# Lie algebroids and some applications to Mechanics

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## Some references

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# Scheme of the talk

1. Unconstrained mechanical systems on Lie algebroids
   - The prolongation of a Lie algebroid over a fibration
   - The Lagrangian formalism on Lie algebroids
   - Examples
   - The Hamiltonian formalism on Lie algebroids
   - The Legendre transformation and equivalence between the Lagrangian and Hamiltonian formalisms
2. Non-holonomic Lagrangian systems on Lie algebroid
   - An standard example
   - Dynamical equations
   - Regular non-holonomic Lagrangian systems
   - The non-holonomic bracket
   - Morphisms and reduction
3. Future work

# The prolongation of a Lie algebroid over a fibration

$$(E, [\![\cdot,\cdot]\!], \rho) \text{ a Lie algebroid over } M, \quad rank\ E = n, \quad \dim M = m$$

$$\pi : M' \to M \text{ a fibration}, \quad \dim M' = m'$$

**The set**

$$\mathcal{T}^E M' = \{(b, v) \in E \times TM' / \rho(b) = (T\pi)(v')\}$$

$$\tau^\pi : \mathcal{T}^E M' \to M'; \quad (b, v') \mapsto \tau_{M'}(v')$$

$$x' \in M' \implies \mathcal{T}^E_{x'} M' = (\tau^\pi)^{-1}(x')$$

$$\dim(\mathcal{T}^E_{x'} M') = n + m' - m, \quad \forall x' \in M'$$

**the vector bundle**

$\mathcal{T}^E M'$ is a vector bundle over $M$ of rank $n + m' - m$

# Lie algebroid structure on $\mathcal{T}^E M' \longrightarrow M'$

Anchor map:

$$\rho^\pi : \mathcal{T}^E M' \to TM', \quad (b, v') \to v'$$

Lie bracket on $\Gamma(\mathcal{T}^E M')$:

$$X \in \Gamma(E), \quad X' \in \mathfrak{X}(M') \quad \tau\text{-projectable on } \rho(X)$$

$$(X, X') \in \Gamma(\mathcal{T}^E M'); \quad (X, X')(x') = (X(\pi(x')), X'(x')), \ \forall x' \in M'$$

$$[\![(X, X'), (Y, Y')]\!]^\pi = ([\![X, Y]\!], [X', Y'])$$

**Prolongation of $E$ over $\pi$ or $E$-tangent bundle to $M'$**

$$(\mathcal{T}^E M', [\![\cdot, \cdot]\!]^\pi, \rho^\pi)$$

# A particular case:

$M' = E, \quad \pi = \tau : E \to M$ the vector bundle projection

$$\mathcal{T}^E E = \{(b, v) \in E \times TE / \rho(b) = (T\tau)(v)\}, \qquad \text{rank } \mathcal{T}^E E = 2n$$

**The vertical endomorphism of $\mathcal{T}^E E$**

$S \in \Gamma(\mathcal{T}^E E \otimes (\mathcal{T}^E E)^*)$

$S(a)(b, v) = (0, b_a^v), \quad a, b \in E, \quad v \in T_a E$

$b_a^v \equiv$ vertical lift of $b$ to $T_a E$

**The Liouville section of $\mathcal{T}^E E$**

$\Delta(a) = (0, a_a^v), \quad a \in E$

# A particular case:

**Second-order differential equations (SODE) on $E$**

$\xi \in \Gamma(\mathcal{T}^E E) \,/ S\xi = \Delta$

$\xi$ SODE $\Rightarrow$ The integral curves of $\rho^\tau(\xi)$ are admissible *

* $\gamma : I \to E$ a curve on $E$

$$\gamma \text{ is admissible } \Leftrightarrow (\gamma(t), \dot\gamma(t)) \in \mathcal{T}^E_{\gamma(t)} E, \quad \forall t$$

$$E = TM \Rightarrow \mathcal{T}^E E = T(TM)$$

standard notions

# A particular case

Local expressions:

$(x^i)$ local coordinates on $U \subseteq M$, $\{e_\alpha\}$ a local basis of $\Gamma(E)$ on $U$

$$\Downarrow$$

$(x^i, y^\alpha)$ local coordinates on $\tau^{-1}(U) \subseteq E$

$\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ a local basis of $\Gamma(\mathcal{T}^E E)$

$$\mathcal{X}_\alpha(a) = (e_\alpha(\tau(a)), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|a}), \qquad \mathcal{V}_\alpha(a) = (0, \frac{\partial}{\partial y^\alpha}_{|a}), \ \forall \alpha$$

$\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ the dual basis of $\Gamma((\mathcal{T}^E E)^*)$

$$\Downarrow$$

**The vertical endomorphism of $\mathcal{T}^E E$**

$S = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$

**The Liouville section of $\mathcal{T}^E E$**

$\Delta = y^\alpha \mathcal{V}_\alpha$

**SODE on $E$**

$\xi = y^\alpha \mathcal{X}_\alpha + \xi^\alpha \mathcal{V}_\alpha$

# The Lagrangian formalism on Lie algebroids

$L : E \to R$ a lagrangian function on $E$

**Poincaré-Cartan 1-section**

$$\Theta_L = S^*(dL) \in \Gamma((\mathcal{T}^E E)^*)$$

**Poincaré-Cartan 2-section**

$$\omega_L = -d\Theta_L \in \Gamma(\wedge^2(\mathcal{T}^E E)^*)$$

**Lagrangian energy**

$$E_L = \rho^\tau(\Delta)(L) - L \in C^\infty(E)$$

$c : I \to E$ a curve on $E$

$c$ is a solution of the Euler-Lagrange (E-L) equations $\iff$

i) $c$ is admissible

ii) $i_{(c(t),\dot{c}(t))}\omega_L(c(t)) = dE_L(c(t)), \quad \forall t$

# The Lagrangian formalism on Lie algebroids

Local expressions:

$$\Theta_L = \frac{\partial L}{\partial y^\alpha} \mathcal{X}^\alpha$$

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + (\frac{1}{2}\frac{\partial L}{\partial y^\alpha} C^\gamma_{\alpha\beta} - \rho^i_\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta}) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta$$

$$E_L = y^\alpha \frac{\partial L}{\partial y^\alpha} - L$$

$c : t \to (x^i(t), y^\alpha(t))$ solution of $E - L$ equations

$$\Updownarrow$$

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha y^\alpha, \quad \forall i \\ \frac{d}{dt}(\frac{\partial L}{\partial y^\alpha}) &= \rho^i_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}, \quad \forall \alpha \end{aligned}$$

# The Lagrangian formalism on Lie algebroids

$L$ regular $\Longleftrightarrow \omega_L$ is non-degenerate

Local condition: $\left(\dfrac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right)$ is a regular matrix

$L$ regular $\Longrightarrow \exists! \xi_L \in \Gamma(\mathcal{T}^E E) / i_{\xi_L}\omega_L = dE_L$

$\xi_L$ is a SODE and the integral sections of $\xi_L$ are solutions of the E-L equations

# Examples

- $E = TM \Rightarrow$ Classical Lagrangian formalism of Mechanics
- $E = \mathfrak{g}$ a real Lie algebra of finite dimension

$y \in \mathfrak{g} \Longrightarrow ad_y : \mathfrak{g} \to \mathfrak{g}, \quad y' \in \mathfrak{g} \to [y, y'] \in \mathfrak{g}$

$ad_y^* : \mathfrak{g}^* \to \mathfrak{g}^*$ the dual linear map

$l : \mathfrak{g} \to \mathbb{R}$ a Lagrangian function

**Euler-Poincaré equations**

E-L equations for $l$: $\frac{d}{dt}(\frac{\partial l}{\partial y}) = ad_y^*(\frac{\partial l}{\partial y})$

- $E = D$ a completely integrable distribution on $M$

Holonomic Lagrangian Mechanics

# Examples

- $\mathfrak{g}$ a real Lie algebra of finite dimension
  $V$ a real vector space of finite dimension
  Linear representation of $\mathfrak{g}$ on $V$

$$\mathfrak{g} \times V \to V, \quad (y, u) \to yu$$

$$\Downarrow$$

linear representation of $\mathfrak{g}$ on $V^*$

$$\mathfrak{g} \times V^* \to V^*, \quad (y, a) \to ya$$

$$(ya)(u) = -a(yu), \quad \forall u \in V$$

$E = \mathfrak{g} \times V^* \to V^*$ action Lie algebroid over $V^*$

# Examples

$l : \mathfrak{g} \times V^* \to \mathbb{R}$ a Lagrangian function
$c : I \to \mathfrak{g} \times V^*, \qquad t \to c(t) = (y(t), a(t)),$

## Euler-Poisson-Poincaré

$c$ a solution of E-L equations for $l$

$$\Updownarrow$$

$$\begin{aligned} \dot{a} &= -ya \\ \frac{d}{dt}\left(\frac{\partial l}{\partial y}\right) &= ad_y^* \frac{\partial l}{\partial y} + \frac{\partial l}{\partial a} \Diamond a \end{aligned}$$

$u \in V, \quad a \in V^* \implies u \Diamond a \in \mathfrak{g}^*$
$(u \Diamond a)(y) = -(ya)(u), \quad \forall y \in \mathfrak{g}$

# Examples

- $$\pi : Q \to M \text{ a principal } G\text{-bundle}$$
  $$\Downarrow$$
  $$\tau_Q|G : TQ/G \to M = Q/G \text{ the Atiyah algebroid}$$
  $$\Gamma(TQ/G) \cong \{X \in \mathfrak{X}(Q)/X \text{ is } G\text{-invariant }\}$$
  $$L : TQ \to \mathbb{R} \text{ a } G\text{-invariant Lagrangian}$$
  $$\Downarrow$$
  $$l : TQ/G \to \mathbb{R} \text{ the reduced Lagrangian}$$

# Examples

$A : TQ \longrightarrow \mathfrak{g}$ a principal connection
$B : TQ \oplus TQ \longrightarrow \mathfrak{g}$ the curvature of $A$
$U \subseteq M$ an open subset of $M$; $\quad (x^i)$

$$\pi^{-1}(U) \cong U \times G$$

$\{\xi_a\}$ a basis of $\mathfrak{g}$, $\quad [\xi_a, \xi_b] = c_{ab}^c \xi_c$

$\xi_a^L$ the corresponding left-invariant vector field on $G$

$$A(\frac{\partial}{\partial x^i}_{|(x,e)}) = A_i^a(x)\xi_a, \qquad B(\frac{\partial}{\partial x^i}_{|(x,e)}, \frac{\partial}{\partial x^j}_{|(x,e)}) = B_{ij}^a(x)\xi_a$$

$$\{\frac{\partial}{\partial x^i} - A_i^a \xi_a^L, \xi_b^L\} \text{ a local basis of } \Gamma(TQ/G)$$

$$\Downarrow$$

$$(x^i; \dot{x}^i, \bar{v}^a) \text{ local coordinates on } TQ/G$$

# Examples

## Lagrange-Poincaré equations for $L$

E-L equations for $l$:

$$\frac{\partial l}{\partial x^j} - \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}^j}\right) = \frac{\partial l}{\partial \bar{v}^a}\left(B^a_{ij}\dot{x}^i + c^a_{db}A^b_j\bar{v}^d\right), \quad \forall j$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \bar{v}^b}\right) = \frac{\partial l}{\partial \bar{v}^a}\left(c^a_{db}\bar{v}^d - c^a_{db}A^d_i\dot{x}^i\right), \quad \forall b$$

# The Hamiltonian formalism on Lie algebroids

$(E, [\![\cdot,\cdot]\!], \rho)$ a Lie algebroid over $M$, $\quad rank E = n, \ \dim M = m$

$\tau^* : E^* \to M$ the vector bundle projection

$\mathcal{T}^E E^* \equiv$ the $E$-tangent bundle to $E^*$

$$\mathcal{T}^E E^* = \{(b, v) \in E \times TE^* / \rho(b) = (T\tau^*)(v)\}$$

$(\mathcal{T}^E E^*, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ a Lie algebroid of rank $2n$ over $E^*$

$(x^i)$ local coordinates on $M$, $\quad \{e_\alpha\}$ a basis of $\Gamma(E)$

$\Downarrow$

$(x^i, y_\alpha)$ local coordinates on $E^*$

$$\begin{aligned} \tilde{e}_\alpha(a^*) &= (e_\alpha(\tau^*(a^*)), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|a^*}), \\ \bar{e}_\alpha(a^*) &= (0, \frac{\partial}{\partial y_\alpha}_{|a^*}) \end{aligned}$$

$\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ a local basis of $\Gamma(\mathcal{T}^E E^*)$

$$E = TM \Rightarrow \mathcal{T}^E E^* = T(T^*M)$$

# The Hamiltonian formalism on Lie algebroids

## The Liouville 1-section

$\lambda_E \in \Gamma((\mathcal{T}^E E^*)^*)$
$\lambda_E(a^*)(b,v) = a^*(b), \quad a^* \in E^*_x, \quad (b,v) \in (\mathcal{T}^E E^*)_{a^*}$

## The canonical symplectic section

$\Omega_E \in \Gamma(\wedge^2(\mathcal{T}^E E^*))$
$\Omega_E = -d\lambda_E$

Local expressions:

$\{\tilde{e}^\alpha, \bar{e}^\alpha\}$ the dual basis of $\{\tilde{e}_\alpha, \bar{e}^\alpha\}$

$\lambda_E = y_\alpha \tilde{e}^\alpha$
$\Omega_E = \tilde{e}_\alpha \wedge \bar{e}^\alpha + \frac{1}{2} C^\gamma_{\alpha\beta} y_\alpha \tilde{e}^\alpha \wedge \tilde{e}^\beta$

# The Hamiltonian formalism on Lie algebroids

$H : E^* \to \mathbb{R}$ a Hamiltonian function

$$\Downarrow$$

$dH \in \Gamma((\mathcal{T}^E E^*)^*)$

$$\Downarrow$$

$\exists! \xi_H \in \Gamma(\mathcal{T}^E E^*) / i_{\xi_H}\Omega_E = dH$

$\xi_H \equiv$ The Hamiltonian section associated with $H$

The integral curves of $\rho^{\tau^*}(\xi_H)$ are the solution of the Hamilton equations associated with $H$

$$\frac{dx^i}{dt} = \rho^i_\alpha \frac{\partial H}{\partial y_\alpha}, \quad \frac{dy_\alpha}{dt} = -(C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i})$$

$i \in \{1, \dots, m\}, \quad \alpha \in \{1, \dots, n\}$

# The Hamiltonian formalism on Lie algebroids

- $E = TM \Rightarrow$ Classical Hamiltonian formalism of Mechanics
- $E = \mathfrak{g}$ a real Lie algebra of finite dimension

Lie-Poisson equations on $\mathfrak{g}^*$

- $E = D$ a complete integrable distribution on $M$

Holonomic Hamiltonian Mechanics

- $E = \mathfrak{g} \times V^* \to V^*$ an action Lie algebroid over $V^*$
  $V$ a real vector space of finite dimension

Lie-Poisson equations on the dual of a semidirect product of Lie algebras

# The Hamiltonian formalism on Lie algebroids

- $\pi : Q \to M = Q/G$ a principal $G$-bundle over $M$

  $\tau_Q|G : TQ/G \to M = Q/G$ the corresponding Atiyah algebroid

Hamilton-Poincaré equations

# The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms

$L : E \to \mathbb{R}$ a Lagrangian function

$$\Downarrow$$

$Leg_L : E \to E^*, \quad Leg_L(a)(b) = \theta_L(a)(z)$

$a, b \in E_x$ and $z \in \mathcal{T}^E_a E / pr_1(z) = b$

$Leg_L \equiv$ *The Legendre transformation associated with L*

$$Leg_L(x^i, y^\alpha) = (x^i, \frac{\partial L}{\partial y^\alpha})$$

$\mathcal{T} Leg_L : \mathcal{T}^E E \to \mathcal{T}^E E^* \quad (b, v) \mapsto (b, (TLeg_L)(v)),$

# The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms

**Theorem**

*The pair $(\mathcal{T}Leg_L, Leg_L)$ is a morphism between the Lie algebroids $(\mathcal{T}^E E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ and $(\mathcal{T}^E E^*, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$. Moreover, if $\theta_L$ and $\omega_L$ (respectively, $\lambda_E$ and $\Omega_E$) are the Poincaré-Cartan 1-section and 2-section associated with $L$ (respectively, the Liouville section and the canonical symplectic section on $\mathcal{T}^{\tau^*}E$) then*

$$(\mathcal{T}Leg_L, Leg_L)^*(\lambda_E) = \theta_L, \quad (\mathcal{T}Leg_L, Leg_L)^*(\Omega_E) = \omega_L$$

# The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms

$L$ regular $\Leftrightarrow$ $Leg_L$ is a local diffeomorphism

$L$ *hyperregular* if $Leg_L$ is a global diffeomorphism

$L$ hyperregular $\Rightarrow H = E_L \circ Leg_L^{-1}$ a Hamiltonian function

**Theorem**

*If the Lagrangian $L$ is hyperregular then the Euler-Lagrange section $\xi_L$ associated with $L$ and the hamiltonian section $\xi_H$ are $(\mathcal{L}Leg_L, Leg_L)$-related, that is,*

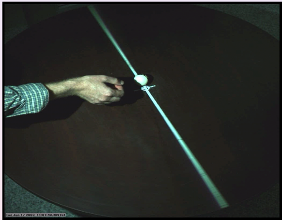
$$\xi_H \circ Leg_L = \mathcal{L}Leg_L \circ \xi_L.$$

*Moreover, if $\gamma : I \to E$ is a solution of the Euler-Lagrange equations associated with $L$, then $\mu = Leg_L \circ \gamma : I \to E^*$ is a solution of the Hamilton equations associated with $H$ and, conversely, if $\mu : I \to E^*$ is a solution of the Hamilton equations for $H$ then $\gamma = Leg_L^{-1} \circ \mu$ is a solution of the Euler-Lagrange equations for $L$*

# An standard example

A motivation: Reduction of standard non-holonomic Lagrangian systems with symmetries

"A ROLLING BALL ON A ROTATING TABLE WITH CONSTANT ANGULAR VELOCITY"



$r \equiv$ the radius of the sphere
$m = 1$ (unit mass)
$k^2 \equiv$ Inertia about any axis
$\Omega \equiv$ the const. angular velocity of the table

- The configuration space: $Q = \mathbb{R}^2 \times SO(3)$
- The phase space of velocities:
  $TQ = T\mathbb{R}^2 \times T(SO(3)) \cong T\mathbb{R}^2 \times (SO(3) \times \mathbb{R}^3)$

$$(x, y, \dot{x}, \dot{y}, \theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi}) \to (x, y, \dot{x}, \dot{y}, \theta, \varphi, \psi, \omega_x, \omega_y, \omega_z)$$

$$\omega_x, \omega_y, \omega_z \equiv \text{ angular velocities}$$

# An standard example

- The Lagrangian function:

$$L = \frac{1}{2}(\dot{x}^2+\dot{y}^2+k^2(\dot{\theta}^2+\dot{\varphi}^2+\dot{\psi}^2+2\dot{\varphi}\dot{\psi}\cos\theta)) = \frac{1}{2}(\dot{x}^2+\dot{y}^2+k^2(\omega_x^2+\omega_y^2+\omega_z^2))$$

- The constraints:

$$\phi_1 \equiv \dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi + \Omega y = 0$$

$$\phi_2 \equiv \dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi - \Omega x = 0$$

$$\Updownarrow$$

$$\phi_1 \equiv \dot{x} - r\omega y + \Omega y = 0, \quad \phi_2 \equiv \dot{y} + r\omega_x - \Omega x = 0$$

$\mathcal{M} = \{v \in TQ / \phi_1(v) = 0, \ \phi_2(v) = 0\}$ the constraint submanifold

$\Omega = 0 \quad \Leftrightarrow \quad$ The constraints are linear
($\Leftrightarrow \mathcal{M}$ is a vector subbundle of $TQ$)

# An standard example

$$Q = \mathbb{R}^2 \times SO(3) \to \mathbb{R}^2 \text{ is a principal } SO(3)\text{-bundle}$$

Action of $SO(3)$ on $TQ \cong T\mathbb{R}^2 \times (SO(3) \times \mathbb{R}^3)$ is the standard action of $SO(3)$ on itself by left-translations

$$\Downarrow$$

The Atiyah algebroid $TQ/SO(3) \to Q/SO(3) = M = \mathbb{R}^2$ is isomorphic to the vector bundle $T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$

# An standard example

$L$ and $\mathcal{M}$ are $SO(3)$-invariant

$L' : T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}$ the reduced Lagrangian

$$L'(x, y, \dot{x}, \dot{y}; \omega_1, \omega_2, \omega_3) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2(\omega_1^2 + \omega_2^2 + \omega_3^2))$$

$\mathcal{M}` = \{v' \in T\mathbb{R}^2 \times \mathbb{R}^3 / \phi_1'(v') = 0, \ \ \phi_2'(v') = 0\}$ the reduced submanifold

$$\phi_1' \equiv \dot{x} - r\omega_2 + \Omega y = 0, \quad \phi_2' = \dot{y} + r\omega_1 - \Omega x = 0$$

**Conclusion**

We have a *Lagrangian system with non-holonomic constraints (which are not, in general, linear) on an Atiyah algebroid*

# Dynamical equations

$\tau : E \to M$ a Lie algebroid

$(\llbracket \cdot, \cdot, \rrbracket, \rho)$ the Lie algebroid structure, $\quad \dim M = m, \quad rank\ E = n$

$\mathcal{M}$ a submanifold of $E$ such that $\pi = \tau_{|\mathcal{M}} : \mathcal{M} \to M$ is a fibration

$\dim \mathcal{M} = r + m$

$\mathcal{M} \equiv$ the constraint submanifold

Linear constraints $\longleftrightarrow$ $\mathcal{M} \to M$ is a vector subbundle $D$ of $E$

# Dynamical equations

**The vector bundle $\mathcal{V} \to \mathcal{M}$ of virtual displacements**

$$a \in \mathcal{M} \Rightarrow \mathcal{V}_a = \{b \in E_{\tau(a)} / b_a^v \in T_a\mathcal{M}\}, \quad rank\mathcal{V} = r$$

**The vector bundle $\Psi \to \mathcal{M}$ of constraint forces**

$$\mathcal{T}^E\mathcal{M} \to \mathcal{M} \text{ the } E\text{-tangent bundle to } \mathcal{M}$$

$$rank(\mathcal{T}^E\mathcal{M}) = 2n - s; \quad s = n - r$$

$$a \in \mathcal{M} \Rightarrow \Psi_a = S^*((\mathcal{T}_a^E\mathcal{M})^o), \qquad rank\Psi = s$$

$$(\mathcal{T}_a^E\mathcal{M})^o = \{\alpha \in (\mathcal{T}_a^E E)^* / <\alpha, z>= 0, \forall z \in \mathcal{T}_a^E\mathcal{M}\}$$

# Dynamical equations

## Problem

We look for curves $t \to c(t)$ on $E$ such that:

1. $c$ is admissible $(\rho(c(t)) = (\tau \circ c)'(t), \text{ for all } t)$
2. $c(t) \in \mathcal{M}$, for all $t$
3. $i_{(c(t),\dot{c}(t))}\omega_L(c(t)) - dE_L(c(t)) \in \Psi(c(t))$, for all $t$

# Dynamical equations

Local expressions:

$(x^i, y^\alpha)$ fibred local coordinates on $E$

$(\rho^i_\alpha, C^\gamma_{\alpha\beta})$ local structure functions of $E$

$\phi^A(x^i, y^\alpha) = 0$ local equations defining $\mathcal{M}$

$$\Downarrow$$

**Lagrange-d'Alembert equations for the constrained system $(L, \mathcal{M})$**

$$\dot{x}^i = \rho^i_\alpha y^\alpha, \quad \text{for all } i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) - \rho^i_\alpha \frac{\partial L}{\partial y^\gamma} + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta = \lambda_A \frac{\partial \phi^A}{\partial y^\alpha}, \quad \forall \alpha$$

$$\phi^A(x^i, y^\alpha) = 0, \quad \forall A = 1, \ldots, s$$

# Dynamical equations

A more geometrical description:

## Dynamical equations

$\xi \in \Gamma(\mathcal{T}^E E)$ such that

$$(i_\xi \omega_L - dE_L)_{|\mathcal{M}} \in \Gamma(\Psi)$$

$$\xi_{|\mathcal{M}} \in \Gamma(\mathcal{T}^E \mathcal{M})$$

Remark: $i)$ $\xi$ solution of our problem $\Rightarrow$ $\xi$ SODE along $\mathcal{M}$

$ii)$ $\pi : \mathcal{M} \to M$ a fibration

$$\Downarrow$$

$S^* : (\mathcal{T}^E \mathcal{M})^o \to \Psi$ is an isomorphism of vector bundles

$iii)$ $E = TM \Rightarrow$ Classical formalism for standard non-holonomic Lagrangian systems

# Regular non-holonomic Lagrangian systems

Two vector bundles over $\mathcal{M}$:

1. $F \to \mathcal{M}$
   $a \in \mathcal{M} \Rightarrow F_a = \omega_L^{-1}(\Psi_a), \quad rankF = s$
2. $\mathcal{T}^\nu\mathcal{M} \to \mathcal{M}$
   $a \in \mathcal{M} \Rightarrow \mathcal{T}_a^\nu\mathcal{M} = \{z \in \mathcal{T}_a^E\mathcal{M} / S(z) \in \mathcal{T}_a^E\mathcal{M}\}, \quad rank(T^\nu\mathcal{M}) = 2r$

**Theorem**

*The following properties are equivalent:*

1. *The constrained Lagrangian system $(L, \mathcal{M})$ is regular, that is, there exists a unique solution of the Lagrange-d'Alembert equations*
2. $\mathcal{T}^E\mathcal{M} \cap F = \{0\}$
3. $\mathcal{T}^\nu\mathcal{M} \cap (\mathcal{T}^\nu\mathcal{M})^\perp = \{0\}$

# Regular non-holonomic Lagrangian systems

Local condition:

The constrained Lagrangian system $(L, \mathcal{M})$ is regular

$$\Updownarrow$$

$$\left(\mathcal{C}^{AB} = \frac{\partial \phi^A}{\partial y^\alpha} W^{\alpha\beta} \frac{\partial \phi^B}{\partial y^\beta}\right)_{A,B=1,\dots,s} \text{ is a regular matrix}$$

$L$ is of mechanical type

$$\Downarrow$$

$(L, \mathcal{M})$ is regular

# Regular non-holonomic Lagrangian systems

$$(2) \Rightarrow (\mathcal{T}^E E)|\mathcal{M} = \mathcal{T}^E\mathcal{M} \oplus F$$

$$P : (\mathcal{T}^E E)|\mathcal{M} \to \mathcal{T}^E\mathcal{M}, \qquad Q : (\mathcal{T}^E E)|\mathcal{M} \to F$$

**Theorem**

*Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system and let $\xi_L$ be the solution of the free dynamics, i.e., $i_{\xi_L}\omega_L = dE_L$. Then, the solution of the constrained dynamics is the SODE $\xi$ obtained as follows*

$$\xi = P(\xi_L|\mathcal{M}).$$

# Regular non-holonomic Lagrangian systems

$$(3) \Rightarrow (\mathcal{T}^E E)_{|\mathcal{M}} = \mathcal{T}^\nu \mathcal{M} \oplus (\mathcal{T}^\nu \mathcal{M})^\perp$$

$$\bar{P} : (\mathcal{T}^E E)_{|\mathcal{M}} \to \mathcal{T}^\nu \mathcal{M}, \quad \bar{Q} : (\mathcal{T}^E E)_{|\mathcal{M}} \to (\mathcal{T}^\nu \mathcal{M})^\perp$$

## Theorem

*Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system, $\xi_L$ (respectively, $\xi$) be the solution of the free (respectively, constrained) dynamics and $\Delta$ be the Liouville section of $\mathcal{T}^E E \to E$. Then, $\xi = \bar{P}(\xi_L|\mathcal{M})$ if and only if the restriction to $\mathcal{M}$ of the vector field $\rho^\tau(\Delta)$ on $E$ is tangent to $\mathcal{M}$.*

## Corollary

*Under the same hypotheses as in the above theorem if $\mathcal{M}$ is a vector subbundle of $E$ (that is, the constraints are linear) then $\xi = \bar{P}(\xi_L|\mathcal{M})$*

# Regular non-holonomic Lagrangian systems

$(L, \mathcal{M})$ a regular constrained Lagrangian system

$$\Downarrow$$

$$\exists! \alpha_{(L,\mathcal{M})} \in \Gamma((\mathcal{T}^E\mathcal{M})^o) / i_{Q\xi_L}\omega_L = S^*(\alpha_{(L,\mathcal{M})})$$

**Theorem (Conservation of the energy)**

*Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system, $\Delta$ be the Liouville section of $\mathcal{T}^E E \to E$ and $\xi$ be the solution of the constrained dynamics. Then, $(d_\xi E_L)|\mathcal{M} = 0$ if and only if $\alpha_{(L,\mathcal{M})}(\Delta|\mathcal{M}) = 0$. In particular,if the restriction to $\mathcal{M}$ of the vector field $\rho^\tau(\Delta)$ on $E$ is tangent to $\mathcal{M}$ then $(d_\xi E_L)|\mathcal{M} = 0$.*

# Example (continued)

$(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_i)_{i=1,\dots,5}$ local coordinates on $TQ = T\mathbb{R}^2 \times T(SO(3))$

$$\begin{cases} \bar{x} = x, \quad \bar{y} = y, \quad \bar{\theta} = \theta, \quad \bar{\varphi} = \varphi, \quad \bar{\psi} = \psi, \\ \pi_1 = r\dot{x} + k^2\dot{q}_2, \quad \pi_2 = r\dot{y} - k^2\dot{q}_1, \quad \pi_3 = k^2\dot{q}_3, \\ \pi_4 = \dfrac{k^2}{(k^2+r^2)}(\dot{x} - r\dot{q}_2 + \Omega y), \quad \pi_5 = \dfrac{k^2}{(k^2+r^2)}(\dot{y} + r\dot{q}_1 - \Omega x), \end{cases}$$

quasi-coordinates

$$\dot{q}_1 = \omega_x, \quad \dot{q}_2 = \omega_y, \quad \dot{q}_3 = \omega_z$$

$$P : (\mathcal{T}^E E)_{|\mathcal{M}} \to \mathcal{T}^E \mathcal{M}, \quad Q : (\mathcal{T}^E E)_{|\mathcal{M}} \to F$$

$$Q = \frac{\partial}{\partial \pi_4} \otimes d\pi_4 + \frac{\partial}{\partial \pi_5} \otimes d\pi_5, \quad P = Id - Q$$

# Example (continued)

## The constrained dynamics

$$\xi = (\dot{x}\frac{\partial}{\partial \bar{x}} + \dot{y}\frac{\partial}{\partial \bar{y}} + \dot{q}_1\frac{\partial}{\partial q_1} + \dot{q}_2\frac{\partial}{\partial q_2} + \dot{q}_3\frac{\partial}{\partial q_3})_{|\mathcal{M}}$$

The energy is not, in general, constant along the solutions

$$(d_\xi E_L)_{|\mathcal{M}} = \frac{\Omega^2 k^2}{(k^2 + r^2)}(x\dot{x} + y\dot{y})_{|\mathcal{M}}$$

$$(d_\xi E_L)_{|\mathcal{M}} = 0 \Leftrightarrow \Omega = 0$$

# The non-holonomic bracket

$(L, \mathcal{M})$ a regular constrained Lagrangian system,
$(\mathcal{T}^E E)_{|\mathcal{M}} = \mathcal{T}^\nu \mathcal{M} \oplus (\mathcal{T}^\nu \mathcal{M})^\perp$

$$\bar{P} : (\mathcal{T}^E E)_{|\mathcal{M}} \to \mathcal{T}^\nu \mathcal{M}, \quad \bar{Q} : (\mathcal{T}^E E)_{|\mathcal{M}} \to (\mathcal{T}^\nu \mathcal{M})^\perp$$

$f, g \in C^\infty(\mathcal{M})$

$$\{f, g\}_{nh} = \omega_L(\bar{P}(X_{\tilde{f}}), \bar{P}(X_{\tilde{g}}))$$

$X_{\tilde{f}}, X_{\tilde{g}}$ hamiltonian sections in $(\mathcal{T}^E E, \omega_L)$ associated with $\tilde{f}$ and $\tilde{g}$

Properties:

1. $\{\cdot, \cdot\}_{nh}$ is skew-symmetric
2. $\{\cdot, \cdot\}_{nh}$ satisfies the Leibniz rule
3. $\{\cdot, \cdot\}_{nh}$ doesn't satisfy, in general, the Jacobi identity
4. $f \in C^\infty(\mathcal{M}) \Rightarrow \dot{f} = \rho^\tau(R_L)(f) + \{f, E_L|\mathcal{M}\}_{nh}$
$$R_L = P(\xi_L|\mathcal{M}) - \bar{P}(\xi_L|\mathcal{M})$$

**Remark:** If $\rho^\tau(\Delta)|\mathcal{M}$ is tangent to $\mathcal{M} \Rightarrow \dot{f} = \{f, E_L|\mathcal{M}\}_{nh}$

# Example (continued)

## The non-holonomic bracket

$$\{x,\pi_1\}_{nh} = r, \quad \{y,\pi_2\}_{nh} = r, \quad \{q_1,\pi_2\}_{nh} = -1$$
$$\{q_2,\pi_1\}_{nh} = 1, \quad \{q_3,\pi_3\}_{nh} = 1, \quad \{\pi_1,\pi_2\}_{nh} = \pi_3$$
$$\{\pi_2,\pi_3\}_{nh} = \frac{k^2}{(k^2+r^2)}\pi_1 + \frac{rk^2\Omega}{(k^2+r^2)}y, \quad \{\pi_3,\pi_1\}_{nh} = \frac{k^2}{(k^2+r^2)}\pi_2 - \frac{rk^2\Omega}{(k^2+r^2)}x$$

## The evolution of an observable

$$\dot{f} = R_L(f) + \{f, L\}_{nh}, \qquad f \in C^\infty(\mathcal{M})$$

$$\begin{aligned} R_L \;=\; & \frac{k^2\Omega}{(k^2+r^2)}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{r\Omega}{(k^2+r^2)}\left(x\frac{\partial}{\partial q_1} + y\frac{\partial}{\partial q_2}\right. \\ & \left. + x(\pi_3 - k^2\Omega)\frac{\partial}{\partial \pi_1} + y(\pi_3 - k^2\Omega)\frac{\partial}{\partial \pi_2} - k^2(\pi_1 x + \pi_2 y)\frac{\partial}{\partial \pi_3}\right) \end{aligned}$$

# Morphisms and reduction

$(L, \mathcal{M})$ a regular constrained Lagrangian system on $\tau : E \to M$

$(L', \mathcal{M}')$ a constrained Lagrangian system on $\tau' : E' \to M'$

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \downarrow \tau & & \downarrow \tau' \\ M & \xrightarrow{\phi} & M' \end{array}$$

epimorphism of Lie algebroids

i) $L = L' \circ \Phi$

ii) $\Phi|\mathcal{M} : \mathcal{M} \to \mathcal{M}'$ is a surjective submersion

iii) $\Phi(\mathcal{V}_a) = \mathcal{V}'_{\Phi(a)}$, for all $a \in \mathcal{M}$

**Remark:** $\mathcal{M} = D, \mathcal{M}' = D'$ are vector subbundles of $E$ and $E'$

$$\Downarrow$$

$$(i), ii) \text{ and } iii) \Leftrightarrow L = L' \circ \Phi, \quad \Phi(D) = D')$$

# Morphisms and reduction

$$\mathcal{T}^{\Phi} : \mathcal{T}^{E}\mathcal{M} \to \mathcal{T}^{E'}\mathcal{M}', \quad (b, v) \to (\Phi(b), (T\Phi)(v))$$

$(\mathcal{T}^{\Phi}\Phi, \Phi)$ is an epimorphim of Lie algebroids

**Theorem ( Reduction of the constrained dynamics)**

*Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system on a Lie algebroid $\tau : E \to M$ and $(L', \mathcal{M}')$ be another constrained Lagrangian system on a second Lie algebroid $\tau' : E' \to M'$. Assume that we have an epimorphism of Lie algebroids $\Phi : E \to E'$ over $\phi : M \to M'$ such that conditions i), ii) and iii) hold. Then:*

1. *The constrained Lagrangian system $(L', \mathcal{M}')$ is regular*
2. *If $\xi$ (respectively, $\xi'$) is the constrained dynamics for $(L, \mathcal{M})$ (respectively, $(L', \mathcal{M}')$) then $\mathcal{T}^{\Phi}\Phi \circ \xi = \xi' \circ \Phi$.*
3. *If $t \to c(t)$ is a solution of Lagrange-d'Alembert equations for $(L, \mathcal{M})$ then $t \to \Phi(c(t))$ is a solution of Lagrange-d'Alembert equations for $(L', \mathcal{M}')$*

$\xi' \equiv$ reduction of the constrained dynamics $\xi$ by the morphism $\Phi$

# Morphisms and reduction

**Theorem ( reduction of the non-holonomic bracket)**

*Under the same hypotheses as in the above theorem, we have that*

$$\{f' \circ \Phi, g' \circ \Phi\}_{nh} = \{f', g'\}_{nh} \circ \Phi,$$

*for* $f', g' \in C^\infty(\mathcal{M}')$*, where* $\{\cdot, \cdot\}_{nh}$ *(respectively,* $\{\cdot, \cdot\}'_{nh}$*) is the non-holonomic bracket for the constrained system* $(L, \mathcal{M})$ *(respectively,* $(L', \mathcal{M}')$*). In other words,* $\Phi : \mathcal{M} \to \mathcal{M}'$ *is an almost Poisson morphism.*

# Morphisms and reduction

A particular case:

$$\phi : Q \to M \text{ a principal } G\text{-bundle}$$

$$\Downarrow$$

$$\tau_Q|G : TQ|G \to M = Q/G \text{ the corresponding Atiyah algebroid}$$

$\Phi : TQ \to TQ/G$ is a fiberwise bijective Lie algebroid morphism over $\phi$

# Morphisms and reduction

$(L, \mathcal{M})$ a regular constrained Lagrangian system on $TQ$

$\mathcal{M}$ a closed submanifold of $TQ$

$L$ and $\mathcal{M}$ are $G$-invariant

$\Downarrow$

- $L' : TQ/G \to \mathbb{R} / L = L' \circ \Phi$
- $\mathcal{M}' = \mathcal{M}|G$ is a closed submanifold of $TQ/G$

$(L', \mathcal{M}')$ is a constrained Lagrangian system on $TQ/G$

Conditions i), ii) and iii) hold for the morphism $\Phi$ and the constrained systems $(L, \mathcal{M})$ and $(L', \mathcal{M}')$

$\Downarrow$

*We may apply the reduction process*

# Example (continued)

$Q = \mathbb{R}^2 \times SO(3) \to \mathbb{R}^2$ is a principal $SO(3)$-bundle

## The reduced Lie algebroid

$E' = TQ/SO(3) \to Q/SO(3) = \mathbb{R}^2$ the Atiyah algebroid

$E' \cong T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$

$([\![\cdot,\cdot]\!]', \rho')$ the Lie algebroid structure

$\{e'_i\}_{i=1,\ldots,5}$ a global basis of $\Gamma(E')$

$$\begin{cases} \rho'(e'_1) = \dfrac{\partial}{\partial x}, \quad \rho'(e'_2) = \dfrac{\partial}{\partial y} \\ \rho'(e'_i) = 0, \quad i = 3,4,5 \\ [\![e'_4, e'_3]\!]' = e'_5, \;\; [\![e'_5, e'_4]\!]' = e'_3, \;\; [\![e'_3, e'_5]\!]' = e'_4 \end{cases}$$

# Example (continued)

The reduced constrained Lagrangian system

The Lagrangian function:

$$L'(x, y, \dot{x}, \dot{y}, \omega_1, \omega_2, \omega_3) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2(\omega_1^2 + \omega_2^2 + \omega_3^3))$$

The constraints:

$$\phi'_1 \equiv \dot{x} - r\omega_2 + \Omega y = 0$$

$$\phi'_2 \equiv \dot{y} + r\omega_1 - \Omega x = 0$$

# Example (continued)

$(x', y', \pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5)$ global coordinates on $E'$

$$x' = x, \qquad y' = y,$$
$$\pi'_1 = r\dot{x} + k^2\omega_2, \qquad \pi'_2 = r\dot{y} - k^2\omega_1, \qquad \pi'_3 = k^2\omega_3,$$
$$\pi'_4 = \frac{k^2}{(k^2+r^2)}(\dot{x} - r\omega_2 + \Omega y), \qquad \pi'_5 = \frac{k^2}{(k^2+r^2)}(\dot{y} + r\omega_1 - \Omega x),$$

$\Phi : TQ \to E' = TQ/SO(3)$ the canonical projection

$$\Phi(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (\bar{x}, \bar{y}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$$

The reduced constrained dynamics

$$(\rho')^{\tau'}(\xi') = (\dot{x}'\frac{\partial}{\partial \dot{x}'} + \dot{y}'\frac{\partial}{\partial \dot{y}'})_{|\mathcal{M}'}$$

# Example (continued)

## The reduced non-holonomic bracket

$$\{x', \pi'_1\}'_{nh} = r, \qquad \{y', \pi'_2\}'_{nh} = r,$$

$$\{\pi'_1, \pi'_2\}'_{nh} = \pi'_3, \qquad \{\pi'_2, \pi'_3\}'_{nh} = \frac{k^2}{(k^2+r^2)}\pi'_1 + \frac{rk^2\Omega}{(k^2+r^2)}y',$$

$$\{\pi'_3, \pi'_1\}'_{nh} = \frac{k^2}{(k^2+r^2)}\pi'_2 - \frac{rk^2\Omega}{(k^2+r^2)}x'$$

## Evolution of an observable

$$\dot{f}' = (\rho')^{\tau'}(R_{L'})(f') + \{f', L'\}'_{nh}, \text{ for } f' \in C^\infty(\mathcal{M}'),$$

$$\begin{aligned}(\rho')^{\tau'}(R_{L'}) = \; & \{\frac{k^2\Omega}{k^2+r^2}(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}) + \frac{r\Omega}{(k^2+r^2)}(x'(\pi'_3 - k^2\Omega)\frac{\partial}{\partial \pi'_1} \\ & + y'(\pi'_3 - k^2\Omega)\frac{\partial}{\partial \pi'_2} - k^2(\pi'_1 x' + \pi'_2 y')\frac{\partial}{\partial \pi'_3})\}_{|\mathcal{M}'}\end{aligned}$$

# Future work

- To develop a Hamiltonian formalism for non-holonomic Mechanics on Lie algebroids and then, using the Legendre transformation, to discuss the equivalence between the Lagrangian and Hamiltonian formalism
- To discuss in more detail the reduction procedure as it has been done in
  Bloch AM, Krishnaprasad PS, Marsden JE and Murray RM: Arch. Rational Mech. Anal. **136** (1996) 21–99

  Cantrijn F, de León M, Marrero JC and Martín de Diego D: Rep. Math. Phys. **42** (1998) 25–45; J. Math. Phys. **40** (1999), 795–820
  for the standard case
- To extend the so-called non-holonomic integrators in
  Cortés J.: Lect. Notes in Math (2002) , nº 1793, Springer-Verlag, Berlin

  Cortés J. and Martínez S.: Nonlinearity, **14** (2001), 1365–1392

  De León M., Martín de Diego D. and Santamaría Merino A.: J. Math. Phys. **45** (3) (2004) 1042-1064
  to the case of non-holonomic Mechanics on Lie algebroids

  Marrero J.C., Martín de Diego D. and Martínez E.: Discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids, Preprint 2005, math.DG/0506299

# The end

THANKS!!!!!