Integrability of Lie Algebroids: Theory and Applications

Rui Loja Fernandes
IST-Lisbon

August, 2005

Lie Algebroids
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Applications of . . .

Main Reference:

M. Crainic and R.L. Fernandes, *Lectures on Integrability of Lie brackets*

soon available on the web page:

http://www.math.ist.utl.pt/∼rfern/

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Plan of the Talk:

1. Lie algebroids
2. Lie groupoids
3. Integrability
4. Applications of integrability

1. Lie Algebroids

Lie algebroids are *geometric* vector bundles:

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1. Lie Algebroids

Lie algebroids are *geometric* vector bundles:

Definition. A **Lie algebroid** over a smooth manifold M is a vector bundle $\pi : A \to M$ with:

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Lemma. *The anchor* $\# : \Gamma(A) \to \mathfrak{X}^1(M)$ *is a Lie algebra homomorphism.*

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Lemma. *The anchor* $\# : \Gamma(A) \to \mathfrak{X}^1(M)$ *is a Lie algebra homomorphism.*

Definition. A **morphism of Lie algebroids** is a bundle map $\phi : A_1 \to A_2$ which preserves anchors and brackets.

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Basic Properties

The kernel and the image of the anchor give basic objects associated with any Lie algebroid:

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- The **isotropy Lie algebra** at $x \in M$:

$$\mathfrak{g}_x \equiv \operatorname{Ker} \#_x .$$

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- The **characteristic foliation** $\mathcal{F}$, which is the singular foliation of M determined by:

$$x \mapsto \mathcal{D}_x \equiv \operatorname{Im} \#_x .$$

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Restricting to a leaf L of $\mathcal{F}$ we have the **short exact sequence of** L:

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \stackrel{\#}{\longrightarrow} TL \longrightarrow 0$$

where $\mathfrak{g}_L = \bigcup_{x \in L} \mathfrak{g}_x$.

EXAMPLES	A
Ordinary Geometry (M a manifold)	$TM \to M$
Lie Theory ($\mathfrak{g}$ a Lie algebra)	$\mathfrak{g} \to \{*\}$
Foliation Theory ($\mathcal{F}$ a regular foliation)	$T\mathcal{F} \to M$
Equivariant Geometry ($\rho : \mathfrak{g} \to \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g} \to M$
Presymplectic Geometry (M presymplectic)	$TM \times \mathbb{R} \to M$
Poisson Geometry (M Poisson)	$T^*M \to M$

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A-Cartan Calculus

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A-Cartan Calculus

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- **A-differential:** $d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$

$$d_A Q(\alpha_0, \ldots, \alpha_r) \equiv \sum_{k=0}^{r+1} (-1)^k \# \alpha_k (Q(\alpha_0, \ldots, \widehat{\alpha}_k, \ldots, \alpha_r))$$

$$+ \sum_{k<l} (-1)^{k+l+1} Q([\alpha_k, \alpha_l], \alpha_0, \ldots, \widehat{\alpha}_k, \ldots, \widehat{\alpha}_l, \ldots, \alpha_r).$$

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- **Lie algebroid cohomology:** $H^\bullet(A) \equiv \frac{\operatorname{Ker} d_A}{\operatorname{Im} d_A}$
  (in general, it is very hard to compute...)

EXAMPLES	A	$H^\bullet(A)$
Ordinary Geometry (M a manifold)	$TM \downarrow M$	de Rham cohomology
Lie Theory ($\mathfrak{g}$ a Lie algebra)	$\mathfrak{g} \downarrow \{*\}$	Lie algebra cohomology
Foliation Theory ($\mathcal{F}$ a regular foliation)	$T\mathcal{F} \downarrow M$	foliated cohomology
Equivariant Geometry ($\rho : \mathfrak{g} \to \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g} \downarrow M$	invariant cohomology
Poisson Geometry (M Poisson)	$T^*M \downarrow M$	Poisson cohomology

2. Groupoids

Definition. A **groupoid** is a small category where every arrow is invertible.

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Definition. A **groupoid** is a small category where every arrow is invertible.

$$\mathcal{G} \equiv \{\text{arrows}\} \qquad M \equiv \{\text{objects}\}.$$

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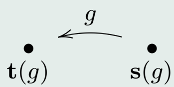
- **source** and **target** maps:



$$\mathcal{G} \underset{\mathbf{s}}{\overset{\mathbf{t}}{\rightrightarrows}} M$$

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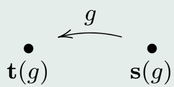

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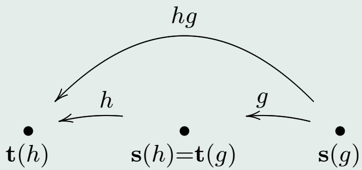

$$\mathcal{G}^{(2)} = \{(h, g) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(h) = \mathbf{t}(g)\}$$

$$m : \mathcal{G}^{(2)} \to \mathcal{G}$$

$$R_g : \mathbf{s}^{-1}(\mathbf{t}(g)) \to \mathbf{s}^{-1}(\mathbf{s}(g))$$

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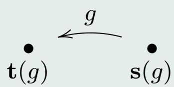

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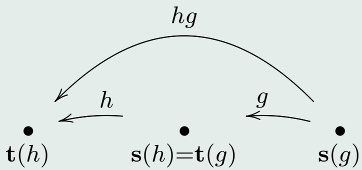

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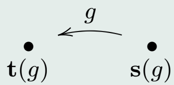

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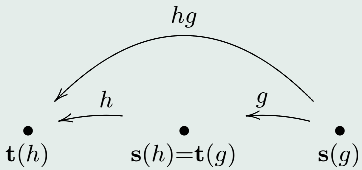

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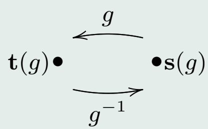
- **identity**: $\epsilon : M \hookrightarrow \mathcal{G}$

- **inverse**: $\iota : \mathcal{G} \longrightarrow \mathcal{G}$



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Groupoids

For any groupoid $\mathcal{G} \underset{\mathbf{s}}{\overset{\mathbf{t}}{\rightrightarrows}} M$ we have:

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Groupoids

For any groupoid $\mathcal{G} \underset{\mathbf{s}}{\overset{\mathbf{t}}{\rightrightarrows}} M$ we have:

- The **isotropy group** at $x \in M$:

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Definition. A **Lie groupoid** is a groupoid where everything is C^∞ and $\mathbf{s}, \mathbf{t} : \mathcal{G} \to M$ are submersions.

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Definition. A **Lie groupoid** is a groupoid where everything is C^∞ and $\mathbf{s}, \mathbf{t} : \mathcal{G} \to M$ are submersions.

Caution: $\mathcal{G}$ may not be Hausdorff, but all other manifolds (M, $\mathbf{s}$ and $\mathbf{t}$-fibers, ...) are.

Lie Groupoids

Proposition. *Every Lie groupoid* $\mathcal{G} \overset{\mathbf{t}}{\underset{\mathbf{s}}{\rightrightarrows}} M$ *determines a Lie algebroid* $\pi : A \longrightarrow M$*, such that:*

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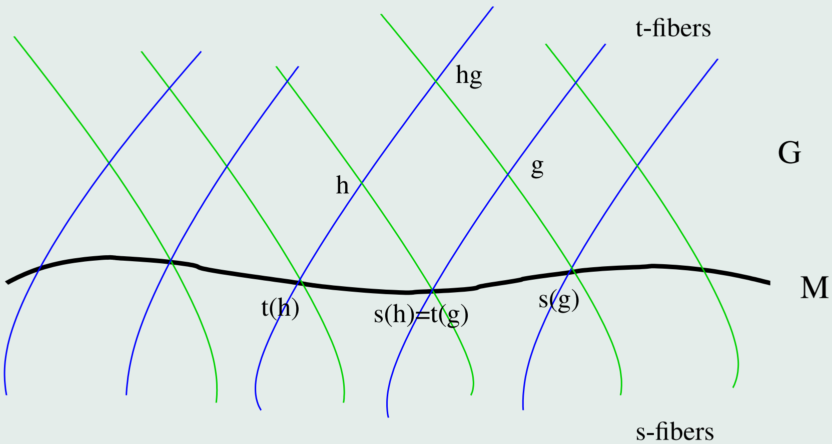

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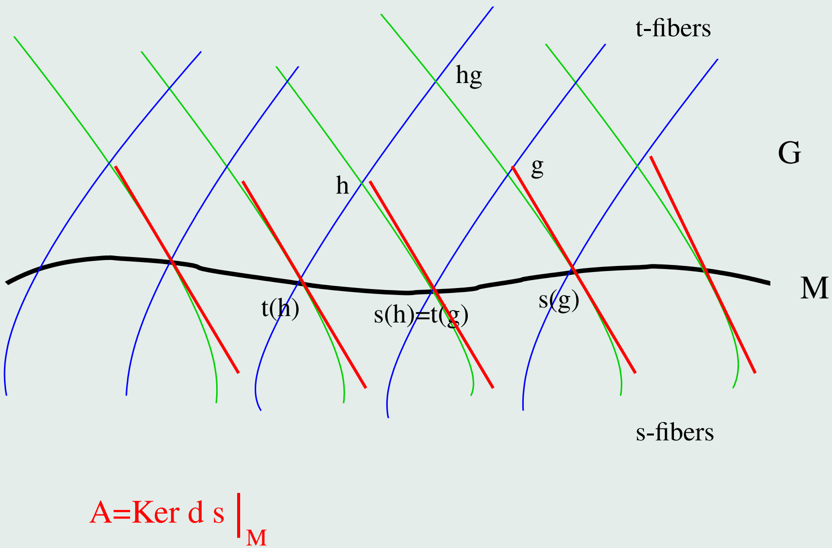

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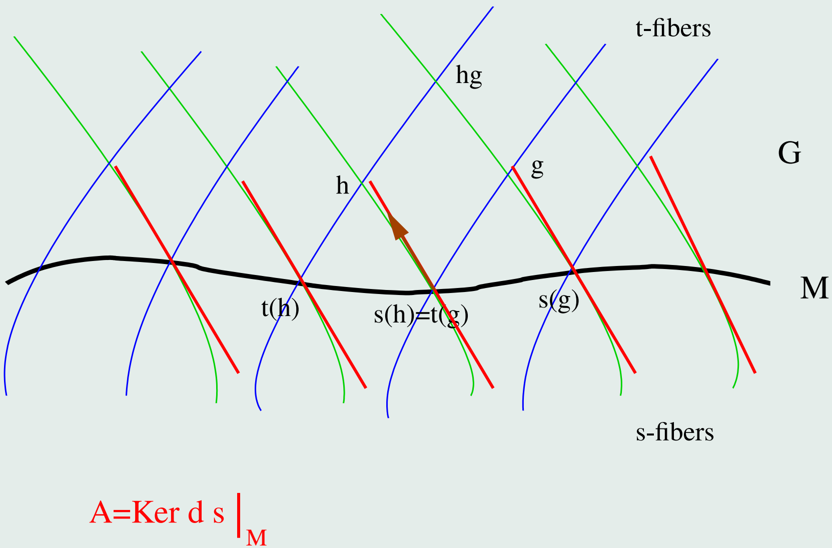

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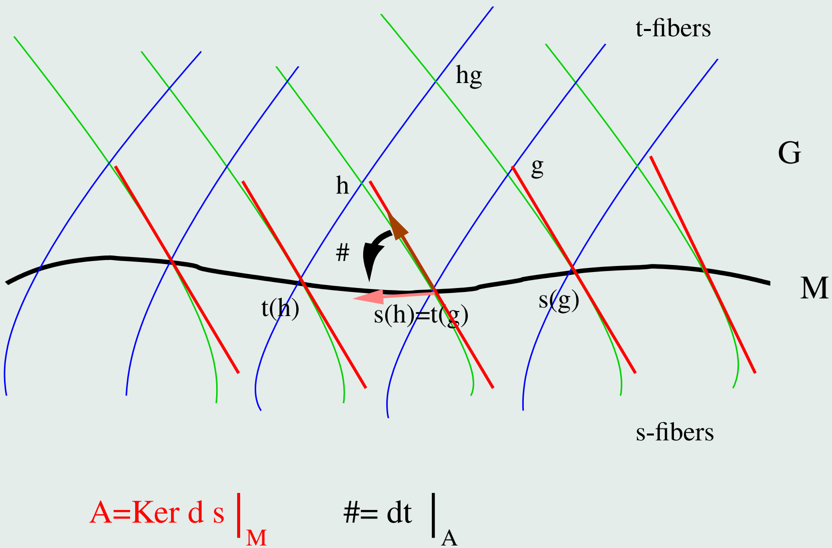

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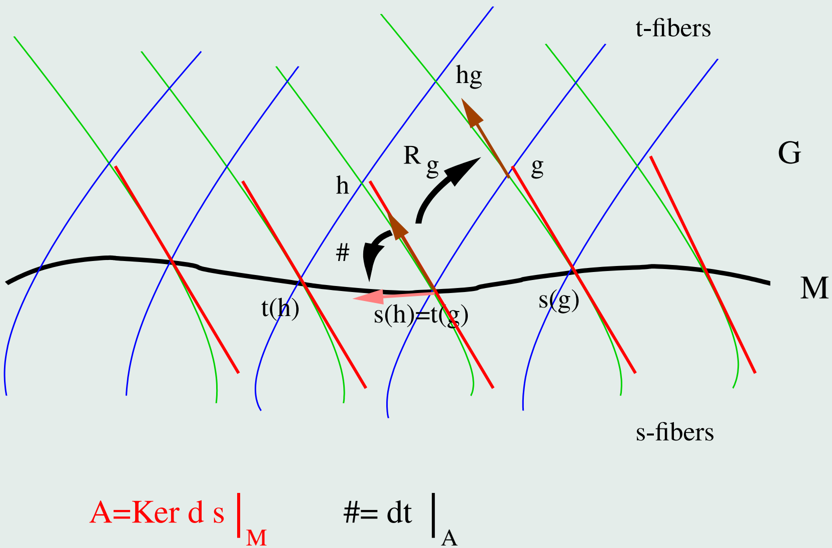

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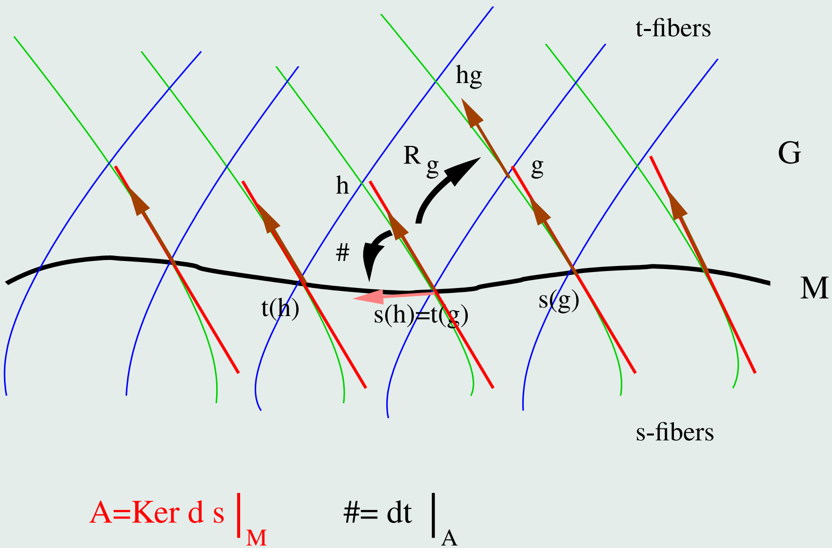

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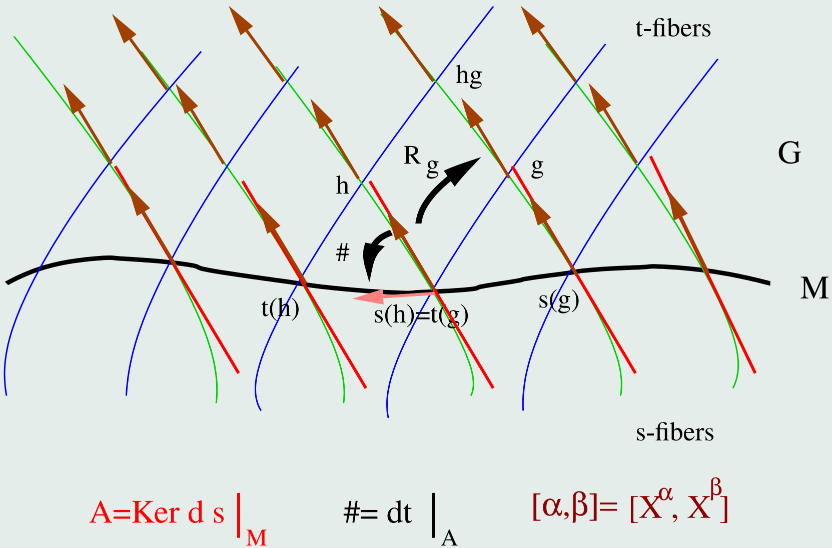

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Lie Theory ($\mathfrak{g}$ a Lie algebra)	$\mathfrak{g} \to \{*\}$	Lie algebra cohomology	$G \rightrightarrows \{*\}$
Foliation Theory ($\mathcal{F}$ a regular foliation)	$T\mathcal{F} \to M$	foliated cohomology	$\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ $\Pi_1(\mathcal{F}) \rightrightarrows M$
Equivariant Geometry ($\rho : \mathfrak{g} \to \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g} \to M$	invariant cohomology	$G \times M \rightrightarrows M$
Poisson Geometry (M Poisson)	$T^*M \to M$	Poisson cohomology	???

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3. Integrability

Problem. *Given a Lie algebroid A is there always a Lie groupoid $\mathcal{G}$ whose associated algebroid is A?*

We will see that the answer is no and we will see why not.

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Uniqueness of integration

Proposition. *For every Lie groupoid* $\mathcal{G}$ *there exists a unique source simply-connected Lie groupoid* $\tilde{\mathcal{G}}$ *with the same associated Lie algebroid.*

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- The product $g \cdot g'$ is defined if $\mathbf{t}(g'(1)) = \mathbf{s}(g(0))$. It is given by:

$$g \cdot g'(t) = \begin{cases} g'(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1)g'(1), & \frac{1}{2} < t \leq 1. \end{cases}$$

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The quotient gives the **monodromy groupoid**:

$$\tilde{\mathcal{G}} \equiv P(\mathcal{G})/\sim \rightrightarrows M$$

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A-paths

Lemma. *The map* $D^R : P(\mathcal{G}) \to P(A)$ *defined by*

$$(D^R g)(t) \equiv \frac{d}{ds} g(s) g^{-1}(t) \Big|_{s=t}$$

is a homeomorphism onto

$$P(A) \equiv \left\{ a : I \to A | \ \frac{d}{dt}\pi(a(t)) = \#a(t) \right\} \qquad (\boldsymbol{A\text{-paths}}).$$

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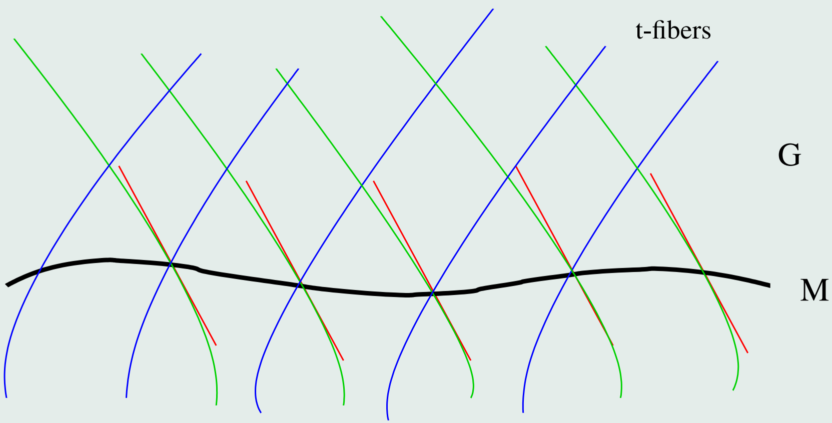

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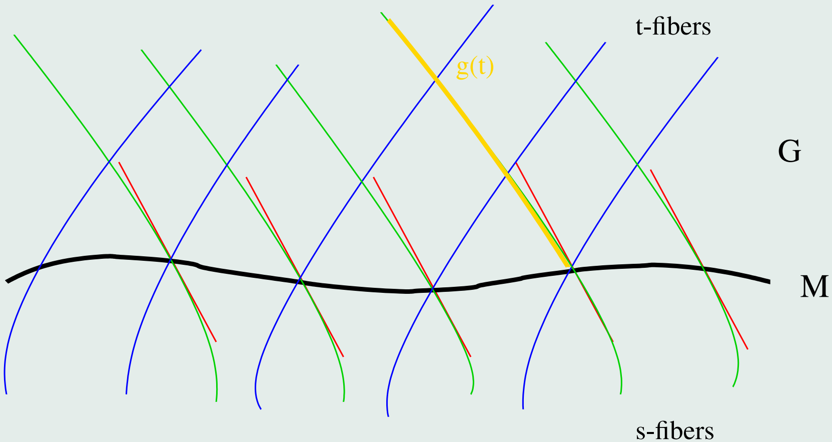

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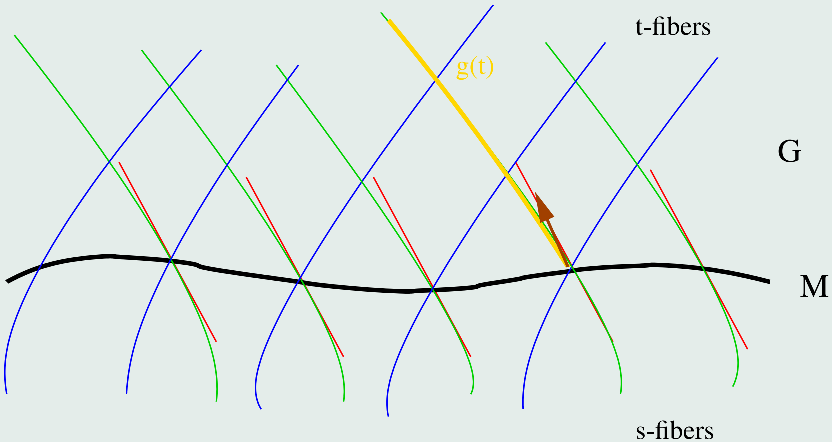

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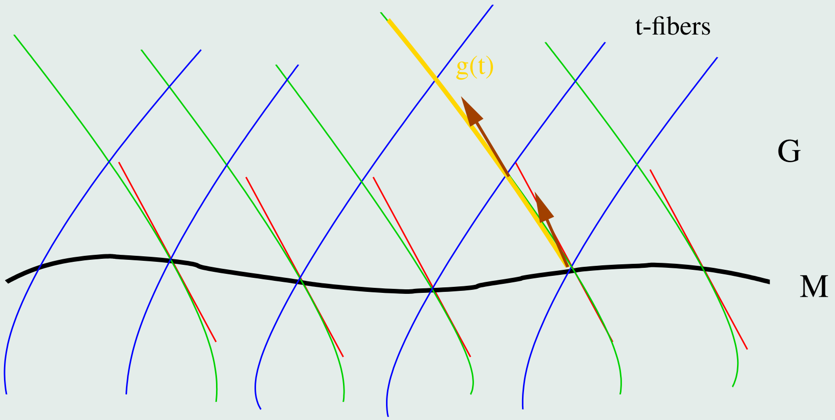

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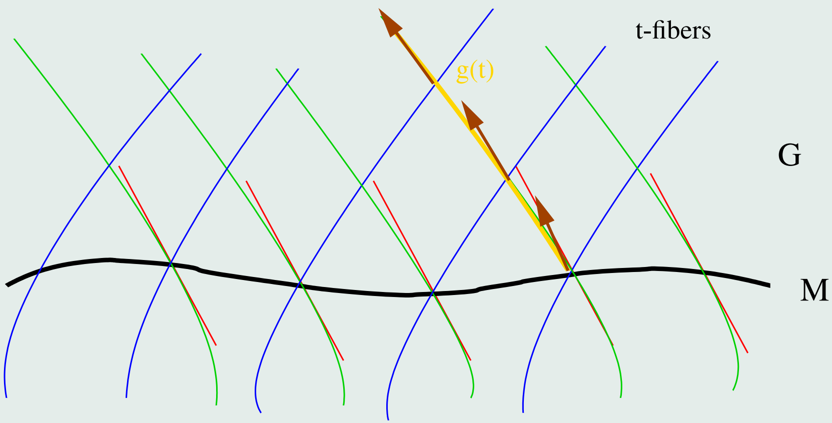

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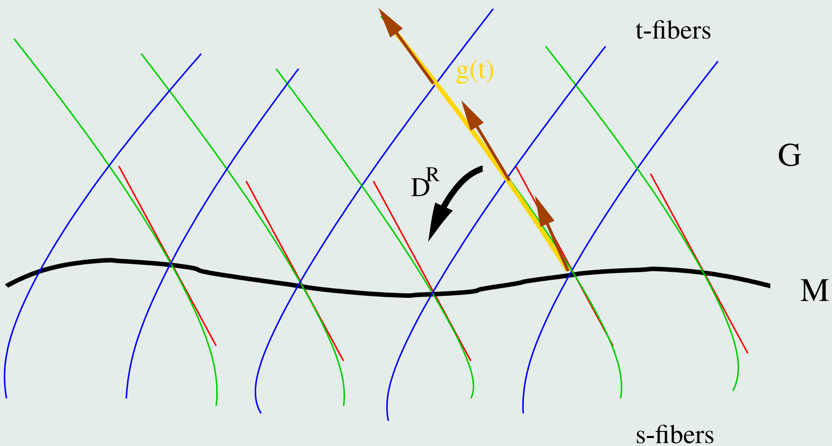

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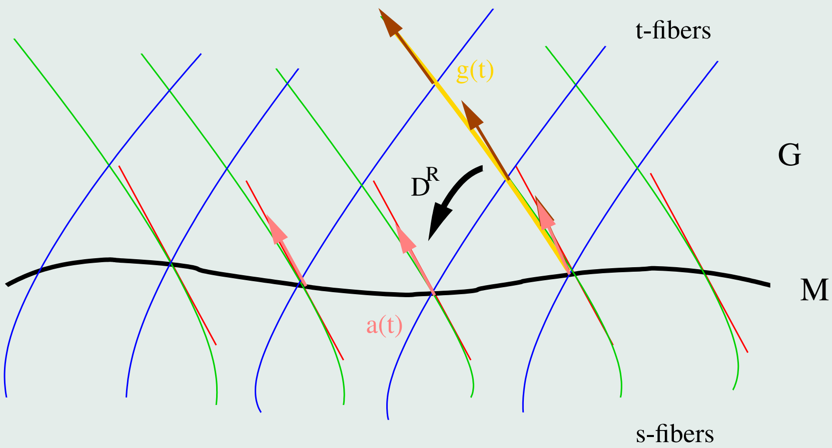

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A-Homotopy

Can transport "$\sim$" and "$\cdot$" to $P(A)$:

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$$a \cdot a'(t) = \begin{cases} 2a'(2t), & 0 \le t \le \frac{1}{2} \\ 2a(2t-1), & \frac{1}{2} < t \le 1. \end{cases}$$

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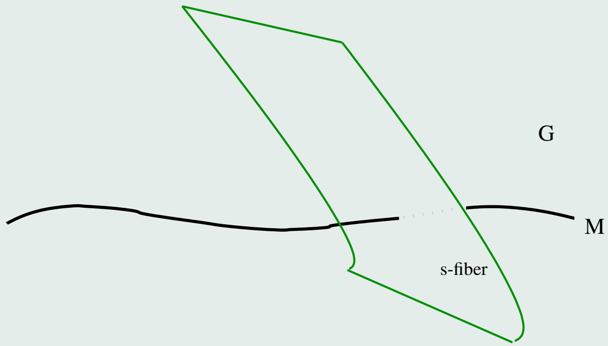

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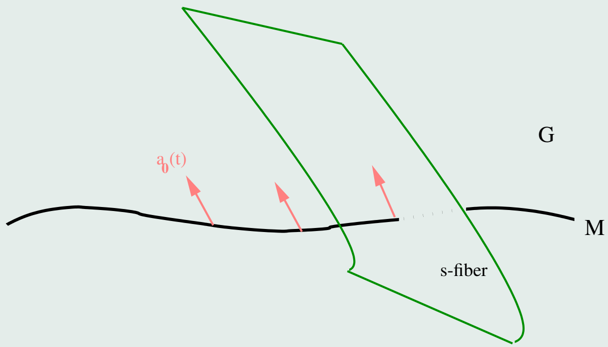

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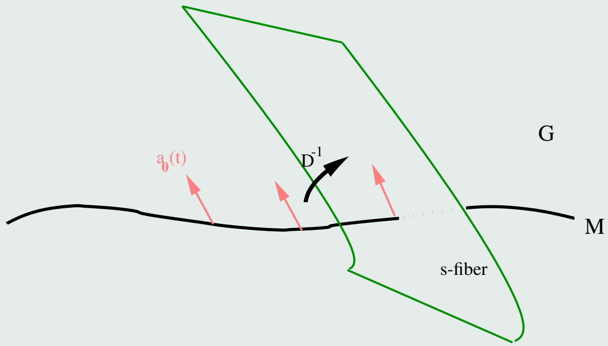

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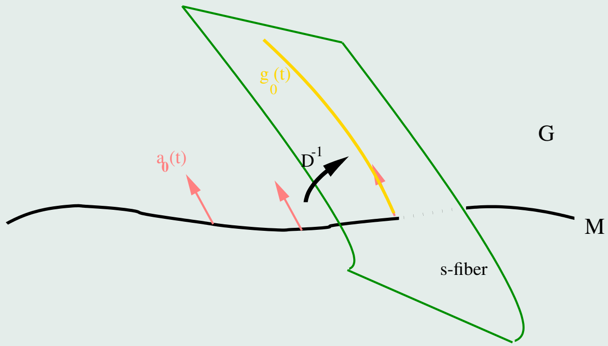

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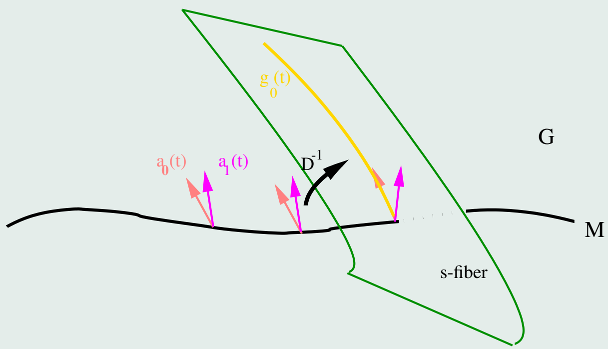

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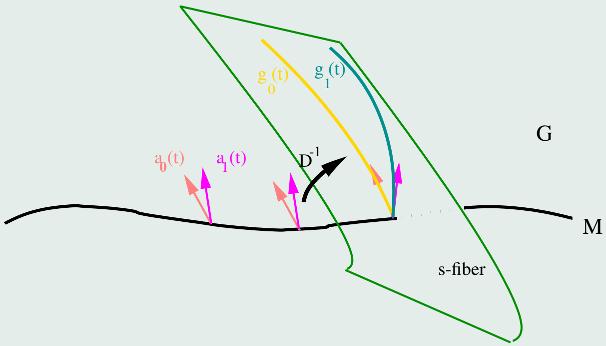

The Universal Groupoid

Observe that:

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The Universal Groupoid

Observe that:

- An A-path is a Lie algebroid map $TI \to A$;

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The Universal Groupoid

Observe that:

- An A-path is a Lie algebroid map $TI \longrightarrow A$;
- An A-homotopy is a Lie algebroid map $T(I \times I) \longrightarrow A$;

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For *any* Lie algebroid A, define a groupoid:

$$\mathcal{G}(A) = P(A)/\sim \quad \text{where} \quad \left|\begin{array}{l} \mathbf{s} : \mathcal{G}(A) \longrightarrow M, \quad [a] \mapsto \pi(a(0)) \\ \\ \mathbf{t} : \mathcal{G}(A) \longrightarrow M, \quad [a] \mapsto \pi(a(1)) \\ \\ M \hookrightarrow \mathcal{G}(A), \quad x \mapsto [0_x] \end{array}\right.$$

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Proposition. *$\mathcal{G}(A)$ is a topological groupoid with source simply-connected fibers.*

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EXAMPLES	A	$H^{\bullet}(A)$	$\mathcal{G}$	$\mathcal{G}(A)$
Ordinary Geometry (M a manifold)	$TM \to M$	de Rham cohomology	$M \times M \rightrightarrows M$	$\Pi_1(M) \rightrightarrows M$
Lie Theory ($\mathfrak{g}$ a Lie algebra)	$\mathfrak{g} \to \{*\}$	Lie algebra cohomology	$G \rightrightarrows \{*\}$	Duistermaat-Kolk construction of G
Foliation Theory ($\mathcal{F}$ a regular foliation)	$T\mathcal{F} \to M$	foliated cohomology	$\mathrm{Hol} \rightrightarrows M$	$\Pi_1(\mathcal{F}) \rightrightarrows M$
Equivariant Geometry ($\rho : \mathfrak{g} \to \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g} \to M$	invariant cohomology	$G \times M \rightrightarrows M$	$\mathcal{G}(\mathfrak{g}) \times M \rightrightarrows M$
Poisson Geometry (M Poisson)	$T^*M \to M$	Poisson cohomology	???	Poisson σ-model (Cattaneo & Felder)

Integrability of Lie Algebroids

A Lie algebroid A is **integrable** if there exists a Lie groupoid $\mathcal{G}$ with A as associated Lie algebroid.

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Lemma. *A is integrable iff $\mathcal{G}(A)$ is a Lie groupoid.*

In general, $\mathcal{G}(A)$ is not smooth: there are obstructions to integrate A.

Obstructions to Integrability

Fix leaf $L \subset M$ and $x \in L$:

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \xrightarrow{\#} TL \longrightarrow 0$$

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$$
\begin{array}{c}
0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \stackrel{\#}{\longrightarrow} TL \longrightarrow 0 \\
\Downarrow \\
\cdots \pi_2(L, x) \stackrel{\partial}{\longrightarrow} \mathcal{G}(\mathfrak{g}_L)_x \longrightarrow \mathcal{G}(A)_x \longrightarrow \pi_1(L, x) \longrightarrow 1
\end{array}
$$

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The **monodromy group** at x is

$$N_x(A) \equiv \operatorname{Im} \partial \subset Z(\mathfrak{g}_L).$$

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$$\mathcal{N}_x(A) \equiv \operatorname{Im} \partial \subset Z(\mathfrak{g}_L).$$

Theorem (Crainic and RLF, 2002). *A Lie algebroid is integrable iff both the following conditions hold:*

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Theorem (Crainic and RLF, 2002)**.** *A Lie algebroid is integrable iff both the following conditions hold:*

(i) Each monodromy group is discrete, and

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Obstructions to Integrability

Fix leaf $L \subset M$ and $x \in L$:

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \xrightarrow{\#} TL \longrightarrow 0$$

$$\Downarrow$$

$$\cdots \pi_2(L,x) \xrightarrow{\partial} \mathcal{G}(\mathfrak{g}_L)_x \longrightarrow \mathcal{G}(A)_x \longrightarrow \pi_1(L,x) \longrightarrow 1$$

The **monodromy group** at x is

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Theorem (Crainic and RLF, 2002)**.** *A Lie algebroid is integrable iff both the following conditions hold:*

(i) Each monodromy group is discrete, and

(ii) The monodromy groups are uniformly discrete.

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Obstructions to Integrability (cont.)

- This theorem allows one to deduce all previous known criteria:

  Lie (1890's), Chevaley (1930's), Van Est (1940's), Palais (1957), Douady & Lazard (1966), Phillips (1980), Almeida & Molino (1985), Mackenzie (1987), Weinstein (1989), Dazord & Hector (1991), Alcade Cuesta & Hector (1995), Debord (2000), Mackenzie & Xu (2000), Nistor (2000).

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- Simple cases:

Corollary. *A Lie algebroid is integrable if, for all leaves* $L \in \mathcal{F}$*, either of the following conditions holds:*

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Obstructions to Integrability (cont.)

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Corollary. *A Lie algebroid is integrable if, for all leaves $L \in \mathcal{F}$, either of the following conditions holds:*

(i) $\pi_2(L)$ is finite (e.g., L is 2-connected);

Obstructions to Integrability (cont.)

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- Simple cases:

Corollary. *A Lie algebroid is integrable if, for all leaves $L \in \mathcal{F}$, either of the following conditions holds:*

(i) $\pi_2(L)$ *is finite (e.g., L is 2-connected);*

(ii) $Z(\mathfrak{g}_L)$ *is trivial (e.g., $\mathfrak{g}_L$ is semi-simple);*

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Computing the Obstructions

In many examples it is possible to compute the monodromy groups:

Computing the Obstructions

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Proposition. *Assume there exists a splitting:*

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \underset{\sigma}{\overset{\#}{\rightleftarrows}} TL \longrightarrow 0$$

with center-valued curvature 2-form

$$\Omega_\sigma(X,Y) = \sigma([X,Y]) - [\sigma(X),\sigma(Y)] \in Z(\mathfrak{g}_L), \qquad \forall X,Y \in \mathfrak{X}(L).$$

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Then:

$$N_x(A) = \left\{ \int_\gamma \Omega : \ [\gamma] \in \pi_2(L,x) \right\}.$$

Example - Presymplectic geometry

Take $A = TM \times \mathbb{R}$ the Lie algebroid of a presymplectic manifold (M, ω):

$$0 \longrightarrow M \times \mathbb{R} \longrightarrow TM \times \mathbb{R} \underset{\sigma}{\overset{\#}{\rightleftarrows}} TM \longrightarrow 0$$

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For the obvious splitting, the curvature is $\Omega_\sigma = \omega$.

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We obtain:

$$N_x = \left\{ \int_\gamma \omega : \ [\gamma] \in \pi_2(L, x) \right\}.$$

Theorem. *$A = TM \times \mathbb{R}$ is integrable iff the group of spherical periods of ω is a discrete subgroup of $\mathbb{R}$.*

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Example: Poisson geometry

Let $(M, \{\ ,\ \})$ be a *regular* Poisson manifold. Fix a symplectic leaf $L \subset M$ and $x \in L$.

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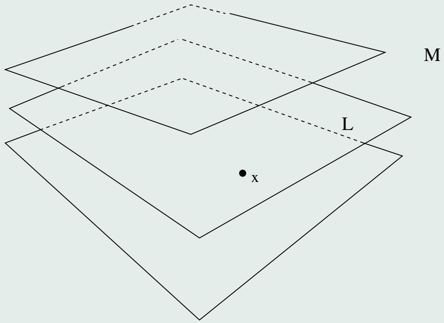

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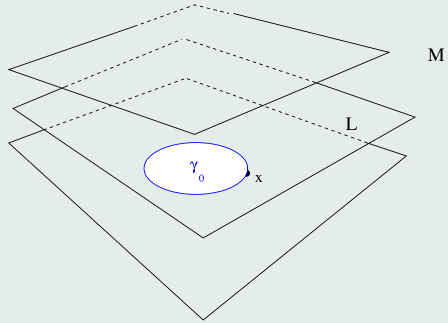

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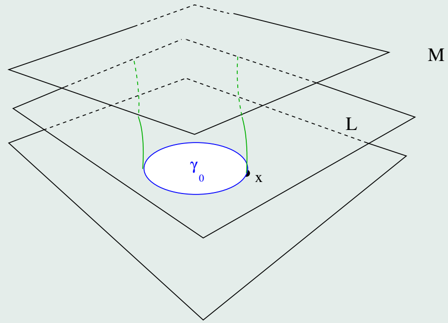

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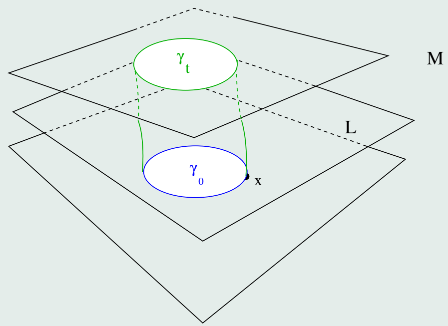

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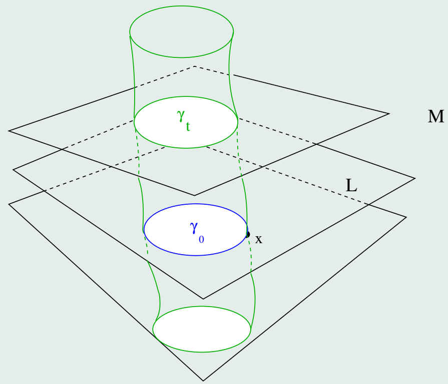

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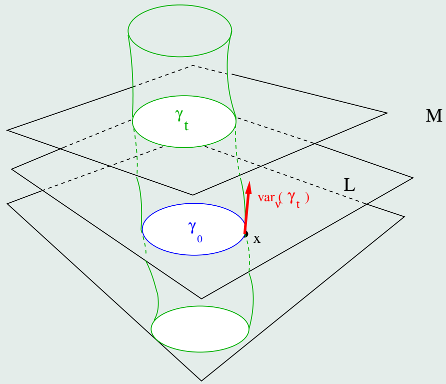

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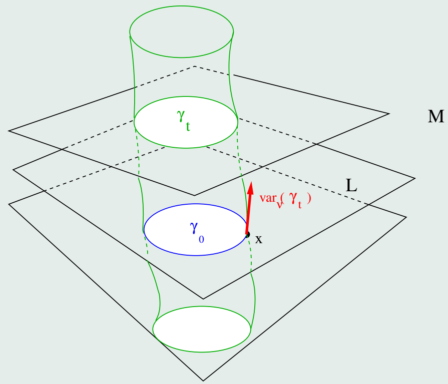

Proposition. *For a foliated family $\gamma_t : \mathbb{S}^2 \to M$, the derivative of the symplectic areas*

$$\frac{d}{dt} A(\gamma_t)\Big|_{x=0},$$

depends only on the class $[\gamma_0] \in \pi_2(L, x)$ and

$$var_\nu(\gamma_t) = [d\gamma_t/dt|_{t=0}] \in \nu(L)_x.$$

Example: Poisson geometry (cont.)

Defining the **variation of symplectic variations** $A'(\gamma_0) \in \nu_x^*(L)$ by

$$\langle A'(\gamma_0), \mathrm{var}_\nu(\gamma_t) \rangle = \frac{d}{dt} A(\gamma_t) \Big|_{t=0}$$

we conclude that:

$$N_x = \{ A'(\gamma) : [\gamma] \in \pi_2(L, x) \} \subset \nu_x^*(L).$$

Example: Poisson geometry (cont.)

- Every two dimensional Poisson manifold is integrable;

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Example: Poisson geometry (cont.)

- Every two dimensional Poisson manifold is integrable;
- A Poisson structure in $M = \mathbb{R}^3 - \{0\}$ with leaves the spheres $x^2+y^2+z^2 =$const. is integrable iff the symplectic areas of the spheres have no critical points.

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- The Reeb foliation of $\mathbb{S}^3$, with the area form on the leaves, is an integrable Poisson manifold.

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- ...

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4. Applications of Integrability

Integrability as many applications:

- Poisson geometry;
- Quantization;
- Cartan's equivalence method;
- . . .

Applications to Poisson geometry: symplectic realizations

Definition. A **symplectic realization** of a Poisson manifold $(M, \{\ ,\ \})$ is a symplectic manifold (S, ω) together with a surjective, Poisson submersion $p : S \to M$.

Applications to Poisson geometry: symplectic realizations

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A **complete symplectic realization** is a symplectic realization for which p is a complete Poisson map.

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Theorem (Karasev, Weinstein (1989))**.** *A Poisson manifold always admits a symplectic realization.*

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Theorem (Karasev, Weinstein (1989)). *A Poisson manifold always admits a symplectic realization.*

Does it admit a *complete* one?

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Applications to Poisson geometry: symplectic realizations

Theorem (Crainic & RLF (2004))**.** *A Poisson manifold admits a complete symplectic realization iff it is integrable.*

Note: One can compute monodromy and decide if it is integrable.

Applications to Poisson geometry: linearization

Let $(M, \{\ ,\ \})$ be a Poisson manifold, such that $\{\ ,\ \}(x_0) = 0$. In local coordinates $(x_1, \ldots, x_m)$ around x_0:

$$\{x_i, x_j\} = c_{ij}^k x_k + O(2).$$

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Definition. $(M, \{\ ,\ \})$ is said to be **linearizable** at x_0 if there exist new coordinates where the higher order terms vanish identically.

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Definition. $(M, \{\ ,\ \})$ is said to be **linearizable** at x_0 if there exist new coordinates where the higher order terms vanish identically.

Linearization problem: When is a Poisson bracket linearizable?

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Applications to Poisson geometry: linearization

Theorem (Conn (1984)). *Assume that the Killing form* $K(X,Y) = c_{il}^{k} c_{kj}^{l} X^i Y^j$ *is negative definite. Then* $\{\ ,\ \}$ *is linearizable.*

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Theorem (Conn (1984)). *Assume that the Killing form* $K(X,Y) = c^k_{il} c^l_{kj} X^i Y^j$ *is negative definite. Then* $\{\ ,\ \}$ *is linearizable.*

- Conn's proof uses hard analysis and no other proof was known.

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- Conn's proof uses hard analysis and no other proof was known.
- A geometric proof can be give using the integrability of Lie algebroids (Crainic & RLF (2004)).

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Applications to quantization:

Let (M, ω) be a simply connected symplectic manifold.

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Applications to quantization:

Let (M, ω) be a simply connected symplectic manifold.

Theorem. $A = TM \times \mathbb{R}$ *is integrable iff*

$$\left\{ \int_\gamma \omega : \gamma \in \pi_2(M) \right\} = r\mathbb{Z} \subset \mathbb{R}, \text{ for some } r \in \mathbb{R}.$$

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This is no accident. . . (Crainic (2005) Cattaneo *et al.* (2005)).

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The Leibniz Identity

For any sections $\alpha, \beta \in \Gamma(A)$ and function $f \in C^\infty(M)$:

$$[\alpha, f\beta] = f[\alpha, \beta] + \#\alpha(f)\beta.$$

The Tangent Lie Algebroid

M - a manifold

- bundle: $A = TM$;
- anchor: $\# : TM \longrightarrow TM$, $\#$ =id;
- Lie bracket $[\ ,\ ] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$, is the usual Lie bracket of vector fields;
- characteristic foliation: $\mathcal{F} = \{M\}$.

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The Lie Algebroid of a Lie Algebra

$\mathfrak{g}$ - a Lie algebra

- bundle: $A = \mathfrak{g} \longrightarrow \{*\}$;
- anchor: $\# = 0$;
- Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, is the given Lie bracket;
- characteristic foliation: $\mathcal{F} = \{*\}$.

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The Lie Algebroid of a Foliation

$\mathcal{F}$ - a regular foliation

- bundle: $A = T\mathcal{F} \to M$;
- anchor: $\# : T\mathcal{F} \hookrightarrow TM$, inclusion;
- Lie bracket: $[\ ,\ ] : \mathfrak{X}(\mathcal{F}) \times \mathfrak{X}(\mathcal{F}) \to \mathfrak{X}(\mathcal{F})$, is the usual Lie bracket restricted to vector fields tangent to $\mathcal{F}$;
- characteristic foliation: $\mathcal{F}$.

The Action Lie Algebroid

$\rho : \mathfrak{g} \to \mathfrak{X}(M)$ - an infinitesimal action of a Lie algebra

- bundle: $A = M \times \mathfrak{g} \to M$;
- anchor: $\# : A \to TM$, $\#(x, v) = \rho(v)|_x$;
- Lie bracket $[,] : C^\infty(M, \mathfrak{g}) \times C^\infty(M, \mathfrak{g}) \to C^\infty(M, \mathfrak{g})$ is:

$$[v, w](x) = [v(x), w(x)] + (\rho(v(x)) \cdot w)|_x - (\rho(w(x)) \cdot v)|_x;$$

- characteristic foliation: orbit foliation.

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The Lie Algebroid of a Presymplectic manifold

M - an presymplectic manifold with closed 2-form ω

- bundle: $A = TM \times \mathbb{R} \to M$;
- anchor: $\# : A \to TM$, $\#(v, \lambda) = v$;
- Lie bracket $\Gamma(A) = \mathfrak{X}(M) \times C^\infty(M)$ is:

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f) - \omega(X, Y));$$

- characteristic foliation: $\mathcal{F} = \{M\}$.

The Cotangent Lie Algebroid

M - a Poisson manifold with Poisson tensor π

- bundle: $A = T^*M$;
- anchor: $\# : TM^* \to TM$, $\#\alpha = i_\pi \alpha$;
- Lie bracket $[\ ,\ ] : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$, is the Kozul Lie bracket:

$$[\alpha, \beta] = \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha - d\pi(\alpha, \beta);$$

- characteristic foliation: the symplectic foliation.

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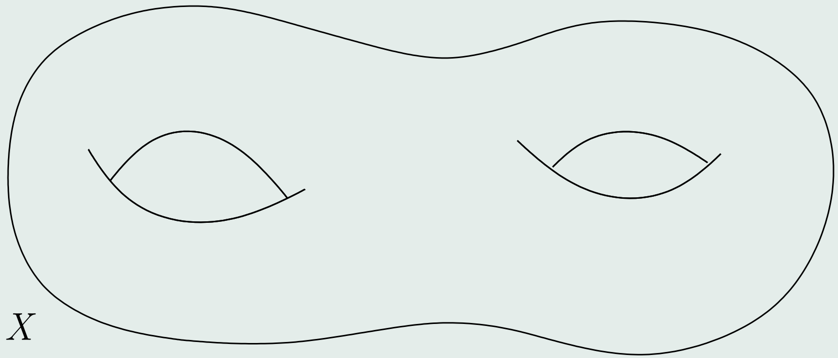
The Pair Groupoid

M - a manifold

- **arrows**: $\mathcal{G} = M \times M$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}(x, y) = x$, $\mathbf{t}(x, y) = y$;
- **product**: $(x, y) \cdot (y, z) = (x, z)$;

The Fundamental Groupoid (of a space)

M - a manifold



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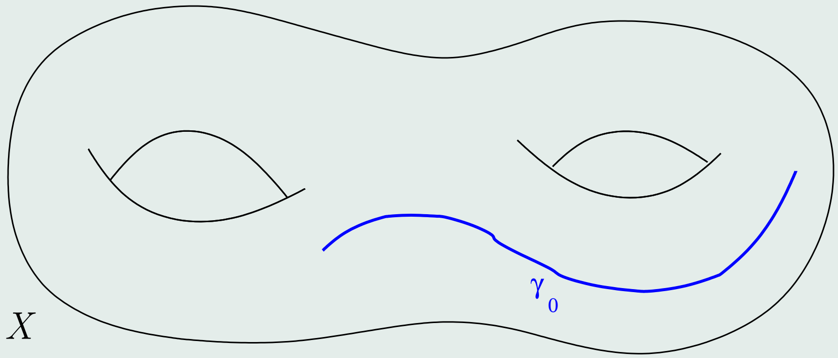

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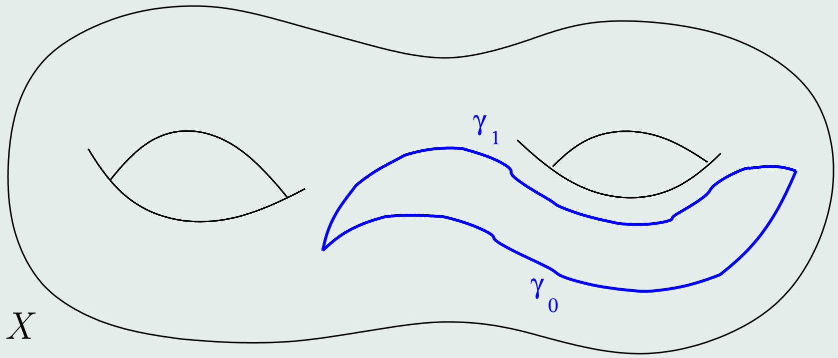

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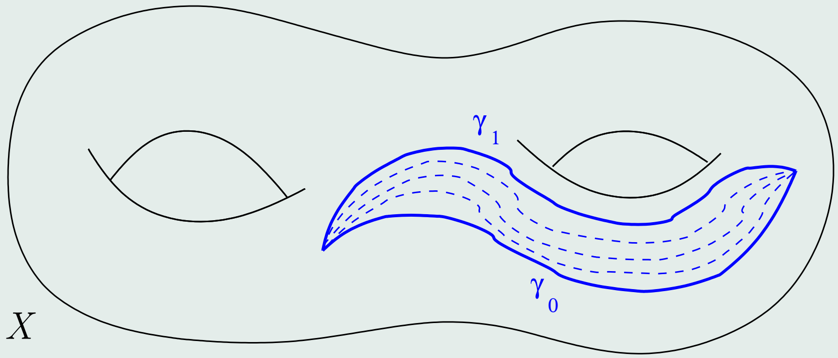

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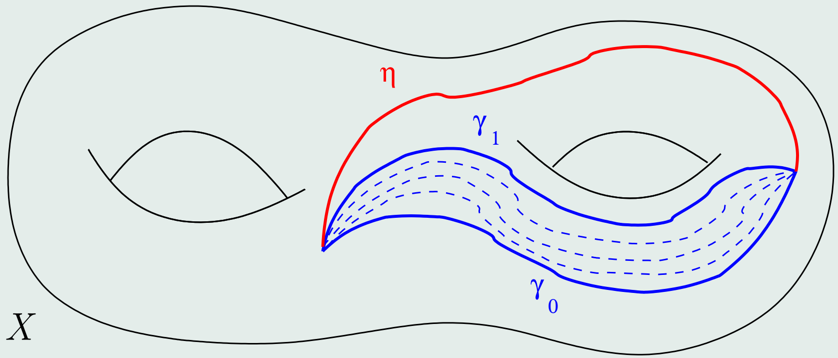

- **arrows**: $\mathcal{G} = \{[\gamma] : \gamma : [0,1] \to M\}$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}([\gamma]) = \gamma(0)$, $\mathbf{t}([\gamma]) = \gamma(1)$;
- **product**: $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$ (concatenation);

The Lie Groupoid of a Lie Group

G - a Lie group

- **arrows**: $\mathcal{G} = G$;
- **objects**: $M = \{*\}$;
- **target** and **source**: $\mathbf{s}(x) = \mathbf{t}(x) = *$;
- **product**: $g \cdot h = gh$;

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The Holonomy Groupoid

$\mathcal{F}$ - a regular foliation in M

- **arrows**: $\mathcal{G} = \{[\gamma] : \text{holonomy equivalence classes}\}$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}([\gamma]) = \gamma(0)$, $\mathbf{t}([\gamma]) = \gamma(1)$;
- **product**: $[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$;

The Fundamental Groupoid (of a foliation)

$\mathcal{F}$ - a regular foliation in M

- **arrows**: $\mathcal{G} = \{[\gamma] : \text{homotopy classes inside leafs}\}$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}([\gamma]) = \gamma(0)$, $\mathbf{t}([\gamma]) = \gamma(1)$;
- **product**: $[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$;

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The Action Groupoid

$G \times M \longrightarrow M$ - an action of a Lie group on M

- **arrows**: $\mathcal{G} = G \times M$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}(g, x) = x$, $\mathbf{t}(g, x) = gx$;
- **product**: $(h, y) \cdot (g, x) = (hg, x)$;

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