Quantization of symplectic symmetric spaces

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joint work with
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I. **Motivations.**

1. WKB-quantization of symplectic symmetric spaces [Karasëv, Weinstein, Zakrzewski, 90']

$$M = G/K$$

Problem: Find $A, S \in C^\infty(M \times M \times M, \mathbb{R})^G$ s.t.

$$a \star b := \int_{M\times M} A\, e^{iS}\, a \otimes b \quad \text{is } \textbf{ASSOCIATIVE}$$

2. Noncommutative symmetric space: Find **function space**

$$\mathcal{D}(M) \subset \mathbb{A} \subset \mathcal{D}'(M)$$

such that

$$(\mathbb{A}, \star) \quad \text{is a } \textbf{TOPOLOGICAL } G\textbf{-ALGEBRA}$$

3. Unify quantization procedures

II. **WKB quantization: cocyclic case.**

Definition 0.1

$$P^q(M) := \{ \ S \in C^\infty(M^q)^G \ ; \ \textit{skew} \ \}$$

$$\delta S(x_0, ..., x_q) := \sum_{i=0}^{q} (-1)^i \ S(x_0, ..., \hat{x}_i, ..., x_q)$$

$S \in P^3(M)$ *is* **admissible** *if*

$$S(x, s_x y, z) = -S(x, y, z)$$

Proposition 0.1

Let $S \in P^3(M)$ *be admissible (*M *oriented).*

$$a \star b := \int_{M \times M} e^{iS} \ a \otimes b$$

then,

$$\delta S = 0 \quad \Rightarrow \quad \star \ \textit{associative}$$

III. **Admissible functions.**

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$$M \;\subset\; \mathfrak{g}^\star \quad \overset{\text{projection: } \pi}{\longrightarrow} \quad \mathfrak{p}^\star$$

M

π

$\mathfrak{p}^\star$

$$S_{\text{can}}(o, x, y)$$

Proposition 0.2

$S \in P^3(M)$ *admissible* $\Rightarrow \exists f \in \text{Fun}(\mathbb{R})_{\text{odd}}$ *such that*

$$S = f \circ S_{\text{can}}$$

IV. **WKB-quantization: solvable case.**

$M = G/K$ with G : **solvable** Lie group (+ conditions)

$$\stackrel{\Phi}{\longrightarrow}$$

Proposition 0.3

(i)

$$a \star_\theta^1 b \; := \; \frac{1}{\theta^2} \int_{M\times M} |\mathrm{Jac}(\Phi)|^{\frac{1}{2}} \; e^{\frac{i}{\theta}\ \mathrm{Area}\,\circ\,\Phi} \; a \otimes b$$

is a **WKB-QUANTIZATION** *of* M

(ii)

$(L^2(M), \star_\theta^1) =$ **HILBERT** G**-ALGEBRA** $\quad (\theta > 0)$

V. **Hermitian symmetric case: strategy.**

1. Curvature **contraction** $\rightsquigarrow$ solvable space

2. **Decontraction**: solvable $\longrightarrow$ semisimple

Key:

evolution of

second order hyperbolic differential operator

VI. **Contracted hyperbolic plane.**

$$\mathbb{D} := SL(2,\mathbb{R})/SO(2) \xrightarrow{\text{contraction}} \mathbb{D}_c := G_c/K_c$$

$$\longrightarrow$$

$$G_c = SO(1,1) \times \mathbb{R}^2 \qquad\qquad K_c = \mathbb{R}$$

Remark: Let

$$1 \to N \to \mathbb{S} \to A \to 1$$

be the '$ax + b$'-group.

$\exists$ canonical symplectic $\mathbb{S}$-equivariant identifications:

$$\mathbb{D} \;=\; \mathbb{D}_c \;=\; \mathbb{S}$$

VII. **The basic operator.**

Proposition 0.4

$\star^c$ $=$ G_c-*invariant* **formal star product** *on* $\mathbb{D}_c$

Then

(i) $\exists$ *essentially* **unique** $D \in \mathtt{Der}(\star^c)$ *such that*

$$\mathbb{R}.D \oplus \mathfrak{s} \simeq \mathfrak{sl}(2,\mathbb{R}) \qquad (\mathfrak{s} := \mathrm{Lie}(\mathbb{S}))$$

(ii)

$$\Box := \mathcal{F}_N \circ D \circ \mathcal{F}_N^{-1}$$

is **second order differential** *operator*
$(\mathbb{S} = AN)$

(iii)

Hyperbolic *principal symbol*

independent *of choice of* $\star^c$

VIII. **Decontraction.**

Proposition 0.5

$u \in \mathcal{D}'(\mathbb{S})[[\theta]]$= *solution of evolution:*

$$-D_{(1)}\left((\epsilon \otimes id) \circ \Delta(u)\right) = F^{\star}_{(2)}\left((\epsilon \otimes id) \circ \Delta(u)\right)$$

with **invertible** *associated convolution operator* ℓ_u *on* $C_c(\mathbb{S})[[\theta]]$.

Then,

(i) $\sharp_u := \ell_u(\star^c)$ *is* $SL(2,\mathbb{R})$*-invariant*

(ii) **every** $SL(2,\mathbb{R})$*-invariant* **star product** *on* $\mathbb{D}$ *is of the form* $\sharp_u$

IX. **Bessel mode solutions—Hilbert algebras.**

Proposition 0.6

$\exists\ \{B_s\}_{s\in\mathbb{R}}$ *invertible solutions s.t.*

(i) **every solution** u *is of the form:*

$$u = \int_{\mathbb{R}} \tilde{u}(s)\ B_s\ \mathrm{d}s$$

with

$$\tilde{u}(s) \in \mathcal{D}'(\mathbb{R})[[\theta]]$$

(ii) $\forall s; \theta > 0:$

$$(L^2(\mathbb{D}), \sharp_{B_s})$$

is a **HILBERT** $SL(2,\mathbb{R})$**-ALGEBRA**

X. **Unterberger calculus revisited: s=0.**

Proposition 0.7

Set

$$a \sharp_{B_O} b =: \int_{\mathbb{D}\times\mathbb{D}} K_\theta^U \, a \otimes b$$

then:

$$K_\theta^U = \int_0^\infty t^2 J_{\frac{1}{\theta}}(t)\, e^{it\, S\mathrm{can}}\, \mathrm{d}t.$$

with

$$S_{\mathrm{can}}(0, z, w) = 4\,\zeta(z)\,\zeta(w)\,\Im(z\overline{w}) \quad (\mathbb{D} \subset \mathbb{C})$$