How the Inverse Problem changed my Life

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1. The inverse problem and its status

2. Compensations

3. Geometric formulation

4. Geometrisation of SODE's

5. Open Questions

1. The Inverse Problem in the Calculus of Variations.

"When are the solutions of

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b) \;\; a, b = 1, \ldots, n$$

the solutions of

$$\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b}\ddot{x}^b + \frac{\partial^2 L}{\partial x^b \partial \dot{x}^a}\dot{x}^b + \frac{\partial^2 L}{\partial t \partial \dot{x}^a} = \frac{\partial L}{\partial x^a}$$

for some $L(t, x^a, \dot{x}^a)$?"

So, find regular g_{ab} (and L) so that

$$g_{ab}(\ddot{x}^b - F^b) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial \dot{x}^a}$$

- The multiplier problem.

Helmholtz conditions (Douglas 1941, Sarlet 1982)

necessary and sufficient conditions on g_{ab}:

$$g_{ab} = g_{ba}, \quad \Gamma(g_{ab}) = g_{ac}\Gamma^c_b + g_{bc}\Gamma^c_a$$

$$g_{ac}\Phi^c_b = g_{bc}\Phi^c_a, \quad \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b}$$

where

$$\Gamma^a_b := \frac{-1}{2}\frac{\partial F^a}{\partial \dot{x}^b}, \quad \Phi^a_b := -\frac{\partial F^a}{\partial x^b} - \Gamma^c_b\Gamma^a_c - \Gamma(\Gamma^a_b)$$

$$\Gamma := \frac{\partial}{\partial t} + u^a\frac{\partial}{\partial x^a} + F^a\frac{\partial}{\partial u^a}$$

Helmholtz conditions: 1st order linear algebraic/differential equations for g_{ab} with data f^a, Γ^a_b, Φ^a_b.

Two approaches/problems:

1. For given F^a find all g's

e.g.

$$\ddot{x} + \dot{y} = 0$$
$$\ddot{y} + y = 0$$

admits no multipliers.

2. For given n classify 2nd order ode's according to existence and multiplicity of solutions of the Helmholtz condition.

Done by Douglas for $n = 2$.

Current Status

- Still not done for $n = 3$.
- done for some cases for arbitrary n, eg $\Phi = \mu I_n$, Φ diagonalisable with distinct e'vals and integrable eigendist'ns
- **pretty poor really!!**

2. Compensations

- geometric theory of SODE's

- geometrisation of Euler-Lagrange eqns and classic thms

- geometry along the projection

- degenerate and constrained systems

- classical mechanics of Riemannian manifolds

- classical mechanics of contact manifolds
- affine geometry of Euler-Lagrange eqns
- Berwald connections and Finsler geometry
- higher order mechanics
- and so on.....

3. Geometric Formulation

Theorem (CPT 1984) Given a SODE Γ, necessary and sufficient conditions for the existence of a Lagrangian whose $E-L$ field is Γ are that there exists $\Omega \in \Lambda^2(E)$:

1. Ω has max'l rank

2. $\Omega(V_1, V_2) = 0 \ \forall V_1, V_2 \in V(E)$

3. $i_\Gamma \Omega = 0$

4. $d\Omega = 0$

Every SODE satisfies these locally except the second one!

Details:

$$E := \mathbb{R} \times TM \qquad (t, x^a, u^a)$$

$$\swarrow t \qquad \downarrow \pi \qquad\qquad dim(E) = 2n + 1$$

$$\mathbb{R} \underset{t}{\longleftarrow} \mathbb{R} \times M \qquad (t, x^a)$$

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b)$$

$$\rightarrow \qquad \Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a} \in \mathfrak{X}(E)$$

E is equipped with

- vertical dist'n $V(E) = Sp\{V_a := \frac{\partial}{\partial u^a}\}$
- contact dist'n
  $\Theta(E) = Sp\{\theta^a := dx^a - u^a dt\}$
- vertical endomorphism $S = V_a \otimes \theta^a$

$$\underbrace{\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a}, \qquad S = V_a \otimes \theta^a}_{\mathcal{L}_\Gamma S}$$

$$TE = Sp\{\Gamma\} \oplus H(E) \oplus V(E)$$

Projectors:

$$P_\Gamma = \Gamma \otimes dt, \quad P_H = H_a \otimes \theta^a, \quad P_V = V_a \otimes \psi^a$$

$$\begin{pmatrix} H_a := \frac{\partial}{\partial x^a} - \Gamma^b_a \frac{\partial}{\partial u^b}, \ \psi^a := du^a - F^a dt + \Gamma^a_b \theta^b \\ [H_a, H_b] = R^d_{ab} V_d \end{pmatrix}$$

Jacobi endomorphism:

$$\Phi := P_V \circ \mathcal{L}_\Gamma P_H$$

When $\ddot{x}^a = F^a(t, x^b, \dot{x}^b)$ are (normalized) Euler-Lagrange equations, then Γ is the unique vectors field on E s.t.

$$i_\Gamma \ d\theta_L = 0, \ \ dt(\Gamma) = 1$$

$$\begin{pmatrix} \theta_L := Ldt + dL \circ S = Ldt + \frac{\partial L}{\partial u^a} \theta^a \\ d\theta_L = \frac{\partial^2 L}{\partial u^a \partial u^b} \psi^a \wedge \theta^b \end{pmatrix}$$

Usually begin the search for Ω by assuming 1., 2., 3 i.e.

$$\Omega = g_{ab}\psi^a \wedge \theta^b, \; |g_{ab}| \neq 0$$

and requiring

$$d\Omega = 0.$$

$d\Omega(X, Y, Z) = 0$ give the Helmholtz conditions e.g.

$$d\Omega(\Gamma, V_a, H_b) = 0 \Leftrightarrow \Gamma(g_{ab}) - g_{bc}\Gamma^c_a - g_{ac}\Gamma^c_b = 0$$
$$d\Omega(\Gamma, V_a, V_b) = 0 \Leftrightarrow g_{ab} = g_{ba}$$
$$d\Omega(\Gamma, H_a, H_b) = 0 \Leftrightarrow g_{ac}\Phi^c_a = g_{ca}\Phi^c_b$$

etc.

4. Geometrisation of SODE's

The model is the auto-parallel eqn of a linear connection with torsion.

$$\nabla_Z Z = 0$$

Shape map:

$$\begin{aligned} A_X(Y) &= \nabla_Y X - T(X,Y) \\ &= \nabla_X Y - [X,Y] \end{aligned}$$

If Z is auto-parallel then A_Z is the key to Raychaudhuri's eqn, Morse theory etc.

Only need ∇_Z for these purposes.

For a given SODE Γ:

- on $\mathbb{R} \times M$ no connection, but:

$$A_Z := \sigma_Z^* P_V$$

Jacobi eqn

$$\nabla_Z^2 X = -\bar{\Phi}(X)$$

This coincides with the autoparallel case.

- on E connection is the Massa and Pagani $\hat{\nabla}$ with

$$\begin{aligned} A_\Gamma &= -P_V \circ \mathcal{L}_\Gamma P_H - P_H \circ \mathcal{L}_\Gamma P_V \\ &= -\Phi - P_H \circ \mathcal{L}_\Gamma P_V \end{aligned}$$

Jacobi eqn

$$\hat{\nabla}_\Gamma^2 X = -\Phi(X)$$

Open Questions

- quantum mechanics of Lagrangian systems

- solution of the Inverse Problem – what is the classification for arbitrary n ?

- Morse theory of SODE's

- relation to contact manifolds

- congruence design problems

- mechanics with quadratic forms

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