]](#); [Dorfman \[1993\]](#); [Courant and Weinstein \[1988\]](#); [van der Schaft and Maschke \[1995\]](#)). On the Lagrangian side, it was shown by [Yoshimura and Marsden \[2006a\]](#) that Dirac structures induced from distributions on configuration manifolds naturally yield a notion of *implicit Lagrangian systems*, allowing for the description of mechanical systems with degenerate Lagrangians and nontrivial constraint distributions.

In order to show the link with the equations (1.2) obtained from the Hamilton-Pontryagin principle, let Δ_Q be a distribution on a manifold Q and consider the canonical symplectic form Ω on T^*Q , and define Ω_M by its pullback to the *Pontryagin bundle* $M := TQ \oplus T^*Q$. The set D , given by

$$D = \{(X, \alpha) \in TM \oplus T^*M \mid \mathbf{i}_X \Omega_M - \alpha \in \Delta_M^\circ, \quad X \in \Delta_M\},$$

where $\Delta_M := T\pi_{Q,M}^{-1}(\Delta_Q)$ (with $\pi_{Q,M} : M \rightarrow Q$ is the Pontryagin bundle projection), is a Dirac structure (see [Yoshimura and Marsden \[2006a,b\]](#)); namely, D is maximally isotropic with respect to the following pairing on $TM \oplus T^*M$:

$$\langle\langle (X, \alpha), (Y, \beta) \rangle\rangle_- = \frac{1}{2}(\mathbf{i}_X \beta + \mathbf{i}_Y \alpha). \tag{1.3}$$

It can then be shown that the *implicit Euler-Lagrange equations*, given by $(X, \mathbf{d}E) \in D$ are given in coordinates by

$$p_i = \frac{\partial L}{\partial v^i}, \quad \dot{p}_i = \frac{\partial L}{\partial q^i} + \lambda_\alpha A_i^\alpha(q), \quad \dot{q}^i = v^i, \quad \text{and} \quad A_i^\alpha(q)v^i = 0; \quad (1.4)$$

see [Cendra, et al \[2010\]](#). Here, the forms $A_i^\alpha(q)dq^i$ form a basis for the annihilator Δ° , and the λ_α are Lagrange multipliers. The equations (1.4) are readily seen to be the nonholonomic Euler-Lagrange equations in nonholonomic form, and when the constraints vanish, $A_i^\alpha = 0$, we recover the implicit Euler-Lagrange equations (1.2).

The Hamilton-Pontryagin Principle for Field Theories. For classical field theories, the Lagrangian $L(x^\mu, y^A, v_\mu^A)$ depends not only on the spacetime coordinates x^μ and the fields y^A , but also on the spacetime velocities v_μ^A , which represent the derivatives of the fields y^A with respect to x^μ . By comparison with (1.1), a candidate Hamilton-Pontryagin principle may be then given in coordinates by

$$S(y^A, y_\mu^A, p_A^\mu) = \int_U \left[p_A^\mu \left(\frac{\partial y^A}{\partial x^\mu} - v_\mu^A \right) + L(x^\mu, y^A, v_\mu^A) \right] d^{n+1}x. \quad (1.5)$$

By taking variations with respect to y^A , v_μ^A and p_A^μ (where δy^A is required to vanish on the boundary ∂U of the integration domain), we obtain the following implicit equations:

$$\frac{\partial p_A^\mu}{\partial x^\mu} = \frac{\partial L}{\partial y^A}, \quad p_A^\mu = \frac{\partial L}{\partial v_\mu^A}, \quad \text{and} \quad \frac{\partial y^A}{\partial x^\mu} = v_\mu^A, \quad (1.6)$$

which clearly generalizes (1.2) to the case of field theories. In section 3, we provide an intrinsic version of this variational principle, using the geometry of the first jet bundle and its dual as our starting point. In contrast to mechanics, the jet bundle is an affine bundle and this makes the definition of the duality pairing used implicitly in (1.5) somewhat more complicated.

Multi-Dirac Structures. In section 4, we describe the implicit field equations (1.6) in terms of multi-Dirac structures. These geometric structures were introduced by [Vankerschaver, Yoshimura, and Leok \[2011\]](#) as a graded version of the standard concept of Dirac structures, and their relevance for field theory lies in the fact that they generalize the concept of “the graph of a multisymplectic structure.”

To make this more precise, we recall that a multisymplectic structure is a form Ω of degree at least two, which is closed and non-degenerate. When the degree of Ω is two, Ω is a symplectic form, while if Ω is a top form, then it is necessarily a volume form. The literature on multisymplectic field theories is by now very extensive, but for fundamental aspects we refer to [Goldschmidt and Sternberg \[1973\]](#); [Kijowski and Tulczyjew \[1979\]](#); [Binz, Śniatycki, and Fischer \[1988\]](#); [Cariñena, Crampin, and Ibort \[1991\]](#); [Gotay, Isenberg, Marsden and Montgomery \[1997\]](#); [Bridges \[1997\]](#) and the references therein.

Just as the graph of a symplectic form is a Dirac structure, the graph of a multisymplectic form Ω on a manifold M turns out to be a multi-Dirac structure.

Here, the graph of Ω is defined as follows. Let $k+1$ be the degree of Ω and consider the mapping whereby an l -multivector field \mathcal{X}_l (with $1 \leq l \leq k$) is contracted with Ω :

$$\mathcal{X}_l \in \wedge^l(TM) \mapsto \mathbf{i}_{\mathcal{X}_l}\Omega \in \wedge^{k+1-l}(T^*M).$$

The graph of this mapping determines a submanifold D_l of $\wedge^l(TM) \times_M \wedge^{k+1-l}(T^*M)$, and by considering the direct sum

$$D = D_1 \oplus \cdots \oplus D_k$$

of all such subbundles, we obtain a *multi-Dirac structure*. This structure is maximally isotropic under a graded antisymmetric version of the pairing (1.3), and multi-Dirac structures can be considered more generally as *graded versions of standard Dirac structures*.

The link between multi-Dirac structures and implicit field equations is provided by theorem 3.4, where we show that the field equations obtained from the Hamilton-Pontryagin principle can be written as $\mathbf{i}_{\mathcal{X}}\Omega_M = (-1)^{n+2}\mathbf{d}E$. In multi-Dirac form this becomes

$$(\mathcal{X}, (-1)^{n+2}\mathbf{d}E) \in D_{n+1}, \quad (1.7)$$

where D_{n+1} is the graph of the multisymplectic form Ω_M , as discussed before. More generally, we arrive in definition 4.5 at the concept of a *Lagrange-Dirac field theory*, which is a triple $(\mathcal{X}, E, D_{n+1})$ satisfying (1.7), where D_{n+1} belongs to a multi-Dirac structure D which does not necessarily come from a multisymplectic form.

Outline of the Paper. After introducing the Hamilton-Pontryagin principle for field theories in section 3, we discuss the link with multi-Dirac structures in section 4. We finish the paper in section 5 with a number of examples of implicit field theories and multi-Dirac structures, taken from mechanics, electromagnetism and elasticity. Among others, we show that the Hu-Washizu principle from linear elastostatics is a particular example of the Hamilton-Pontryagin principle.

The focus throughout this paper is on the Hamilton-Pontryagin principle and the application of multi-Dirac structures to field theories. In the companion paper [Vankerschaver, Yoshimura, and Leok \[2011\]](#), we develop the algebraic and geometric properties of multi-Dirac structures.

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2 The Geometry of Jet Bundles

In this section, we provide a quick overview of the geometry of jet bundles for the treatment of Lagrangian field theories. Most of the material in this section is standard, and can be found in [Saunders \[1989\]](#); [Gotay, Isenberg, Marsden and Montgomery \[1997\]](#) and the references therein.

Jet Bundles. Let X be an oriented manifold with volume form η , which in many examples is spacetime, and let $\pi_{XY} : Y \rightarrow X$ be a finite-dimensional fiber bundle which we call the *covariant configuration bundle*. The physical fields are sections of this bundle, which is the covariant analogue of the configuration space in classical mechanics. For future reference, we suppose that the dimension of X is $n + 1$ and that π_{XY} is a bundle of rank N , so that $\dim Y = n + N + 1$. Coordinates on X are denoted x^μ , $\mu = 1, 2, \dots, n + 1$, and fiber coordinates Y are denoted by y^A , $A = 1, \dots, N$ such that a section $\phi : X \rightarrow Y$ of π_{XY} has coordinate representation $\phi(x) = (x^\mu, y^A(x))$.

The analogue in classical field theory of the tangent bundle in mechanics is the *first jet bundle* J^1Y , which consists of equivalence classes of local sections of π_{XY} , where we say that two local sections ϕ_1, ϕ_2 of Y are equivalent at $x \in X$ if their Taylor expansions around x agree to the first order. In other words, ϕ_1 and ϕ_2 are equivalent if $\phi_1(x) = \phi_2(x)$ and $T_x\phi_1 = T_x\phi_2$. It follows that an equivalence class $[\phi]$ of local sections can be identified with a linear map $\gamma : T_xX \rightarrow T_yY$ such that $T\pi_{XY} \circ \gamma = \text{Id}_{T_xX}$. As a consequence, J^1Y is a fiber bundle over Y , where the projection $\pi_{Y, J^1Y} : J^1Y \rightarrow Y$ is defined as follows: let $\gamma : T_xX \rightarrow T_yY$ be an element of J^1Y , then $\pi_{Y, J^1Y}(\gamma) := y$. Coordinates on J^1Y are denoted by (x^μ, y^A, v_μ^A) , where the fiber coordinates v_μ^A represent the first-order derivatives of a section. They are defined by noting that any $\gamma \in J^1Y$ is locally of the form

$$\gamma = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + v_\mu^A \frac{\partial}{\partial y^A} \right). \quad (2.1)$$

The first jet bundle J^1Y is an *affine* bundle over Y , with underlying vector bundle $L(TX, VY)$ of linear maps from the tangent space TX into the vertical bundle VY , defined as

$$V_yY = \{v \in T_yY \mid T\pi_{XY}(v) = 0\}, \quad \text{for } y \in Y.$$

Given any section $\phi : X \rightarrow Y$ of π_{XY} , its tangent map $T_x\phi$ at $x \in X$ is an element of J_y^1Y , where $y = \phi(x)$. Thus, the map $x \mapsto T_x\phi$ defines a section of J^1Y , where now J^1Y is regarded as a bundle over X . This section is denoted $j^1\phi$ and is called the *first jet prolongation* of ϕ . In coordinates, $j^1\phi$ is given by

$$x^\mu \mapsto (x^\mu, y^A(x^\mu), \partial_\nu y^A(x^\mu)),$$

where $\partial_\nu = \partial/\partial x^\nu$. A section of the bundle $J^1Y \rightarrow X$ which is the first jet prolongation of a section $\phi : X \rightarrow Y$ is said to be *holonomic*.

Dual Jet Bundles. Next, we consider the field-theoretic analogue of the cotangent bundle. We define the *dual jet bundle* J^1Y^* to be the vector bundle over Y whose fiber at $y \in Y_x$ is the set of affine maps from J_y^1Y to $\Lambda_x^{n+1}X$, where $\Lambda^{n+1}X$ denotes the bundle of $(n+1)$ -forms on X . Note that since the space of affine maps from an affine space into a vector space forms a vector space, J^1Y^* is a vector bundle despite the fact that J^1Y is only an affine bundle. A smooth section of J^1Y^* is therefore an affine bundle map of J^1Y to $\Lambda^{n+1}X$. Any affine map from J_y^1Y to $\Lambda_x^{n+1}X$ can locally be written as

$$v_\mu^A \mapsto (p + p_A^\mu v_\mu^A) d^{n+1}x,$$

where $d^{n+1}x := dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n+1}$ is a coordinate representation of the volume form η , so that coordinates on J^1Y^* are given by (x^μ, y^A, p_A^μ, p) .

Throughout this paper we will employ another useful description of J^1Y^* . Consider again the bundle $\Lambda^{n+1}Y$ of $(n+1)$ -forms on Y and let $Z \subset \Lambda^{n+1}Y$ be the subbundle whose fiber over $y \in Y$ is given by

$$Z_y = \{z \in \Lambda_y^{n+1}Y \mid \mathbf{i}_v \mathbf{i}_w z = 0 \text{ for all } v, w \in V_y Y\},$$

where \mathbf{i}_v denotes left interior multiplication by v . In other words, the elements of Z vanish when contracted with two or more vertical vectors. The bundle Z is canonically isomorphic as a vector bundle over Y to J^1Y^* ; this can be easily understood from the fact that the elements of Z can locally be written as

$$z = p d^{n+1}x + p_A^\mu dy^A \wedge d^n x_\mu,$$

where $d^n x_\mu := \partial_\mu \lrcorner d^{n+1}x$.

From now on, we will silently identify J^1Y^* with Z . The duality pairing between elements $\gamma \in J_{y_x}^1 Y$ and $z \in Z_{y_x}$ can then be written as

$$\langle z, \gamma \rangle = \gamma^* z \in \Lambda_x^{n+1}X.$$

In coordinates, $\langle z, \gamma \rangle = (p_A^\mu v_\mu^A + p) d^{n+1}x$.

Canonical Multisymplectic Forms. Analogous to the canonical symplectic forms on a cotangent bundle, there are canonical forms on J^1Y^* . Let us first define the *canonical* $(n+1)$ -*form* Θ on $J^1Y^* \cong Z$ by

$$\begin{aligned} \Theta(z)(u_1, \dots, u_{n+1}) &= z(T\pi_{Y, J^1Y^*}(u_1), \dots, T\pi_{Y, J^1Y^*}(u_{n+1})) \\ &= (\pi_{Y, J^1Y^*}^* z)(u_1, \dots, u_{n+1}), \end{aligned} \quad (2.2)$$

where we have interpreted $z \in Z \cong J^1Y^*$ as before as an $(n+1)$ -form on Y , and $u_1, \dots, u_{n+1} \in T_z Z$. The *canonical multisymplectic* $(n+2)$ -*form* Ω on J^1Y^* is now defined as

$$\Omega = -\mathbf{d}\Theta.$$

In coordinates, one has the following expression for Θ :

$$\Theta = p_A^\mu dy^A \wedge d^n x_\mu + p d^{n+1}x,$$

while Ω is locally given by

$$\Omega = dy^A \wedge dp_A^\mu \wedge d^n x_\mu - dp \wedge d^{n+1} x. \quad (2.3)$$

It is easy to show (see [Cantrijn, Ibort, and de Leon \[1999\]](#)) that Ω is non-degenerate in the sense that $\mathbf{i}_X \Omega = 0$ implies that $X = 0$. Moreover, Ω is trivially closed: $\mathbf{d}\Omega = 0$. Forms which are both closed and non-degenerate are referred to as ***multisymplectic*** forms. If the degree of Ω is two, Ω is just a symplectic form, while if Ω is of maximal degree, Ω is a volume form. More examples of multisymplectic manifolds can be found in [Cantrijn, Ibort, and de Leon \[1999\]](#).

A form which is non-degenerate but not necessarily closed is also referred to as an ***almost multisymplectic*** form. On the other hand, in the context of field theory, we will often refer to a closed but possibly degenerate form as a ***pre-multisymplectic*** form.

Lagrangian Densities and the Covariant Legendre Transformation. A ***Lagrangian density*** is a smooth map $\mathcal{L} = L\eta : J^1 Y \rightarrow \Lambda^{n+1} X$. In local coordinates, we may write

$$\mathcal{L}(\gamma) = L(x^\mu, y^A, v_\mu^A) d^{n+1} x,$$

where L is a function on $J^1 Y$ to which we also refer as the Lagrangian.

The corresponding ***covariant Legendre transformation*** for a given Lagrangian density $\mathcal{L} : J^1 Y \rightarrow \Lambda^{n+1} X$ is a fiber preserving map $\mathbb{F}\mathcal{L} : J^1 Y \rightarrow J^1 Y^*$ over Y , which is given by the first order vertical Taylor approximation to \mathcal{L} :

$$\langle \mathbb{F}\mathcal{L}(\gamma), \gamma' \rangle = \mathcal{L}(\gamma) + \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{L}(\gamma + \epsilon(\gamma' - \gamma)),$$

where $\gamma, \gamma' \in J^1 Y$. In coordinates, $\mathbb{F}\mathcal{L}(x^\mu, y^A, v_\mu^A) = (x^\mu, y^A, p_A^\mu, p)$, where

$$p_A^\mu = \frac{\partial L}{\partial v_\mu^A}, \quad p = L - \frac{\partial L}{\partial v_\mu^A} v_\mu^A. \quad (2.4)$$

3 The Hamilton-Pontryagin Principle for Field Theories

In this section, we introduce the Hamilton-Pontryagin action principle for classical field theories. Let us first introduce the Pontryagin bundle

$$M := J^1 Y \times_Y Z,$$

as the fibered product over Y of the jet bundle $J^1 Y$ and its dual Z (recall that we identified $J^1 Y^*$ with Z). This bundle plays a similar role as the Pontryagin bundle $TQ \oplus T^*Q$ over a configuration manifold Q for the case of classical mechanics. Note also that M is a bundle over Y and also over X .

The Generalized Energy Density. Let $\mathcal{L} : J^1Y \rightarrow \Lambda^{n+1}X$ be a Lagrangian density which is possibly degenerate. We define the *generalized energy density* associated to \mathcal{L} by the map $\mathcal{E} : M \rightarrow \Lambda^{n+1}X$ defined as

$$\mathcal{E}(\gamma, z) := E(\gamma, z)\eta = \langle z, \gamma \rangle - \mathcal{L}(\gamma) \quad (3.1)$$

for $(\gamma, z) \in M$. In local coordinates $(x^\mu, y^A, v_\mu^A, p, p_A^\mu)$ on M , the generalized energy density is represented by

$$\begin{aligned} \mathcal{E} &= E(x^\mu, y^A, v_\mu^A, p, p_A^\mu) d^{n+1}x \\ &= (p + p_A^\mu v_\mu^A - L(x^\mu, y^A, v_\mu^A)) d^{n+1}x, \end{aligned} \quad (3.2)$$

where $E = p + p_A^\mu v_\mu^A - L(x^\mu, y^A, v_\mu^A)$ is called the *generalized energy* on M .

Pre-Multisymplectic Forms on M . Recall the definition of the canonical forms Θ and $\Omega := -d\Theta$ on the dual jet bundle $Z \cong J^1Y^*$, and let $\pi_{ZM} : M \rightarrow Z$ be the projection onto the second factor. The forms Θ and Ω can be pulled back along π_{ZM} to yield corresponding forms on M :

$$\Theta_M := \pi_{ZM}^* \Theta \quad \text{and} \quad \Omega_M := \pi_{ZM}^* \Omega.$$

Note that Ω_M cannot be multisymplectic since it has a non-trivial kernel: for all $v \in TJ^1Y$ we have that $\mathbf{i}_v \Omega_M = 0$. As a result, we will refer to Ω_M as the *pre-multisymplectic $(n+2)$ -form* on M .

Finally, for any Lagrangian density \mathcal{L} with associated energy density \mathcal{E} , we introduce another pre-multisymplectic $(n+2)$ -form on M by

$$\Omega_{\mathcal{E}} = \Omega_M + d\mathcal{E}.$$

The Hamilton-Pontryagin Principle. Using the pre-multisymplectic forms Ω_M and $\Omega_{\mathcal{E}}$, we establish a *Hamilton-Pontryagin variational principle* for classical field theories which is similar to the expression (1.1) for mechanical systems outlined in the introduction.

Definition 3.1. Consider a Lagrangian density \mathcal{L} with associated generalized energy density \mathcal{E} . The Hamilton-Pontryagin action functional is defined as

$$S(\psi) = \int_X \psi^*(\Theta_M - \mathcal{E}), \quad (3.3)$$

where ψ is a section of $\pi_{XM} : M \rightarrow X$. In other words, ψ can be written as $\psi = (\gamma, z)$ where γ is a section of $\pi_{X, J^1Y} : J^1Y \rightarrow X$ and z is a section of $\pi_{XZ} : Z \rightarrow X$.

We have defined the Hamilton-Pontryagin principle in terms of the $(n+1)$ -form Θ_M . By using the definition (2.2) and (3.1), we can rewrite the Hamilton-Pontryagin action functional as

$$S(\psi) = \int_X (\langle z, j^1\phi \rangle - \langle z, \gamma \rangle + \mathcal{L}(\gamma)),$$

where ϕ is a section of π_{XY} as before.

A **vertical variation** of a section ψ of $\pi_{XM} : M \rightarrow X$ is a one-parameter family $\psi_\lambda : X \rightarrow M$ of sections, such that $\psi_0 = \psi$, and ψ_λ agrees with ψ outside a compact set U of X . This can be obtained by considering diffeomorphisms $\nu_\lambda : M \rightarrow M$ such that ν_0 is the identity, ν_λ is the identity outside of $\pi_{XM}^{-1}(U)$, where U is a compact subset of X , and ν_λ preserves the fibration $\pi_{XM} : M \rightarrow X$, or in other words $\pi_{XM} \circ \nu_\lambda = \pi_{XM}$. Then, the composition $\psi_\lambda = \nu_\lambda \circ \psi$ is a vertical variation of ψ .

On the infinitesimal level, an infinitesimal variation of ψ is a π_{XM} -vertical vector field $\mathcal{V}_M : \text{Im } \psi \rightarrow VM$, defined on the image of ψ in M , and where VM is the vertical subbundle of TM defined by

$$VM = \{w \in TM \mid T\pi_{XM}(w) = 0\}.$$

Let us denote by $\mathfrak{X}^V(M)$ the module of the π_{XM} -vertical vector fields on M .

We say that ψ is a **critical point** of the action (3.3) if $\delta S(\psi) = 0$ for all variations of ψ , where

$$\begin{aligned} \delta S(\psi) &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} S(\psi_\lambda) \\ &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \int_U \psi_\lambda^*(\Theta_M - \mathcal{E}), \end{aligned}$$

where U is a compact subset of X that depends on the variation ψ_λ .

Proposition 3.2. *A section ψ of M is a critical point of the Hamilton-Pontryagin action functional if ψ satisfies the **implicit Euler-Lagrange equations***

$$\psi^*(\mathcal{V}_{M \lrcorner} \Omega_{\mathcal{E}}) = 0, \quad \text{for any } \mathcal{V}_M \in \mathfrak{X}^V(M). \quad (3.4)$$

Proof. Recall that a variation of ψ is given by $\psi_\lambda = \nu_\lambda \circ \psi$ where $\nu_\lambda : M \rightarrow M$ is the flow of a vertical vector field \mathcal{V}_M . Then, it follows that

$$\begin{aligned} \left. \frac{d}{d\lambda} \right|_{\lambda=0} S(\psi_\lambda) &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \int_U \psi_\lambda^*(\Theta_M - \mathcal{E}) \\ &= \int_U \psi^* \mathcal{L}_{\mathcal{V}_M}(\Theta_M - \mathcal{E}) \\ &= - \int_U \psi^*(\mathcal{V}_{M \lrcorner} (\Omega_M + \mathbf{d}\mathcal{E})) + \int_U \mathbf{d}(\psi^*(\mathcal{V}_{M \lrcorner} (\Theta_M - \mathcal{E}))) \\ &= - \int_U \psi^*(\mathcal{V}_{M \lrcorner} \Omega_{\mathcal{E}}) + \int_{\partial U} \psi^*(\mathcal{V}_{M \lrcorner} (\Theta_M - \mathcal{E})) \\ &= - \int_U \psi^*(\mathcal{V}_{M \lrcorner} \Omega_{\mathcal{E}}) \end{aligned}$$

for all \mathcal{V}_M , where we utilized Cartan's magic formula and Stokes' theorem and the fact that the π_{XM} -vertical vector field $\mathcal{V}_M : M \rightarrow VM$ is compactly supported in $U \subset X$. A standard argument then shows that ψ is a critical point of S if and only if (3.4) holds. ■

Coordinate Expressions. Employing local coordinates $(x^\mu, y^A, v_\mu^A, p, p_A^\mu)$ on M , the action functional is denoted by

$$\begin{aligned} S(\gamma) &= \int_U \psi^*(\Theta_M - \mathcal{E}) \\ &= \int_U \left(p + p_A^\mu \frac{\partial y^A}{\partial x^\mu} \right) d^{n+1}x - \{ (p + p_A^\mu v_\mu^A) d^{n+1}x - L(x^\mu, y^A, v_\mu^A) d^{n+1}x \} \\ &= \int_U \left\{ p_A^\mu \left(\frac{\partial y^A}{\partial x^\mu} - v_\mu^A \right) + L(x^\mu, y^A, v_\mu^A) \right\} d^{n+1}x. \end{aligned} \quad (3.5)$$

A small coordinate calculation, similar to what was done in the introduction, then shows that the implicit Euler-Lagrange equations (3.4) are locally given by

$$\frac{\partial y^A}{\partial x^\mu} = v_\mu^A, \quad \frac{\partial p_A^\mu}{\partial x^\mu} = \frac{\partial L}{\partial y^A}, \quad p_A^\mu = \frac{\partial L}{\partial v_\mu^A}. \quad (3.6)$$

Generalized Energy Constraint. The coordinates on the Pontryagin bundle are $(x^\mu, y^A, v_\mu^A, p, p_A^\mu)$, and the Hamilton-Pontryagin principle gives equations for the fields y^A , y_μ^A and p_A^μ , but leaves p undetermined. In order to fix p , we impose the *generalized energy constraint*

$$E \equiv p + p_A^\mu v_\mu^A - L(x^\mu, y^A, v_\mu^A) = 0.$$

Together with the third equation in (3.6), this naturally recovers the covariant Legendre transformation (2.4). We summarize the results up to this point in the following theorem.

Theorem 3.3. *The following statements for a section $\psi : X \rightarrow M$ of π_{XM} satisfying $E(\psi) = 0$ are equivalent:*

- (1) ψ is a critical point of the Hamilton-Pontryagin action functional (3.3).
- (2) $\psi^*(\mathcal{V}_{M\perp} \Omega_{\mathcal{E}}) = 0$ for all π_{XM} -vertical vector fields \mathcal{V}_M on M .
- (3) ψ satisfies the implicit Euler-Lagrange equations (3.6) together with the covariant Legendre transformation (2.4).

Implicit Field Equations Using Multivector Fields. We now rewrite the field equations (3.6) in a slightly different way, which will make the link with multi-Dirac structures in the next section clearer. Up to this point, we have viewed the field equations as coming from the Hamilton-Pontryagin variational principle. We now show how these equations can be understood directly in terms of the multisymplectic form. A quick introduction to the geometry of multivector fields can be found in appendix A.

To this end, we will use multivector fields on M (see Marle [1997]; Echeverría-Enríquez, Muñoz-Lecanda, and Román-Roy [2002]; Forger, Paufler, and Römer [2005]), which can be seen as the infinitesimal counterparts of sections $\psi : X \rightarrow M$

of the Pontryagin bundle, in the sense that a multivector field gives rise to a set of first-order PDEs, whose integral sections are sections of the Pontryagin bundle. For the purpose of this paper, we will restrict our attention to multivector fields that are locally of the form

$$\mathcal{X} = \bigwedge_{\mu=1}^{n+1} \mathcal{X}_\mu = \bigwedge_{\mu=1}^{n+1} \left(\frac{\partial}{\partial x^\mu} + C_\mu^A \frac{\partial}{\partial y^A} + C_{A\mu}^\nu \frac{\partial}{\partial p_A^\nu} + C_\mu \frac{\partial}{\partial p} \right), \quad (3.7)$$

where C_μ^A , $C_{A\mu}^\nu$ and C_μ are local component functions of \mathcal{X} on M . Note in particular that the component of the multivector field in the v_μ^A -direction is zero. We will refer to multivector fields \mathcal{X} which are locally given by (3.7) as **partial multivector fields**.¹

A section $\psi : X \rightarrow M$, given in local coordinates by

$$\psi(x) = (x^\mu, y^A(x), y_\mu^A(x), p(x), p_A^\mu(x)),$$

is an integral section of \mathcal{X} if it satisfies the following system of PDEs:

$$\frac{\partial y^A}{\partial x^\mu} = C_\mu^A, \quad \frac{\partial p_A^\nu}{\partial x^\mu} = C_{A\mu}^\nu, \quad \frac{\partial p}{\partial x^\mu} = C_\mu, \quad (3.8)$$

which express the condition that the image of ψ is tangent to \mathcal{X} . We now claim that the implicit field equations (3.6) can be expressed using multivector fields and the generalized energy (3.2) as follows:

Theorem 3.4. *Let E be the generalized energy (3.2) and consider a multivector field \mathcal{X} on M of the form (3.7) such that*

$$\mathbf{i}_\mathcal{X} \Omega_M = (-1)^{n+2} \mathbf{d}E, \quad (3.9)$$

where $\dim X = n + 1$, and let $\psi : X \rightarrow M$ be an integral section of \mathcal{X} . In local coordinates, where $\psi(x) = (x^\mu, y^A(x), y_\mu^A(x), p(x), p_A^\mu(x))$, we then have that ψ satisfies the following set of local equations:

$$\frac{\partial p_A^\mu}{\partial x^\mu} = \frac{\partial L}{\partial y^A}, \quad \frac{\partial y^A}{\partial x^\mu} = v_\mu^A, \quad p_A^\mu = \frac{\partial L}{\partial v_\mu^A}, \quad (3.10)$$

together with

$$\frac{\partial}{\partial x^\mu} (p + p_A^\nu v_\nu^A - L) = 0. \quad (3.11)$$

Proof. To show this, we compute the differential of the generalized energy (3.2):

$$\begin{aligned} \mathbf{d}E &= \frac{\partial E}{\partial x^\mu} dx^\mu + \frac{\partial E}{\partial y^A} dy^A + \frac{\partial E}{\partial v_\mu^A} dv_\mu^A + \frac{\partial E}{\partial p_A^\mu} dp_A^\mu + \frac{\partial E}{\partial p} dp \\ &= \left(-\frac{\partial L}{\partial x^\mu} \right) dx^\mu + \left(-\frac{\partial L}{\partial y^A} \right) dy^A + \left(p_A^\mu - \frac{\partial L}{\partial v_\mu^A} \right) dv_\mu^A + v_\mu^A dp_A^\mu + dp, \end{aligned}$$

¹The characterization (3.7) can be recast into an intrinsic form by noting that a partial multivector field is *decomposable, normalized* (in the sense that $\mathbf{i}_\mathcal{X}(\pi_{X^*}^* \eta) = 1$ with η the volume form on X), and satisfies $\bigwedge^{n+1} \tau_{J^1 Y, M} \circ \mathcal{X} = 0$, where $\bigwedge^{n+1} \tau_{J^1 Y, M} : \bigwedge^{n+1}(TM) \rightarrow \bigwedge^{n+1}(TJ^1 Y)$ is the natural projection.

and the contraction of the multivector field \mathcal{X} given in (3.7) with the multisymplectic form Ω_M :

$$\mathbf{i}_{\mathcal{X}}\Omega_M = (-1)^{n+2} \left[(C_{\mu}^A C_{A\lambda}^{\lambda} - C_{\lambda}^A C_{A\mu}^{\lambda} - C_{\mu}) dx^{\mu} + C_{\mu}^A dp_A^{\mu} + C_{A\mu}^{\mu} dy^A + dp \right]. \quad (3.12)$$

As a result, we have that (3.9) holds if and only if

$$C_{\mu}^A C_{A\lambda}^{\lambda} - C_{\lambda}^A C_{A\mu}^{\lambda} - C_{\mu} = -\frac{\partial L}{\partial x^{\mu}} \quad (3.13)$$

as well as

$$C_{\mu}^A = v_{\mu}^A, \quad C_{A\mu}^{\mu} = -\frac{\partial L}{\partial y^A}, \quad p_A^{\mu} = \frac{\partial L}{\partial v_{\mu}^A}. \quad (3.14)$$

Now, let ψ be an integral section, so that locally (3.8) holds. After a few simple algebraic manipulations, we then obtain that ψ satisfies (3.10) and (3.11). ■

4 Multi-Dirac Structures and Field Theories

In this section, we show how the implicit field equations $\mathbf{i}_{\mathcal{X}}\Omega_M = (-1)^{n+2}\mathbf{d}E$ of theorem 3.4 can be described in terms of *multi-Dirac structures*, which are a graded analogue of the concept of Dirac structures introduced by Courant [1990] (for applications of Dirac structures to mechanics, see Dalsmo and van der Schaft [1999]; Yoshimura and Marsden [2006a] and the references therein). In order to keep the technical side of the paper to a minimum, we focus on the applications of multi-Dirac structures to field theories, and we leave proofs and a more thorough discussion to Vankerschaver, Yoshimura, and Leok [2011]. We also point out that there are alternative approaches to the construction of a “field-theoretic analogue” of Dirac structures; see for instance Baez, Hoffnung, and Rogers [2010] and Zambon [2010].

Pairings between Multivectors and Forms. While multi-Dirac structures can in principle be defined on an arbitrary manifold M , we will focus for now on multi-Dirac structures on the Pontryagin bundle $M \rightarrow X$. Recall that the dimension of the base space X is written as $n + 1$.

Consider the spaces $\bigwedge^l(TM)$ of l -multivector fields on M and $\bigwedge^k(T^*M)$ of k -forms on M . For $k \geq l$, there is a natural pairing between elements of $\Sigma \in \bigwedge^k(T^*M)$ and $\mathcal{X} \in \bigwedge^l(TM)$ given by

$$\langle \Sigma, \mathcal{X} \rangle := \mathbf{i}_{\mathcal{X}}\Sigma \in \bigwedge^{k-l}(T^*M). \quad (4.1)$$

We now introduce the *graded Pontryagin bundle of degree r* over M as follows:

$$P_r = \bigwedge^r(TM) \oplus \bigwedge^{n+2-r}(T^*M), \quad (4.2)$$

where $r = 1, \dots, n + 1$.

Using the pairing in equation (4.1), let us define the following antisymmetric and symmetric pairings between the elements of P_r and P_s as follows. For $(\mathcal{X}, \Sigma) \in P_r$ and $(\bar{\mathcal{X}}, \bar{\Sigma}) \in P_s$, where $r, s = 1, \dots, n+1$, we put

$$\langle\langle (\mathcal{X}, \Sigma), (\bar{\mathcal{X}}, \bar{\Sigma}) \rangle\rangle_- := \frac{1}{2} (\mathbf{i}_{\bar{\mathcal{X}}}\Sigma - (-1)^{rs}\mathbf{i}_{\mathcal{X}}\bar{\Sigma}) \quad (4.3)$$

and

$$\langle\langle (\mathcal{X}, \Sigma), (\bar{\mathcal{X}}, \bar{\Sigma}) \rangle\rangle_+ := \frac{1}{2} (\mathbf{i}_{\bar{\mathcal{X}}}\Sigma + (-1)^{rs}\mathbf{i}_{\mathcal{X}}\bar{\Sigma}), \quad (4.4)$$

each of which takes values in $\bigwedge^{n+2-r-s}(T^*M)$. Hence, both of these pairing are identically zero whenever $n+2 < r+s$.

Let V_s be a subbundle of P_s . The r -*orthogonal complementary subbundle* of V_s associated to the antisymmetric paring $\langle\langle \cdot, \cdot \rangle\rangle_-$ is the subbundle $(V_s)^{\perp, r}$ of P_r defined by

$$(V_s)^{\perp, r} = \{(\mathcal{X}, \Sigma) \in P_r \mid \langle\langle (\mathcal{X}, \Sigma), (\bar{\mathcal{X}}, \bar{\Sigma}) \rangle\rangle_- = 0 \text{ for all } (\bar{\mathcal{X}}, \bar{\Sigma}) \in V_s\}.$$

Note that $(V_s)^{\perp, r}$ is a subbundle of P_r , and that $(V_s)^{\perp, r} = P_r$ whenever $n+2 < r+s$.

Almost Multi-Dirac Structures on Manifolds. The definition of an *almost multi-Dirac structure* on M mimics the standard definition of Courant [1990] of Dirac structures.

Definition 4.1. *An almost multi-Dirac structure of degree $n+1$ on M is a direct sum $D_1 \oplus \dots \oplus D_{n+1}$ of subbundles*

$$D_r \subset P_r \quad (\text{for } r = 1, \dots, n+1),$$

*which is $(n+1)$ -Lagrangian; namely, the sequence of the bundles D_r are **maximally $(n+1)$ -isotropic**:*

$$D_r = (D_s)^{\perp, r} \quad (4.5)$$

for $r, s = 1, \dots, n+1$ and where $r+s \leq n+2$.

When no confusion can arise, we will refer to the direct sum $D_1 \oplus \dots \oplus D_{n+1}$ simply as D . In the companion paper Vankerschaver, Yoshimura, and Leok [2011], we define a graded version of the Courant bracket, and we refer to *integrable multi-Dirac structures* as almost multi-Dirac structures which are closed under the graded Courant bracket.

For the case of classical field theories, it will turn out that only the orthogonal complements of the multi-Dirac structure of the “lowest” and “highest” order (namely, $r = 1$ and $r = n+1$ respectively) play an essential role to formulate the field equations.

Proposition 4.2. *Let $D = D_1 \oplus \dots \oplus D_{n+1}$ be a multi-Dirac structure of degree $n+1$. For any $(\mathcal{X}, \Sigma) \in D_r$ and $(\bar{\mathcal{X}}, \bar{\Sigma}) \in D_s$, the following relation holds:*

$$\mathbf{i}_{\bar{\mathcal{X}}}\Sigma - (-1)^{rs}\mathbf{i}_{\mathcal{X}}\bar{\Sigma} = 0. \quad (4.6)$$

Proof. This is clear from the r -Lagrangian (maximally r -isotropic) property of D_r . ■

The above r -Lagrangian property of the multi-Dirac structure D_r in equation (4.6) may be understood as the field-theoretic analogue of the *virtual work principle* in mechanics and is related to *Tellegen's theorem* in electric circuits.

Example. Let M be a manifold with a multi-Dirac structure D_1 of degree 1. The isotropy property then becomes

$$D_1 = (D_1)^{\perp,1},$$

where $P_1 = TM \oplus T^*M$ and the pairing is given by

$$\langle\langle (\mathcal{X}, \Sigma), (\bar{\mathcal{X}}, \bar{\Sigma}) \rangle\rangle_- = \frac{1}{2} (\mathbf{i}_{\bar{\mathcal{X}}}\Sigma + \mathbf{i}_{\mathcal{X}}\bar{\Sigma}).$$

for all $(\mathcal{X}, \Sigma), (\bar{\mathcal{X}}, \bar{\Sigma}) \in P_1$. This is nothing but the definition of an (almost) Dirac structure developed by Courant [1990]. Our concept of multi-Dirac structures also includes the so-called *higher-order Dirac structures* of Zambon [2010].

Multi-Dirac Structures Induced by Differential Forms. Consider an $(n+2)$ -form Ω_M on a manifold M . We will show that the graph of Ω_M (in the sense defined below) defines a multi-Dirac structure of degree $n+1$. This example of a multi-Dirac structure will be fundamental in our subsequent treatment of classical field theories, where Ω_M will be the canonical multisymplectic form, but for now Ω_M can be an arbitrary form.

Proposition 4.3. *Let Ω_M be an arbitrary $(n+2)$ -form on M and define the following subbundles D_1, \dots, D_{n+1} , where $D_r \subset P_r$:*

$$D_r = \{(\mathcal{X}, \Sigma) \in P_r \mid \mathbf{i}_{\mathcal{X}}\Omega_M = \Sigma\} \quad (4.7)$$

for $r = 1, \dots, n+1$. Then $D = D_1 \oplus \dots \oplus D_{n+1}$ is a multi-Dirac structure of degree $n+1$ on M .

Proof. To prove that $D = D_1 \oplus \dots \oplus D_{n+1}$ is a multi-Dirac structure of degree $n+1$, we need to check the isotropy property (4.5), namely namely, $D_r = D_s^{\perp, r}$ for all $r, s = 1, \dots, n+1$, with $r + s \leq n+1$.

Let us first show that $D_r \subset D_s^{\perp, r}$. Let $(\mathcal{X}, \Sigma) \in D_r$ and $(\bar{\mathcal{X}}, \bar{\Sigma}) \in D_s$. Then, it follows that

$$\begin{aligned} \langle\langle (\mathcal{X}, \Sigma), (\bar{\mathcal{X}}, \bar{\Sigma}) \rangle\rangle_- &= \frac{1}{2} \{ \mathbf{i}_{\bar{\mathcal{X}}}\Sigma + (-1)^{rs+1} \mathbf{i}_{\mathcal{X}}\bar{\Sigma} \} \\ &= \frac{1}{2} \{ \mathbf{i}_{\bar{\mathcal{X}}}\mathbf{i}_{\mathcal{X}}\Omega_M + (-1)^{rs+1} \mathbf{i}_{\mathcal{X}}\mathbf{i}_{\bar{\mathcal{X}}}\Omega_M \} \\ &= 0, \end{aligned}$$

since $\mathbf{i}_{\bar{\mathcal{X}}}\mathbf{i}_{\mathcal{X}}\Omega_M = (-1)^{rs}\mathbf{i}_{\mathcal{X}}\mathbf{i}_{\bar{\mathcal{X}}}\Omega_M$. Thus, $D_r \subset D_s^{\perp,r}$.

Next, let us show that $D_s^{\perp,r} \subset D_r$. Let $(\bar{\mathcal{X}}, \bar{\Sigma}) \in D_s^{\perp,r}$. By definition of $D_s^{\perp,r}$,

$$\mathbf{i}_{\bar{\mathcal{X}}}\Sigma + (-1)^{rs+1}\mathbf{i}_{\mathcal{X}}\bar{\Sigma} = 0$$

for all $(\mathcal{X}, \Sigma) \in D_r$, i.e. $\mathcal{X} \in \bigwedge^r(TZ)$ such that $\mathbf{i}_{\mathcal{X}}\Omega_M = \Sigma$. Then, it follows

$$\begin{aligned} \mathbf{i}_{\bar{\mathcal{X}}}\Sigma + (-1)^{rs+1}\mathbf{i}_{\mathcal{X}}\bar{\Sigma} &= \mathbf{i}_{\bar{\mathcal{X}}}\mathbf{i}_{\mathcal{X}}\Omega_M + (-1)^{rs+1}\mathbf{i}_{\mathcal{X}}\bar{\Sigma} \\ &= \mathbf{i}_{\mathcal{X}}\{(-1)^{rs}\mathbf{i}_{\bar{\mathcal{X}}}\Omega_M + (-1)^{rs+1}\bar{\Sigma}\} \\ &= 0 \end{aligned}$$

for all $\mathcal{X} \in \bigwedge^r(TZ)$ and with $r + s \leq n + 2$. Therefore, one has

$$\mathbf{i}_{\bar{\mathcal{X}}}\Omega_M = \bar{\Sigma}.$$

Thus, $D_s^{\perp,r} \subset D_r$ and it follows that $D_r = D_s^{\perp,r}$, so that $D = D_1 \oplus \cdots \oplus D_{n+1}$ is a multi-Dirac structure of degree $n + 1$ on M . \blacksquare

Local Characterization of Multi-Dirac Structures. In his paper on higher-order Dirac structures, Zambon [2010] has given a complete local characterization of multi-Dirac structures. The following theorem is a combination of corollary 3.9 and lemma 4.3 in that paper.

Theorem 4.4. *Let T be an arbitrary vector space and consider a form $\omega \in \bigwedge^{n+1}T^*$, and a subspace $S \subset T$ such that $\dim S \leq \dim T - n$ or $S = T$. Then the sets*

$$D_r = \{(X, \Sigma) \mid X \in S \wedge (\bigwedge^{r-1}T), \quad \Sigma - \mathbf{i}_X\omega \in \bigwedge^{n+1-r}S^\circ\},$$

$r = 1, \dots, n$, form a multi-Dirac structure. Conversely, each multi-Dirac structure on T is of this form, for a suitable choice of S and ω .

Applications to Implicit Field Theories. We now consider Lagrangian field theories in the context of multi-Dirac structures. Consider a fiber bundle $\pi_{XY} : Y \rightarrow X$ and let $M = J^1Y \times_Y Z$ be the Pontryagin bundle. Let \mathcal{L} be a Lagrangian density on J^1Y and denote the corresponding generalized energy density by \mathcal{E} . Let Ω be the canonical multisymplectic structure on Z and recall the pre-multisymplectic $(n + 2)$ -form defined on the Pontryagin bundle by $\Omega_M = \pi_{ZM}^*\Omega$.

Using Ω_M we can define a multi-Dirac structure $D = D_1 \oplus \cdots \oplus D_{n+1}$ of degree $n + 1$ on M by the construction of proposition 4.3. We refer to D as the **canonical multi-Dirac structure** on M . Explicitly, $D = D_1 \oplus \cdots \oplus D_{n+1}$ is given by

$$D_r = \{(\mathcal{X}, \mathbf{i}_{\mathcal{X}}\Omega_M) \mid \mathcal{X} \in \bigwedge^r(TM)\}, \quad (4.8)$$

for $r = 1, \dots, n + 1$. As $\mathbf{d}\Omega_M = 0$, this multi-Dirac structure turns out to be *integrable*; see Vankerschaver, Yoshimura, and Leok [2011].

From theorem 3.4, we have that the implicit field equations can be written in terms of a partial multivector field \mathcal{X} and the generalized energy E as

$$\mathbf{i}_{\mathcal{X}}\Omega_M = (-1)^{n+2}\mathbf{d}E.$$

This can be expressed in terms of the multi-Dirac structure induced by Ω_M by the condition $(\mathcal{X}, (-1)^{n+2}\mathbf{d}E) \in D_{n+1}$. Note that the other components D_1, \dots, D_n of the multi-Dirac structure are *not needed to write down the field equations*; we will comment on their possible significance later on.

Taking our cue from this observation, we now define a special class of field theories, whose field equations are specified in terms of a Lagrangian \mathcal{L} , a multivector field \mathcal{X} , and a multi-Dirac structure D on the Pontryagin bundle M .

Definition 4.5. *Let $D = D_1 \oplus \dots \oplus D_{n+1}$ be a multi-Dirac structure of degree $n+1$ on the Pontryagin bundle M and consider a Lagrangian density \mathcal{L} with associated generalized energy E given by (3.2). A **Lagrange-Dirac system for field theories** is defined by a triple $(\mathcal{X}, E, D_{n+1})$, where \mathcal{X} is a normalized, decomposable partial vector field, so that*

$$(\mathcal{X}, (-1)^{n+2}\mathbf{d}E) \in D_{n+1}. \quad (4.9)$$

Note that in the definition of a Lagrange-Dirac system, only the highest-order component D_{n+1} of the multi-Dirac structure appears.

Theorem 4.6. *For any Lagrange-Dirac system $(\mathcal{X}, E, D_{n+1})$, the condition of **conservation of the generalized energy***

$$\mathcal{X} \lrcorner \mathbf{d}E = 0 \quad (4.10)$$

is satisfied. In other words, the generalized energy E is constant along the solutions of the Lagrange-Dirac system $(\mathcal{X}, E, D_{n+1})$ (i.e. the integral manifolds of \mathcal{X}).

Proof. The proof relies on lemma A.2 for decomposable vector fields. Using the field equations, the inner product on the left-hand side of (4.10) can be written as

$$\mathcal{X} \lrcorner \mathbf{d}E = \mathcal{X} \lrcorner (\mathcal{X} \lrcorner \Omega_M) = 0,$$

according to Corollary A.3. ■

To see why the previous theorem implies energy conservation, decompose the multivector field \mathcal{X} as in (3.7). Using lemma A.2, the interior product can then be written as

$$\mathcal{X} \lrcorner \mathbf{d}E = \sum_{\mu=1}^k (-1)^{\mu+1} \langle \mathcal{X}_{\mu}, \mathbf{d}E \rangle \hat{\mathcal{X}}_{\mu}$$

where $\hat{\mathcal{X}}_{\mu}$ is the n -multivector field obtained by deleting \mathcal{X}_{μ} from \mathcal{X} , i.e.,

$$\hat{\mathcal{X}}_{\mu} = \bigwedge_{\substack{\lambda=1 \\ \lambda \neq \mu}}^{n+1} \mathcal{X}_{\lambda}.$$

Since the multivector fields $\hat{\mathcal{X}}_{\mu}$ are linearly independent, the energy conservation equation (4.10) then implies that $\langle \mathcal{X}_{\mu}, \mathbf{d}E \rangle = 0$ for $\mu = 1, \dots, n+1$. In other words, the function E is constant on the integral sections of \mathcal{X} .

5 Examples

In this section, we shall present some examples of Lagrange-Dirac field theories in the context of multi-Dirac structures. Namely, we show that the two examples of *scalar nonlinear wave equations* and *electromagnetism* can be described as standard Lagrange-Dirac field theories. Furthermore, as an example of the Lagrange-Dirac field theories with nonholonomic constraints, we consider *time-dependent mechanical systems with affine constraints*. Lastly, we show that the *Hu-Washizu variational principle* of linear elastostatics can be understood in the context of the Hamilton-Pontryagin principle.

5.1 Nonlinear Wave Equations

Let us consider the scalar nonlinear wave equation as discussed in Bridges [1997]. The configuration bundle is $\pi_{XY} : Y \rightarrow X$, with $X = \mathbb{R}^2$, so that $n = \dim X - 1 = 1$, and $Y = X \times \mathbb{R}$. Local coordinates are given by (t, x) for $U \subset X$ and (t, x, ϕ) for Y , and hence (t, x, ϕ, v_t, v_x) for J^1Y and $(t, x, \phi, p, p^t, p^x)$ for $Z \cong J^1Y^*$. The nonlinear wave equation is given by

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - V'(\phi) = 0,$$

where ϕ is a section of π_{XY} and $V : Y \rightarrow \mathbb{R}$ is a nonlinear potential. The Lagrangian for this equation is given by

$$L(t, x, \phi, v_t, v_x) = \frac{1}{2} (v_t^2 - v_x^2) + V(\phi)$$

and the Lagrangian density is of course given by $\mathcal{L} = L dt \wedge dx$. The generalized energy is then $E = p + p^t v_t + p^x v_x - L(t, x, \phi, v_t, v_x)$.

Hamilton-Pontryagin Principle. The Hamilton-Pontryagin action functional (3.3) for the scalar field is given by

$$S(\psi) = \int_U \left(\frac{1}{2} (v_t^2 - v_x^2) + V(\phi) + p_t(\phi, t - v_t) + p_x(\phi, x - v_x) \right) dt \wedge dx.$$

Taking variations of ϕ, v_t, v_x, p^t, p^x (with the condition that $\delta\phi$ vanishes on the boundary of U), we obtain the implicit field equations

$$\frac{\partial \phi}{\partial t} = v_t, \quad \frac{\partial \phi}{\partial x} = v_x, \quad p^t = v_t, \quad p^x = -v_x, \quad \frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} = V'(\phi), \quad (5.1)$$

and imposing the generalized energy constraints $E = 0$, one has

$$p = L(t, x, \phi, v_t, v_x) - p^t v_t - p^x v_x. \quad (5.2)$$

The equations (5.1) are precisely the covariant equations obtained by Bridges [1997].

Multi-Dirac Structures. The canonical pre-multisymplectic 3-form on M is given by

$$\Omega_M = -dp^t \wedge d\phi \wedge dx + dp^x \wedge d\phi \wedge dt - dp \wedge dt \wedge dx,$$

and one can define a multi-Dirac structure $D = D_1 \oplus D_2$ of degree 2 associated with Ω_M on M by the standard prescription, where we set $n = 1$. We omit the definition of D_1 , since as discussed previously, only D_2 is important for the formulation of the dynamics:

$$D_2 = \left\{ (\mathcal{X}, \Sigma) \in P_2 = \bigwedge^2(TM) \oplus T^*M \mid \mathbf{i}_{\mathcal{X}}\Omega_M = \Sigma \right\}.$$

Here, \mathcal{X} is a partial multivector field locally given by

$$\mathcal{X} = \bigwedge_{\mu=0}^1 \left(\frac{\partial}{\partial x^\mu} + C_\mu \frac{\partial}{\partial \phi} + C_\mu^\nu \frac{\partial}{\partial p^\nu} + c_\mu \frac{\partial}{\partial p} \right),$$

where the variables x^μ , $\mu = 0, 1$, are given by $x^0 = t$ and $x^1 = x$. The field equations can now be expressed as

$$(\mathcal{X}, -\mathbf{d}E) \in D_2,$$

or in other words $\mathbf{i}_{\mathcal{X}}\Omega_M = -\mathbf{d}E$. In coordinates, these equations become

$$\begin{aligned} v_0 = C_0, \quad v_1 = C_1, \quad C_0^0 + C_1^1 &= \frac{\partial L}{\partial \phi}, \quad p^0 = \frac{\partial L}{\partial v_0}, \quad p^1 = \frac{\partial L}{\partial v_1} \\ c_0 + C_0^1 C_1 - C_0 C_0^1 &= \frac{\partial L^0}{\partial x}, \quad c_1 + C_0^0 C_1 - C_0 C_1^0 = \frac{\partial L}{\partial x^1}, \end{aligned}$$

where

$$C_\mu = \frac{\partial \phi}{\partial x^\mu}, \quad C_\mu^\nu = \frac{\partial p^\nu}{\partial x^\mu}, \quad c_\mu = \frac{\partial p}{\partial x^\mu}.$$

Hence, by simple rearrangements, we can obtain implicit Euler-Lagrange equations (5.1) for nonlinear waves, together with

$$\frac{\partial p}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} (L(x^0, x^1, \phi, v_0, v_1) - p^0 v_0 - p^1 v_1).$$

By imposing the generalized energy constraint $E = 0$, we again obtain (5.2).

5.2 Electromagnetism

The multisymplectic description of electromagnetism can be found, among others, in [Gotay, Isenberg, Marsden and Montgomery \[1997\]](#). Here, we highlight the role of the Hamilton-Pontryagin variational principle and the associated multi-Dirac structure. The electromagnetic potential $A = A_\mu dx^\mu$ is a section of the bundle $Y = T^*X$ of one-forms on spacetime X . For the sake of simplicity, we let X be \mathbb{R}^4 with the Minkowski metric, but curved spacetimes can be treated equally well. The bundle Y has coordinates (x^μ, A_μ) while J^1Y has coordinates $(x^\mu, A_\mu, A_{\mu,\nu})$. The electromagnetic Lagrangian density is given by

$$\mathcal{L}(A, j^1A) = -\frac{1}{4} \mathbf{d}A \wedge * \mathbf{d}A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where the Hodge star operator $*$ is associated to the Minkowski metric and where $F_{\mu\nu}$ is the anti-symmetrization $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$.

Hamilton-Pontryagin Principle. The Hamilton-Pontryagin action principle is given in coordinates by

$$S = \int_U \left[p^{\mu,\nu} \left(\frac{\partial A_\mu}{\partial x^\nu} - A_{\mu,\nu} \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] d^4x,$$

where U is an open subset of X . The Hamilton-Pontryagin equations are given by

$$p^{\mu,\nu} = F^{\mu\nu}, \quad A_{\mu,\nu} = \frac{\partial A_\mu}{\partial x^\nu}, \quad \frac{dp^{\mu,\nu}}{dx^\nu} = 0, \quad (5.3)$$

and by eliminating $p^{\mu,\nu}$ lead to Maxwell's equations: $\partial_\nu F^{\mu\nu} = 0$.

Multi-Dirac Structures. The canonical multisymplectic form for electromagnetism is given in coordinates by

$$\Omega_M = dA_\mu \wedge dp^{\mu,\nu} \wedge d^n x_\nu - dp \wedge d^{n+1}x.$$

Let $D = D_1 \oplus \cdots \oplus D_4$ be the multi-Dirac structure induced by this form and consider in particular the component D_4 of the highest degree. In terms of this bundle, the implicit Euler-Lagrange equations can be described by

$$(\mathcal{X}, -\mathbf{d}E) \in D_4,$$

where \mathcal{X} is a partial vector field of degree 4 given locally by

$$\mathcal{X} = \bigwedge_{\mu=1}^4 \left(\frac{\partial}{\partial x^\mu} + C_{\nu\mu} \frac{\partial}{\partial A_\nu} + C_\mu^{\kappa\nu} \frac{\partial}{\partial p^{\kappa,\nu}} + C_\mu \frac{\partial}{\partial p} \right)$$

and E is the generalized energy density given by

$$E(x^\mu, A_\mu, A_{\mu,\nu}, p^{\mu,\nu}, p) = p + p^{\mu,\nu} A_{\mu,\nu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

As before, a coordinate computation then shows that the coefficients of \mathcal{X} are given by

$$C_{\mu\nu} = A_{\mu,\nu}, \quad C_\nu^{\mu\nu} = \frac{\partial L}{\partial A_\mu} = 0, \quad p^{\mu,\nu} = \frac{\partial L}{\partial A_{\mu,\nu}} = F^{\mu\nu}$$

and

$$C_\mu + C_\mu^{\kappa\lambda} C_{\kappa\lambda} - C_{\nu\mu} C_\lambda^{\nu\lambda} = \frac{\partial L}{\partial x^\mu} = 0,$$

where

$$C_{\nu\mu} = \frac{\partial A_\nu}{\partial x^\mu}, \quad C_\mu^{\kappa\lambda} = \frac{\partial p^{\kappa,\lambda}}{\partial x^\mu}, \quad \text{and} \quad C_\mu = \frac{\partial p}{\partial x^\mu}.$$

It is easy to show that these equations are equivalent to Maxwell's equations (5.3) in implicit form, together with the energy constraint (3.11).

5.3 Time-Dependent Mechanical Systems with Affine Constraints

We now show how time-dependent mechanics with linear nonholonomic constraints can be interpreted as a special case of Lagrange-Dirac field theories. Let Q be the configuration space of a mechanical system and consider the bundle $\pi_{XY} : Y \rightarrow X$, where $X = \mathbb{R}$ (coordinatized by the time t) and $Y = \mathbb{R} \times Q$, with coordinates (t, q^i) . The first jet bundle is given by $J^1Y \cong \mathbb{R} \times TQ$, with local coordinates (t, q^i, v^i) . Let $\mathcal{L} : J^1Y \rightarrow \Lambda^1(X)$ be a Lagrangian density and put $\mathcal{L}(\gamma) = L(t, q^i, v^i)dt$. The dual jet bundle Z is a bundle over Y with local coordinates (t, q^i, p_t, p^i) and can be identified with $T^*Y \cong T^*\mathbb{R} \times T^*Q$. The canonical one-form on Z is given by $\Theta = p_i dq^i + p_t dt$ and the canonical two-form is given by $\Omega = -d\Theta = dq^i \wedge dp_i - dp_t \wedge dt$.

Non-Holonomic Affine Constraints. Consider now a distribution Δ on Y which is *weakly horizontal*, i.e., there exists a vertical bundle $W \subset VY$ such that $\Delta \oplus W = TY$ (see Krupková [2005]). In local coordinates, let

$$\varphi^\alpha := A_i^\alpha(t, q) dq^i + B^\alpha(t, q) dt \quad (5.4)$$

with $\alpha = 1, \dots, k$, be a basis for the annihilator Δ° . The distribution Δ gives rise to an affine subbundle of J^1Y , denoted by \mathcal{C} and given in coordinates by

$$\mathcal{C} = \{(t, q^i, v^i) \in J^1Y \mid A_i^\alpha(t, q)v^i + B^\alpha(t, q) = 0\}.$$

Intrinsically, \mathcal{C} can be defined as follows. Recall that a jet $\gamma \in J^1Y$ can be viewed as a map from $T_t\mathbb{R}$ to $T_{(t,q)}Y$ which is locally of the form $\gamma = dt \otimes (\partial/\partial t + v^i \partial/\partial q^i)$ (see (2.1)). With this interpretation, \mathcal{C} is the set of all $\gamma \in J^1Y$ such that $\text{Im } \gamma \subset \Delta$.

The distribution Δ can now be lifted to the Pontryagin bundle $M = J^1Y \oplus Z$ as follows. First, let $\pi_{Y, J^1Y} : J^1Y \rightarrow Y$ be the canonical jet bundle projection, given in coordinates by $\pi_{Y, J^1Y}(t, q^i, v^i) = (t, q^i)$, and define now

$$\Delta_{J^1Y} = T\pi_{Y, J^1Y}^{-1}(\Delta) \subset T(J^1Y)$$

This is a distribution defined on the whole of J^1Y , but only its values along \mathcal{C} will be needed. We therefore denote the restriction of this bundle to \mathcal{C} by $\Delta_{\mathcal{C}}$. If the annihilator of Δ is given by (5.4), then an easy coordinate computation shows that $\Delta_{\mathcal{C}}^\circ$ is spanned by the k contact forms

$$\tilde{\varphi}^\alpha := A_i^\alpha(t, q)(dq^i - v^i dt), \quad (5.5)$$

with $\alpha = 1, \dots, k$. Finally, we let Δ_M be the distribution defined along points of $\mathcal{C} \times J^1Y^* \subset M$, given by $\Delta_M := T\pi_{J^1Y, M}^{-1}(\Delta_{\mathcal{C}})$, where $\pi_{J^1Y, M} : M \rightarrow J^1Y$. The annihilator Δ_M° is spanned by the pullback to M of the one-forms $\tilde{\varphi}^\alpha$ in (5.5). This is but one way of constructing $\Delta_{\mathcal{C}}$; other approaches (which, unlike our construction, can be used for nonlinear constraints as well) can be found in de León and Martín de Diego [1996]; Cantrijn, de León, and Martín de Diego [1999], and the references therein.

Dirac Structures. We now construct a Dirac structure on the Pontryagin bundle M as follows:

$$D_M = \{(\mathcal{X}, \Sigma) \in TM \oplus T^*M \mid \Sigma - \mathbf{i}_{\mathcal{X}}\Omega_M \in \Delta_M^\circ \text{ for } \mathcal{X} \in \Delta_M\}, \quad (5.6)$$

where $\Omega_M := \pi_{ZM}^*\Omega$. It can be shown directly that D_M is a Dirac structure (see [Yoshimura and Marsden \[2006a\]](#)); alternatively, note that D_M is of the form described in the theorem 4.4 and hence automatically determines a (multi-)Dirac structure.

Now, let $\mathcal{X} : M \rightarrow TZ$ be a partial vector field as in (3.7). In local coordinates, we have

$$\mathcal{X} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \dot{p}_t \frac{\partial}{\partial p_t}.$$

On the other hand, the generalized energy \mathcal{E} is given by $\mathcal{E} = E dt$, where

$$E(t, q^i, v^i, p_t, p_i) = p_t + p_i v^i - L(t, q^i, v^i)$$

and the differential of E is locally given by

$$\mathbf{d}E(t, q^i, v^i, p_t, p_i) = \left(-\frac{\partial L}{\partial t}\right) dt + \left(-\frac{\partial L}{\partial q^i}\right) dq^i + \left(p_i - \frac{\partial L}{\partial v^i}\right) dv^i + dp_t + v^i dp_i.$$

Thus, the *nonholonomic Lagrange-Dirac system for time-dependent mechanics* is given by a triple (\mathcal{X}, E, D) which satisfies the condition

$$(\mathcal{X}, \mathbf{d}E) \in D. \quad (5.7)$$

This is in turn equivalent to the intrinsic nonholonomic Lagrange-Dirac equations:

$$\mathbf{i}_{\mathcal{X}}\Omega - \mathbf{d}E \in \Delta_M^\circ \text{ and } \mathcal{X} \in \Delta_M.$$

In coordinates, one obtains

$$\dot{q}^i = v^i, \quad \dot{p}_i = \frac{\partial L}{\partial q^i} + \lambda_r A_i^r, \quad \dot{p}_i = \frac{\partial L}{\partial v^i}, \quad (5.8)$$

together with the affine constraints $A_i^r(t, q)v^i + B^i(t, q) = 0$ and the energy conservation law

$$\frac{d}{dt} (p_t + p_i v^i - L(t, q^i, v^i)) = 0.$$

By imposing the constraint that $E = 0$, we find that $p_t = L - p_i v^i$. Note that the equations of motion (5.8) are the analogues (for affine constraints) of the Dirac equations (1.4).

One can apply a similar approach to construct multi-Dirac structures for classical field theories with nonholonomic constraints. After constructing the distribution Δ_M in a suitable way (see e.g. [Vankerschaver, Cantrijn, de Leon and Matín de Diego \[2005\]](#)), the nonholonomic multi-Dirac structure will then be given again by theorem 4.4 and the nonholonomic field equations will be given by (5.7). At this

point, we emphasize that in the current description of nonholonomic multi-Dirac structures the constraints are assumed to be *linear/affine* in the multi-velocities, while Δ_M can be constructed for nonlinear constraints indeed. Most physically relevant nonholonomic field theories, however, have nonlinear constraints, and the study of nonlinear nonholonomic field theories using multi-Dirac structures will be the subject of future work.

5.4 Elastostatics

To conclude, we show how the **Hu-Washizu principle** from linear elastostatics (see Washizu [1968]) can be understood as a Hamilton-Pontryagin variational principle. Let \mathcal{B} be an open subset of \mathbb{R}^3 representing an elastic medium and consider the bundle $\pi : Y := \mathcal{B} \times \mathbb{R}^3 \rightarrow \mathcal{B}$. We denote coordinates on J^1Y by (X^I, ϕ^i, F_I^i) , where the fiber coordinates F_I^i represent the **deformation gradient** of the material. The dual bundle Z has coordinates $(X^I, \phi^i, p, \pi_i^I)$, where π_i^I will be seen to represent the first **Piola-Kirchhoff tensor**.

Furthermore, we let $W : J^1Y \rightarrow \mathbb{R}$ be the **stored-energy function** of the material, and we assume that W is a function of F_I^i only (this assumption can easily be relaxed). We denote the external body forces acting on the medium by $b_i(X)$ and we assume that there are traction boundary conditions acting on the boundary $\partial\mathcal{B}$, given by $\pi_i^I n_I = \tau_i$, where n_I is the unit normal to the boundary and $\tau_i(X)$ is a prescribed traction vector. Dirichlet boundary conditions can easily be incorporated by restricting the space of fields as in Marsden and Hughes [1994], though we will not do so here.

Taking W as our Lagrangian, the Hamilton-Pontryagin principle then becomes

$$S(\phi, F, \pi) = \int_{\mathcal{B}} \left[\pi_i^I \left(\frac{\partial \phi^i}{\partial X^I} - F_I^i \right) + W(F) - b_i(X) \phi^i(X) \right] dV + \int_{\partial\mathcal{B}} \tau_i(X) \phi^i(X) dS, \quad (5.9)$$

where we have incorporated the traction boundary conditions through the surface integral. Taking variations with respect to ϕ , F , and π , we then obtain the following equations:

$$\frac{\partial \pi_i^I}{\partial X^I} + b_i = 0, \quad F_I^i = \frac{\partial \phi^i}{\partial X^I}, \quad \text{and} \quad \pi_i^I = \frac{\partial W}{\partial F_I^i}, \quad (5.10)$$

together with the boundary conditions $\pi_i^I n_I = \tau_i$. The last of these equations shows that π_i^I can be identified with the first Piola-Kirchhoff tensor, the second equation is nothing but the definition of the deformation gradient, while the first equation is the Navier-Cauchy equation in the absence of external forces. To incorporate external forces and boundary conditions, the variational principle (5.9) can be extended along the lines of Marsden and Hughes [1994].

The variational principle (5.9) is not yet the Hu-Washizu variational principle. To obtain the latter, we work in the context of linear elasticity. Write the elastic field ϕ^i as $\phi^i(x) = x^i + \mathbf{u}(x)$, where \mathbf{u} is the **displacement field**, and where material

and spatial points are both denoted by x^i . For the stored energy function, we take a quadratic expression of the form

$$W(e) = \frac{1}{2} c_{ik}^{jl} e_j^i e_l^k \quad \text{with} \quad e_j^i = \frac{1}{2} (u_{,j}^i + u_{,i}^j),$$

where e_j^i are referred to as the **strain tensor**, while the **constitutive tensor** c_{ik}^{jl} characterizes the elastic medium. We now substitute these expressions into (5.9). Up to a global sign and ignoring boundary conditions and external forces for the sake of simplicity, we obtain the Hu-Washizu principle for linear elastostatics:

$$S(u, e, p) = \int_{\mathcal{B}} \left[p_i^j \left(e_j^i - \frac{1}{2} (u_{,j}^i + u_{,i}^j) \right) + W(e) \right] dV.$$

Here, we have used the fact that the stress tensor $p_i^j = \partial W(e) / \partial e_j^i$ is symmetric in both indices.

It goes without saying that this derivation relies heavily on the Euclidian structure of the space in which the body moves, and that the linear approximation is necessary. Nevertheless, it is possible to derive a Hu-Washizu principle for *nonlinear* elasticity in terms of the **Green tensor** which is valid on arbitrary Riemannian manifolds. This modified principle can be obtained from the Hamilton-Pontryagin principle by means of symmetry reduction and will be the subject of a forthcoming paper.

6 Conclusions and Future Work

In this paper, we have developed the Hamilton-Pontryagin principle for field theories and we have studied the relation between the field equations obtained from the Hamilton-Pontryagin principle and Lagrange-Dirac field theories, formulated in terms of multi-Dirac structures. These ideas are shown to be a natural extension of Dirac structures and Lagrange-Dirac dynamical systems in mechanics. Finally, we have given a few illustrative examples of Lagrange-Dirac field theories: the Klein-Gordon equation, Maxwell's equations, the case of time-dependent mechanics with affine nonholonomic constraints, and elastostatics.

Space-Time Slicings and the Adjoint Formalism. The approach in this paper are *covariant* in the sense that no distinction is made between time and the spatial variables on the base space. In fact, as we have pointed out before, the configuration bundle $\pi_{XY} : Y \rightarrow X$ can be arbitrary and in particular X does not have to represent spacetime. While this approach has various advantages (see Kijowski and Tulczyjew [1979]; Bridges [1997]; Gotay, Isenberg, Marsden and Montgomery [1997] for an overview), not in the least that the resulting structures are all finite-dimensional, it is often necessary to single out one coordinate direction as time, for instance when considering the initial-value problem for field theories.

In related work (see Castrillón-López et al [2010]) we will show that when such a space-time slicing is chosen (as in Gotay, Isenberg and Marsden [1999]; Binz,

Śniatycki, and Fischer [1988]), the Hamilton-Pontryagin variational principle yields the field equations in *adjoint form*, bringing out the distinction between the *dynamical* variables of the theory (the physical fields) and the *kinematic* variables, which are present due to gauge freedom. For instance, in the case of relativity, the former are the metric and the extrinsic curvature on a spatial hypersurface, the latter are the lapse and shift, and the adjoint formalism reduces to the familiar ADM equations of relativity.

Furthermore, the choice of a space-time splitting induces a correspondence between multi-Dirac structures on the jet bundle and certain infinite-dimensional Dirac structures on the space of all fields. The latter are in turn related to the *Stokes-Dirac structures* of van der Schaft and Maschke [2002] used in boundary control theory. The precise link between Stokes-Dirac structures and infinite-dimensional Dirac structures involves Poisson reduction and was established in Vankerschaver, Yoshimura, Leok, and Marsden [2010].

Discretizations of Multi-Dirac Structures and Field Theories. Leok and Ohsawa [2011] recently proposed a concept of *discrete Dirac structure* in mechanics. At the same time, it has been shown by Bou-Rabee and Marsden [2009] that discrete versions of the Hamilton-Pontryagin principle can be used to derive very accurate geometric integrators for mechanical systems. We propose to explore the field-theoretic version of these discretizations. Since multi-Dirac structures and variational principles are a natural way to describe field theories even for degenerate Lagrangians, this approach is especially promising. In particular, we will focus on field theories whose covariance group includes the diffeomorphism group of the base space X , focusing on the link with the adjoint formalism.

Symmetry and Reduction of Multi-Dirac Structures. The presence of non-trivial symmetry groups is an essential feature of many classical field theories, such as electromagnetism, Yang-Mills theories, and elasticity. More complex examples include the theory of perfect fluids and Einstein's theory of relativity. It is often useful to obtain a form of the field equations from which the symmetry has been factored out; this is for instance what happens in writing Maxwell's equations in terms of the field strength $F_{\mu\nu}$ rather than the vector potential A_μ . From a bundle-theoretic point of view, various approaches of symmetry reduction for field theories have been proposed (see Castrillon-Lopez and Marsden [2003]; Castrillón López and Ratiu [2003]; Gay-Balmaz and Ratiu [2010] and the references therein). We propose to develop a reduction theory for multi-Dirac structures along the lines of the theory of symmetry reduction for Dirac structures in classical mechanics developed by Blankenstein and Ratiu [2004]; Yoshimura and Marsden [2007, 2009].

A Generalities about Multi-Vector Fields

A.1 Definitions

Let M be a manifold of dimension n . A k -multivector field (where $k \leq n$) on M is a section \mathcal{X} of the k -fold exterior power $\bigwedge^k(TM)$ of the tangent bundle. The module of all k -multivector fields is denoted by $\mathfrak{X}^k(M)$.

We define the **interior product** of multivector fields and forms as follows. Let $\mathcal{X} \in \mathfrak{X}^k(M)$ and $\alpha \in \Omega^l(M)$, where $k \leq l$. The **left interior product** $\mathcal{X} \lrcorner \alpha$ is then the unique $(l - k)$ -form such that

$$(\mathcal{X} \lrcorner \alpha)(\mathcal{X}') = \alpha(\mathcal{X} \wedge \mathcal{X}')$$

for all $(l - k)$ -multivectors \mathcal{X}' . Similarly, the **right interior product** of a multivector $\mathcal{X} \in \mathfrak{X}^k(M)$ and a form $\beta \in \Omega^m(M)$, with $k \geq m$, is the unique $(k - m)$ -multivector $\mathcal{X} \lrcorner \beta$ with the property that

$$(\mathcal{X} \lrcorner \beta) \lrcorner \gamma = (\beta \wedge \gamma)(\mathcal{X})$$

for all $(k - m)$ -forms γ on M .

Further information about multivectors and interior products can be found in (for instance) Tulczyjew [1974]; Marle [1997]. From these papers, we quote the following two results on the relation between the interior product and the wedge product of vector fields and forms, respectively.

Lemma A.1 (Tulczyjew [1974]). *Let $\mathcal{X} \in \mathfrak{X}^k(M)$ and $\alpha \in \Omega^1(M)$. Then*

$$(\mathcal{X} \wedge \mathcal{X}') \lrcorner \alpha = (\mathcal{X} \lrcorner \alpha) \wedge \mathcal{X}' + (-1)^k \mathcal{X} \wedge (\mathcal{X}' \lrcorner \alpha)$$

for any multivector \mathcal{X}' , and

$$\mathcal{X} \lrcorner (\alpha \wedge \beta) = (\mathcal{X} \lrcorner \alpha) \lrcorner \beta + (-1)^k \alpha \wedge (\mathcal{X} \lrcorner \beta)$$

for any form β .

When the multi-vector field is decomposable, the right interior product with a one-form can easily be computed by means of the following formula, the proof of which follows from the previous theorem.

Lemma A.2. *Let \mathcal{X} be a k -multi-vector field which is **decomposable** in the sense that $\mathcal{X} = \bigwedge_{\mu=1}^k \mathcal{X}_\mu$, where \mathcal{X}_μ ($\mu = 1, \dots, k$) are vector fields and consider a one-form α . Then*

$$\mathcal{X} \lrcorner \alpha = \sum_{\mu=1}^k (-1)^{\mu+1} \langle \mathcal{X}_\mu, \alpha \rangle \hat{\mathcal{X}}_\mu,$$

where $\hat{\mathcal{X}}_\mu$ is the $(k - 1)$ -vector field obtained by deleting \mathcal{X}_μ , i.e.

$$\hat{\mathcal{X}}_\mu = \bigwedge_{\substack{\lambda=1 \\ \lambda \neq \mu}}^{n+1} \mathcal{X}_\lambda.$$

Corollary A.3. *Let \mathcal{X} be a decomposable k -multivector and α an l -form, where $l \geq k$. Then $\mathcal{X} \lrcorner (\mathcal{X} \lrcorner \alpha) = 0$.*

Proof. Let $\beta = \mathcal{X} \lrcorner \alpha$. With the notations of the previous lemma, we have that

$$\mathcal{X} \lrcorner \beta = \sum_{\mu=1}^k (-1)^{\mu+1} \langle \mathcal{X}_\mu, \beta \rangle \hat{\mathcal{X}}_\mu,$$

but $\langle \mathcal{X}_\mu, \beta \rangle = 0$ for all $\mu = 1, \dots, k$, since $\langle \mathcal{X}_\mu, \beta \rangle = \mathbf{i}_{\mathcal{X}_\mu} \mathbf{i}_{\bigwedge_{\lambda=1}^{n+1} \mathcal{X}_\lambda} \alpha = 0$. \blacksquare

In this paper, we also let $\hat{\mathcal{X}}_{\mu\nu}$ (where $\mu \neq \nu$) be the $(k-2)$ -multivector obtained by deleting both \mathcal{X}_μ and \mathcal{X}_ν :

$$\hat{\mathcal{X}}_{\mu\nu} = \bigwedge_{\substack{\lambda=1 \\ \lambda \neq \mu, \nu}}^{n+1} \mathcal{X}_\lambda.$$

If $\mu < \nu$, we then have that

$$\mathcal{X}_\lambda \wedge \hat{\mathcal{X}}_{\mu\nu} = (-1)^{\mu+1} \delta_{\mu\lambda} \hat{\mathcal{X}}_\nu - (-1)^{\nu+1} \delta_{\nu\lambda} \hat{\mathcal{X}}_\mu.$$

If $\mu > \nu$, note that $\hat{\mathcal{X}}_{\mu\nu} = \hat{\mathcal{X}}_{\nu\mu}$. The above formula can then be applied.

A.2 Proof of (3.12)

The computation of equation (3.12) is somewhat involved and depends on a number of coordinate identities, listed here. First, recall that \mathcal{X} is an $(n+1)$ -multivector field with local expression (3.7), and that the multi-symplectic form Ω is locally given by equation (2.3).

Recall the volume form η on X is locally expressed by $d^{n+1}x$ and denote

$$d^n x_\mu := \frac{\partial}{\partial x^\mu} \lrcorner d^{n+1}x, \quad d^{n-1}x_{\mu\nu} := \frac{\partial}{\partial x^\nu} \lrcorner d^n x_\mu, \quad \dots$$

The following contractions will be useful:

$$\hat{\mathcal{X}}_\mu \lrcorner d^n x_\nu = (-1)^{\nu+n} \delta_{\mu\nu} \tag{A.1}$$

and, for $\mu < \nu$,

$$\hat{\mathcal{X}}_{\mu\nu} \lrcorner d^n x_\lambda = (-1)^{n+\mu+\nu} (\delta_{\nu\lambda} dx^\mu + \delta_{\mu\lambda} dx^\nu). \tag{A.2}$$

Now return to the calculation of equation (3.12) and the contraction of \mathcal{X} with Ω_M is now given by

$$\mathcal{X} \lrcorner \Omega_M = \mathcal{X} \lrcorner (dy^A \wedge dp_A^\mu \wedge d^n x_\mu) - \mathcal{X} \lrcorner (dp \wedge d^{n+1}x),$$

and the two terms on the right-hand side will be calculated separately.

We begin with the first term. Using lemma A.1, we have

$$\mathcal{X} \lrcorner (dy^A \wedge dp_A^\mu \wedge d^n x_\mu) = (\mathcal{X} \lrcorner dy^A) \lrcorner (dp_A^\mu \wedge d^n x_\mu) + (-1)^{n+1} dy^A \wedge (\mathcal{X} \lrcorner (dp_A^\mu \wedge d^n x_\mu)).$$

Both terms can be calculated using lemma A.2 and using (A.1) and (A.2). After some rearrangements, the result is that

$$\mathcal{X} \lrcorner (dy^A \wedge dp_A^\mu \wedge d^n x_\mu) = (-1)^{n+2} \left[(C_\mu^A C_{A\lambda}^\lambda - C_\lambda^A C_{A\mu}^\lambda) dx^\mu + C_\mu^A dp_A^\mu \right].$$

For the second term, we use the same techniques to conclude that

$$\mathcal{X} \lrcorner (dp \wedge d^{n+1} x) = (-1)^{n+1} [dp - C_\mu dx^\mu].$$

Putting the results for both terms together, we obtain (3.12).

References

- Baez, J. C., A. E. Hoffnung, and C. L. Rogers [2010], Categorified symplectic geometry and the classical string, *Comm. Math. Phys.* **293**, 701–725.
- Binz, E., J. Śniatycki, and H. Fischer [1988], *Geometry of Classical Fields*, volume 154 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam.
- Blankenstein, G. and T. S. Ratiu [2004], Singular reduction of implicit Hamiltonian systems. *Rep. Math. Phys.* **53**(2), 211–260.
- Bou-Rabee, N. and J. E. Marsden [2009], Hamilton-Pontryagin integrators on Lie groups. I. Introduction and structure-preserving properties, *Found. Comput. Math.* **9**, 197–219.
- Bridges, T. J. [1997], Multi-symplectic structures and wave propagation, *Math. Proc. Camb. Phil. Soc.* **121**, 147–190.
- Cantrijn, F., L. A. Ibort, and M. de León [1999], On the geometry of multisymplectic manifolds, *J. Austral. Math. Soc. (Series A)* **66**, 303–330.
- Cantrijn, F., M. de León, and D. Martín de Diego [1999], On almost-Poisson structures in nonholonomic mechanics, *Nonlinearity* **12**, 721–737.
- Cariñena, J. F., M. Crampin, and L. A. Ibort [1991], On the multisymplectic formalism for first order field theories, *Diff. Geom. Appl.* **1**, 345–374.
- Castrillón López, M., J. Vankerschaver, M. J. Gotay, J. E. Marsden and H. Yoshimura [2010], The Hamilton-Pontryagin principle and the adjoint formalism, *preprint*.
- Castrillón-López, M. and J. E. Marsden [2003], Some remarks on Lagrangian and Poisson reduction for field theories, *J. Geom. and Phys.* **48**, 52–83.
- Castrillón López, M. and T. S. Ratiu [2003], Reduction in principal bundles: covariant Lagrange-Poincaré equations, *Comm. Math. Phys.* **236**, 223–250.
- Cendra, H., J. E. Marsden, T. S. Ratiu, and H. Yoshimura [2010], Dirac bundle reduction for mechanical systems with symmetry, *preprint*.
- Courant, T. J. [1990], Dirac manifolds, *Trans. Amer. Math. Soc.* **319**, 631–661.
- Courant, T. and A. Weinstein [1988], Beyond Poisson structures. In *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*, volume 27 of *Travaux en Cours*, pages 39–49. Hermann, Paris.

- Dalsmo, M. and A. van der Schaft [1999], On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM J. Control Optim.*, **37**, nr. 1, 54–91 (electronic), 1999.
- de León, M. and D. Martín de Diego [1996], On the geometry of non-holonomic Lagrangian systems, *J. Math. Phys.* **37**, 3389–3414.
- Dirac, P. A. M. [1950], Generalized Hamiltonian dynamics, *Canadian J. Math.* **2**, 129–148.
- Dorfman, I. [1993], *Dirac Structures and Integrability of Nonlinear Evolution Equations*. Number 94j:58081 in *Nonlinear Science: Theory and Applications*. John Wiley & Sons Ltd., Chichester.
- Echeverría-Enríquez, A., M. C. Muñoz-Lecanda, and N. Román-Roy [2002], Multivector fields and connections. Applications to field theories, *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **96**, 83–93.
- Forger, M., C. Paufler, and H. Römer [2005], Hamiltonian multivector fields and Poisson forms in multisymplectic field theory, *J. Math. Phys.* **46**, 112903, 29.
- Gay-Balmaz, F. and T. S. Ratiu [2010], A new Lagrangian dynamic reduction in field theory, To appear in *Ann. Inst. Fourier, Grenoble*, 2010.
- Gitman, D. M. and I. V. Tyutin [1990], *Quantization of Fields with Constraints*. Springer Series in Nuclear and Particle Physics. Springer-Verlag, Berlin. Translated from the Russian.
- Goldschmidt, H. and S. Sternberg [1973], The Hamilton-Cartan formalism in the calculus of variations, *Ann. Inst. Fourier (Grenoble)* **23**(1), 203–267.
- Gotay, M., J. Isenberg, J. E. Marsden and R. Montgomery [1997], Momentum maps and classical relativistic fields, Part I: Covariant field theory (66 pages). [www.arxiv.org: \[2004\] physics/9801019](http://www.arxiv.org/abs/2004physics/9801019).
- Gotay, M., J. Isenberg, and J. E. Marsden [1999], Momentum maps and classical relativistic fields, Part II: Canonical analysis of field theories (70 pages). [www.arxiv.org: \[2004\] math-ph/0411032](http://www.arxiv.org/abs/2004math-ph/0411032).
- Kijowski, J. and W. Tulczyjew [1979], *A Symplectic Framework for Field Theories*. Springer Lect. Notes in Physics, Volume 107.
- Krupková, O. [2005], Partial differential equations with differential constraints, *J. Diff. Eq.* **220**, 354–395.
- Leok, M. and T. Ohsawa [2011], Discrete Dirac Structures and Variational Discrete Dirac Mechanics, *Foundations of Computational Mathematics*. (accepted, [arXiv:0810.0740](https://arxiv.org/abs/0810.0740) [[math.SG](https://arxiv.org/abs/0810.0740)]).
- Livens, G. H. [1919], On Hamilton’s principle and the modified function in analytical dynamics, *Proc. Roy. Soc. Edinburgh* **39**, 113.
- Marle, C. M. [1997], The Schouten-Nijenhuis bracket and interior products, *J. Geom. Phys.*, **23**, 350–359.
- Marsden, J. E. and T. J. Hughes [1994], *Mathematical Foundations of Elasticity*. Dover Publications Inc., New York. Corrected reprint of the 1983 original.

- Marsden, J. E., G. W. Patrick, and S. Shkoller [1998], Multisymplectic geometry, variational integrators and nonlinear PDEs, *Comm. Math. Phys.* **199**, 351–395.
- Marsden, J. E. and S. Shkoller [1999], Multisymplectic geometry, covariant Hamiltonians and water waves, *Math. Proc. Camb. Phil. Soc.* **125**, 553–575.
- Saunders, D. J. [1989], *The Geometry of Jet Bundles*, volume 142 of *London Mathematical Society Lecture Note Series*. Cambridge University Press.
- Schwinger, J., L. L. DeRaad, K. A. Milton, and W.-y. Tsai [1998], *Classical Electrodynamics*. Perseus Books, Reading, Mass.
- Tulczyjew, W. M. [1974], The graded Lie algebra of multivector fields and the generalized Lie derivative of forms, *Bull. Acad. Pol. Sci.* **22**, 937–942.
- van der Schaft, A. J. and B. M. Maschke [1995], The Hamiltonian formulation of energy conserving physical systems with external ports, *Archiv für Elektronik und Übertragungstechnik* **49**, 362–371.
- van der Schaft, A. J. and B. M. Maschke [2002], Hamiltonian formulation of distributed-parameter systems with boundary energy flow, *J. Geom. Phys.* **42**, 166–194.
- Vankerschaver, J. [2007], A class of nonholonomic kinematic constraints in elasticity, *J. Phys. A: Math. Theor.* **40**(14), 3389–3913.
- Vankerschaver, J., F. Cantrijn, M. de Leon, and D. Martín de Diego [2005], Geometric aspects of nonholonomic field theories, *Rep. Math. Phys.* **56**, 387–411.
- Vankerschaver, J., H. Yoshimura, and M. Leok [2011], *Multi-Dirac Structures and Gerstenhaber Algebras*, *J. Geom. Phys.* **61**, 1415–1425.
- Vankerschaver, J., H. Yoshimura, M. Leok, and J. E. Marsden [2010], Stokes-Dirac Structures through Reduction of Infinite-Dimensional Dirac Structures, Proc. 49th Conf. on Decision and Control, 6265–6270, 2010.
- Washizu, K. [1968], *Variational Methods in Elasticity and Plasticity*. Pergamon Press, Oxford. International Series of Monographs in Aeronautics and Astronautics, Vol. 9.
- Yoshimura, H. and J. E. Marsden [2006a], Dirac structures in Lagrangian mechanics Part I: Implicit Lagrangian systems, *J. Geom. and Phys.* **57**, 133–156.
- Yoshimura, H. and J. E. Marsden [2006b], Dirac structures in Lagrangian mechanics Part II: Variational structures, *J. Geom. and Phys.* **57**, 209–250.
- Yoshimura, H. and J. E. Marsden [2007], Reduction of Dirac structures and the Hamilton-Pontryagin principle, *Rep. Math. Phys.* **60**, 381–426.
- Yoshimura, H. and J. E. Marsden [2009], Dirac cotangent bundle reduction, *J. Geom. Mech.* **1**(1), 87–158.
- Zambon, M. [2010], L-infinity algebras and higher analogues of Dirac structures, *preprint*, 2010.