Two separately developed theories,

- theory of geodesically equivalent metrics and

- theory of quadratically integrable Hamiltonian systems and separations of variables

study essentially the same object.

We apply methods of one in the other

Benenti-systems, L-systems, cofactor systems, quasi-bi-hamiltonian systems, systems admitting special conformal Killing tensor

Levi-Civita Painlevé Eisenhart

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Definition: Two Riemannian metrics on one manifold are **geodesically equivalent**, if every geodesic of the first metric is a (probably, re-parametrised) geodesic of the second metric.

Example 1: The geodesics of the Klein model of the hyperbolic space are straight lines

Example 2: Given $A \in SL(n+1) \setminus O(n+1)$, Beltrami 1865 constructed a diffeomorphism $a : S^n \to S^n$ of the standard sphere $S^n \subset \mathbb{R}^{n+1}$ that is

- not an isometry
- but takes geodesics (great circles) to geodesics

Example 2: $A \xrightarrow{\text{we construct}} a,\ a(x) := \frac{A(x)}{|A(x)|}$

a takes great circles to great circles and is not an isometry

Example 3: **Levi-Civita's Theorem 1896:** Let g, $\bar{g}$ be two metrics on M^n. Assume the roots of $P(t) := det(g - t\bar{g})$ are all simple at $x \in M^n$.

Then, the metrics are geodesically equivalent near x if and only there exist coordinates $x_1, x_2, ..., x_n$ in some neighbourhood of x such that in these coordinates the metrics have the following model form:

$$\begin{aligned} ds_g^2 &= \Pi_1 dx_1^2 + \Pi_2 dx_2^2 + \cdots + \Pi_n dx_n^2, \\ ds_{\bar{g}}^2 &= \rho_1 \Pi_1 dx_1^2 + \rho_2 \Pi_2 dx_2^2 + \cdots + \rho_n \Pi_n dx_n^2, \end{aligned}$$

where the functions Π_i and ρ_i are given by

$$\begin{aligned} \Pi_i &\stackrel{\text{def}}{=} \prod_{\substack{j \neq i \\ j = 1}}^{n} |(\lambda_i - \lambda_j)| \\ \rho_i &\stackrel{\text{def}}{=} \frac{1}{\lambda_1 \lambda_2 ... \lambda_{n-1}} \frac{1}{\lambda_i}. \end{aligned}$$

where, for each i, the function λ_i is a smooth function of the variable x_i.

Relation with integrable systems and separation of variables

For $g,\ \bar g$ on $M^n \xrightarrow{\text{we construct}} L := \bar g^{-1} g \cdot \left(\frac{\det \bar g}{\det g}\right)^{\frac{1}{n+1}}$

$\xrightarrow{\forall t \in \mathbb{R} \text{ define}} S_t := (L - t\cdot \mathbf{Id})^{-1} \cdot det(L - t \cdot \mathbf{Id})$

$\xrightarrow{\text{consider}} I_t : TM^n \to \mathbb{R},\ I_t(\xi) := g(S_t(\xi), \xi).$

Theorem (Topalov, Matveev 1998):
If $g \sim \bar{g}$, then, $\forall t_1, t_2 \in \mathbb{R}$, the functions I_{t_i} are commuting integrals for the geodesic flow of g (i.e. for the Hamiltonian $H(\xi) := g(\xi, \xi)$)

The family contains n integrals which are functionally independent almost everywhere, if and only if there exists a point where all roots of $P(t) := det(g - t\bar{g})$ are simple.

Theorem (Topalov, Matveev 2001):
If $g \sim \bar{g}$, then, $\forall t_1, t_2 \in \mathbb{R}$, the operators

$$\mathcal{I}_{t_i} := \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_\alpha} \sqrt{\det g}\ g^{\alpha\gamma}\ \underset{\textcircled{t_i}}{S}{}^{\beta}_{\gamma}\ \frac{\partial}{\partial x_\beta}$$

commute with the Laplacian of g and mutually commute.

There is no problem to introduce potential energy in the picture

Plan:

- Geometric sense of the integrals

- First application of geodesic equivalence to integrable systems: Sinjukov-Topalov hierarchy

- Second application of geodesic equivalence to integrable systems: superintegrability

- One application of integrable systems to geodesic equivalence: topology

- Probably one more application of integrable systems to geodesic equivalence: proof of Lichnerowicz-Obata conjecture

Symplectic nature of the integrals (Topalov, M∼)

Consider Hamiltonian systems

(N^{2n}, ω, H, X_H) and $(\bar{N}^{2n}, \bar{\omega}, \bar{H}, X_{\bar{H}})$

and their energy surfaces

$Q^{2n-1} := \{H(x) = h\}$ and $\bar{Q}^{2n-1} := \{\bar{H}(x) = \bar{h}\}$

Suppose there exists $m : Q^{2n-1} \to \bar{Q}^{2n-1}$

such that $\mathrm{d}m(X_H) = \lambda(x) X_{\bar{H}}$

Then we can construct integrals for X_H:

indeed: consider $\sigma := \omega_{|Q}, \bar{\sigma} := \bar{\omega}_{|\bar{Q}}$ and the pull-back $m^*\bar{\sigma}$.

Lemma: The flow of X_H preserves σ, $m^*\bar{\sigma}$.

Proof: $L_{X_H} m^*\bar{\sigma} = \mathrm{d}\,[\imath_{X_H} m^*\bar{\sigma}] + \imath_{X_H}\mathrm{d}\,[m^*\bar{\sigma}] = 0$.

Since the forms σ, $m^*\bar{\sigma}$ are preserved by the flow, a function constructed invariantly by using these forms must automatically be an integral. So the coefficients of the characteristic polynomial of one form with respect to the second are integrals.

The procedure does not guarantee that the integrals commute. The proof of commutativity is a separate result. Bihamiltonian approach which also implies commutativity is due to Ibort, Magri, Marmo 2000

First applications in integrable systems: We can construct many new examples of (quantum) integrable systems:

Given g, $\bar{g}$ let us construct L as above.

For every $(1,1)$-tensor B, define:

$$g_B(\xi,\eta) := g(B(\xi),\eta)$$
$$\bar{g}_B(\xi,\eta) := \bar{g}(B(\xi),\eta)$$

Theorem (Topalov, Matveev 2001): Assume $g \sim \bar{g}$. For every real-analytic function F, the metrics $g_{F(L)}$ and $\bar{g}_{F(L)}$ are geodesically equivalent.

The example of Beltrami gives us a pair of geodesically equivalent metrics. If we apply the above Theorem to it for functions $F(x) = x$ and $F(x) = x^2$, we get the metrics of the ellipsoid and of the Poisson spheres. Thus, the metrics of the ellipsoid and of the Poisson sphere are (quantum) integrable

Second application: superintegrable and superseparable systems:

Problems:

1. How big can be the dimension of the integrals of a certain form.
   - Locally, near a point
   - or globally, on a compact or a complete manifold
2. To construct all natural Hamiltonian superintegrable systems.
3. Given a metric, to decide whether its geodesic flow is superintegrable

Second application: superintegrable and superseparable systems:

“Reformulation” of (1) for geodesically equivalent metrics:

How big can be the degree of mobility of a metric?

(the degree of mobility is the dimension of the space of metrics, geodesically equivalent to the given one).

Answers locally (Lie 1882 Fubini 1903 Egorov 1939 Solodovnikov 1956 Mikes 1982 Shandra 2000):

In dim 2, the degree of mobility can be 1,2,3,4,6 (locally and globally) only

If $dim(M) \geq 3$ then, locally, the degree of mobility of a metric of nonconstant curvature can take the values

$$\frac{m(m+1)}{2} + l$$

only, where $1 \leq m \leq n$ and $1 \leq l \leq \left[\frac{n+1-m}{3}\right]$.

Globally, the following theorem is true:

Theorem (Matveev 2004): *Let (M^n, g), $n \geq 2$, be a connected complete irreducible Riemannian manifold of nonconstant sectional curvature. Then the degree of mobility of g is ≤ 2.*

Second application: superintegrable and superseparable systems:

“Reformulation” of (2) for geodesically equivalent metrics:

To obtain a list of all metrics whose degree of mobility is ≥ 3.

For dimensions ≥ 3 the local version of the problem was solved by Solodovnikov in 1956–1969 and Shandra in 2001.

In dimension 2, the local version is nontrivial and is not solved yet.

Concerning the global version, the compact variant is due to Kolokoltsov 1986 and Kiyohara 1991

Theorem: (Matveev 2004) *Suppose the degree of mobility of a compete metric on R^2 of the form $\lambda(x,y)(dx^2+dy^2)$ be ≥ 3. Then the metric is isomorphic to one of the following metrics:*

1. $(x^2+y^2+C)(dx^2+dy^2)$,
2. $(x^2+y^2/4+C)(dx^2+dy^2)$,
3. dx^2+dy^2,

where C is a constant.

Second application: superintegrable and superseparable systems:

"Reformulation" of (3) for geodesically equivalent metrics:

Given a metric to decide how big is the space of metrics geodesically equivalent to a given one

For dimensions ≥ 3 the problem was almost solved by Solodovnikov in 1956–1969 and Shandra in 2000.

In dimension 2,

Theorem: (Manno, Matveev 2005) *If the dimension of the space of quadratic integrals is precisely* 4*, then there exists three independent projective vector fields.*

(A vector field is projective if its flow sends geodesics to geodesics).

In 1996 Romanovskii constructed a differential operator which decides whether a affine connection admits three projective vector fields. Combining his result with the result above we obtain a differential operator which decides whether a metric is Darboux-superintegrable.

Application of integrable systems in geodesic equivalence

Geodesic rigidity problem (generalisation of Beltrami 1865): What closed manifolds admit geodesically equivalent nonproportional metrics.

Theorem (Matveev 2006) *Suppose M is closed connected. Let Riemannian metrics g and $\bar{g}$ on M be geodesically equivalent and nonproportional. Then the manifold can be covered by the sphere, or it admits a metric with reducible holonomy group.*

Corollary 1 (Topalov, Matveev, 2001): A closed orientable surface admitting nonproportional geodesically equivalent metrics is S^2 or T^2.

Corollary (Matveev 2003): A closed 3-manifold admitting nonproportional geodesically equivalent metrics is $L_{p,q}$ or Seifert manifold with zero Euler number. ($L_{p,q}$ are covered by S^3, Seifert manifold with zero Euler number are 3-manifolds admitting metrics with reducible holonomy groups.)

Proof of Corollary 1: In dimension 2, the integral I_0 is

$$I_0(\xi) := \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{3}} \bar{g}(\xi, \xi).$$

Because of topology, there exists x_0 such that $g_{|x_0} = \bar{g}_{|x_0}$. We assume $g_{|x_1} \neq \bar{g}_{|x_1}$ and find a contradiction.

Explanation of Corollary 2 Assume $dim(M) = 3$

Case 1: There exists a point of the manifold such that that the polynomial $det(g - \lambda\bar{g})$ has 3 different roots. Then, the geodesic flow of g is Liouville-integrable.

Theorem (Kruglikov, Matveev 2005): *Then, the topological entropy of g vanishes.*

(And therefore modulo the Poincare conjecture the manifold can be covered by S^3, $S^2 \times S^1$ or by $S^1 \times S^1 \times S^1$.)

Case 2: At every point the number of roots of the polynomial is ≤ 2.

Then precisely the same trick as in dimension 2 works.

Another application in global differential geometry

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a Lie group (of dim*$\geq$ 1*) act on a closed Riemannian manifold by geodesic-preserving transformations. Then, the manifold is covered by the round sphere, or the group acts by isometries.*

History of L-O-S conjecture:

first examples Beltrami 1865

first paper of Lie groups of geodesic transformations Lie 1882.

first local results Fubini 1903

formulated as a question Schouten 1924

proved under different tensor assumptions 1950–1980

proved assuming $dim(M^n) \geq 3$ and that all objects are real analytic Solodovnikov 1969

Explanation if there exists a point such that the roots of $P(t) := det(g - t\bar{g})$ are all simple.

Recall

- **Levi-Civita 1896:** locally the metrics are :

$$ds_g^2 = \sum_i \prod_{j\neq i} |\lambda_i(x_i) - \lambda_j(x_j)| dx_i^2$$

$$ds_{\bar g}^2 = \sum_i \left[\frac{1}{\lambda_i(x_i)} \prod_\alpha \frac{1}{\lambda_\alpha(x_\alpha)} \prod_{j\neq i} |\lambda_i(x_i) - \lambda_j(x_j)| dx_i^2 \right].$$

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Matveev 2004: *If the manifold is not (covered by) the round sphere, the degree of mobility* ≤ 2.

Combining the facts above, we obtain that an essential projective vector field v must have the entries $(v_1(x_1), v_2(x_2), ..., v_n(x_n))$.

Then a projective transformation gives a system of ODE. One can analyse the system and prove the conjecture.

Open problems (Benenti, Rauch-Wojciechowski, Matveev)

1. How to decide?

   - To construct a differential operator that decides whether the geodesic flow of a metric admits an additional quadratic in momenta integral.

   - To construct a differential operator that decides whether a metrics admits a geodesically equivalent one.

2. Superintegrable systems

- To construct all metrics whose space of quadratic in momenta integrals has dimension 3.

- To introduce the potential energy in Solodovnikov's results

3. To construct global theory of geodesically equivalent pseudo-Riemannian metrics.
   - To understand topology of closed manifolds carrying geodesically equivalent pseudo-Riemannian metrics.
   - To solve pseudo-Riemannian analog of projective L-O-S conjecture.