A GEOMETRIC GLANCE AT THE HAMILTON-JACOBI EQUATION

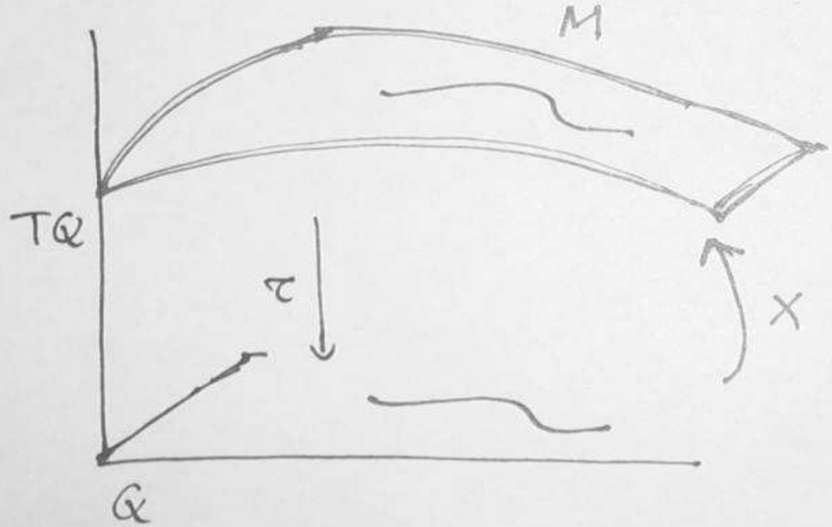
Gent 2005

Beppe, Eduardo, Miguel, Narciso, Pepín, Xavier

$\Gamma \in \mathfrak{X}(TQ)$ second-order vector field

describe integral curves of Γ as

integral curves of a family of first-order vector fields $X \in \mathfrak{X}(Q)$



n-dimensional submanifold $M \subset TQ$

invariant by Γ

transversal to the projection τ

$\Downarrow$

defines X

there may be alternative lagrangians L_1, L_2 for Γ

$M = X(Q)$ may be a lagrangian submanifold for ω_{L_1} but not for ω_{L_2}

Ⓐ

$$\begin{tikzcd} TM \arrow[r, "T\alpha"] \arrow[d] & TN \arrow[d] \\ M \arrow[u, bend left, "X"] \arrow[r, "\alpha"] & N \arrow[u, bend right, "Y"'] \end{tikzcd}$$

X, Y, α

(1) γ integral curve of X $\Rightarrow$ $\delta = \alpha \circ \gamma$ integral curve of Y

(2) $T\alpha \circ X = Y \circ \alpha$ $\quad (X \underset{\alpha}{\sim} Y)$

(3) $\alpha \circ F_X^t = F_Y^t \circ \alpha$ whenever defined $\quad F_X$ flow of X

(B)

$$\begin{tikzcd}
TM \arrow[r, "T\alpha"] \arrow[d] & TN \arrow[d] \\
M \arrow[r, "\alpha"] \arrow[u, bend left, "X"] & N \arrow[u, bend right, "Y"']
\end{tikzcd}$$

Y

α <u>embedding</u>

$\Downarrow$

$\alpha_0 : M \longrightarrow \alpha(M) \subset N$ diffeomorphism

$\uparrow$ regular submanifold

X is determined by condition (2)

(4) Y is tangent to $\alpha(M)$

Then $\boxed{X = \alpha_0^*\left[Y|_{\alpha(M)} \right]}$

(1') the integral curves of Y with initial condition in $\alpha(M) \subset N$ correspond through α to the integral curves of X

C

$$\begin{tikzcd}
TM \arrow[r] \arrow[d] & TN \arrow[d] \\
M \arrow[u, bend left, "X"] \arrow[r, "\alpha"] & N \arrow[u, bend right, "Y"'] \arrow[l, bend left, "\pi"]
\end{tikzcd}$$

Y

α is a <u>section</u> of a bundle map

$$N \xrightarrow{\pi} M$$

i.e. $\pi \circ \alpha = Id_M$

Then from (2) we have $\boxed{X = T\pi \circ Y \circ \alpha}$

(5) $T\alpha \circ T\pi \circ Y \underset{\alpha(M)}{=} Y$

(Dv)

$$\begin{tikzcd} TQ \arrow[r, "T\alpha"] \arrow[d] & T(TQ) \arrow[d] \\ Q \arrow[r, "\alpha"] \arrow[u, bend left, "X"] & TQ \arrow[u, bend right, "Y"'] \end{tikzcd}$$

Y <u>second-order</u> vector field

α section of $TQ \xrightarrow{\tau} Q$ (<u>vector field</u>)

Y second-order vector field $\Leftrightarrow$ $T\tau \circ Y = Id_{TQ}$

Then $X = \alpha$

$$\begin{tikzcd} TQ \arrow[r, "TX"] \arrow[d] & T(TQ) \arrow[d] \\ Q \arrow[r, "X"] \arrow[u, bend left, "X"] & TQ \arrow[u, bend right, "Y"'] \end{tikzcd}$$

(2v) $TX \circ X = Y \circ X$

Ev

$L : TQ \longrightarrow \mathbb{R}$ <u>reguler lagrangian</u>

$E_L : TQ \longrightarrow \mathbb{R}$

$\theta_L \in \Omega^1(TQ) \qquad \omega_L = -d\theta_L$

$i_{\Gamma_L} \omega_L = dE_L$ <u>lagrangian vector field</u> Γ_L

X

$$\begin{tikzcd} TQ \arrow[r, "TX"] \arrow[d] & T(TQ) \arrow[d] \\ Q \arrow[r, "X"] \arrow[u, bend left, "X"] & TQ \arrow[u, bend right, "\Gamma_L"'] \end{tikzcd}$$

generalised lagrangian Hamilton-Jacobi problem

(2v') $\quad TX \circ X = \Gamma_L \circ X$

(6v) $\quad i_X X^*(\omega_L) = X^*(dE_L)$

$$\boxed{F_v}\quad \begin{tikzcd} TQ \arrow[r, "TX"] \arrow[d, shift left] & T(TQ) \arrow[d, shift right] \\ Q \arrow[u, bend left, "X"] \arrow[r, "X"] & TQ \arrow[u, bend right, "\Gamma_L"'] \end{tikzcd}$$

X such that $\underline{X^*(\omega_L) = 0}$

i.e. $X^*(\theta_L)$ closed 1-form

$\quad\quad \| \;$ locally

$\quad\quad dW$

lagrangian Hamilton-Jacobi problem

(7v) $\quad X^*(dE_L) = 0$

(7v') $\quad X^*(E_L) =$ locally constant

$\quad\quad \|$

$\quad\quad E_L \circ X$

$X(Q) \subset TQ$ lagrangian submanifold

Em $H : T^*Q \longrightarrow \mathbb{R}$ hamiltonian

$i_{Z_H} \omega_Q = dH$ hamiltonian vector field Z_H

α

$$\begin{tikzcd} TQ \arrow[r, "T\alpha"] \arrow[d, shift left] & T(T^*Q) \arrow[d, shift right] \\ Q \arrow[r, "\alpha"] \arrow[u, shift left, "X"] & T^*Q \arrow[u, shift right, "Z_H"'] \end{tikzcd}$$

generalised hamiltonian Hamilton-Jacobi problem

$$X = T\pi \circ Z_H \circ \alpha = \mathcal{F}H \circ \alpha$$

$\mathcal{F}H : T^*Q \longrightarrow TQ$ fibre derivative

(2m) $T\alpha \circ X = Z_H \circ \alpha$

$E \longrightarrow Q$ vector bundle

$vl : E \times_Q E \longrightarrow VE \subset TE$
vertical lift

$$vl\left(\alpha, i_X d\alpha + \alpha^*(dH)\right) = T\alpha \circ X - Z_H \circ \alpha$$

(6m) $i_X d\alpha + \alpha^*(dH) = 0$

Fm

$$\begin{tikzcd} TQ \arrow[r, "T\alpha"] \arrow[d, shift left] & T(T^*Q) \arrow[d, shift left] \\ Q \arrow[u, shift left, "X"] \arrow[r, "\alpha"] & T^*Q \arrow[u, shift left, "Z_H"'] \end{tikzcd}$$

α such that $\underline{d\alpha = 0}$ (closed 1-form)

i.e. $\alpha \overset{\text{locally}}{=} dW$

hamiltonian Hamilton-Jacobi problem

(7m) $\quad \alpha^*(dH) = 0$

(7m') $\quad \alpha^*(H) = H \circ \alpha$ locally constant

$\alpha(Q) \subset T^*Q$ lagrangian submanifold

Equivalence between lagrangian and hamiltonian ...

Let $L : TQ \to \mathbb{R}$ be a hyperregular lagrangian

The map

$$X \longmapsto \alpha = \mathcal{F}L \circ X$$

is a bijection between the solutions of the [generalized] lagrangian Hamilton-Jacobi problem and the solutions of the [generalized] hamiltonian Hamilton-Jacobi problem

- complete solutions $\sim$ invariant foliations

  $X : Q \times \Lambda \longrightarrow TQ$

  $\alpha : Q \times \Lambda \longrightarrow T^*Q$

- singular case
  - without lagrangian constraints

- time-dependent case

- examples
  - distance on pseudoriemannian manifolds
  - Lie groups

A GEOMETRIC GLANCE AT THE HAMILTON-JACOBI EQUATION

Gen. 2009

Borja, Eduardo, Miguel, Narciso, Sofía, Xavier