Geometrical mechanics on algebroids

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References

- *E. Martínez*: Lagrangian Mechanics on Lie Algebroids, Acta Appl. Math. vol. **67**, 2001 (295–320).
- *E. Martínez*: Geometric formulation of Mechanics on Lie algebroids, in Proceedings of the VIII Fall Workshop on Geometry and Physics, Medina del Campo, 1999, Publicaciones de la RSME, vol. **2**, 2001, (209–222).
- *M. de León, J.C. Marrero, E. Martínez*: Lagrangian submanifolds and dynamics on Lie algebroids, J. Phys. A: Math. Gen. vol. **38**, 2005 (R241–R308).

References

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- *J. Grabowski and P. Urbański*: Lie algebroids and Poisson-Nijenhuis structures, Rep. Math. Phys. vol. **40**, 1997,(195–208).
- *J. Grabowski and P. Urbański*: Algebroids – general differential calculi on vector bundles, J. Geom. Phys. vol. **31**, 1999, (111-141).

Notation and coordinates

$$
\begin{array}{cccc}
TM & T^*M & E & E^* \\
\Big\downarrow{\scriptstyle \tau_M} & \Big\downarrow{\scriptstyle \pi_M} & \Big\downarrow{\scriptstyle \tau} & \Big\downarrow{\scriptstyle \pi} \\
M & M & M & M \\
(x^a, \dot{x}^b) & (x^a, p_b) & (x^a, y^i) & (x^a, \xi_i)
\end{array}
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$TE \quad (x^a, y^i, \dot{x}^b, \dot{y}^j),$

$TE^* \quad (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j),$

$$\left.\begin{array}{ll} T^*E & (x^a, y^i, p_b, \pi_j), \\ T^*E^* & (x^a, \xi_i, p_b, \varphi^j). \end{array}\right\} \text{ isomorphic as double vector bundles}$$

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$$R_\tau : T^*E^* \longrightarrow T^*E$$

$$(x^a, y^i, p_b, \pi_j) \circ \mathcal{R}_\tau = (x^a, \varphi^i, -p_b, \xi_j).$$

More notation

For a bundle $\tau : E \longrightarrow M$ and its dual $\pi : E^* \longrightarrow M$ we denote:

- the tensor product over M: $\otimes^k_M(E) = E \otimes_M \cdots \otimes_M E$,
- the set of sections of the above tensor bundle: $\otimes^k(\tau)$,

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$$\iota(K) : \otimes^k(\pi) \ni a \longmapsto \langle K(m), a\rangle \in \mathbb{R},$$

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- for X being a section of τ we have the operator of insertion $i_X : \otimes^{k+1}(\pi) \ni \mu_1 \otimes \cdots \otimes \mu_{k+1} \mapsto$

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$$\langle X, \mu_1 \rangle \mu_2 \otimes \cdots \otimes \mu_{k+1} \in \otimes^k(\pi),$$
- in particular, for $\Lambda \in \otimes^2(\tau)$ we have $\widetilde{\Lambda} : E \longrightarrow E^*$, given by $\widetilde{\Lambda} \circ X = i_X \Lambda$.

Lagrange formalism for $\mathsf{T}M$ (according to W.M.Tulczyjew)

M - the configuration manifold, T^*M - the phase space,
$L : \mathsf{T}M \longrightarrow \mathbb{R}$ - the Lagrangian.

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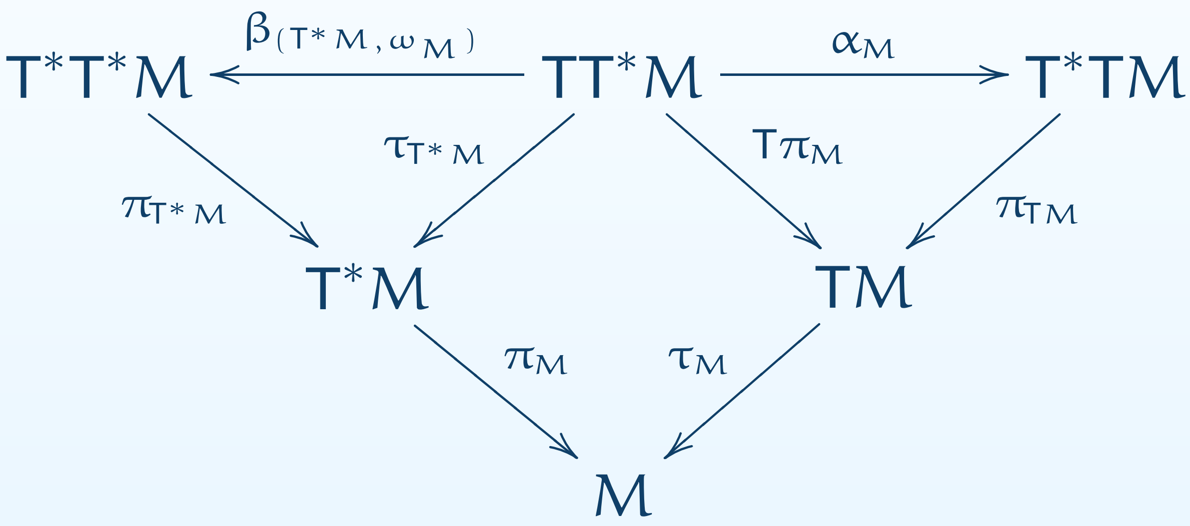
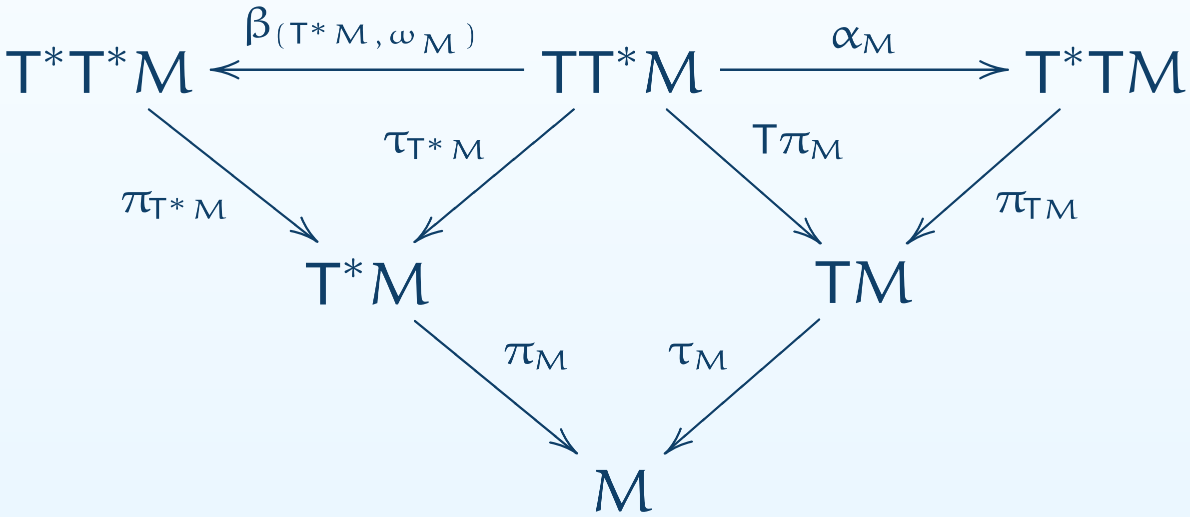


Lagrange formalism for TM (according to W.M.Tulczyjew)

M - the configuration manifold, T^*M - the phase space,
$L : TM \longrightarrow \mathbb{R}$ - the Lagrangian.



The dynamics is a first-order differential equation $D \subset TT^*M$.
In the regular case

$$D = \alpha_M^{-1}(dL(TM)).$$

Lagrange formalism for $\mathsf{T}M$ (according to W.M.Tulczyjew)

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Trajectories in the configuration space (i.e. $\pi_M \circ \gamma$) are solutions of the second order Euler-Lagrange equation.

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The Euler-Lagrange equation can be obtained directly from D:

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The Euler-Lagrange equation can be obtained directly from D:

- Take the prolongation $\mathsf{P}D = \mathsf{T}D \cap \mathsf{T}^2\mathsf{T}^*M$
- Project to T^2M by $\mathsf{T}^2\pi_M : \mathsf{T}^2\mathsf{T}^*M \longrightarrow \mathsf{T}^2M$:

$$E_L = \mathsf{T}^2\pi_M(\mathsf{P}D).$$

Lie algebroids as double vector bundle morphisms

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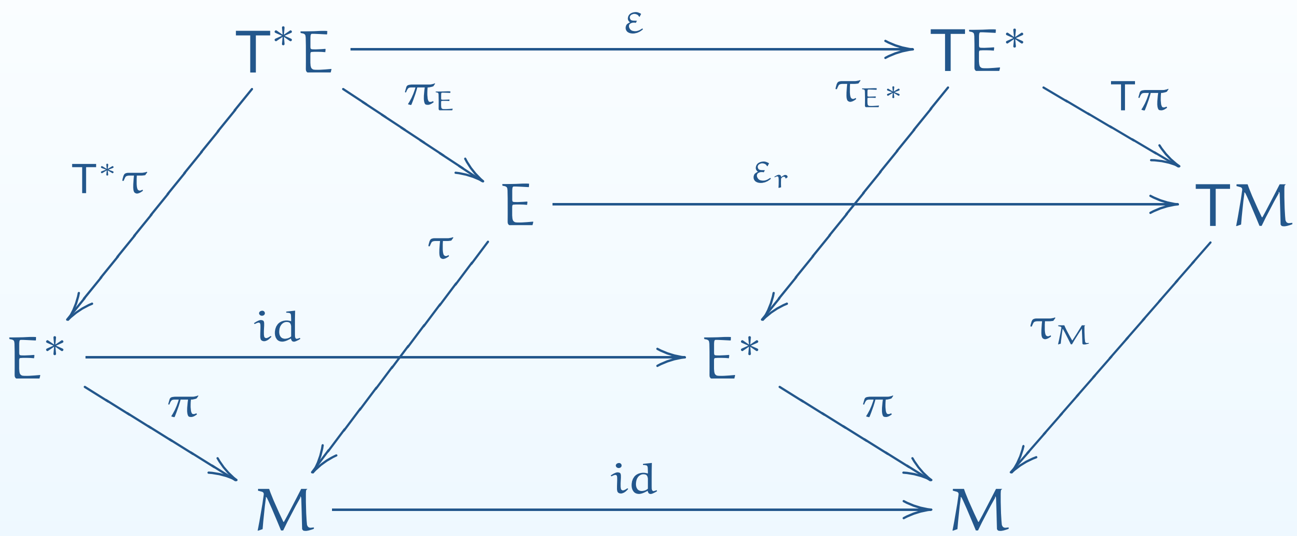
$$
\begin{array}{ccc}
T^*E^* & \xrightarrow{\widetilde{\Lambda}} & TE^* \\
{\scriptstyle R_\tau}\downarrow & \nearrow{\scriptstyle \varepsilon} & \\
T^*E & &
\end{array}
$$

$\widetilde{\Lambda}$ is the double vector bundle morphism given by the 2-contravariant tensor Λ canonically associated with the bracket.

Algebroids as double vector bundle morphisms



On the cores $\qquad \varepsilon_c : T^*M \longrightarrow E^*$

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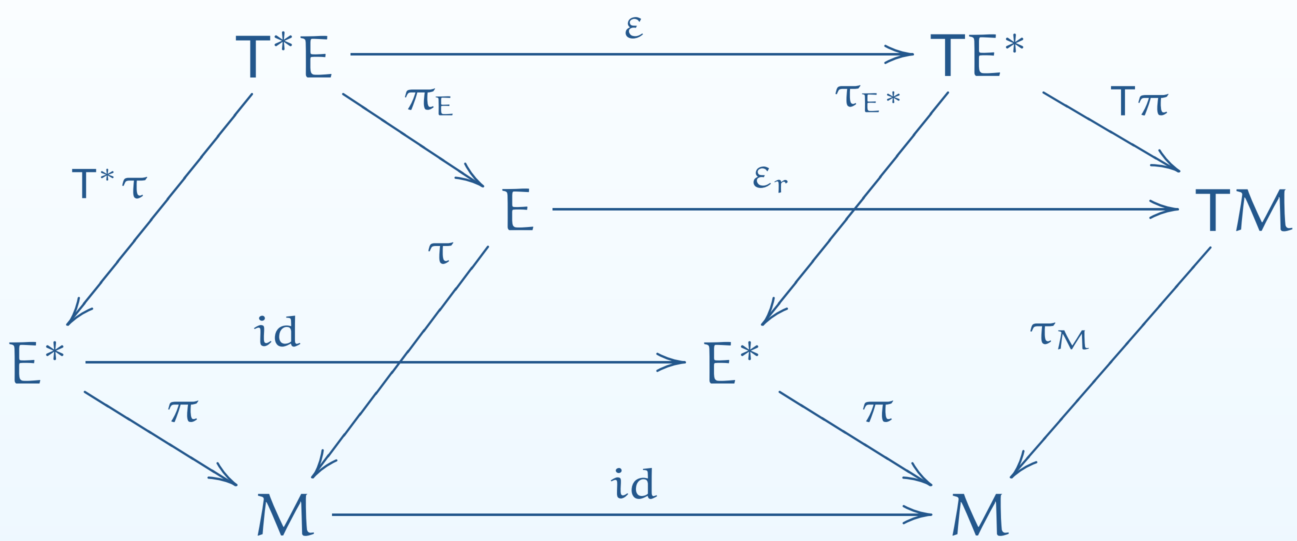

On the cores $\quad \varepsilon_c : T^*M \longrightarrow E^*$

$$\Lambda : \quad \Lambda = c^k_{ij}(x)\xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma^a_j(x)\partial_{x^a} \otimes \partial_{\xi_j}$$

$$\varepsilon : \quad (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \circ \varepsilon = (x^a,\ \pi_i,\ \rho^b_k(x)y^k,\ \ c^k_{ij}(x)y^i\pi_k + \sigma^a_j(x)p_a)$$

$$\varepsilon_r : \quad (x^a,\ \dot{x}^b) \circ \varepsilon_r = (x^a,\ \rho^b_k(x)y^k),$$

$$\varepsilon_c : \quad (x^a,\ \xi_i) \circ \varepsilon_c = (x^a,\ \sigma^b_i(x)p_b).$$

Algebroids as double vector bundle morphisms

Remark: An algebroid (E, ε) is a Lie algebroid if and only if the corresponding tensor Λ_ε is a Poisson tensor. It means in particular that $\varepsilon_r = (\varepsilon_c)^*$ ($\rho = \sigma$).

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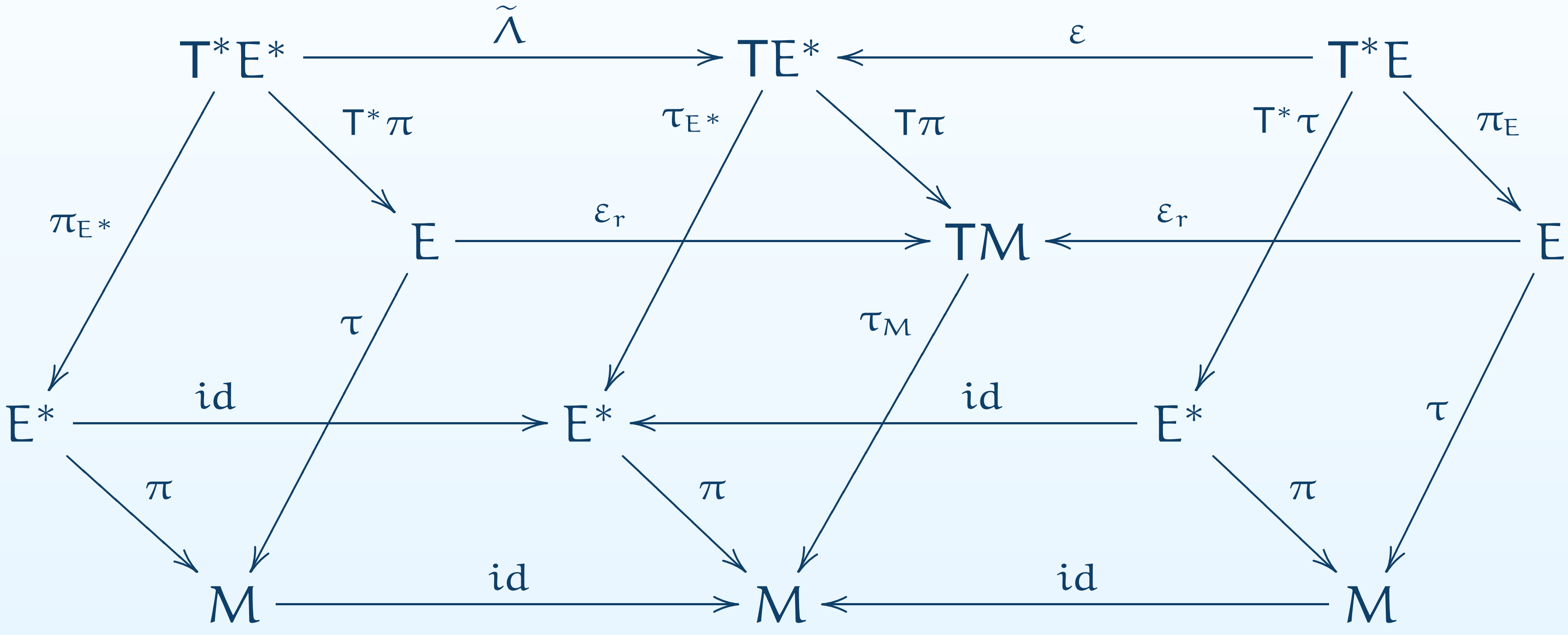
Canonical example: Canonical algebroid of TM is described by the Tulczyjew ${\alpha_M}^{-1}$

$$\alpha_M : \mathsf{T}^*\mathsf{T}M \longrightarrow \mathsf{T}\mathsf{T}^*M$$

Lagrange formalism for general algebroid

The algebroid analogue for the Tulczyjew triple. E replaces TM.



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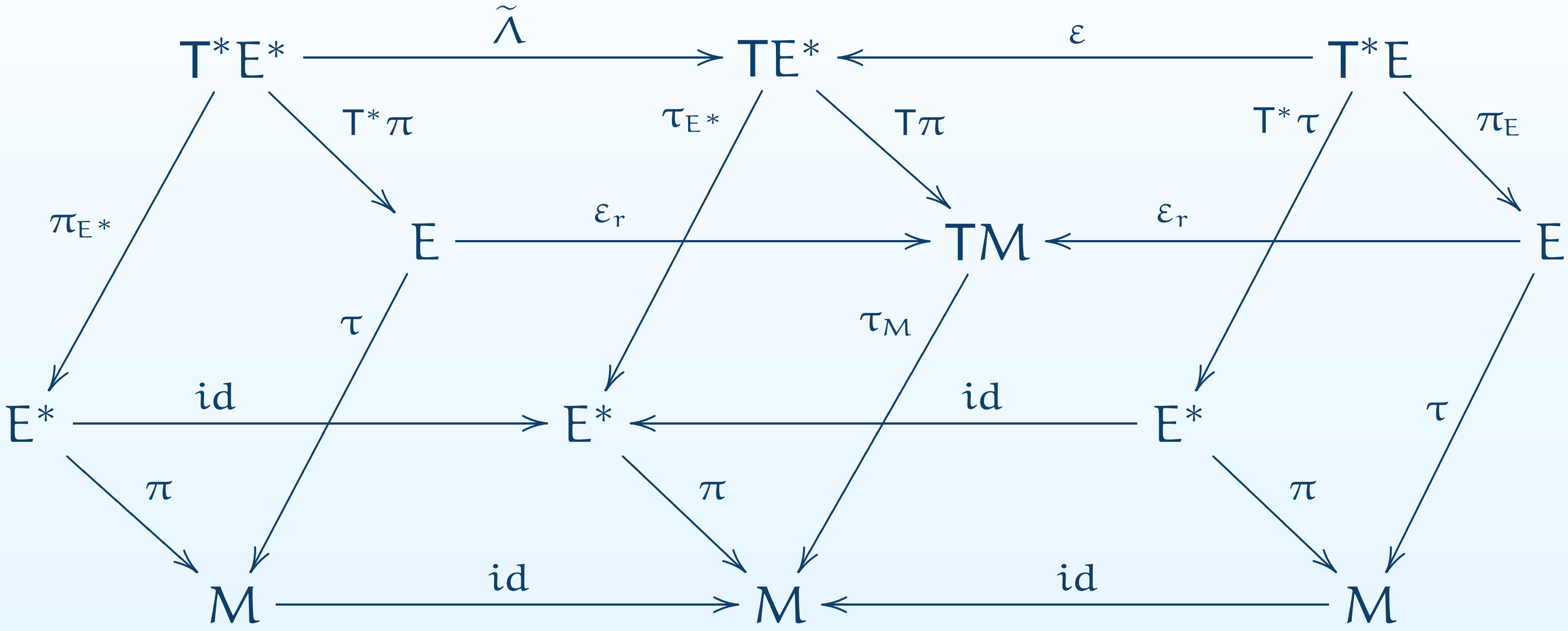

Any function ("Lagrangian") $L : E \longrightarrow \mathbb{R}$ defines $\mathrm{d}L(E) \subset T^*E$ and

$$D = \varepsilon(\mathrm{d}L(E)) \subset TE^*.$$

Lagrange formalism for general algebroid

The Lagrangian defines two smooth maps:

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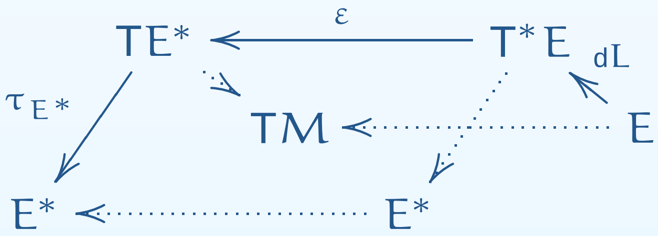
(1) The Legendre mapping:

$$\mathcal{L}eg : E \longrightarrow E^* \quad \text{defined by} \quad \mathcal{L}eg = \tau_{E^*} \circ \varepsilon \circ dL$$



Lagrange formalism for general algebroid

The Lagrangian defines two smooth maps:

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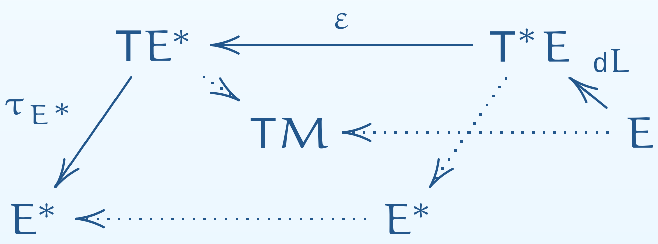
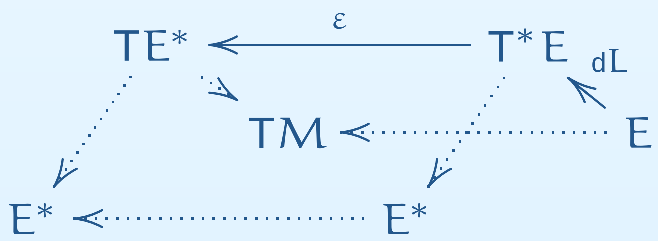
$$\mathcal{L}_{eg} : E \longrightarrow E^* \quad \text{defined by} \quad \mathcal{L}_{eg} = \tau_{E^*} \circ \varepsilon \circ dL$$



(2) $$\widetilde{\mathcal{L}}_{eg} : E \longrightarrow TE^* \quad \text{defined by} \quad \widetilde{\mathcal{L}}_{eg} = \varepsilon \circ dL$$



Lagrange formalism for general algebroid

For general algebroid we have constructed two equations for curves $\gamma : I \to E$:

(1) The equation $\mathcal{E}^1_L$ given by

$$\mathcal{E}^1_L = (\mathsf{T}\mathcal{L}_{eg})^{-1}(\mathcal{D}).$$

The curve γ is a solution of $\mathcal{E}^1_L$ if and only if $\mathcal{L}_{eg} \circ \gamma$ is a solution of $\mathcal{D}$,
i.e. $\mathsf{T}(\mathcal{L}_{eg} \circ \gamma) \subset \mathcal{D}$.

$\mathcal{E}^1_L$ corresponds to the construction of de Léon and Lacomba:

M. de León, E. Lacomba: Lagrangian submanifolds and higher-order mechanical systems, J.Phys. A: Math. Gen. vol. 22, 1989, (3809–3820).

Lagrange formalism for general algebroid

(2) The equation E^2_L given by

$$E^2_L = \{v \in \mathsf{T}E : \mathsf{T}\widetilde{L}_{eg}(v) \in \mathsf{T}^2E^*\},$$

where $\mathsf{T}^2E^* \subset \mathsf{TT}E^*$ is the subset of holonomic vectors w, i.e. $w \in \mathsf{TT}E^*$ such that $\tau_{\mathsf{T}E^*}(w) = \mathsf{T}\tau_{E^*}(w)$.

The curve γ is a solution of E^2_L if the tangent prolongation $\mathsf{T}(L_{eg} \circ \gamma)$ is exactly $\widetilde{L}_{eg} \circ \gamma$. Since $\widetilde{L}_{eg} \circ \gamma$ is in D by definition, we have

$$E^2_L \Longrightarrow E^1_L.$$

E^2_L is the equation given in works of A. Weinstein, E. Martinez, ...

Lagrange formalism for general algebroid

The coordinate expression for E_L^2 is

$$\frac{dx^a}{dt} = \rho^a_k(x) y^k, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial y^j}\right) = c^k_{ij}(x) y^i \frac{\partial L}{\partial y^k}(x,y) + \sigma^a_j(x) \frac{\partial L}{\partial x^a}(x,y)$$

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The coordinate expression for E_L^1 is

$$\frac{dx^a}{dt} = \rho_k^a(x)y^k, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial y^j}\right) = c_{ij}^k(x)y_0^i \frac{\partial L}{\partial y^k}(x,y) + \sigma_j^a(x)\frac{\partial L}{\partial x^a}(x,y).$$

for certain $(x, y_0) \in E$ satisfying $L_{eg}(x, y_0) = L_{eg}(x, y)$ and $\varepsilon_r(x, y_0) = \varepsilon_r(x, y)$, i.e.

$$\frac{\partial L}{\partial y^i}(x,y) = \frac{\partial L}{\partial y^i}(x,y_0), \qquad \rho_k^b(x)y^k = \rho_k^b(x)y_0^k.$$

Noether Theorem

- The vertical lift: $K \in \bigotimes^k(\tau)$, $\mathsf{v}_\tau(K) \in \bigotimes(\tau_E)$, in coordinates

$$\mathsf{v}_\tau(f^{i_1 \cdots k} e_{i_1} \otimes \cdots \otimes e_{i_k}) = f^{i_1 \cdots k} \partial_{y^{i_1}} \otimes \cdots \otimes \partial_{y^{i_k}} .$$

- The algebroid lift:

$$\iota(\mathrm{d}_T^\varepsilon(K)) = \mathrm{d}_T(\iota(K)) \circ \varepsilon^{\otimes k}.$$

Noether Theorem

- The vertical lift: $K \in \bigotimes^k(\tau)$, $v_\tau(K) \in \bigotimes(\tau_E)$, in coordinates

$$v_\tau(f^{i_1\cdots k} e_{i_1} \otimes \cdots \otimes e_{i_k}) = f^{i_1\cdots k} \partial_{y^{i_1}} \otimes \cdots \otimes \partial_{y^{i_k}}.$$

- The algebroid lift:

$$\iota(d_T^\varepsilon(K)) = d_T(\iota(K)) \circ \varepsilon^{\otimes k}.$$

In local coordinates, for functions:

$$d_T^\varepsilon(f(x)) = y^i \rho_i^a(x) \frac{\partial f}{\partial x^a}(x)$$

and for sections of τ:

$$d_T^\varepsilon(f^i(x) e_i) = f^i(x) \sigma_i^a(x) \partial_{x^a} + \left(y^i \rho_i^a(x) \frac{\partial f^k}{\partial x^a}(x) + c_{ij}^k(x) y^i f^j(x) \right) \partial_{y^k}.$$

Noether Theorem

Theorem. If X is a section of E and f is a function on M, then

$$d_T^{\varepsilon}(X)L = d_T^{\varepsilon}(f) \tag{1}$$

if and only if the function $(\iota(X) - v_{\pi}(f)) \circ L_{eg}$ on E is a first integral of the equation (E_L^2). In particular, if (1) is satisfied, then $(\iota(X) - v_{\pi}(f)) \circ L_{eg}$ is a first integral of the equation (E_L^1).

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Remark. One can consider the Hamiltonian part of the Tulczyjew triple for an arbitrary algebroid and prove the standard correspondence between the Lagrangian and the Hamiltonian in the regular case.
To this end, dealing with a Lie and not with an arbitrary algebroid played no role, contrary to the common opinion that geometrical mechanics is a Lie algebroid theory.
No brackets have been really used!