Non-holonomic reduction by stages

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The Lagrange-d'Alembert principle

A non-holonomic system is a mechanical system $q(t) \in Q$ subjected to some velocity-dependent (i.e. non-holonomic) constraints $a_k^\alpha(q)\dot{q}^k = 0$.

The dynamics is determined by a Lagrangian $L : TQ \longrightarrow \mathbb{R}$ (kinetic minus potential energy) and a distribution $D_q \subset T_qQ$, representing the constraints.

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1. satisfies the constraints, i.e. $\dot{q}(t) \in D_{q(t)}$.

2. satisfies $\delta \int_b^a L(q, \dot{q})dt = 0$, for all variations satisfying $\delta q(t) \in D_{q(t)}$, $\forall t \in [a, b]$.

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$$\Longleftrightarrow \quad \begin{cases} a_k^\alpha \, \dot{q}^k = 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}^k} - \dfrac{\partial L}{\partial q^k} = \lambda_\alpha a_k^\alpha \end{cases}$$

for some Lagrangian multipliers λ_α.

The above equations are the Lagrange-d'Alembert equations!

Lagrange-d'Alembert equations

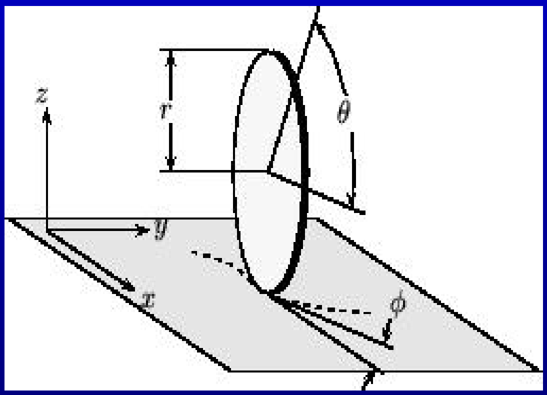
Choose coordinates $(q^k) = (s^\alpha, r^I)$ such that $a_k^\alpha\,\dot q^k = 0$ can be rewritten as $\dot s^\alpha + A_I^\alpha \dot r^I = 0$.

Then: Lagrange multipliers can be eliminated:

$$\left\{\begin{array}{l} a_k^\alpha\,\dot q^k = 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot q^k} - \dfrac{\partial L}{\partial q^k} = \lambda_\alpha a_k^\alpha \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{rcl} \dot s^\alpha &=& -A_I^\alpha \dot r^I, \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot r^I} - \dfrac{\partial L}{\partial r^I} &=& A_I^\alpha\left(\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot s^\alpha} - \dfrac{\partial L}{\partial s^\alpha}\right) \end{array}\right.$$

Some examples of non-holonomic systems[1]

- The vertically rolling disk:



Configuration space Q is $SE(2) \times S^1 = \mathbb{R}^2 \times S^1 \times S^1$, with Lagrangian:

$$L(x, y, \phi, \theta, \dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2$$

Non-holonomic constraint = rolling without slipping

$$\begin{cases} \dot{x} & = & r \cos\phi\, \dot{\theta} \\ \dot{y} & = & r \sin\phi\, \dot{\theta} \end{cases}$$

[1] All pictures were stolen from the internet!

- The snakeboard

- The snakeboard

- The rattleback

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- The roller racer

Symmetry of the vertically rolling disk

The Lagrangian $L(x, y, \phi, \theta, \dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2$ and the constraint $\left\{ \begin{array}{lcl} \dot{x} & = & r\cos\phi\,\dot{\theta} \\ \dot{y} & = & r\sin\phi\,\dot{\theta} \end{array} \right.$ are invariant

1. under the $SE(2)$-action on $(Q = SE(2) \times S^1, TQ)$, given by

$$\left\{ \begin{array}{ll} (a, b, \alpha) & \times(x, y, \theta, \phi) \mapsto (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \theta, \phi + \alpha) \\ & \times(\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) \mapsto (\dot{x}\cos\alpha - \dot{y}\sin\alpha, \dot{x}\sin\alpha + \dot{y}\cos\alpha, \dot{\theta}, \dot{\phi}) \end{array} \right.$$

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2. under the $\mathbb{R}^2 \times S^1$-action on (Q, TQ), given by

$$\left\{ \begin{array}{ll} (\lambda, \mu, \beta) & \times(x, y, \theta, \phi) \mapsto (x + \lambda, y + \mu, \theta + \beta, \phi) \\ & \times(\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) \mapsto (\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) \end{array} \right.$$

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- Lagrange-d'Alembert equations on TQ for L and D drop to equations on the quotient TQ/G for $\overline{L}$ and D/G = the so-called Lagrange-d'Alembert-Poincaré equations.

Reduction by stages

Example: For the rolling disk example with $Q = SE(2) \times S^1$ and $G = \mathbb{R}^2 \times S^1$-action:

1. $L \in C^\infty(TQ)$ and D

$$\downarrow \qquad \downarrow$$

Reduction by $G = \mathbb{R}^2 \times S^1$

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The dynamics obtained after reduction by G and after reduction by N and H should be equivalent!!!

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1. Lie algebroids and quotient Lie algebroids
2. Prolongation bundles and quotients of prolongation bundles

Common idea: Define a category of systems that are stable under reduction.

1. Lie algebroids

Definition 1. *A Lie algebroid is a vector bundle* $\tau : \mathsf{V} \longrightarrow M$*, which comes equipped with*

- *a bracket operation* $[\cdot,\cdot] : \mathrm{Sec}(\tau) \times \mathrm{Sec}(\tau) \longrightarrow \mathrm{Sec}(\tau)$*,*
- *a linear bundle map* $\rho : \mathsf{V} \longrightarrow TM$ *(and its extension* $\rho : \mathrm{Sec}(\tau) \longrightarrow \mathcal{X}(M)$*),*

which are related in such a way that

1. $[\cdot,\cdot]$ *is a real Lie algebra bracket on the vector space* $\mathrm{Sec}(\tau)$ *(skew-symmetry, bi-linear and Jacobi identity);*

2. ρ *satisfies for all* $\mathsf{s}, \mathsf{r} \in \mathrm{Sec}(\tau)$*,* $f \in C^\infty(M)$*:* $[\mathsf{s}, f\mathsf{r}] = f[\mathsf{s}, \mathsf{r}] + \rho(\mathsf{s})(f)\,\mathsf{r}$

Standard Example: Tangent bundle: $TM \to M$ with bracket of vector fields and anchor $\rho = id : TM \to TM$

Exterior derivative of a Lie algebroid

k-Forms are skew-symmetric, $C^\infty(M)$-linear maps

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The exterior derivative on forms is a map $d : k$-form $\longmapsto (k+1)$-form which is such that:

- On functions, $df(\mathsf{r}) = \rho(\mathsf{r})f$.
- On 1-forms, $d\theta(\mathsf{r},\mathsf{s}) = \rho(\mathsf{r})(\theta(\mathsf{s})) - \rho(\mathsf{s})(\theta(\mathsf{r})) - \theta([\mathsf{r},\mathsf{s}])$.
- For a k-form ω and a l-form φ, $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^{kl}\omega \wedge d\varphi$.

$\rightsquigarrow$ d satisfies $d^2 = 0$

Quotients of vector bundles

Suppose $\overline{\pi}^M : M \to \overline{M} = M/G$ is a principal fibre bundle, with action $\psi^M : (g, m) \mapsto gm$.

Definition 2. *An action $\psi^{\mathsf{V}} : G \times \mathsf{V} \to \mathsf{V}; (g, \mathsf{v}) \mapsto g\mathsf{v}$ such that for each $g \in G$ the map $\psi^{\mathsf{V}}_g : \mathsf{V}_m \to \mathsf{V}_{gm}$ is an isomorphism (over ψ^M_g) and such that τ is equivariant (meaning that $\tau \circ \psi^{\mathsf{V}}_g = \psi^M_g \circ \tau$) is called a vector bundle action.*

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$\rightsquigarrow$ At this stage, it is possible to define the quotient vector bundle $\overline{\tau} : \overline{\mathsf{V}} = \mathsf{V}/G \to \overline{M} = M/G$, with

$$\mathsf{v}'_{m'} \in [\mathsf{v}_m] \text{ if } \exists\, g \in G, \text{ such that } m' = gm \text{ and } \mathsf{v}' = g\mathsf{v}$$

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$\mathsf{s} \in \mathrm{Sec}(\tau)$ is an invariant section if $\forall\, g$, $\mathsf{s}(gm) = g\mathsf{s}(m)$. Notation: $\mathsf{s} \in \mathrm{Sec}^I(\tau)$.

$\rightsquigarrow$ Invariant sections are in 1-1 correspondence with sections of the quotient bundle $\overline{\tau} : \overline{\mathsf{V}} \to \overline{M}$!

Lie algebroid morphisms and quotient Lie algebroids

Let $\tau : \mathsf{V} \to M$ and $\tau' : \mathsf{V}' \to M'$ be two Lie algebroids and $\Phi : \mathsf{V} \to \mathsf{V}'$ a linear bundle map over $\phi : M \to M'$ (so Φ is a morphism of vector bundles).

Then, for $\theta' \in \bigwedge^k(\tau')$, define $\Phi^*\theta' \in \bigwedge^k(\tau)$ given by

$$\Phi^*\theta'(m)(\mathsf{v}_1, \ldots, \mathsf{v}_k) = \theta'(\phi(m))(\Phi(\mathsf{v}_1), \ldots \Phi(\mathsf{v}_k)), \qquad \mathsf{v}_i \in \mathsf{V}_m,$$

Definition 3. Φ *is called a Lie algebroid morphism if*

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Definition 4. *A vector bundle action ψ^{V} is a Lie algebroid action if ψ^{V}_g is a Lie algebroid isomorphism over ψ^M_g for all $g \in G$.*

Lemma 1. *For a Lie algebroid action* ψ^{V}, $\mathrm{Sec}^I(\tau)$ *is a Lie subalgebra of* $\mathrm{Sec}(\tau)$ *and the reduction by the group* G *yields a Lie algebroid structure on the quotient* $\overline{\tau}$ *with bracket*

$$[\bar{\mathsf{r}}, \bar{\mathsf{s}}] = \left([\bar{\mathsf{r}}^I, \bar{\mathsf{s}}^I]\right)_I.$$

and anchor

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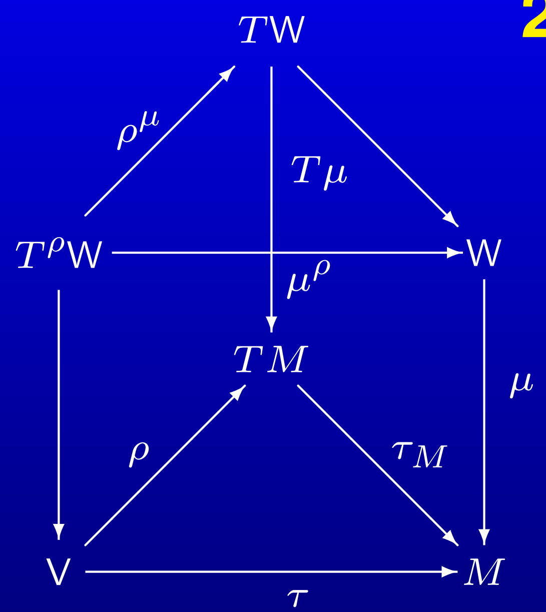
EXAMPLE: The standard Lie algebroid TM with action $T\psi^M$ (for any G-action ψ^M on M) gives rise to a quotient Lie algebroid TM/G, the so-called Atiyah Lie algebroid.

(basically the Lie algebra of invariant vector fields)

2. Prolongation bundles



- τ : V $\to M$ is a vector bundle and ρ : V $\to TM$ is a linear map
- μ : W $\to M$ is a second fibre bundle.

The prolongation is a (vector) bundle μ^ρ : T^ρW $\longrightarrow$ W, with

(i) $T^\rho \mathsf{W} = \rho^* T\mathsf{W} = \{(\mathsf{v}, X_\mathsf{w}) \in \mathsf{V} \times T\mathsf{W} \mid \rho(\mathsf{v}) = T\mu(X_\mathsf{w})\}$;

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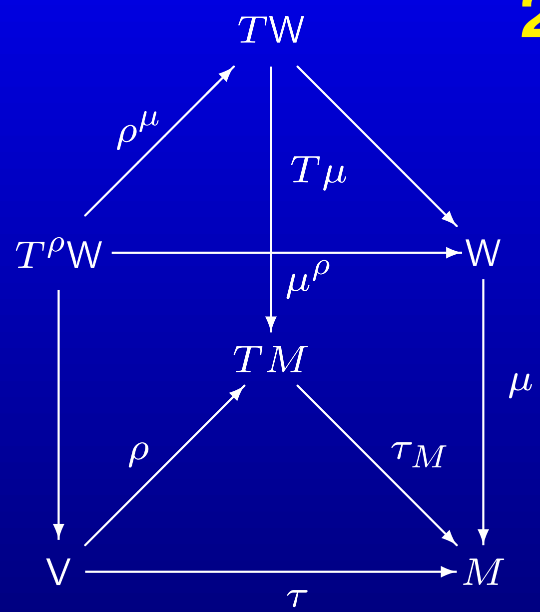

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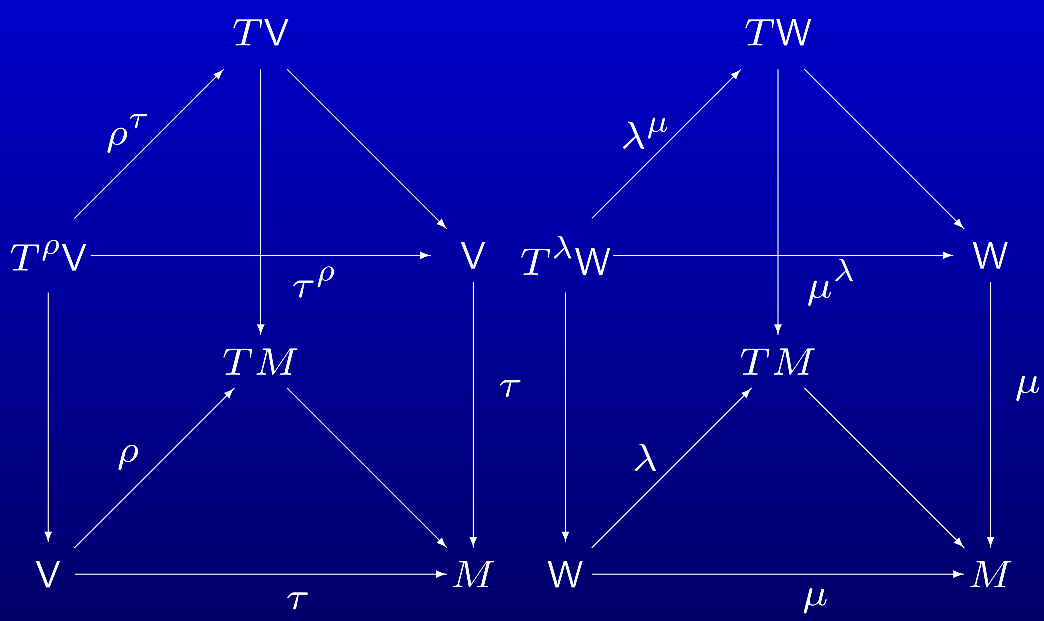
Important property: If τ is a Lie algebroid, then so is the prolongation μ^ρ:

1) its anchor is $\rho^\mu : T^\rho\mathsf{W} \to T\mathsf{W}$, $(\mathsf{v}, X_\mathsf{w}) \mapsto X_\mathsf{w}$

2) its bracket is $[\mathcal{Z}_1, \mathcal{Z}_2] = ([\mathsf{r}_1, \mathsf{r}_2], [X_1, X_2])$ for projectable sections $\mathcal{Z} = (\mathsf{r} \in \mathrm{Sec}(\tau), X \in \mathcal{X}(\mathsf{W}))$.

Two prolongations of interest for non-holonomic systems



We suppose now:

- $\tau : \mathsf{V} \to M$ is a Lie algebroid with anchor ρ.
- $L \in C^\infty(\mathsf{V})$ is a (regular) Lagrangian
- $\mu : \mathsf{W} \to M$ is a vector subbundle of τ with injection $i : \mathsf{W} \to \mathsf{V}$ and $\lambda = \rho \circ i : \mathsf{W} \to TM$.

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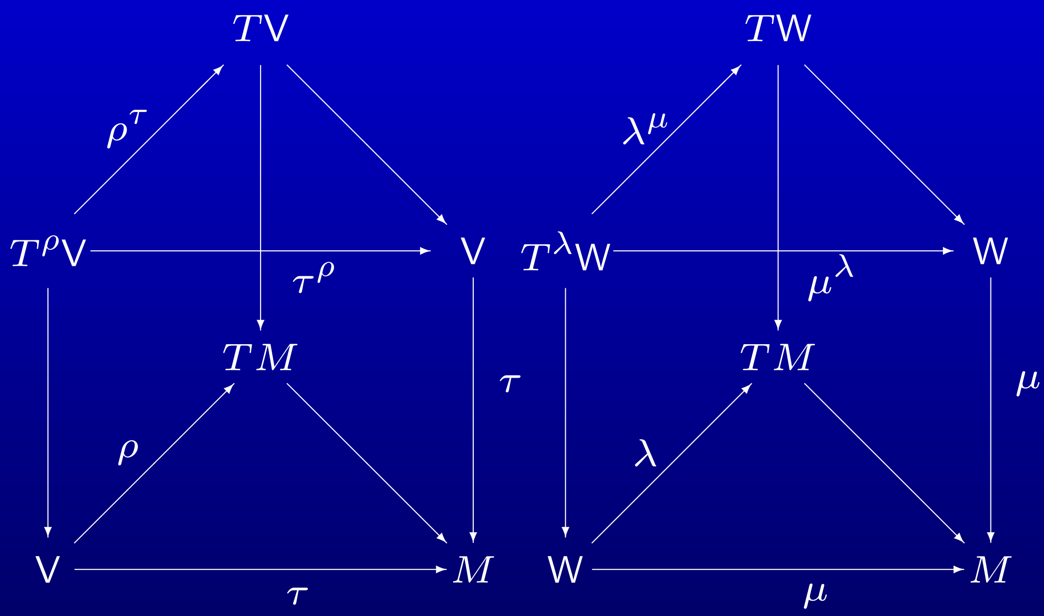

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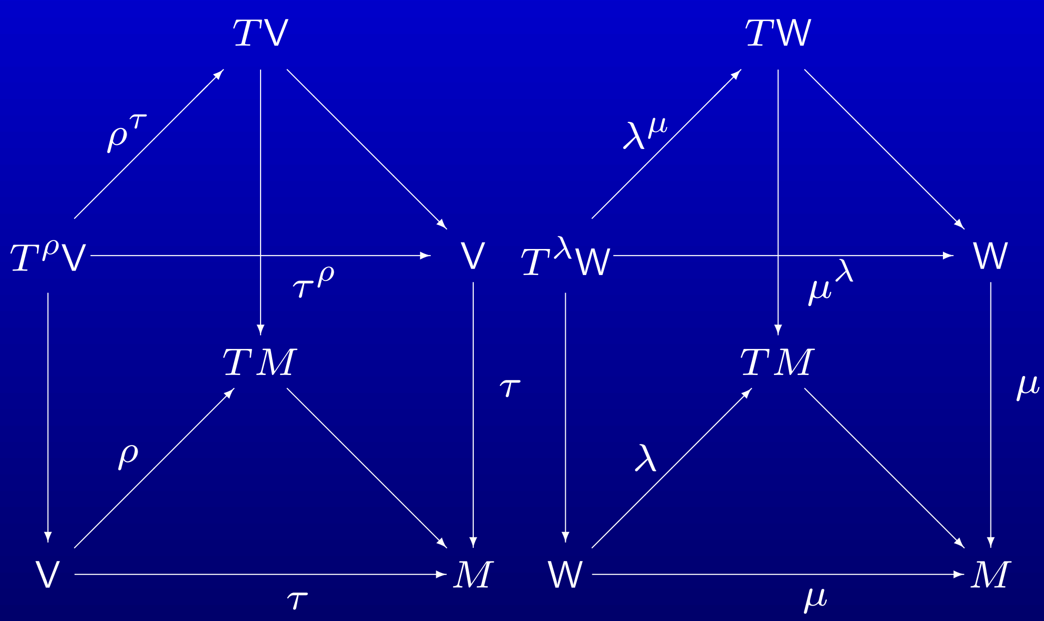

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$\mu^\lambda : T^\lambda$W $\to$ W is a vector subbundle of $\tau^\rho : T^\rho$V $\to$ V with injection:

$$\mathcal{T}^i i : T^\lambda\mathsf{W} \to T^\rho\mathsf{V}, (\mathsf{w}_1, X_{\mathsf{w}_2}) \mapsto (i(\mathsf{w}_1), Ti(X_{\mathsf{w}_2}))$$

Prolongation Lie algebroid τ^ρ is similar to $TTM \to TM$

On τ^ρ, it is possible to define the usual canonical objects:

- Vertical sections: those whose projection on V gives zero.
- Vertical endomorphism $S^\tau : \mathrm{Sec}(\tau^\rho) \to \mathrm{Sec}(\tau^\rho)$.
- Liouville Section $\mathcal{C}^\tau \in \mathrm{Sec}(\tau^\rho)$

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The dynamics requires a mix of objects on τ^ρ and μ^λ

The Lagrangian $L \in C^\infty(\mathsf{V})$ determines

- $\theta_L = S^\tau(dL) \in \bigwedge^1(\tau^\rho)$
- $E_L = \rho^\tau(\mathcal{C}^\tau)L - L \in C^\infty(\mathsf{V})$.

Recall the injection $\mathcal{T}^i i : T^\lambda \mathsf{W} \to T^\rho \mathsf{V}$.

Definition 5. *Let* $d : \bigwedge^k(\tau^\rho) \to \bigwedge^{k+1}(\tau^\rho)$ *be the exterior derivative of the Lie algebroid* τ^ρ *and put*

$$\Delta = (\mathcal{T}^i i)^* \circ d : \bigwedge^k(\tau^\rho) \to \bigwedge^{k+1}(\mu^\lambda).$$

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If Γ is the section of the prolongation bundle μ^λ, determined by

$$i_\Gamma \Delta\theta_L = -\Delta E_L,$$

the vector field $\lambda^\mu(\Gamma) \in \mathcal{X}(\mathsf{W})$ is said to define the Lagrangian system on the subbundle μ of the Lie algebroid τ, associated to the given Lagrangian L on V.

Two Important cases

- Non-holonomic systems: If $\tau = \tau_Q : TQ \longrightarrow Q$, $\mu : D \longrightarrow Q$ distribution, $L \in C^\infty(TQ)$ and $i : D \longrightarrow TQ$, then we get the Lagrange-d'Alembert equations.

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- Non-holonomic systems with symmetry: If $\tau : TQ/G \to Q/G$, $\mu : D/G \to Q/G$, $L \in C^\infty(TQ/G)$ and $i : D/G \to TQ/G$, then we get the Lagrange-d'Alembert-Poincaré equations.

Non-holonomic systems on Lie algebroids with symmetry

From now on we suppose

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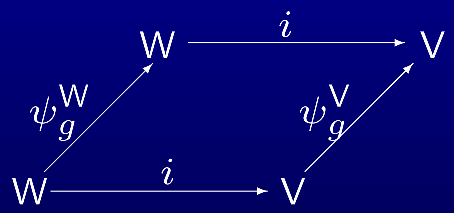

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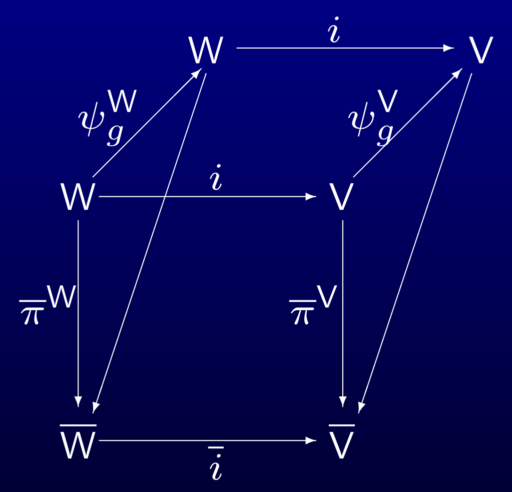

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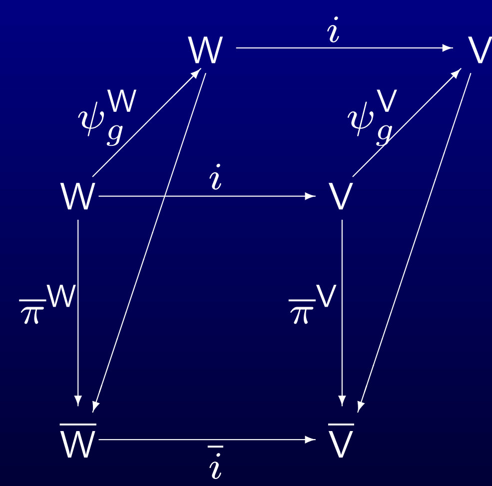

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How can we perform reduction?

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Lie algebroid V $\rightarrow M$ with

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Proposition 1. *They are isomorphic as Lie algebroids. Moreover,* $\mathrm{Sec}(\overline{\tau^{\rho}})$, $\mathrm{Sec}(\overline{\tau}^{\overline{\rho}})$ *and* $\mathrm{Sec}^{I}(\tau^{\rho})$ *are isomorphic as Lie algebras.*

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Proposition 2. *If* $W \subset V$ *and* ψ^W *is a constrained Lie algebroid action, then the bundles* $\overline{\mu^\lambda} : \overline{T^\lambda W} \to \overline{W}$ *and* $\overline{\mu}^{\overline{\lambda}} : T^{\overline{\lambda}}\overline{W} \to \overline{W}$ *are isomorphic as vector bundles.*

Reduction of non-holonomic systems on Lie algebroids

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Proposition 3. *If L is a regular invariant Lagrangian on V, then also $\overline{L}$ is regular. Moreover the Lagrangian section $\Gamma \in \mathrm{Sec}(\mu^\lambda)$ is invariant and the solutions of the non-holonomic equations on L (i.e. the integral curves of $\lambda^\mu(\Gamma)$) project to those for the reduced Lagrangian $\overline{L}$ (i.e. the integral curves of $\overline{\lambda}^{\overline{\mu}}(\overline{\Gamma})$).*

PROOF. Define the map $\mathcal{T}^{\overline{\pi}^{\mathsf{W}}}\overline{\pi}^{\mathsf{W}} : T^{\lambda}\mathsf{W} \to T^{\overline{\lambda}}\overline{\mathsf{W}}$ as

$$\mathcal{T}^{\overline{\pi}^{\mathsf{W}}}\overline{\pi}^{\mathsf{W}}(\mathsf{w}_1, X_{\mathsf{w}_2}) = (\overline{\pi}^{\mathsf{W}}(\mathsf{w}_1), T\overline{\pi}^{\mathsf{W}}(X_{\mathsf{w}_2})) \in T^{\overline{\lambda}}_{\overline{\pi}^{\mathsf{W}}(\mathsf{w})}\overline{\mathsf{W}}.$$

Then

$$\mathcal{T}^{\overline{\pi}^{\mathsf{W}}}\overline{\pi}^{\mathsf{W}}(\Gamma(\mathsf{w})) = \overline{\Gamma}(\overline{\pi}^{\mathsf{W}}(\mathsf{w})).$$

Reduction by stages

$N \subset G$ normal subgroup

Proposition 4. *The dynamics obtained by a twofold reduction (by N and H) is equivalent with the one obtained from a reduction by G directly.*

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Reduction by G

$\downarrow \quad \downarrow$

2. $\overline{\Gamma}$

or

1. Γ invariant under $N \subset G$ normal

$\downarrow \quad \downarrow$

Reduction by N

$\downarrow \quad \downarrow$

2. $\hat{\Gamma}$ invariant under $H = G/N$

$\downarrow \quad \downarrow$

Reduction by H

$\downarrow \quad \downarrow$

3. $\hat{\hat{\Gamma}}$

$\overline{\Gamma} \simeq \hat{\hat{\Gamma}}$

PROOF. There is an isomorphism $\beta^{\mathsf{W}} : \overline{\mathsf{W}} \to \hat{\hat{\mathsf{W}}}, [\mathsf{w}]_G \to [[\mathsf{w}]_N]_H$.

Then $\mathcal{T}^{\beta^{\mathsf{W}}}\beta^{\mathsf{W}} : T^{\overline{\lambda}}\overline{\mathsf{W}} \to T^{\hat{\hat{\lambda}}}\hat{\hat{\mathsf{W}}}$ with

$$\mathcal{T}^{\beta^{\mathsf{W}}}\beta^{\mathsf{W}}([\mathsf{w}_1]_G, X_{[\mathsf{w}_2]_G}) = (\beta^{\mathsf{W}}([\mathsf{w}_1]_G), T\beta^{\mathsf{W}}(X_{[\mathsf{w}_2]_G})).$$

is an isomorphism and

$$\mathcal{T}^{\beta^{\mathsf{W}}}\beta^{\mathsf{W}}(\overline{\Gamma}([\mathsf{w}]_G) = \hat{\hat{\Gamma}}(\beta^{\mathsf{W}}([\mathsf{w}]_G)).$$

Multiple reduction: e.g. for $\{e\} \subset \ldots \subset N_2 \subset N_1 \subset G$.

$$
\begin{array}{ccccccc}
\mathrm{Sec}^{I,G}(\tau) & \longrightarrow & \mathrm{Sec}^{I,N_1}(\tau) & \longrightarrow & \mathrm{Sec}^{I,N_2}(\tau) & \longrightarrow & \mathrm{Sec}(\tau) \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Sec}^{I,H_{02}}(\tau/N_2) & \longrightarrow & \mathrm{Sec}^{I,H_{12}}(\tau/N_2) & \longrightarrow & \mathrm{Sec}(\tau/N_2) & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{Sec}^{I,H_{01}}(\tau/N_1) \simeq & \longrightarrow & \mathrm{Sec}(\tau/N_1) \simeq & & & & \\
\mathrm{Sec}^{I,H_{01}}((\tau/N_2)/H_{12}) & & \mathrm{Sec}((\tau/N_2)/H_{12}) & & & & \\
\downarrow & & & & & &
\end{array}
$$

$$
\mathrm{Sec}(\tau/G) \simeq \mathrm{Sec}\big((\tau/N_2)/H_{02}\big) \simeq \mathrm{Sec}((\tau/N_1)/H_{01}) \simeq \mathrm{Sec}\big(((\tau/N_2)/H_{12})/H_{01}\big)
$$