Conic Distributions and Accessible sets

B. Langerock
Department of Mathematical Physics and Astronomy
Ghent University

Conic Distributions and Accessible sets

B. Langerock
Department of Mathematical Physics and Astronomy
Ghent University

1. Introduction and Motivation

2. Definitions and Examples

3. Some results on topology of accessible sets

4. Outlook

Introduction and Motivation

Topological/smooth structure of subsets associated to a family of vector fields.

Introduction and Motivation

Topological/smooth structure of subsets associated to a family of vector fields.

Convex conic distributions generalise linear distributions on manifolds

Introduction and Motivation

Topological/smooth structure of subsets associated to a family of vector fields.

Convex conic distributions generalise linear distributions on manifolds

'Cones in tangent spaces' are encountered in *non-linear control theory*
$\dot{x}(t) = f(x(t), u(t))$.

Introduction and Motivation

Topological/smooth structure of subsets associated to a family of vector fields.

Convex conic distributions generalise linear distributions on manifolds

'Cones in tangent spaces' are encountered in *non-linear control theory*
$\dot{x}(t) = f(x(t), u(t))$.

Example : On $\mathbb{R}^2$
$X_1 = \partial/\partial x$ and $X_2 = \partial/\partial x + x\partial/\partial y$

Introduction and Motivation

Topological/smooth structure of subsets associated to a family of vector fields.

Convex conic distributions generalise linear distributions on manifolds

'Cones in tangent spaces' are encountered in *non-linear control theory*
$\dot{x}(t) = f(x(t), u(t))$.

Example : On $\mathbb{R}^2$
$X_1 = \partial/\partial x$ and $X_2 = \partial/\partial x + x\partial/\partial y$

$\mathrm{Acc}_x(X_1, X_2) := \{\phi^{i_k}_{t_k} \circ \cdots \circ \phi^{i_1}_{t_1}(x)$
$\mid i_k = 1, 2,\ (t_1, \ldots t_k) \in \mathbb{R}^k_+\}$

Introduction and Motivation

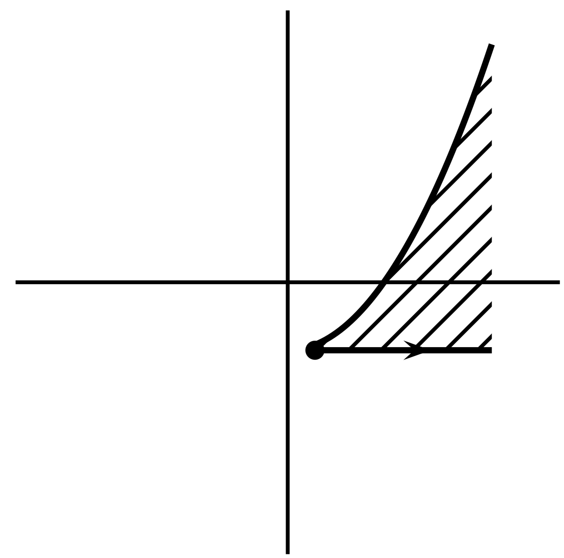
Topological/smooth structure of subsets associated to a family of vector fields.

Convex conic distributions generalise linear distributions on manifolds

'Cones in tangent spaces' are encountered in *non-linear control theory*
$\dot{x}(t) = f(x(t), u(t))$.

Example : On $\mathbb{R}^2$
$X_1 = \partial/\partial x$ and $X_2 = \partial/\partial x + x\partial/\partial y$

Acc$_x(X_1, X_2) := \{\phi^{i_k}_{t_k} \circ \cdots \circ \phi^{i_1}_{t_1}(x)$
$\mid i_k = 1, 2,\ (t_1, \ldots t_k) \in \mathbb{R}^k_+\}$



Introduction and Motivation

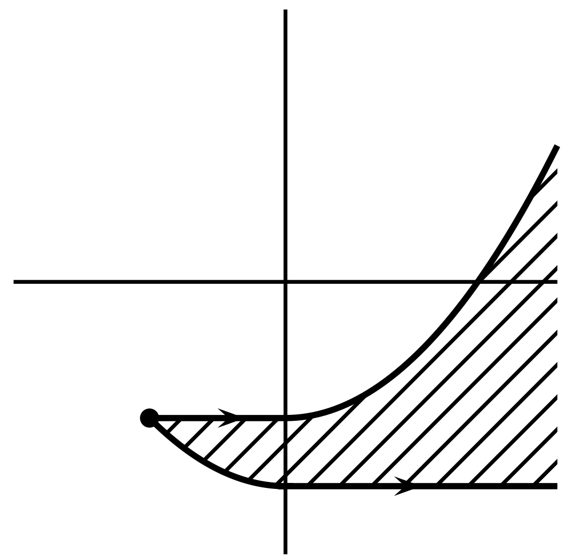
Topological/smooth structure of subsets associated to a family of vector fields.

Convex conic distributions generalise linear distributions on manifolds

'Cones in tangent spaces' are encountered in *non-linear control theory*
$\dot{x}(t) = f(x(t), u(t))$.

Example : On $\mathbb{R}^2$
$X_1 = \partial/\partial x$ and $X_2 = \partial/\partial x + x\partial/\partial y$

$\mathrm{Acc}_x(X_1, X_2) := \{\phi_{t_k}^{i_k} \circ \cdots \circ \phi_{t_1}^{i_1}(x)$
$\mid i_k = 1, 2,\ (t_1, \ldots t_k) \in \mathbb{R}_+^k\}$



Introduction and Motivation

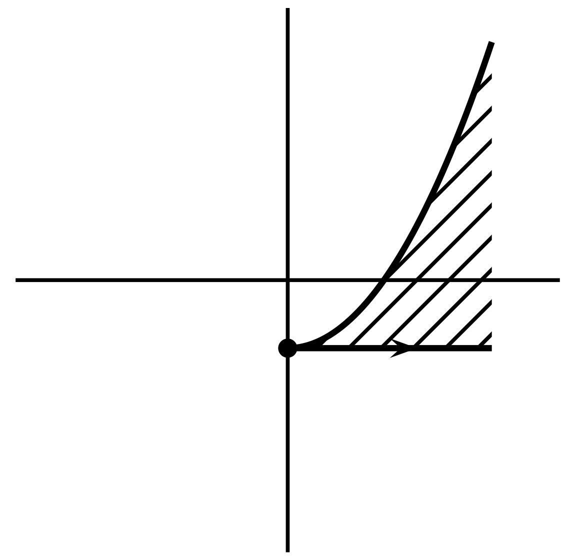
Topological/smooth structure of subsets associated to a family of vector fields.

Convex conic distributions generalise linear distributions on manifolds

'Cones in tangent spaces' are encountered in *non-linear control theory*
$\dot{x}(t) = f(x(t), u(t))$.

Example : On $\mathbb{R}^2$
$X_1 = \partial/\partial x$ and $X_2 = \partial/\partial x + x\partial/\partial y$

Acc$_x(X_1, X_2) := \{\phi^{i_k}_{t_k} \circ \cdots \circ \phi^{i_1}_{t_1}(x)$
$\mid i_k = 1, 2,\ (t_1, \ldots t_k) \in \mathbb{R}^k_+\}$



$\rightarrow$ What is the relation with control theory ?

Given an initial point x_0, what other points x_1 can be accessed with a given control law ?

→ What is the relation with control theory ?

Given an initial point x_0, what other points x_1 can be accessed with a given control law ?

Consider the family of vector fields $f(x, u(x))$ where $u(x)$ is an arbitrary function.

$\rightarrow$ What is the relation with control theory ?

Given an initial point x_0, what other points x_1 can be accessed with a given control law ?

Consider the family of vector fields $f(x, u(x))$ where $u(x)$ is an arbitrary function.

The accessible set corresponding to this family equals the accessible set from control theory.

→ What is the relation with control theory ?

Given an initial point x_0, what other points x_1 can be accessed with a given control law ?

Consider the family of vector fields $f(x, u(x))$ where $u(x)$ is an arbitrary function.

The accessible set corresponding to this family equals the accessible set from control theory.

Integral curves of $x \mapsto f(x, u(x))$ are controlled curves. Endpoints of a concatenation of such integral curves is precisely a point of the accessible set.

$\rightarrow$ What is the relation with control theory ?

Given an initial point x_0, what other points x_1 can be accessed with a given control law ?

Consider the family of vector fields $f(x, u(x))$ where $u(x)$ is an arbitrary function.

The accessible set corresponding to this family equals the accessible set from control theory.

Integral curves of $x \mapsto f(x, u(x))$ are controlled curves. Endpoints of a concatenation of such integral curves is precisely a point of the accessible set.

$\rightarrow$ What is the structure of the set of accessible points ? A smooth manifold, singular points, ...

$\rightarrow$ What is the relation with control theory ?

Given an initial point x_0, what other points x_1 can be accessed with a given control law ?

Consider the family of vector fields $f(x, u(x))$ where $u(x)$ is an arbitrary function.

The accessible set corresponding to this family equals the accessible set from control theory.

Integral curves of $x \mapsto f(x, u(x))$ are controlled curves. Endpoints of a concatenation of such integral curves is precisely a point of the accessible set.

$\rightarrow$ What is the structure of the set of accessible points ? A smooth manifold, singular points, . . .

$\rightarrow$ What are abnormal paths ?

Linear Distributions

1. A distribution on a manifold N is a subset D of TN such that for all $x \in N$ the set $D_x = D \cap T_xN$ has the structure of a linear subspace of T_xN.

Linear Distributions

1. A distribution on a manifold N is a subset D of TN such that for all $x \in N$ the set $D_x = D \cap T_xN$ has the structure of a linear subspace of T_xN.

2. The differentiable distribution $D(\mathcal{F})$ generated by an everywhere defined family of vector fields $\mathcal{F}$ on N is defined by

$$D_x(\mathcal{F}) = \left\{ \sum_{i=1}^{\ell} \lambda^i X_i(x) \mid \ell \in \mathbb{N}, (\lambda_\ell, \ldots, \lambda_1) \in \mathbb{R}^\ell, X_i \in \mathcal{F}, i = 1, \ldots, \ell \right\}.$$

Linear Distributions

1. A distribution on a manifold N is a subset D of TN such that for all $x \in N$ the set $D_x = D \cap T_x N$ has the structure of a linear subspace of $T_x N$.

2. The differentiable distribution $D(\mathcal{F})$ generated by an everywhere defined family of vector fields $\mathcal{F}$ on N is defined by

$$D_x(\mathcal{F}) = \left\{ \sum_{i=1}^{\ell} \lambda^i X_i(x) \mid \ell \in \mathbb{N}, (\lambda_\ell, \ldots, \lambda_1) \in \mathbb{R}^\ell, X_i \in \mathcal{F}, i = 1, \ldots, \ell \right\}.$$

3. The *orbit* $L_x(\mathcal{F})$ through x of the everywhere defined family of (local) vector fields $\mathcal{F}$ is the subset of N defined by

$$L_x(\mathcal{F}) = \{\mathcal{X}_T(x) \mid \ell \in \mathbb{N}, \mathcal{X} = (X_\ell, \cdots, X_1), X_i \in \mathcal{F}, i = 1, \ldots, \ell, T \in \mathbb{R}^\ell\}.$$

Conic Distributions

1. A *conic distribution* on a manifold N is a subset C of TN such that for all $x \in N$ the set $C_x = C \cap T_xN$ has the structure of a convex cone in T_xN.

Conic Distributions

1. A *conic distribution* on a manifold N is a subset C of TN such that for all $x \in N$ the set $C_x = C \cap T_xN$ has the structure of a convex cone in T_xN.

2. The differentiable conic distribution $C(\mathcal{F})$ generated by a family of vector fields $\mathcal{F}$

$$C_x(\mathcal{F}) = \left\{ \sum_{i=1}^{\ell} \lambda^i X_i(x) \mid \ell \in \mathbb{N}, (\lambda_\ell, \ldots, \lambda_1) \in \boxed{\mathbb{R}_+^\ell}, X_i \in \mathcal{F}, i = 1, \ldots, \ell \right\}.$$

Conic Distributions

1. A *conic distribution* on a manifold N is a subset C of TN such that for all $x \in N$ the set $C_x = C \cap T_x N$ has the structure of a convex cone in $T_x N$.

2. The differentiable conic distribution $C(\mathcal{F})$ generated by a family of vector fields $\mathcal{F}$

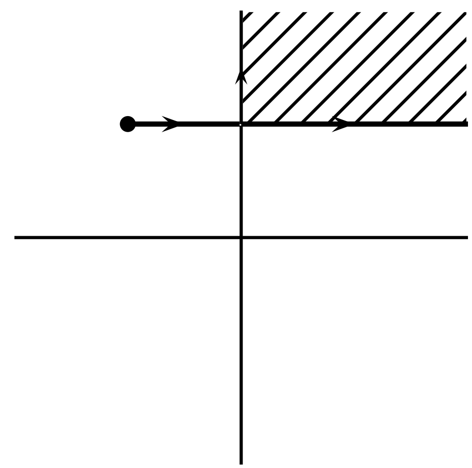
$$C_x(\mathcal{F}) = \left\{ \sum_{i=1}^{\ell} \lambda^i X_i(x) \mid \ell \in \mathbb{N}, (\lambda_\ell, \ldots, \lambda_1) \in \boxed{\mathbb{R}_+^\ell}, X_i \in \mathcal{F}, i = 1, \ldots, \ell \right\}.$$

3. The *accessible set* $\mathrm{Acc}_x(\mathcal{F})$ from x of the everywhere defined family of (local) vector fields $\mathcal{F}$ is the subset of N defined by

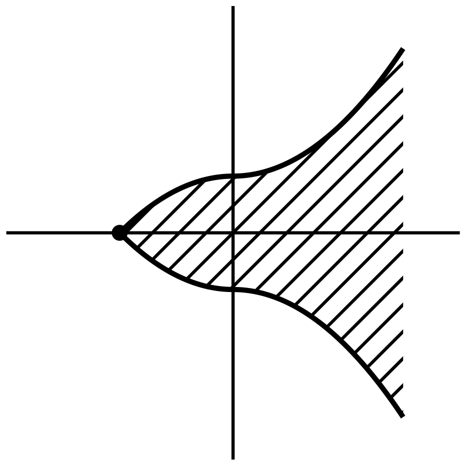
$$\mathrm{Acc}_x(\mathcal{F}) = \{\mathcal{X}_T(x) \mid \ell \in \mathbb{N}, \mathcal{X} = (X_\ell, \cdots, X_1), X_i \in \mathcal{F}, i = 1, \ldots, \ell, T \in \boxed{\mathbb{R}_+^\ell}\}.$$

More Examples

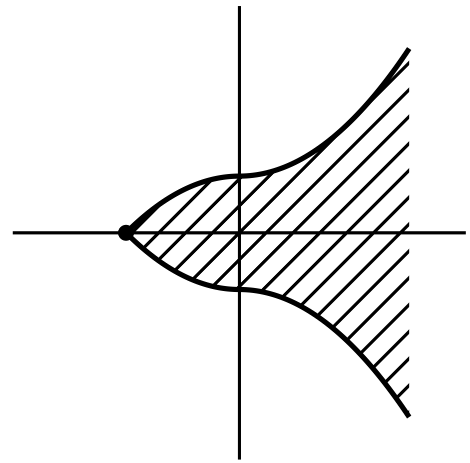
$N = \mathbb{R}^2, \mathcal{F} = \{$
$X_1 = \partial/\partial x$ on $\mathbb{R}^2$,
$X_2 = \partial/\partial y$ on $\mathbb{R}^2 \setminus \{(x,y)|x \leq 0\}\}$



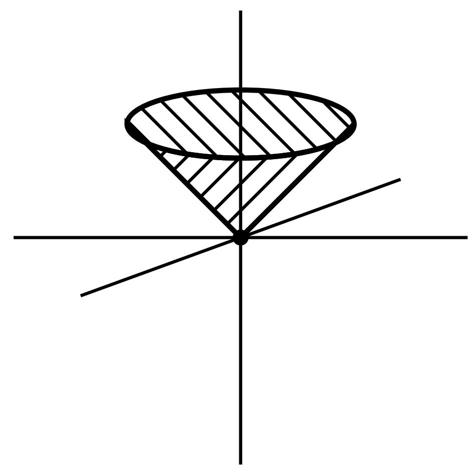
$N = \mathbb{R}^2, \mathcal{F} = \{$
$X_1 = \partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$,
$X_2 = \partial/\partial x - x\partial/\partial y$ on $\mathbb{R}^2\}$



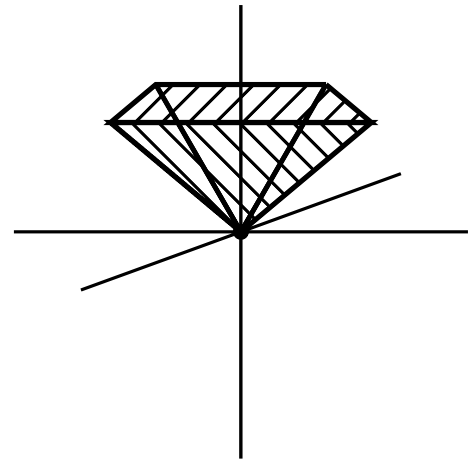
$N = \mathbb{R}^2, \mathcal{F} = \{$
$X_1 = \partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$,
$X_2 = \partial/\partial x - x\partial/\partial y$ on $\mathbb{R}^2\}$



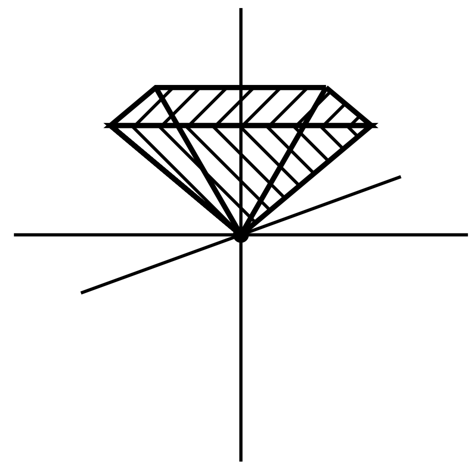
$N = \mathbb{R}^3, \mathcal{F} = \{$
$X_1 = \partial/\partial z + \cos\theta\partial/\partial x +$
$\sin\theta\partial/\partial y |\ \theta \in [0, 2\pi[\}$



$N = \mathbb{R}^3, \mathcal{F} = \{$
$X_1, X_2, X_3, X_4\}$
'polyhedral conic distribution'



$N = \mathbb{R}^3, \mathcal{F} = \{$
$X_1, X_2, X_3, X_4\}$
'polyhedral conic distribution'



Let (N, g) be a Lorentz manifold (i.e. g is a metric with signature $(+ - --)$) and such that N is equipped with a global time direction. The subset C of TN consisting of all time-like future oriented tangent vectors v such that $g(v, v) > 0$ defines an open conic distribution on N.

Results on topology of accessible sets

Key Idea: Construct a notion of 'tangent space' to the admissible set ?

Results on topology of accessible sets

Key Idea: Construct a notion of 'tangent space' to the admissible set ?

1.

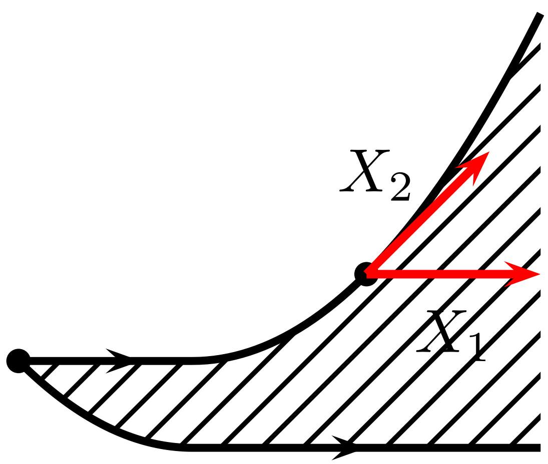
This tangent space contains the original convex cone $C(\mathcal{F})$

Results on topology of accessible sets

Key Idea: Construct a notion of 'tangent space' to the admissible set ?

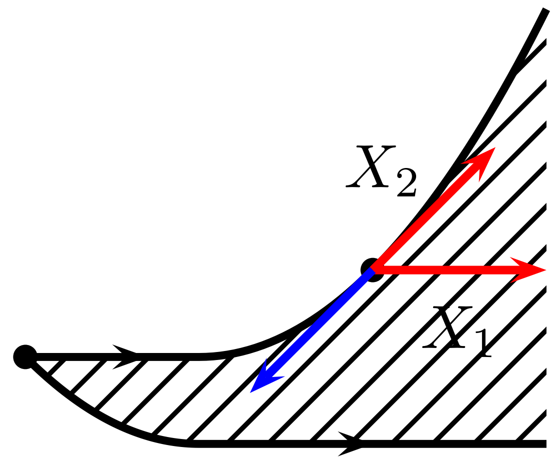
1.

This tangent space contains the original convex cone $C(\mathcal{F})$



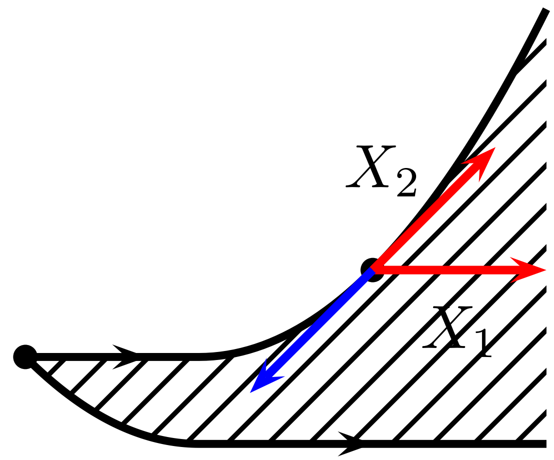
2.

This tangent space contains $-X_2$



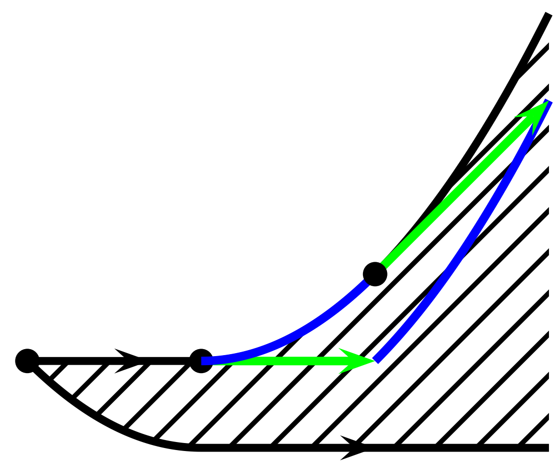
2.

This tangent space contains $-X_2$



3.

This tangent space contains push forward of X_1



Results on topology of accessible sets

The convex cone spanned by the above defined set is a ‘tangent space’ to the accessible set: any tangent in this cone is a tangent to a curve in $\mathrm{Acc}_x(\mathcal{F})$.

Results on topology of accessible sets

The convex cone spanned by the above defined set is a 'tangent space' to the accessible set: any tangent in this cone is a tangent to a curve in $\mathrm{Acc}_x(\mathcal{F})$.

1. If this extended cone equals the entire tangent space
   $\Rightarrow$ an interior point of $\mathrm{Acc}_x(\mathcal{F})$

Results on topology of accessible sets

The convex cone spanned by the above defined set is a ‘tangent space’ to the accessible set: any tangent in this cone is a tangent to a curve in $\mathrm{Acc}_x(\mathcal{F})$.

1. If this extended cone equals the entire tangent space
   $\Rightarrow$ an interior point of $\mathrm{Acc}_x(\mathcal{F})$

2. Any tangent vector in the interior of this extended cone
   points to interior points of $\mathrm{Acc}_x(\mathcal{F})$.

Results on topology of accessible sets

The convex cone spanned by the above defined set is a ‘tangent space’ to the accessible set: any tangent in this cone is a tangent to a curve in $\mathrm{Acc}_x(\mathcal{F})$.

1. If this extended cone equals the entire tangent space
   $\Rightarrow$ an interior point of $\mathrm{Acc}_x(\mathcal{F})$

2. Any tangent vector in the interior of this extended cone
   points to interior points of $\mathrm{Acc}_x(\mathcal{F})$.

3. Given two families $\mathcal{F}$ and $\mathcal{F}'$ such that $C(\mathcal{F}) = C(\mathcal{F}')$ and have maximal rank then

Results on topology of accessible sets

The convex cone spanned by the above defined set is a ‘tangent space’ to the accessible set: any tangent in this cone is a tangent to a curve in $\text{Acc}_x(\mathcal{F})$.

1. If this extended cone equals the entire tangent space
   $\Rightarrow$ an interior point of $\text{Acc}_x(\mathcal{F})$

2. Any tangent vector in the interior of this extended cone
   points to interior points of $\text{Acc}_x(\mathcal{F})$.

3. Given two families $\mathcal{F}$ and $\mathcal{F}'$ such that $C(\mathcal{F}) = C(\mathcal{F}')$ and have maximal rank then

   * $\text{cl}(\text{Acc}_x(\mathcal{F})) = \text{cl}(\text{Acc}_x(\mathcal{F}'))$;
   * $\text{int}(\text{Acc}_x(\mathcal{F})) = \text{int}(\text{Acc}_x(\mathcal{F}')) = \text{Acc}_x(\text{int}\, C(\mathcal{F}))$.

4. Abnormal extremals $\Leftrightarrow$ this extended cone does not equal the entire tangent space. Boundary points are abnormal

Outlook

$*$ Controllability of $\mathcal{F}$? $\rightarrow$ related to accessibility w.r.t $-\mathcal{F}$. Sussmann ?

Outlook

- Controllability of $\mathcal{F}$? $\rightarrow$ related to accessibility w.r.t $-\mathcal{F}$. Sussmann ?
- Can we define a smooth structure on $\text{Acc}_x(\mathcal{F})$?

Outlook

$*$ Controllability of $\mathcal{F}$? $\rightarrow$ related to accessibility w.r.t $-\mathcal{F}$. Sussmann ?

$*$ Can we define a smooth structure on $\text{Acc}_x(\mathcal{F})$?

Under *strong conditions* then the structure of a manifold with corners...

Outlook

- $*$ Controllability of $\mathcal{F}$? $\to$ related to accessibility w.r.t $-\mathcal{F}$. Sussmann ?

- $*$ Can we define a smooth structure on $\text{Acc}_x(\mathcal{F})$?

  Under *strong conditions* then the structure of a manifold with corners...

- $*$ Modeling ? What family provides the most 'interesting' way of controlling/steering a given system