Lagrangian submanifolds and dynamics on Lie affgebroids

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D. IGLESIAS, J.C. MARRERO, E. PADRÓN, D. SOSA, Lagrangian submanifolds and dynamics on Lie affgebroids, Preprint, math.DG/0505117.

Scheme of the talk

1. Motivation
2. Lie affgebroids
   - Lie affgebroids morphism
   - Lie affgebroid structure on $\mathcal{T}^A A$
3. Hamiltonian and Lagrangian formalism on Lie affgebroids
   - The Hamiltonian formalism
   - The Lagrangian formalism
   - The Legendre transformation and the equivalence between the Hamiltonian and Lagrangian formalisms
4. The prolongation of a symplectic Lie affgebroid
5. Lagrangian submanifolds and dynamics on a Lie affgebroid

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Motivation

A. Weinstein
Lagrangian Mechanics and groupoids
Fields Inst. Comm. **7** (1996), 207-231.

Motivation

E. Martínez
Lagrangian Mechanics on Lie algebroids
Acta Appl. Math. **67** (2001), 295-320.

Motivation

M. de León, J.C. Marrero, E. Martínez
Lagrangian submanifolds and dynamics on Lie algebroids
J.Phys.A: Math.Gen. **38** (2005), 241-308.

Motivation

W. Tulczyjew
Les sous-variétés lagrangiennes et la dynamique hamiltonienne
C.R. Acad. Sci., Paris **283** (1976), 15-18.

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Les sous-variétés lagrangiennes et la dynamique hamiltonienne
C.R. Acad. Sci., Paris **283** (1976), 675-678.

Motivation

E. Martínez, T. Mestdag, W. Sarlet
Lie algebroid structures and Lagrangian systems on affine bundles
J. Geom. Phys. **44** (2002), 70-95.

J. Grabowski, K. Grabowska, P. Urbanski
Lie brackets on affine bundles
Ann. Glob. Anal. Geom., **24** (2003), 101-130.

Lie affgebroids

- Notation

$$\tau_A : A \to M, \quad \tau_V : V \to M$$

$$\tau_{A^+} : A^+ = Aff(A, \mathbb{R}) \to M, \quad 1_A \in \Gamma(\tau_{A^+})$$

$$\tau_{\widetilde{A}} : \widetilde{A} = (A^+)^* \to M$$

$$i_A : A \to \widetilde{A} \quad i_A(a)(\varphi) = \varphi(a), \quad i_V : V \to \widetilde{A}$$

Lie affgebroids

Definition

Lie affgebroid structure on A:

$$[\![\cdot,\cdot]\!]_V : \Gamma(\tau_V) \times \Gamma(\tau_V) \to \Gamma(\tau_V) \quad \text{Lie bracket}$$
$$D : \Gamma(\tau_A) \times \Gamma(\tau_V) \to \Gamma(\tau_V) \quad \mathbb{R}\text{-linear action}$$
$$\rho_A : A \to TM \quad \text{affine map, the } \textit{anchor map}$$

such that

$$D_X [\![\bar{Y}, \bar{Z}]\!]_V = [\![D_X \bar{Y}, \bar{Z}]\!]_V + [\![\bar{Y}, D_X \bar{Z}]\!]_V$$
$$D_{X+\bar{Y}} \bar{Z} = D_X \bar{Z} + [\![\bar{Y}, \bar{Z}]\!]_V$$
$$D_X (f\bar{Y}) = f D_X \bar{Y} + \rho_A(X)(f)\bar{Y}$$

for $X \in \Gamma(\tau_A)$, $\bar{Y}, \bar{Z} \in \Gamma(\tau_V)$, $f \in C^\infty(M)$

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Lie affgebroids

- Lie algebroid $(E, [\![\cdot,\cdot]\!], \rho)$ is Lie affgebroid with $D = [\![\cdot,\cdot]\!]$

Lie affgebroids

- $(A, [\![\cdot,\cdot]\!]_V, D, \rho_A)$ Lie affgebroid $\Rightarrow$ $(V, [\![\cdot,\cdot]\!]_V, \rho_V)$ Lie algebroid

Lie affgebroids

- $(A, [\![\cdot,\cdot]\!]_V, D, \rho_A)$ Lie affgebroid

$$\Downarrow$$

$(\widetilde{A}, [\![\cdot,\cdot]\!]_{\widetilde{A}}, \rho_{\widetilde{A}})$ Lie algebroid $+\ 1_A \in \Gamma(\tau_{A^+})$ 1-cocycle

Conversely, $(U, [\![\cdot,\cdot]\!]_U, \rho_U)$ Lie algebroid and $\phi : U \to \mathbb{R}$ 1-cocycle, $\phi_{/U_x} \neq 0$

$$\Downarrow$$

$A = \phi^{-1}\{1\}$ Lie affgebroid with $(\widetilde{A}, [\![\cdot,\cdot]\!]_{\widetilde{A}}, \rho_{\widetilde{A}}) \approx (U, [\![\cdot,\cdot]\!]_U, \rho_U)$, $1_A \approx \phi$ and $V = \phi^{-1}\{0\}$

Lie affgebroids

Lie affgebroids morphism

Definition

$((F, f), (F^l, f))$ affine morphism between two Lie affgebroids $(A, [\![\cdot,\cdot]\!]_V, D, \rho_A)$ and $(A', [\![\cdot,\cdot]\!]_{V'}, D', \rho_{A'})$ is a *Lie affgebroid morphism* if:

$i)$ (F^l, f) Lie algebroid morphism between $(V, [\![\cdot,\cdot]\!]_V, \rho_V)$ and $(V', [\![\cdot,\cdot]\!]_{V'}, \rho_{V'})$

$ii)$ $Tf \circ \rho_A = \rho_{A'} \circ F$

$iii)$ $F^l \circ D_X \bar{Y} = (D'_{X'} \bar{Y}') \circ f$

$X \in \Gamma(\tau_A)$, $X' \in \Gamma(\tau_{A'})$, $\bar{Y} \in \Gamma(\tau_V)$, $\bar{Y}' \in \Gamma(\tau_{V'})$: $X' \circ f = F \circ X$ and $\bar{Y}' \circ f = F^l \circ \bar{Y}$

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Lie affgebroids
Lie affgebroids morphism

- $((F, f), (F^l, f))$ Lie affgebroid morphism

$$\begin{array}{ccc} A & \xrightarrow{F} & A' \\ \tau_A \downarrow & & \downarrow \tau_{A'} \\ M & \xrightarrow{f} & M' \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{F^l} & V' \\ \tau_V \downarrow & & \downarrow \tau_{V'} \\ M & \xrightarrow{f} & M' \end{array}$$

$$\Downarrow$$

$(\widetilde{F}, f)$ Lie algebroid morphism

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\widetilde{F}} & \widetilde{A}' \\ \tau_{\widetilde{A}} \downarrow & & \downarrow \tau_{\widetilde{A}'} \\ M & \xrightarrow{f} & M' \end{array}$$

$$\begin{array}{rcl} \widetilde{F}(\widetilde{a})(\varphi') & = & \widetilde{a}(\varphi' \circ F) \\ (\widetilde{F}, f)^* 1_{A'} & = & 1_A \end{array}$$

Lie affgebroids
Lie affgebroids morphism

- Conversely, $(\widetilde{F}, f)$ Lie algebroid morphism

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{F}} & U' \\ \tau_U \downarrow & & \downarrow \tau_{U'} \\ M & \xrightarrow{f} & M' \end{array} \qquad \begin{array}{c} \phi \in \Gamma(\tau_U), \phi' \in \Gamma(\tau_{U'}) \\ \phi(x) \neq 0, \; \phi'(x') \neq 0 \\ (\widetilde{F}, f)^* \phi' = \phi \end{array}$$

$$\Downarrow$$

$((F, f), (F^l, f))$ Lie affgebroid morphism

$$\begin{array}{ccc} A & \xrightarrow{F} & A' \\ \tau_A \downarrow & & \downarrow \tau_{A'} \\ M & \xrightarrow{f} & M' \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{F^l} & V' \\ \tau_V \downarrow & & \downarrow \tau_{V'} \\ M & \xrightarrow{f} & M' \end{array}$$

$$\begin{array}{lll} A = \phi^{-1}\{1\} & V = \phi^{-1}\{0\} & \tau_A = (\tau_U)_{/A} \\ A' = (\phi')^{-1}\{1\} & V' = (\phi')^{-1}\{0\} & \tau_{A'} = (\tau_{U'})_{/A'} \end{array}$$

Lie affgebroids

Lie affgebroid structure on $\mathcal{T}^A A$

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$$\mathcal{T}^A A = \{(a, v) \in A \times TA / \rho_A(a) = (T\tau_A)(v)\}$$

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$\phi_0 \in \Gamma((\tau^{\tau_A}_{\tilde{A}})^*)$, $\quad \phi_0 : \mathcal{T}^{\tilde{A}} A \to \mathbb{R} \quad \phi_0(\tilde{a}, v) = 1_A(\tilde{a})$

- ϕ_0 1-cocycle, $\quad (\phi_0)_{|(\mathcal{T}^{\tilde{A}} A)_a} \neq 0$
- $(\phi_0)^{-1}\{1\} = \mathcal{T}^A A, \quad (\phi_0)^{-1}\{0\} = \mathcal{T}^V A$

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$\tau^{\tau_A}_{A} : \mathcal{T}^A A \to A$ admits a Lie affgebroid structure with bidual Lie algebroid $(\mathcal{T}^{\tilde{A}} A, [\![\cdot,\cdot]\!]^{\tau_A}_{\tilde{A}}, \rho^{\tau_A}_{\tilde{A}})$ and modelled on $(\mathcal{T}^V A, [\![\cdot,\cdot]\!]^{\tau_A}_{V}, \rho^{\tau_A}_{V})$

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

$(\tau_A : A \to M, \tau_V : V \to M, ([\![\cdot,\cdot]\!]_V, D, \rho_A))$ Lie affgebroid

$(\tau^{\tau_V^*}_{\widetilde{A}} : \mathcal{T}^{\widetilde{A}}V^* \to V^*, [\![\cdot,\cdot]\!]^{\tau_V^*}_{\widetilde{A}}, \rho^{\tau_V^*}_{\widetilde{A}})$

(x^i) local coordinates on M

$\{e_0, e_\alpha\}$ local basis of $\Gamma(\tau_{\widetilde{A}})$ adapted to 1_A $(1_A(e_0) = 1, 1_A(e_\alpha) = 0)$

$$[\![e_0, e_\alpha]\!]_{\widetilde{A}} = C^\gamma_{0\alpha} e_\gamma \quad [\![e_\alpha, e_\beta]\!]_{\widetilde{A}} = C^\gamma_{\alpha\beta} e_\gamma$$

$$\rho_{\widetilde{A}}(e_0) = \rho^i_0 \frac{\partial}{\partial x^i} \quad \rho_{\widetilde{A}}(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$$

$$\Downarrow$$

(x^i, y^0, y^α) local coordinates on $\widetilde{A}$

(x^i, y_0, y_α) the dual coordinates on A^+

(x^i, y_α) local coordinates on V^*

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

$(\tau_A : A \to M, \tau_V : V \to M, ([\![\cdot,\cdot]\!]_V, D, \rho_A))$ Lie affgebroid

$(\tau_{\widetilde{A}}^{\tau_V^*} : \mathcal{T}^{\widetilde{A}}V^* \to V^*, [\![\cdot,\cdot]\!]_{\widetilde{A}}^{\tau_V^*}, \rho_{\widetilde{A}}^{\tau_V^*})$

(x^i) local coordinates on M

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$$[\![e_0, e_\alpha]\!]_{\widetilde{A}} = C_{0\alpha}^{\gamma} e_\gamma \quad [\![e_\alpha, e_\beta]\!]_{\widetilde{A}} = C_{\alpha\beta}^{\gamma} e_\gamma$$

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

$\{\tilde{e}_0, \tilde{e}_\alpha, \bar{e}_\alpha\}$ local basis of $\Gamma(\tau_{\widetilde{A}}^{\tau_V^*})$

$$\tilde{e}_0(\psi) = (e_0(\tau_V^*(\psi)), \rho_0^i \frac{\partial}{\partial x^i}_{|\psi})$$

$$\tilde{e}_\alpha(\psi) = (e_\alpha(\tau_V^*(\psi)), \rho_\alpha^i \frac{\partial}{\partial x^i}_{|\psi}) \qquad \bar{e}_\alpha(\psi) = (0, \frac{\partial}{\partial y_\alpha}_{|\psi})$$

$$\Downarrow$$

$(x^i, y_\alpha; z^0, z^\alpha, v_\alpha)$ local coordinates on $\mathcal{T}^{\widetilde{A}} V^*$

- $\mu : A^+ \to V^*$ the canonical projection
  $\mu(\varphi) = \varphi^l$ linear map associated with φ
- $h : V^* \to A^+$ Hamiltonian section of μ
  $h(x^i, y_\alpha) = (x^i, -H(x^j, y_\beta), y_\alpha)$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

- $$\mathcal{T}h : \mathcal{T}^{\widetilde{A}}V^* \to \mathcal{T}^{\widetilde{A}}A^+$$

$$\mathcal{T}h(\tilde{a}, X_\alpha) = (\tilde{a}, (T_\alpha h)(X_\alpha))$$

$$\Downarrow$$

$(\mathcal{T}h, h)$ Lie algebroid morphism

$$\lambda_h = (\mathcal{T}h, h)^*(\lambda_{\widetilde{A}}) \quad \Omega_h = (\mathcal{T}h, h)^*(\Omega_{\widetilde{A}})$$

$\lambda_{\widetilde{A}}$ and $\Omega_{\widetilde{A}}$ are the Liouville section and the canonical symplectic section associated with $\widetilde{A}$

$$\Downarrow$$

$$\lambda_h \in \Gamma((\tau_{\widetilde{A}}^{\mathcal{T}V^*})^*) \quad \Omega_h \in \Gamma(\Lambda^2(\mathcal{T}^{\widetilde{A}}V^*)^*)$$

$$\Omega_h = -d^{\mathcal{T}^{\widetilde{A}}V^*}\lambda_h$$

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The Hamiltonian formalism

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

- $pr_1 : \mathcal{T}^{\widetilde{A}}V^* \to \widetilde{A}$ the canonical projection on the first factor

$$\Downarrow$$

(pr_1, τ_V^*) Lie algebroid morphism

$\eta : \mathcal{T}^{\widetilde{A}}V^* \to \mathbb{R}$ $\quad (pr_1, \tau_V^*)^*(1_A) = \eta$

$\eta(\tilde{a}, X_\alpha) = 1_A(\tilde{a})$ $\quad 1_A$ is a 1-cocycle $\Rightarrow \eta$ is a 1-cocycle

$$\Downarrow$$

(Ω_h, η) is a cosymplectic structure on $\tau_{\widetilde{A}}^{\tau_V^*} : \mathcal{T}^{\widetilde{A}}V^* \to V^*$:

$$\{\eta \wedge \Omega_h \wedge \ldots^{(n} \cdots \wedge \Omega_h\}(\alpha) \neq 0, \quad \text{for all} \quad \alpha \in V^*$$

$$d^{\mathcal{T}^{\widetilde{A}}V^*}\eta = 0 \quad d^{\mathcal{T}^{\widetilde{A}}V^*}\Omega_h = 0$$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

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$$d^{\mathcal{T}^{\widetilde{A}}V^*}\eta = 0 \quad d^{\mathcal{T}^{\widetilde{A}}V^*}\Omega_h = 0$$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

- $R_h \in \Gamma(\tau_{\tilde{A}}^{\mathcal{T}V^*})$ the Reeb section of (Ω_h, η): $i_{R_h}\Omega_h = 0, \;\; i_{R_h}\eta = 1$

$$R_h = \tilde{e}_0 + \frac{\partial H}{\partial y_\alpha}\tilde{e}_\alpha - (C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i} - C^\gamma_{0\alpha} y_\gamma)\bar{e}_\alpha$$

$$\Downarrow$$

the integral sections of R_h (i.e., the integral curves of the vector field $\rho_{\tilde{A}}^{\mathcal{T}V^*}(R_h)$) are just *the solutions of the Hamilton equations* for h

$$\frac{dx^i}{dt} = \rho^i_0 + \frac{\partial H}{\partial y_\alpha}\rho^i_\alpha \qquad \frac{dy_\alpha}{dt} = -\rho^i_\alpha \frac{\partial H}{\partial x^i} + y_\gamma (C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha}\frac{\partial H}{\partial y_\beta})$$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Hamiltonian formalism

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Lagrangian formalism

$(\tau_A : A \to M, \tau_V : V \to M, ([\![\cdot,\cdot]\!]_V, D, \rho_A))$ Lie affgebroid

$(\tau_{\widetilde{A}}^{\tau_A} : \mathcal{T}^{\widetilde{A}}A \to A, [\![\cdot,\cdot]\!]_{\widetilde{A}}^{\tau_A}, \rho_{\widetilde{A}}^{\tau_A})$

(x^i) local coordinates on M

$\{e_0, e_\alpha\}$ local basis of sections of $\tau_{\widetilde{A}}$ adapted to 1_A

$$\Downarrow$$

$\{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\}$ local basis of sections of $\tau_{\widetilde{A}}^{\tau_A}$

$$\tilde{T}_0(a) = (e_0(\tau_A(a)), \rho_0^i \frac{\partial}{\partial x^i}_{|a})$$

$$\tilde{T}_\alpha(a) = (e_\alpha(\tau_A(a)), \rho_\alpha^i \frac{\partial}{\partial x^i}_{|a}) \qquad \tilde{V}_\alpha(a) = (0, \frac{\partial}{\partial y^\alpha}_{|a})$$

$$\Downarrow$$

$(x^i, y^0, y^\alpha, z^\alpha)$ local coordinates on $\mathcal{T}^{\widetilde{A}}A$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Lagrangian formalism

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$(x^i, y^0, y^\alpha, z^\alpha)$ local coordinates on $\mathcal{T}^{\widetilde{A}}A$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Lagrangian formalism

- $\{\tilde{T}^0, \tilde{T}^\alpha, \tilde{V}^\alpha\}$ the dual basis of $\{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\}$

$$\Downarrow$$

$\tilde{T}^0 = \phi_0$ is globally defined and it is a 1-cocycle

• $\gamma : I \subseteq \mathbb{R} \to A$ is *admissible* if $\widehat{(\tau_A \circ \gamma)} = \rho_{\tilde{A}} \circ i_A \circ \gamma$

or locally if $\gamma(t) = (x^i(t), y^\alpha(t))$ and $\dfrac{dx^i}{dt} = \rho_0^i + \rho_\alpha^i y^\alpha$

• $\xi \in \Gamma(\tau_{\tilde{A}}^{\tau_A})$ is a *second order differential equation* (SODE) on A if the integral sections of ξ, that is, the integral curves of the vector field $\rho_{\tilde{A}}^{\tau_A}(\xi)$, are admissible.

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Lagrangian formalism

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Lagrangian formalism

- $L : A \to \mathbb{R}$ Lagrangian function

the Poincaré-Cartan 1-section and 2-section

$$\Theta_L = L\phi_0 + (d^{\mathcal{T}^{\widetilde{A}}A}L) \circ S \in \Gamma((\tau_{\widetilde{A}}^{\mathcal{T}A})^*)$$
$$\Omega_L = -d^{\mathcal{T}^{\widetilde{A}}A}\Theta_L \in \Gamma(\wedge^2(\tau_{\widetilde{A}}^{\mathcal{T}A})^*)$$

the vertical endomorphism $S : A \to \mathcal{T}^{\widetilde{A}}A \otimes (\mathcal{T}^{\widetilde{A}}A)^*$

$$S = (\tilde{T}^\alpha - y^\alpha \tilde{T}^0) \otimes \tilde{V}_\alpha$$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Lagrangian formalism

- $\gamma : I \subseteq \mathbb{R} \to A$ is *a solution of the Euler-Lagrange equations* iff
  i) γ is admissible
  ii) $i_{(i_A(\gamma(t)),\dot{\gamma}(t))}\Omega_L(\gamma(t)) = 0$

or locally $\gamma(t) = (x^i(t), y^\alpha(t))$ and

$$\frac{dx^i}{dt} = \rho^i_0 + \rho^i_\alpha y^\alpha \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \frac{\partial L}{\partial x^i} + (C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} y^\beta)\frac{\partial L}{\partial y^\gamma}$$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Lagrangian formalism

• L is *regular* iff the matrix $(W_{\alpha\beta}) = (\dfrac{\partial^2 L}{\partial y^\alpha \partial y^\beta})$ is regular or, equivalently, (Ω_L, ϕ_0) is a cosymplectic structure on $\mathcal{T}^{\tilde{A}}A$

• If L is regular

$\Downarrow$

the Reeb section of (Ω_L, ϕ_0), R_L, is the unique Lagrangian SODE associated with L

$\Downarrow$

the integral curves of the vector field $\rho^{\mathcal{T}A}_{\tilde{A}}(R_L)$ are solutions of the Euler-Lagrange equations associated with L

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Legendre transformation and the equivalence between these formalisms

$L : A \to \mathbb{R}$ Lagrangian function
$\Theta_L \in \Gamma((\tau^{\mathcal{T}A}_{\tilde{A}})^*)$ the Poincaré-Cartan 1-section

the extended Legendre transformation

$$Leg_L : A \to A^+ \quad Leg_L(a)(b) = \Theta_L(a)(z)$$

$$a, b \in A_x, z \in (\mathcal{T}^{\tilde{A}}A)_a : pr_1(z) = i_A(b)$$

the Legendre transformation

$$leg_L : A \to V^* \quad leg_L = \mu \circ Leg_L$$

$$\Downarrow$$

$$\mathcal{T}leg_L : \mathcal{T}^{\tilde{A}}A \to \mathcal{T}^{\tilde{A}}V^* \qquad (\mathcal{T}leg_L)(\tilde{b}, X_a) = (\tilde{b}, (T_a leg_L)(X_a))$$

Hamiltonian and Lagrangian formalism on Lie affgebroids
The Legendre transformation and the equivalence between these formalisms

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Hamiltonian and Lagrangian formalism on Lie affgebroids
The Legendre transformation and the equivalence between these formalisms

Proposition

The Lagrangian L is regular if and only if the Legendre transformation $leg_L : A \to V^*$ is a local diffeomorphism.

- L is *hyperregular* if leg_L is a global diffeomorphism

- If L is hyperregular

$$\Downarrow$$

$(\mathcal{T}leg_L, leg_L)$ is a Lie algebroid isomorphism

$$\Downarrow$$

$$h : V^* \to A^+ \qquad h = Leg_L \circ leg_L^{-1}$$

Hamiltonian and Lagrangian formalism on Lie affgebroids
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Hamiltonian and Lagrangian formalism on Lie affgebroids
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Theorem

If the Lagrangian L is hyperregular then the Euler-Lagrange section R_L associated with L and the Hamiltonian section R_h associated with h satisfy the following relation

$$R_h \circ leg_L = \mathcal{T} leg_L \circ R_L.$$

Moreover, if $\gamma : I \to A$ is a solution of the Euler-Lagrange equations associated with L, then $leg_L \circ \gamma : I \to V^*$ is a solution of the Hamilton equations associated with h and, conversely, if $\bar{\gamma} : I \to V^*$ is a solution of the Hamilton equations for h then $\gamma = leg_L^{-1} \circ \bar{\gamma}$ is a solution of the Euler-Lagrange equations for L.

The prolongation of a symplectic Lie affgebroid

Definition

Let be a Lie affgebroid $\tau_A : A \to M$ modelled on the Lie algebroid $\tau_V : V \to M$. It is said to be a *symplectic Lie affgebroid* if $\tau_V : V \to M$ admits a symplectic section Ω, that is, Ω is a section of the vector bundle $\wedge^2 V^* \to M$ such that:

i) For all $x \in M$, the 2-form $\Omega(x) : V_x \times V_x \to \mathbb{R}$ on the vector space V_x is non-degenerate and

ii) Ω is a 2-cocycle, i.e., $d^V \Omega = 0$.

The prolongation of a symplectic Lie affgebroid

Example

$\tau_A : A \to M$ Lie affgebroid modelled $\tau_V : V \to M$

$\eta : \mathcal{T}^{\tilde{A}} V^* \to \mathbb{R}$, $\eta(\tilde{a}, X_\alpha) = 1_A(\tilde{a})$, η is 1-cocycle

$$\eta^{-1}\{1\} = \rho_A^*(TV^*) \qquad \eta^{-1}\{0\} = \mathcal{T}^V V^*$$

$$\Downarrow$$

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$\widetilde{\pi_{V^*}} : \rho_A^*(TV^*) \to V^*$ is symplectic Lie affgebroid

The prolongation of a symplectic Lie affgebroid

Example

$\tau_A : A \to M$ Lie affgebroid modelled $\tau_V : V \to M$

$\eta : \mathcal{T}^{\widetilde{A}} V^* \to \mathbb{R}$, $\eta(\tilde{a}, X_\alpha) = 1_A(\tilde{a})$, η is 1-cocycle

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The prolongation of a symplectic Lie affgebroid

$(\tau_A : A \to M, \tau_V : V \to M, ([\![\cdot,\cdot]\!]_V, D, \rho_A))$ Lie affgebroid

- $f \in C^\infty(M)$ *the complete and vertical lift f^c and f^v of f to A*

$$f^c(a) = \rho_A(a)(f) \qquad f^v(a) = f(\tau_A(a)) \quad a \in A$$

- $\tilde{X} \in \Gamma(\tau_{\tilde{A}}) \Rightarrow \tilde{X}^c, \tilde{X}^v \in \mathfrak{X}(\tilde{A}) \Rightarrow \tilde{X}^{\mathbf{c}}, \tilde{X}^{\mathbf{v}} \in \Gamma(\tau_{\tilde{A}}^{\tau_{\tilde{A}}})$:

$$\tilde{X}^{\mathbf{c}}(\tilde{a}) = (\tilde{X}(\tau_{\tilde{A}}(\tilde{a})), \tilde{X}^c(\tilde{a})) \qquad \tilde{X}^{\mathbf{v}}(\tilde{a}) = (0(\tau_{\tilde{A}}(\tilde{a})), \tilde{X}^v(\tilde{a}))$$

- $X \in \Gamma(\tau_V) \Rightarrow i_V \circ X \in \Gamma(\tau_{\tilde{A}}) \Rightarrow (i_V \circ X)^c_{|A} \in \mathfrak{X}(A)$

$$(i_V \circ X)^v_{|A} \in \mathfrak{X}(A)$$

$\Rightarrow$ *the complete and vertical lift of X*

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The prolongation of a symplectic Lie affgebroid

Proposition

If α is a section of the vector bundle $\wedge^k V^* \to M$, then there exists a unique section $\alpha^{\mathbf{c}}$ of the vector bundle $\wedge^k (\mathcal{T}^V A)^* \to A$ such that

$$\begin{aligned} \alpha^{\mathbf{c}}(X_1^{\mathbf{c}}, \ldots, X_k^{\mathbf{c}}) &= \alpha(X_1, \ldots, X_k)^c \\ \alpha^{\mathbf{c}}(X_1^{\mathbf{v}}, X_2^{\mathbf{c}}, \ldots, X_k^{\mathbf{c}}) &= \alpha(X_1, X_2, \ldots, X_k)^v \\ \alpha^{\mathbf{c}}(X_1^{\mathbf{v}}, \ldots, X_s^{\mathbf{v}}, X_{s+1}^{\mathbf{c}}, \ldots, X_k^{\mathbf{c}}) &= 0 \quad \text{if} \quad 2 \leq s \leq k \end{aligned}$$

for $X_1, \ldots, X_k \in \Gamma(\tau_V)$. Moreover, $d^{\mathcal{T}^V A}\alpha^{\mathbf{c}} = (d^V \alpha)^{\mathbf{c}}$.

The section $\alpha^{\mathbf{c}}$ of the vector bundle $\wedge^k (\mathcal{T}^V A)^* \to A$ is called *the complete lift of* α

The prolongation of a symplectic Lie affgebroid

Theorem

Let $\tau_A : A \to M$ be a symplectic Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to M$ and Ω be a symplectic section of $\tau_V : V \to M$. Then, the prolongation $\mathcal{T}^A A$ of the Lie affgebroid A over the projection $\tau_A : A \to M$ is a symplectic Lie affgebroid and the complete lift $\Omega^{\mathbf{c}}$ of Ω to the prolongation $\mathcal{T}^V A$ is a symplectic section of $\tau_V^{\tau_A} : \mathcal{T}^V A \to A$.

Lagrangian submanifolds and dynamics on a Lie affgebroid

Definition

Let S be a submanifold of the symplectic Lie affgebroid A and $i : S \to A$ be the canonical inclusion. Denote by $\tau_A^S : S \to M$ the map given by $\tau_A^S = \tau_A \circ i$ and suppose that $\rho_V(V_{\tau_A^S(a)}) + (T_a\tau_A^S)(T_aS) = T_{\tau_A^S(a)}M$, for all $a \in S$. Then, S is said to be *Lagrangian submanifold* if the corresponding Lie subaffgebroid $(\widetilde{\pi_S} : \rho_A^*(TS) \to S, \tau_V^{\tau_A^S} : \mathcal{T}^V S \to S)$ of the symplectic Lie affgebroid $(\tau_A^{\tau_A} : \mathcal{T}^A A \to A, \tau_V^{\tau_A} : \mathcal{T}^V A \to A)$ is Lagrangian*.

A *Lie subaffgebroid* of A is a Lie affgebroid morphism $((j : A' \mapsto A, i : M' \mapsto M),$ $(j^l : V' \mapsto V, i : M' \mapsto M))$: j is injective and i is an injective inmersion

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- $(\tau_A : A \to M, \tau_V : V \to M)$ Lie affgebroid

  $h : V^* \to A^+$ Hamiltonian section

  (Ω_h, η) cosymplectic structure on $\mathcal{T}^{\widetilde{A}} V^*$

  $R_h \in \Gamma(\tau^{\tau^*_V}_{\widetilde{A}})$ the Reeb section

  - $\eta(R_h) = 1 \Rightarrow R_h(V^*) \subseteq \rho^*_A(TV^*)$

Theorem

$S_h = R_h(V^*)$ Lagrangian submanifold of $\rho^*_A(TV^*)$

Ψ_h: {curves in V^*} $\longleftrightarrow$ { curves in S_h}

$c : I \to V^* \quad \longmapsto \quad R_h \circ c : I \to S_h$

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Lagrangian submanifolds and dynamics on a Lie affgebroid

- $\gamma : I \to S_h \subseteq \rho_A^*(TV^*) \subseteq \mathcal{T}^{\widetilde{A}}V^* \subseteq \widetilde{A} \times TV^*$
  $t \mapsto (\gamma_1(t), \gamma_2(t))$

is *admissible* if $\gamma_2 : I \to TV^*, \quad \gamma_2(t) = \dot{c}(t)$
$c : I \to V^*, c = \pi_{V^*} \circ \gamma_2$
$\pi_{V^*} : TV^* \to V^*$ the canonical projection

Theorem

Under the bijection Ψ_h, the admissible curves in the Lagrangian submanifold S_h correspond with the solutions of the Hamilton equations for h.

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- $L : A \to \mathbb{R}$ Lagrangian function

  $A_A : \rho_A^*(TV^*) \to (\mathcal{T}^V A)^*$ the canonical isomorphism between $pr_{1|\rho_A^*(TV^*)} : \rho_A^*(TV^*) \to A$ and $(\tau_V^{\mathcal{T}A})^* : (\mathcal{T}^V A)^* \to A$

Theorem

$S_L = (A_A^{-1} \circ d^{\mathcal{T}^V A} L)(A)$ *Lagrangian submanifold of* $\rho_A^*(TV^*)$

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  $\Theta_L, \quad \Omega_L = -d^{\mathcal{T}^{\tilde{A}}A}\Theta_L$ the Poincaré-Cartan sections

$$(i_V, Id) : \mathcal{T}^V A \to \mathcal{T}^{\tilde{A}}A$$
$$(i_V, Id)(v, X_a) = (i_V(v), X_a)$$
$$\Downarrow$$

(i_V, Id) is a Lie algebroid morphism over the identity of A

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(i_V, Id) is a Lie algebroid morphism over the identity of A

(Ω_L, ϕ_0) cosymplectic structure on $\mathcal{T}^{\tilde{A}}A$: $(i_V, Id)^*\phi_0 = 0$

$$\Downarrow$$

$(i_V, Id)^*\Omega_L$ is a symplectic section of $\tau_V^{\mathcal{T}A} : \mathcal{T}^V A \to A$

$$\Downarrow$$

$(\tau_A^{\mathcal{T}A} : \mathcal{T}^A A \to A, \tau_V^{\mathcal{T}A} : \mathcal{T}^V A \to A)$ is a symplectic Lie affgebroid

Lagrangian submanifolds and dynamics on a Lie affgebroid

- R_L the Reeb section of (Ω_L, ϕ_0): $\phi_0(R_L) = 1$

$$\Downarrow$$

$$R_L \in \Gamma(\tau_A^{\mathcal{T}A})$$

Theorem

$S_{R_L} = R_L(A)$ Lagrangian submanifold of $\mathcal{T}^A A$

$$\Psi_{S_{R_L}} : \{\text{ curves in } S_{R_L}\} \longleftrightarrow \{\text{ curves in } A\}$$

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Lagrangian submanifolds and dynamics on a Lie affgebroid

- $\gamma : I \to S_{R_L} \subseteq \mathcal{T}^A A \subseteq A \times TA, \quad t \mapsto (\gamma_1(t), \gamma_2(t))$

is *admissible* if $\gamma_2 : I \to TA$, $\gamma_2(t) = \dot{c}(t)$

$c = \pi_A \circ \gamma_2 : I \to A$

Theorem

If the Lagrangian L is hyperregular then under the bijection $\Psi_{S_{R_L}}$ the admissible curves in the Lagrangian submanifold S_{R_L} correspond with the solutions of the Euler-Lagrange equations for L.

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Lagrangian submanifolds and dynamics on a Lie affgebroid

$L : A \to \mathbb{R}$ hyperregular $\Rightarrow leg_L : A \to V^*$ global diffeom.

$h : V^* \to A^+$ Hamiltonian section $\quad h = Leg_L \circ leg_L^{-1}$

- The Lagrangian submanifolds S_L and S_h of the symplectic Lie affgebroid $\rho_A^*(TV^*)$
- The Lagrangian submanifold S_{R_L} of the symplectic Lie affgebroid $\mathcal{T}^A A$

Theorem

If the Lagrangian function $L : A \to \mathbb{R}$ is hyperregular and $h : V^* \to A^+$ is the corresponding Hamiltonian section then the Lagrangian submanifolds S_L and S_h are equal and

$$\mathcal{T} leg_L(S_{R_L}) = S_L = S_h$$