

On projective connections: the affine case

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Abstract

We compare the approaches of E. Cartan and of T.Y. Thomas and J.H.C. Whitehead to the study of ‘projective connections’. Although the quoted phrase has quite different meanings in the two contexts considered, we are able to show that a class of projectively equivalent symmetric affine connections on a manifold (the latter meaning) gives rise, in a global way, to a unique Cartan connection on a principal bundle over the manifold, defining a development of curves in the manifold to curves in projective space (the former meaning). The unparametrized geodesics of the affine connections are identical to the geodesics of the Cartan connection. The principal bundle on which the Cartan connection is defined is itself a geometric object, and exists independently of any particular connection.

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1 Introduction

Elie Cartan’s paper on projective connections [2], published in 1924, was one of a series intended to extend the idea of an affine connection as formulated by Levi-Civita and Weyl to a more general, non-vector, situation. Cartan imagined, attached to each point of a manifold, a projective space of the same dimension, together with a mechanism whereby the spaces at two infinitely-neighbouring points could be ‘connected’. Such a connection would define geodesics as those curves in the manifold which could be ‘developed’ into straight lines in the connected projective spaces.

A modern interpretation of Cartan's idea can be found in the recent book by Sharpe [14]. According to Sharpe, the fruitful way to view Cartan's theory of connections is to think of it as a generalization of Klein's concept of geometry. In this approach, each 'Cartan geometry' is based upon a model geometry called a 'Klein geometry'. A Klein geometry is a homogeneous space G/H of a Lie group G ; G itself is a principal H -bundle over G/H and comes equipped with a \mathfrak{g} -valued 1-form (where \mathfrak{g} is the Lie algebra of G), its Maurer-Cartan form. A Cartan geometry on a manifold M , corresponding to a Klein geometry for which G/H has the same dimension, is a principal H -bundle $P \rightarrow M$ together with a \mathfrak{g} -valued 1-form on P which is called the 'connection form' and is intended to generalize the Maurer-Cartan form. A construction of this kind is called a Cartan connection. For a Cartan projective connection on an m -dimensional manifold the model geometry is m -dimensional real projective space P^m . To realise this as a homogeneous space G/H we take for G the group of projective transformations of P^m , which is $\text{PGL}(m+1)$, the quotient of $\text{GL}(m+1)$ by non-zero multiples of the identity; and for H we take the subgroup $H_{m+1} \subset \text{PGL}(m+1)$ which is the stabilizer of the point $[1, 0, \dots, 0] \in P^m$.

Cartan projective connections differ in concept and in practice from the type of connection on a principal bundle introduced in 1950 by Ehresmann. Ehresmann's definition of a connection is based on the idea of parallel transport originally formulated by Levi-Civita; on the face of it, there is no notion of parallelism associated with a Cartan projective connection. The practical differences show up in the fact that the connection form of a Cartan connection takes its values in \mathfrak{g} while that of an Ehresmann connection takes its values in the Lie algebra of H , the group of the principal bundle.

Cartan's work on connections predates the formulation of the concept of a fibre bundle, of course, so though he discusses in detail the projective connection as a local object, there is no direct hint in [2] of what the principal H_{m+1} -bundle on which the global connection form should live might be (except of course that it should embody the notion of attaching a projective space to each point of the manifold). The same is true, in a sense, of [14]: while Sharpe gives the general procedure for constructing the bundle implicitly by inferring its transition functions from the local connection forms, he does not carry it out in the particular case of the projective connection, let alone give an explicit definition of the bundle.

Around the time that Cartan published his paper on projective connections a somewhat different line of research, also described as a theory of projective connections, was being pursued by several other authors, including T.Y. Thomas [15, 16] and J.H.C. Whitehead [17]. This second theory is concerned with the relationship between two affine connections whose geodesics, although having different parametrizations, are geometrically the same; two such connections are said to be projectively related. Here the concept of connection is that of Ehresmann. A brief history of the development of these ideas up to 1930, which names the mathematicians principally involved, can be found in the introductory section of Whitehead's paper.

There has recently been a resurgence of interest in both of these approaches, with a view

to applications, and for purely mathematical reasons. Cartan's approach to connection theory and the equivalence of geometric structures has been found to be relevant to the programme of research in general relativity which has been carried out over the last dozen years by E.T. Newman and his co-workers (see [6]). As a consequence Cartan's theory of projective connections has been subject to new scrutiny (see [11] and [12]). The approach of Thomas and Whitehead, on the other hand, has been discussed in [1], also from a relativistic perspective. So far as purely mathematical interest in Cartan is concerned there is the book of Sharpe [14] which has already been mentioned; whereas a modern version of the method used by Thomas and Whitehead, captured in the concept of a Thomas-Whitehead projective connection, has been given by Roberts in [13] and developed in [7].

Our plan in the present paper is to provide a geometrical formulation of projective connections which unifies these ideas. The key result is the explicit construction of a principal H_{m+1} -bundle over any manifold M which serves as the bundle for the global Cartan projective connection according to Sharpe's interpretation. We call this bundle the Cartan bundle $\mathcal{CM} \rightarrow M$. The Cartan bundle is defined independently of any particular connection; any Cartan projective connection form can be realised as a form on it. We further show how a Thomas-Whitehead projective connection, representing a projective equivalence class of connections on M , gives rise in a natural way to a Cartan projective connection on \mathcal{CM} having the same unparametrized geodesics, thus establishing the exact relationship between the theories of Thomas and Whitehead and of Cartan.

In Section 2 we recall relevant properties of projective equivalence classes of affine connections, and describe the Thomas-Whitehead theory in the formulation due to Roberts. We discuss the theory of Cartan projective connections in Section 3; in particular, we show how the transformation properties of a Cartan connection specify the transition functions of the principal bundle on which it is defined. In both Sections 2 and 3, indeed, the exposition, though carried out in local terms, is made with a view to global properties. In Section 4 we give the construction of the Cartan bundle, and in Section 5 we describe the construction of a Cartan connection on \mathcal{CM} from a projective equivalence class via the Thomas-Whitehead theory.

The material in Sections 2 and 3, while broadly familiar, contains some new insights; that in Sections 4 and 5 is to the best of our knowledge new in its entirety. In a second paper [4] we will extend our results to what Douglas [5] called the general geometry of paths, that is, from affine sprays (which are treated here) to general sprays. This involves both a completely new version of the Thomas-Whitehead theory, and a major extension of the Cartan theory (Cartan dealt only with the 2-dimensional case). While the present paper is, we believe, of interest in its own right, it also serves an important introductory function for this second paper.

We use the Einstein summation convention for repeated indices. Indices a, b, \dots range and sum from 1 to m and indices α, β, \dots from 0 to m .

2 Projective equivalence classes of affine connections

In this section we review the theory of projective transformations of a symmetric affine connection.

2.1 The fundamental descriptive invariant

A symmetric affine connection ∇ on an m -dimensional manifold M with coordinates (x^a) has, as its geodesic field, an affine spray S on the tangent bundle TM (with coordinates (x^a, u^a)) given by

$$S = u^a \frac{\partial}{\partial x^a} - \Gamma_{ab}^c u^a u^b \frac{\partial}{\partial u^c}.$$

Two sprays S, \hat{S} are projectively equivalent if $\hat{S} - S = -2\alpha\Delta$, where Δ is the Liouville field $u^a \partial / \partial u^a$ and the function α is linear in the u^a , $\alpha = \alpha_a u^a$, with $\alpha_a dx^a$ a 1-form on M . From this transformation rule it follows that

$$\hat{\Gamma}_{ab}^c = \Gamma_{ab}^c + (\alpha_a \delta_b^c + \alpha_b \delta_a^c).$$

By taking a trace and writing $\Gamma_a = \Gamma_{ab}^b$, $\hat{\Gamma}_a = \hat{\Gamma}_{ab}^b$ we obtain $\hat{\Gamma}_a = \Gamma_a + (m+1)\alpha_a$, so that the quantities

$$\Pi_{ab}^c = \Gamma_{ab}^c - \frac{1}{m+1} (\Gamma_a \delta_b^c + \Gamma_b \delta_a^c)$$

are projectively invariant, that is, unchanged under a projective transformation, and therefore associated with a whole equivalence class of projectively related sprays rather than with any individual spray. These quantities were introduced by T.Y. Thomas [15, 16], who called them collectively the projective connection; to use that terminology in the present context would be to invite confusion, so we have adopted another. Douglas [5] used a generalized form of the same quantities, and called them collectively the fundamental descriptive invariant of a projective equivalence class of geodesics; this is the term we will use. One particularly important property of the fundamental descriptive invariant is that, defining $\Pi_a = \Pi_{ab}^b = \Pi_{ba}^b$, we have $\Pi_a = 0$.

On the face of it, if we take

$$\alpha_a = -\frac{1}{m+1} \Gamma_a$$

then the transformed spray has Π_{ab}^c for its connection coefficients. However, the Γ_a are not the components of a 1-form: their transformation law involves the determinant of the Jacobian of the coordinate transformation; consequently the Π_{ab}^c are not, in general, the components of a connection. In fact if $\Pi_{ab}^c, \hat{\Pi}_{ab}^c$ are the components of the fundamental descriptive invariant with respect to coordinates $(x^a), (\hat{x}^a)$ then

$$\hat{\Pi}_{ab}^c = \bar{J}_a^d \bar{J}_b^e (J_f^c \Pi_{de}^f - J_{de}^c) + \frac{1}{m+1} \frac{\partial \log |J|}{\partial x^d} (\bar{J}_a^d \delta_b^c + \bar{J}_b^d \delta_a^c),$$

where $J_b^a = \partial \hat{x}^a / \partial x^b$ are the elements of the Jacobian matrix of the coordinate transformation, \bar{J}_b^a those of its inverse, $J_{ab}^c = \partial J_a^c / \partial x^b = \partial J_b^c / \partial x^a$, and J is the Jacobian determinant.

We will nevertheless follow Douglas in taking the Π_{ab}^c as fundamental in describing a certain kind of path space, that is, a manifold together with a collection of paths (unparametrized curves) with the property that there is exactly one path of the collection through a given point in a given direction. Douglas, in [5], deals with a more general type of path space, and calls a path space of the kind discussed here a restricted path space. In fact we could define a restricted path space as an assignment, to each coordinate patch on a manifold, of a set of functions Π_{ab}^c , symmetric in a and b , transforming under a change of coordinates according to the formula given above. The paths are defined by

$$\ddot{x}^c + \Pi_{ab}^c \dot{x}^a \dot{x}^b \propto \dot{x}^c,$$

a condition which is invariant both under coordinate transformations and under change of parametrization. It is not strictly necessary to impose the condition that $\Pi_{ab}^b = 0$, since if the Π_{ab}^c transform as specified then the Π_{ab}^b are components of a 1-form, and if

$$\tilde{\Pi}_{ab}^c = \Pi_{ab}^c - \frac{1}{m+1} (\Pi_a \delta_b^c + \Pi_b \delta_a^c)$$

then $\tilde{\Pi}_{ab}^c$ transforms in the same way, defines the same paths, and does satisfy $\tilde{\Pi}_a = 0$. Nevertheless we will reserve the term fundamental descriptive invariant for the Π_{ab}^c which satisfy $\Pi_a = 0$. Clearly if $\Pi_a = 0$ and $\hat{\Pi}_{ab}^c$ is related to Π_{ab}^c by the transformation formula given above then $\hat{\Pi}_a = 0$ also.

Every affine connection defines a restricted path space in this sense, with projectively equivalent ones defining the same path space. As it happens the converse also holds, so the concept of restricted path space is not more general; however, this is not immediately apparent, so we will work with restricted path spaces for the moment, though we will prove the converse shortly.

2.2 The projective curvature tensor

We denote by \mathfrak{R}_{cab}^d the curvature ‘tensor’ derived from the Π_{ab}^c (we use gothic type to indicate that it is not in fact a tensor):

$$\mathfrak{R}_{cab}^d = \frac{\partial \Pi_{bc}^d}{\partial x^a} - \frac{\partial \Pi_{ac}^d}{\partial x^b} + \Pi_{ae}^d \Pi_{bc}^e - \Pi_{be}^d \Pi_{ac}^e;$$

its trace, the corresponding Ricci ‘tensor’ $\mathfrak{R}_{bc} = \mathfrak{R}_{bdc}^d$, is given, since $\Pi_a = 0$, by

$$\mathfrak{R}_{bc} = \frac{\partial \Pi_{bc}^d}{\partial x^d} - \Pi_{be}^d \Pi_{cd}^e;$$

it is symmetric. Despite appearances, the quantities P_{cab}^d defined by

$$P_{cab}^d = \mathfrak{R}_{cab}^d - \frac{1}{m-1} (\mathfrak{R}_{bc}\delta_a^d - \mathfrak{R}_{ac}\delta_b^d)$$

are the components of a tensor; it has the same symmetries as the curvature tensor, and is in addition completely trace-free; it is projectively invariant, and its vanishing is the necessary and sufficient condition, for $m \geq 3$, for the paths of the restricted path space to be rectifiable, that is, for there to be local coordinates with respect to which all the paths are straight lines. This tensor is called the projective curvature tensor.

Since the case $m = 2$ is somewhat special, and has been discussed elsewhere [3], we will for the remainder of this section assume that $m \geq 3$.

2.3 Connections on projective space

The fundamental example of a restricted path space is projective space \mathbf{P}^m itself. As a manifold, \mathbf{P}^m is the quotient of $\mathbf{R}^{m+1} - \{0\}$ under the multiplicative action of $\mathbf{R} - \{0\}$; the infinitesimal generator of this action is the radial vector field given in Cartesian coordinates by $x^\alpha \partial_\alpha = \Upsilon$. We may represent objects on \mathbf{P}^m as objects on $\mathbf{R}^{m+1} - \{0\}$ transforming appropriately under the action; for convenience this will be expressed in terms of the Lie derivative with respect to Υ , together with invariance under the reflection map $j : x \mapsto -x$. So functions on \mathbf{P}^m may be represented by functions f on $\mathbf{R}^{m+1} - \{0\}$ satisfying $\Upsilon f = 0$ and $j^*(f) = f$: call the set of such functions \mathcal{F}_Υ . Similarly, vector fields on \mathbf{P}^m may be represented by equivalence classes of vector fields X on $\mathbf{R}^{m+1} - \{0\}$ satisfying $\mathcal{L}_\Upsilon X \propto \Upsilon$ and $j_* X = X$, with equivalence $Y \equiv X$ if $Y - X \propto \Upsilon$. Let \mathfrak{X}_Υ denote the set of such vector fields, and for $X \in \mathfrak{X}_\Upsilon$ let $[[X]]$ denote the equivalence class of X . The set $[[\mathfrak{X}_\Upsilon]]$ of equivalence classes $[[X]]$ for $X \in \mathfrak{X}_\Upsilon$ is a Lie algebra over the module \mathcal{F}_Υ , with $[[[X]], [[Y]]] = [[[X, Y]]]$. Furthermore, for any $f \in \mathcal{F}_\Upsilon$, $Xf \in \mathcal{F}_\Upsilon$ if $X \in \mathfrak{X}_\Upsilon$ and $Yf = Xf$ if $Y \equiv X$: thus $[[X]]f$ is well-defined (as Xf); $[[\mathfrak{X}_\Upsilon]]$ acts as derivations on \mathcal{F}_Υ ; and the Lie bracket of equivalence classes is the commutator of the corresponding derivations.

We will define a covariant derivative operator on $[[\mathfrak{X}_\Upsilon]]$ as a map $\nabla : [[\mathfrak{X}_\Upsilon]] \times [[\mathfrak{X}_\Upsilon]] \rightarrow [[\mathfrak{X}_\Upsilon]]$ which is \mathbf{R} -bilinear, \mathcal{F}_Υ -linear in the first variable, and satisfies

$$\nabla_{[[X]]}(f[[Y]]) = f\nabla_{[[X]]}[[Y]] + ([[X]]f)[Y].$$

A covariant derivative is symmetric if

$$\nabla_{[[X]]}[[Y]] - \nabla_{[[Y]]}[[X]] = [[[X]], [[Y]]].$$

We now relate such operators to the standard covariant derivative D on \mathbf{R}^{m+1} , by the device of choosing a representative of each equivalence class. Let ϑ be a 1-form on $\mathbf{R}^{m+1} - \{0\}$ such that $\langle \Upsilon, \vartheta \rangle = 1$ and $j^*\vartheta = \vartheta$, and for any vector field X set $\tilde{X} =$

$X - \langle X, \vartheta \rangle \Upsilon$. Then if $Y \equiv X$, $\tilde{Y} = \tilde{X}$; and if $X \in \mathfrak{X}_\Upsilon$, $\tilde{X} \in \mathfrak{X}_\Upsilon$ also. Thus such a 1-form ϑ enables one to select a representative of each equivalence class, in fact by the condition $\langle X, \vartheta \rangle = 0$. If, furthermore, $\mathcal{L}_\Upsilon \vartheta = 0$ then $\mathcal{L}_\Upsilon \tilde{X} = 0$. Now Υ is an infinitesimal affine transformation of D , and so when $\mathcal{L}_\Upsilon \tilde{X} = \mathcal{L}_\Upsilon \tilde{Y} = 0$

$$\mathcal{L}_\Upsilon(D_{\tilde{X}}\tilde{Y}) = D_{\mathcal{L}_\Upsilon\tilde{X}}\tilde{Y} + D_{\tilde{X}}(\mathcal{L}_\Upsilon\tilde{Y}) = 0$$

also. Furthermore, j is an affine transformation, so when $j_*\tilde{X} = \tilde{X}$ and $j_*\tilde{Y} = \tilde{Y}$, $j_*(D_{\tilde{X}}\tilde{Y}) = D_{\tilde{X}}\tilde{Y}$. So for any choice of ϑ such that $\langle \Upsilon, \vartheta \rangle = 1$, $\mathcal{L}_\Upsilon \vartheta = 0$ and $j^*\vartheta = \vartheta$ we may set

$$\nabla_{[[X]]}^\vartheta[[Y]] = [[D_{\tilde{X}}\tilde{Y}]];$$

then ∇^ϑ is a symmetric connection on \mathfrak{X}_Υ .

We may now consider the geodesics of ∇^ϑ . First of all, a geodesic will be a 2-surface Σ in $\mathbf{R}^{m+1} - \{0\}$ invariant under the action generated by Υ , that is, ruled by rays. It will be defined by any curve in it transverse to the rays, and among such curves we can choose those σ whose tangent vectors satisfy $\langle \dot{\sigma}, \vartheta \rangle = 0$. Such curves are mapped to each other by the action generated by Υ , so it is enough to consider one of them. Then Σ will be a geodesic of ∇^ϑ if and only if $[[D_{\dot{\sigma}}\dot{\sigma}] \propto [[\dot{\sigma}]]$, that is, if and only if $D_{\dot{\sigma}}\dot{\sigma} = \ddot{\sigma}$ is a linear combination of $\dot{\sigma}$ and $\Upsilon|_{\dot{\sigma}}$. But this means that the tangent planes to Σ at all points on it are parallel to one another, and therefore that Σ is itself a plane. Thus the geodesics of ∇^ϑ are the straight lines in \mathbf{P}^m . We note for future reference that Υ has the property that $D\Upsilon = \text{id}$, where id is the identity tensor; and indeed this determines Υ up to the addition of a constant vector field.

We can describe the idea behind the construction of Roberts [13] as follows: to introduce for any manifold M , a manifold \mathcal{VM} of one higher dimension, whose role in relation to M is to be like that of $\mathbf{R}^{m+1} - \{0\}$ in relation to \mathbf{P}^m ; and on \mathcal{VM} to define a covariant derivative operator whose role in relation to a projective equivalence class of connections on M is to be like that of D in relation to \mathbf{P}^m as described above. We can motivate the construction of \mathcal{VM} by introducing a particular way of thinking about $\mathbf{R}^{m+1} - \{0\}$ in this context.

Let Ω be the standard volume form on \mathbf{R}^{m+1} . Let π be the projection $\mathbf{R}^{m+1} \rightarrow S^m$, where S^m , the m -sphere, is the quotient of $\mathbf{R}^{m+1} - \{0\}$ by the action generated by Υ , so that \mathbf{P}^m is obtained from S^m by identifying diametrically opposite points. Then for any point $p \in \mathbf{R}^{m+1}$, $p \neq 0$, we can define an m -covector θ at $\pi(p) \in S^m$ as follows: let ξ_a be any m elements of $T_{\pi(p)}S^m$, and let v_a be any m elements of $T_p\mathbf{R}^{m+1}$ such that $\pi_{p*}v_a = \xi_a$; set

$$\theta(\xi_1, \xi_2, \dots, \xi_m) = \Omega_p(\Upsilon_p, v_1, v_2, \dots, v_m);$$

θ is clearly well-defined since adding a multiple of Υ_p to any v_a doesn't change the value of the right-hand side. Now take any $s \in \mathbf{R}$, $s > 0$, and carry out the same construction but starting at sp . It is clear that the right-hand side gets multiplied by s^{m+1} . There is therefore a map $\varphi : \mathbf{R}^{m+1} - \{0\} \rightarrow \wedge^m S^m$ such that $\varphi(sp) = s^{m+1}\varphi(p)$. In this case φ

will be a diffeomorphism of $\mathbf{R}^{m+1} - \{0\}$ with either of the two classes of oriented volume forms on S^m . There is no need to take this any further here: our aim was just to suggest that it will be profitable to consider volume forms.

A somewhat similar account is to be found in [7], but as an application of Roberts's construction rather than as motivation for it.

2.4 The volume bundle

The basic idea of the Thomas-Whitehead theory of projectively equivalent connections is to represent a projective equivalence class on an m -dimensional manifold by a single connection on a manifold of dimension $m + 1$, extending the approach of the previous subsection from projective space to a more general manifold M . We start by describing the appropriate $(m + 1)$ -dimensional manifold, broadly following Roberts [13] but diverging from him over some details.

We start with the non-zero volume elements $\theta \in \wedge^m T^*M$; the set of pairs $[\pm\theta]$ of such elements will be called the volume bundle of M (strictly speaking it should be called the unoriented volume bundle but we will normally omit the prefix 'unoriented') and denoted by \mathcal{VM} . It is indeed a bundle, with projection $\nu : \mathcal{VM} \rightarrow M$, defined by $\nu([\pm\theta]) = x$ whenever $\theta, -\theta \in \wedge^m T_x^*M$. If x^a are coordinates on M then a candidate for the fibre coordinate on the (one-dimensional) fibre of ν is $|v|$, where v satisfies

$$\theta = v(\theta) \left(dx^1 \wedge \dots \wedge dx^m \right)_x$$

for any $\theta \in \wedge^m T^*M$; however, in view of the discussion in the previous subsection we choose instead to use $x^0 = |v|^{1/(m+1)}$ as the fibre coordinate, with the convention that the positive root is to be taken if m is odd so that $x^0 > 0$. The local trivializations defined in this way describe a principal \mathbf{R}_+ -bundle structure on ν (\mathbf{R}_+ is the multiplicative group of positive reals). We will let $\mu : \mathcal{VM} \times \mathbf{R}_+ \rightarrow \mathcal{VM}$ denote the corresponding (right) action $[\pm\theta] \mapsto [\pm s^{m+1}\theta]$ of \mathbf{R}_+ on the fibres of ν , and also write $\mu_s : \mathcal{VM} \rightarrow \mathcal{VM}$ for the map defined by $\mu_s([\pm\theta]) = \mu([\pm\theta], s)$. The fundamental vector field of this bundle will be denoted by Υ ; in coordinates

$$\Upsilon = x^0 \frac{\partial}{\partial x^0}.$$

Although our construction of the volume bundle is similar to that used by Roberts [13], it is not quite the same. A small difference is that Roberts uses the structure of an \mathbf{R} -bundle rather than an \mathbf{R}_+ -bundle, by exploiting the exponential isomorphism. More significant is that his bundle is built from m -vectors rather than m -covectors — the two bundles are isomorphic, but the natural fibre coordinate is $|v|^{-1}$ rather than $|v|$. Another significant difference is that our \mathbf{R}_+ -bundle structure uses multiplication by s^{m+1} rather than multiplication by s as the right action, and so our fundamental vector field Υ is $-(m + 1)$ times the one used by Roberts.

The volume bundle has some additional natural structure, a so-called odd scalar density, which is defined in the following way. Observe first that $\wedge^m T^*M$, as a bundle of m -covectors, has a tautological m -form Θ ; in coordinates

$$\Theta = v dx^1 \wedge \dots \wedge dx^m.$$

The differential $d\Theta$ is a natural volume form on $\wedge^m T^*M$ defining, at each point $[\pm\theta] \in \mathcal{VM}$, a pair of $(m+1)$ -covectors differing only in sign; this is the odd scalar density we require. We will denote it by $|d\Theta|$. In the coordinates on \mathcal{VM} , this may be written as

$$\pm(m+1)(x^0)^m dx^0 \wedge dx^1 \wedge \dots \wedge dx^m.$$

2.5 TW -connections

Roberts's version of the Thomas-Whitehead theory is based on his notion of a Thomas-Whitehead projective connection, or TW -connection for short. A TW -connection is a symmetric affine connection $\tilde{\nabla}$ on the volume bundle \mathcal{VM} which is invariant under the \mathbf{R}_+ action on $\nu : \mathcal{VM} \rightarrow M$ and which satisfies the condition that $\tilde{\nabla}\Upsilon = \text{id}$, where id is the identity tensor on \mathcal{VM} . The invariance condition is equivalent to saying that Υ is an infinitesimal affine transformation of $\tilde{\nabla}$, and we will generally use it in this form. (The definition above is essentially the one given in [13], with the difference that the formulæ there differ from ours by the constant factor $-(m+1)$ as we use a different fundamental vector field). These conditions on $\tilde{\nabla}$, when expressed in terms of its connection coefficients $\tilde{\Gamma}_{\alpha\beta}^\gamma$ with respect to coordinates (x^α) , adapted to the bundle structure, give $\tilde{\Gamma}_{a0}^0 = \tilde{\Gamma}_{00}^\gamma = 0$, $\tilde{\Gamma}_{a0}^b = (x^0)^{-1}\delta_a^b$; furthermore, the $\tilde{\Gamma}_{ab}^c$ are functions on M transforming as the components of the fundamental descriptive invariant of a restricted path space (though not necessarily satisfying $\tilde{\Gamma}_{ab}^a = 0$), while the $\tilde{\Gamma}_{ab}^0$ are of the form $x^0\alpha_{ab}$ where the α_{ab} are functions on M , transforming appropriately. There is therefore a many-one correspondence between TW -connections and restricted path spaces. The geodesic equations for a TW -connection are

$$\ddot{x}^c + \tilde{\Gamma}_{ab}^c \dot{x}^a \dot{x}^b = -2(\dot{x}^0/x^0)\dot{x}^c, \quad \ddot{x}^0 + x^0 \alpha_{ab} \dot{x}^a \dot{x}^b = 0.$$

The first of these defines the paths on M . The second equation tells us that the terms α_{ab} in the connection essentially determine a preferred parametrization of the paths. Suppose that we wish to make a change of parametrization so that with respect to the new parameter the equations become

$$\ddot{x}^c + \tilde{\Gamma}_{ab}^c \dot{x}^a \dot{x}^b = 0.$$

Then from the first equation s must satisfy $\ddot{s} = -2(\dot{x}^0/x^0)\dot{s}$, and from the second

$$\frac{d}{dt} \left(\frac{\ddot{s}}{\dot{s}} \right) - \frac{1}{2} \left(\frac{\ddot{s}}{\dot{s}} \right)^2 = 2\alpha_{ab} \dot{x}^a \dot{x}^b.$$

The left-hand side of this equation is the Schwarzian derivative of s , sometimes denoted by $S(s)$. It is known that if f is a Möbius function of t ,

$$f(t) = \frac{at + b}{ct + d}$$

for some constants a, b, c and d , then $S(s \circ f) = S(s)$; thus if s is a reparametrization to the new parameter so is $s \circ f$ for any Möbius function f .

The importance of a TW -connection on the volume bundle is that it gives rise to a family of connections on M . It is shown in [13] that given a TW -connection $\tilde{\nabla}$, with the aid of any 1-form ϑ on \mathcal{VM} which is \mathbf{R}_+ -invariant and satisfies $\langle \Upsilon, \vartheta \rangle = 1$ one can construct a symmetric affine connection ∇^ϑ on M whose geodesics are the paths of the restricted path space corresponding to $\tilde{\nabla}$, just as we showed for P^m earlier. Such a 1-form ϑ is the connection form of a connection on the principal bundle $\mathcal{VM} \rightarrow M$. It is well known (see for example [9] Chapter II, Theorem 2.1) that every principal bundle over a paracompact manifold admits a global connection. It follows that every restricted path space on a paracompact manifold M is the space of geodesic paths of some symmetric affine connection on M .

In fact $\tilde{\nabla}$ gives rise in this way to a projective equivalence class $[\nabla]$ of symmetric affine connections on M , the different members of the class corresponding to different choices of ϑ ; the difference $\vartheta' - \vartheta$ of two such 1-forms on \mathcal{VM} is the pull-back of a 1-form on M , which determines the projective transformation relating the two corresponding connections on M . Conversely, each such projective equivalence class $[\nabla]$ gives rise to many TW -connections, and in particular to a unique TW -connection $\tilde{\nabla}$ satisfying the additional conditions that $\tilde{\nabla}(|d\Theta|) = 0$ and that the Ricci curvature of $\tilde{\nabla}$ vanishes. In coordinates,

$$\tilde{\nabla}_0(\partial_0) = 0, \quad \tilde{\nabla}_0(\partial_b) = \tilde{\nabla}_b(\partial_0) = (x^0)^{-1}\partial_b, \quad \tilde{\nabla}_a(\partial_b) = \Pi_{ab}^c \partial_c - \frac{1}{m-1} x^0 \mathfrak{R}_{ab} \partial_0,$$

where as before Π_{ab}^c is the fundamental descriptive invariant of the equivalence class $[\nabla]$ and \mathfrak{R}_{ab} its Ricci ‘tensor’. More generally, those TW -connections for which $\tilde{\nabla}(|d\Theta|) = 0$ take the form

$$\tilde{\nabla}_0(\partial_0) = 0, \quad \tilde{\nabla}_0(\partial_b) = \tilde{\nabla}_b(\partial_0) = (x^0)^{-1}\partial_b, \quad \tilde{\nabla}_a(\partial_b) = \Pi_{ab}^c \partial_c - x^0 \alpha_{ab} \partial_0.$$

That is to say, the condition $\tilde{\nabla}(|d\Theta|) = 0$ forces $\tilde{\Gamma}_{ab}^c$ to be Π_{ab}^c , that is, to have vanishing trace. We will accordingly call such a TW -connection trace-free, and we will call the trace-free TW -connection whose Ricci curvature vanishes the normal TW -connection for the given projective equivalence class.

It is also the case that if M is paracompact, $\nu : \mathcal{VM} \rightarrow M$ admits global sections ([9] Chapter I, Theorem 5.7). A global section σ determines a connection on the principal bundle, which is integrable, and whose connection 1-form is exact, say $d\varphi$; the function φ satisfies $\Upsilon\varphi = 1$, and the horizontal submanifolds are the level sets of φ . Such a global section is called a choice of projective scale in [1]. The corresponding affine connection

$\nabla^{d\varphi}$ has the property that its Ricci tensor is symmetric, and any connection in the projective equivalence class with this property is determined in this way. The projective transformation relating two connections with this property is given by an exact 1-form on M .

We can also relate this construction to that of the so-called tractor bundle introduced by Bailey *et al* [1]. Suppose given a connection form ϑ ; let H_a be the corresponding horizontal lifts of the ∂_a from M to \mathcal{VM} . The invariance of the connection form implies that $\mathcal{L}_\Upsilon H_a = 0$. Now consider the vector fields X on \mathcal{VM} such that $\tilde{\nabla}_\Upsilon X = 0$; call them Υ -vectors. We may equivalently write the defining condition as $\mathcal{L}_\Upsilon X = -X$. The Υ -vectors form a module over functions on M . For any Υ -vector X and for any Y such that $\mathcal{L}_\Upsilon Y \propto \Upsilon$ (i.e. any projectable vector field Y), $\tilde{\nabla}_Y X$ is also a Υ -vector, by virtue of the rules for a TW -connection. If X is a Υ -vector and ϑ is a connection form then the horizontal component of X , that is, $X - \langle X, \vartheta \rangle \Upsilon$, is also a Υ -vector, as is its vertical component $\langle X, \vartheta \rangle \Upsilon$. We can write a vertical Υ -vector as $\mu^0 \Upsilon$; then $\Upsilon \mu^0 = -\mu^0$, from which it follows that μ^0 is a scalar density on M of weight $-1/(m+1)$. We can write a horizontal Υ -vector as $\mu^a H_a$; then similarly the μ^a are components of a contravariant vector density of weight $-1/(m+1)$.

We can now define a covariant derivative operator on Υ -vectors, with respect to vector fields on M , by restricting the arguments of the TW -connection to be respectively projectable vector fields and Υ -vectors. For a trace-free TW -connection the representation of this covariant derivative with respect to an exact connection form $\vartheta = d\varphi$ coincides with the formulæ given in [1].

2.6 TW -connections and sprays

A symmetric affine connection determines and is determined by its corresponding affine spray; it is therefore not surprising that we can specify TW -connections, and in particular the normal TW -connection, entirely in terms of sprays.

The defining conditions for a TW -connection, when expressed in terms of the corresponding spray \tilde{S} on $T(\mathcal{VM})$, turn out to be

$$\mathcal{L}_{\Upsilon^C} \tilde{S} = 0; \quad \mathcal{L}_{\Upsilon^V} \tilde{S} = \Upsilon^C - 2\tilde{\Delta}$$

where Υ^C and Υ^V are respectively the complete and vertical lifts of Υ to $T(\mathcal{VM})$, and $\tilde{\Delta}$ is the Liouville field on $T(\mathcal{VM})$. The first of these conditions is equivalent to the requirement that Υ is an infinitesimal affine transformation of the TW -connection, and the second to the requirement that $\tilde{\nabla} \Upsilon = \text{id}$. The second condition may be reformulated in terms of the horizontal lift Υ^H of Υ to $T(\mathcal{VM})$: since for any vector field X on \mathcal{VM}

$$X^H = \frac{1}{2}(\mathcal{L}_{X^V} \tilde{S} + X^C),$$

we have

$$\Upsilon^C - \Upsilon^H = \tilde{\Delta}.$$

A variant of this formula will be important later.

Both of the claims above are easily confirmed by the following general coordinate calculations. Consider a manifold (\mathcal{VM} for example) equipped with a symmetric affine connection ∇ and corresponding affine spray S . The condition in coordinates (x^α) for a vector field X to be an affine transformation of ∇ is

$$\frac{\partial^2 X^\gamma}{\partial x^\alpha \partial x^\beta} + \frac{\partial X^\delta}{\partial x^\alpha} \Gamma_{\delta\beta}^\gamma + \frac{\partial X^\delta}{\partial x^\beta} \Gamma_{\delta\alpha}^\gamma - \frac{\partial X^\gamma}{\partial x^\delta} \Gamma_{\alpha\beta}^\delta + X^\delta \frac{\partial \Gamma_{\alpha\beta}^\gamma}{\partial x^\delta} = 0,$$

while the condition that $\mathcal{L}_X S = 0$ is just this contracted with u^α and u^β . The condition that $\nabla X = \text{id}$ is

$$\frac{\partial X^\alpha}{\partial x^\beta} + \Gamma_{\beta\gamma}^\alpha X^\gamma = \delta_\beta^\alpha,$$

while

$$\begin{aligned} \mathcal{L}_X S &= \left[X^\alpha \frac{\partial}{\partial u^\alpha}, u^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma \frac{\partial}{\partial u^\alpha} \right] \\ &= X^\alpha \frac{\partial}{\partial x^\alpha} - \left(u^\beta \frac{\partial X^\alpha}{\partial x^\beta} + 2\Gamma_{\beta\gamma}^\alpha u^\beta X^\gamma \right) \frac{\partial}{\partial u^\alpha} \\ &= X^C - 2u^\beta \left(\frac{\partial X^\alpha}{\partial x^\beta} + \Gamma_{\beta\gamma}^\alpha X^\gamma \right) \frac{\partial}{\partial u^\alpha}. \end{aligned}$$

The odd scalar density on \mathcal{VM} defines a volume form vol on $T(\mathcal{VM})$ by ‘squaring’ (and ignoring a constant factor):

$$\text{vol} = (x^0)^{2m} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^m \wedge du^0 \wedge du^1 \wedge \cdots \wedge du^m.$$

It is easy to see that the necessary and sufficient condition for a TW -connection to be trace-free is that the corresponding spray satisfies $\mathcal{L}_S \text{vol} = 0$.

For any affine spray S , the vertical component of $\mathcal{L}_S X^H$ is

$$R_{\beta\gamma\delta}^\alpha u^\beta X^\gamma u^\delta \frac{\partial}{\partial u^\alpha}$$

where $R_{\beta\gamma\delta}^\alpha$ is the curvature of the corresponding connection; the quantity $R_{\beta\gamma\delta}^\alpha u^\beta u^\delta$, which is a type(1,1) tensor field along the projection $\tau_M : TM \rightarrow M$ in component form, is often called the Jacobi endomorphism of the spray. The trace of the Jacobi endomorphism is just $R_{\beta\delta}^\beta u^\beta u^\delta$, a function on TM formed out of the Ricci curvature of the connection. In this way the Ricci curvature can be expressed entirely in terms of the spray. Then a trace-free TW -connection is the normal TW -connection if and only if the trace of its Jacobi endomorphism vanishes.

It can be shown that given a projective equivalence class of affine sprays on a manifold M there is a unique affine spray \tilde{S} on $T(\mathcal{VM})$, whose integral curves when projected into M belong to the path space determined by the projective class, such that

- $\mathcal{L}_{\Upsilon^C} \tilde{S} = 0$;
- $\Upsilon^C - \Upsilon^H = \tilde{\Delta}$;
- $\mathcal{L}_{\tilde{S}} \text{vol} = 0$;
- the Jacobi endomorphism of \tilde{S} has vanishing trace.

This spray is given in coordinates adapted to \mathcal{VM} by

$$\tilde{S} = u^\alpha \frac{\partial}{\partial x^\alpha} - (\Pi_{bc}^a u^b u^c + (x^0)^{-1} u^0 u^a) \frac{\partial}{\partial u^a} + \frac{1}{(m-1)} x^0 \mathfrak{R}_{cd} u^c u^d \frac{\partial}{\partial u^0}.$$

We can define the normal TW -connection as the symmetric affine connection determined by this spray.

Any other affine spray differs from this by a vertical vector field of the form

$$T = T_{\beta\gamma}^\alpha u^\beta u^\gamma \frac{\partial}{\partial u^\alpha}$$

where $T_{\beta\gamma}^\alpha$ are the components of a tensor field on \mathcal{VM} , symmetric in its lower indices. The new spray will continue to define a TW -connection if and only if T satisfies

$$\mathcal{L}_{\Upsilon^C} T = \mathcal{L}_{\Upsilon^V} T = 0,$$

and a trace-free TW -connection if and only if T satisfies in addition

$$\mathcal{L}_T \text{vol} = 0.$$

From the first two conditions we find that $T_{0\beta}^\alpha = T_{\beta 0}^\alpha = 0$, T_{bc}^a is independent of x^0 , and $T_{ab}^0 = x^0 T_{ab}$ where T_{ab} is again independent of x^0 . This determines the general form of a TW -connection. For a trace-free TW -connection we must have $T_{ab}^a = T_{ba}^a = 0$.

We will develop these ideas in our second paper [4], where we will base our generalization of the concept of a TW -connection on the spray approach; the proof of the result above will be given there.

3 Cartan projective geometry

We now turn to Cartan's theory.

3.1 The projective group

The projective group $\text{PGL}(m+1)$ is the quotient of $\text{GL}(m+1)$ by non-zero multiples of the identity. When m is even, $\text{PGL}(m+1) \cong \text{SL}(m+1)$; when m is odd, on the other

hand, elements of $\mathrm{PGL}(m+1)$ may be identified with equivalence classes containing pairs of matrices $\pm g$ where $\det g = \pm 1$ according as the corresponding element of $\mathrm{PGL}(m+1)$ consists of matrices with positive or with negative determinant. We will take particular care in the discussion below to identify any differences between the two cases. We will in fact represent elements of $\mathrm{PGL}(m+1)$ by matrices g with $|\det g| = 1$, but we will bear in mind that for m odd such a matrix is determined only up to sign.

Other authors take G be the group $\mathrm{PSL}(m+1)$ instead: when m is even this the same as $\mathrm{PGL}(m+1)$, but when m is odd the latter group is not connected; $\mathrm{PSL}(m+1)$ is then its identity component. Using this subgroup when m is odd amounts to choosing an orientation for the model geometry (recall that \mathbb{P}^m is orientable in this case); a corresponding Cartan geometry can then be constructed only when M is orientable. The use of $\mathrm{PGL}(m+1)$ avoids this restriction.

In any event, the Lie algebra of G is $\mathfrak{sl}(m+1)$.

The other group of importance in the definition of a Cartan projective connection is the subgroup $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$ which is the stabilizer of the point $[1, 0, \dots, 0] \in \mathbb{P}^m$. In matrix representation its elements are matrices whose first column is zero below the diagonal.

3.2 Cartan projective connections

A Cartan projective geometry, in the sense of Sharpe ([14], Definition 5.3.1), consists of a suitable principal H_{m+1} -bundle $P \rightarrow M$ and an $\mathfrak{sl}(m+1)$ -valued 1-form ω on P , the connection form, satisfying the following conditions:

1. the map $\omega_p : T_p P \rightarrow \mathfrak{sl}(m+1)$ is an isomorphism for each $p \in P$;
2. $R_h^* \omega = \mathrm{ad}(h^{-1})\omega$ for each $h \in \mathrm{H}_{m+1}$; and
3. $\langle A^\dagger, \omega \rangle = A$ for each $A \in \mathfrak{h}_{m+1}$, where \mathfrak{h}_{m+1} is the Lie algebra of H_{m+1} and where A^\dagger is the fundamental vector field corresponding to A .

Though the global, bundle definition of a Cartan connection is the most satisfying, in practice one usually works locally, in a gauge (as indeed Cartan himself did, in effect). By a gauge we simply mean a local section, say κ , of $P \rightarrow M$; the connection form in that gauge is $\kappa^* \omega$, a locally-defined $\mathfrak{sl}(m+1)$ -valued 1-form on M .

It follows from the conditions on ω itemized above that given two overlapping local gauges κ and $\hat{\kappa}$, the corresponding locally-defined matrices of forms $\kappa^* \omega$ and $\hat{\kappa}^* \omega$ on M are related by the transformation rule $\hat{\kappa}^* \omega = \mathrm{ad}(h^{-1})(\kappa^* \omega) + h^*(\theta_{\mathrm{H}_{m+1}})$, where $\theta_{\mathrm{H}_{m+1}}$ is the Maurer-Cartan form on H_{m+1} and h is the local H_{m+1} -valued function relating the two gauges κ and $\hat{\kappa}$. If the domain of h is simply connected we can consistently choose a matrix-valued function to represent it, in which case the transformation rule may be

written as

$$\hat{\kappa}^*\omega = h^{-1}(\kappa^*\omega)h + h^{-1}dh;$$

since h enters this equation quadratically, the possible sign indeterminacy in its matrix representation has no effect.

Conversely, given a covering of M by local gauges and local matrices of forms satisfying this transformation rule, it is possible to reconstruct the principal bundle in terms of transition functions, as we will explain more fully below.

One advantage of working in a gauge is that it may be possible to select a particularly simple gauged connection form, and this is certainly the case for a projective connection. We start with an arbitrary gauged connection form $\kappa^*\omega$ which we assume is defined in a coordinate patch. We can write $\kappa^*\omega$ as a matrix-valued form as follows:

$$\kappa^*\omega = \begin{pmatrix} \omega_0^0 & \omega_b^0 \\ \omega_0^a & \omega_b^a \end{pmatrix};$$

each entry in the matrix is a locally defined 1-form on M . It is a consequence of the defining conditions for a connection form that the map $T_xM \rightarrow \mathbf{R}^m$ defined by the elements ω_0^a below the diagonal in the first column of $\kappa^*\omega$ is an isomorphism, or in other words if we set $\omega_0^a = \omega_{0b}^a dx^b$ then the $m \times m$ matrix (ω_{0b}^a) is nonsingular. We will show that by a change of gauge we can transform ω_0^a to dx^a . To see this, note first that if h is a matrix of the form

$$h = \begin{pmatrix} h_0^0 & h_b^0 \\ 0 & h_b^a \end{pmatrix},$$

then its inverse is given by

$$h^{-1} = \begin{pmatrix} \bar{h}_0^0 & -\bar{h}_0^0 h_c^0 \bar{h}_b^c \\ 0 & \bar{h}_b^a \end{pmatrix}$$

where the overbar signifies (an element of) the inverse matrix ($m \times m$ or 1×1 as the case may be). Note that $\det h = h_0^0 \det(h_b^a)$. We denote the matrix elements of $\hat{\kappa}^*\omega = h^{-1}(\kappa^*\omega)h + h^{-1}dh$ by $\hat{\omega}_\beta^\alpha$, so that $\hat{\omega}_\beta^\alpha = \bar{h}_\gamma^\alpha \omega_\delta^\gamma h_\beta^\delta + \bar{h}_\gamma^\alpha dh_\beta^\gamma$; then $\hat{\omega}_0^a = h_0^0 \bar{h}_b^a \omega_0^b$. In order to make $\hat{\omega}_0^a = dx^a$ we must therefore solve the equations

$$h_0^0 \bar{h}_c^a \omega_{0b}^c = \delta_b^a$$

for elements h_0^0, h_b^a of a matrix h representing an element of \mathbf{H}_{m+1} . From these equations we obtain, by taking determinants,

$$(h_0^0)^{m+1} (\det h)^{-1} \det \omega_0 = 1,$$

where $\omega_0 = (\omega_{0b}^a)$ and $\det \omega_0 \neq 0$. If m is even we require that $\det h = 1$, so $h_0^0 = (\det \omega_0)^{-1/(m+1)}$; this solution is unique. On the other hand, if m is odd we require only that $|\det h| = 1$; then a necessary condition for a solution to exist is that $\det h$ and $\det \omega_0$ have the same sign: if $\det \omega_0 > 0$ then we must take $\det h = 1$ and so $h_0^0 = \pm (\det \omega_0)^{-1/(m+1)}$, whereas if $\det \omega_0 < 0$ then we must take $\det h = -1$ and so

$h_0^0 = \pm(\det(-\omega_0))^{-1/(m+1)}$. In either case, we obtain a unique solution for h_0^0 modulo sign. If we set $h_b^a = h_0^0 \omega_{0b}^a$ we obtain (for any choice of h_a^0) an element of H_{m+1} such that $\hat{\omega}_0^a = dx^a$.

We can combine the solutions for even and odd m in one formula by setting

$$h_0^0 = \frac{\det \omega_0}{|\det \omega_0|} |\det \omega_0|^{-1/(m+1)};$$

it must be understood that when m is odd both $(m+1)$ -th roots must be taken.

There is still some freedom in the choice of gauge, which we can eliminate as follows. The gauge transformation rule gives

$$\hat{\omega}_0^0 = \omega_0^0 - h_b^0 \bar{h}_a^b \omega_0^a + \bar{h}_0^0 dh_0^0;$$

so if we define h_a^0 by

$$h_a^0 dx^a = h_0^0 \omega_0^0 + dh_0^0,$$

we will have $\hat{\omega}_0^0 = 0$. Therefore, for any projective connection on a manifold M there is a covering of M by coordinate patches and for each patch a unique choice of gauge with respect to which the gauged connection form is

$$\begin{pmatrix} 0 & \omega_b^0 \\ dx^a & \omega_b^a \end{pmatrix},$$

where $\omega_a^a = 0$. We call such a gauge the standard gauge for those coordinates.

We can use the standard gauges to find transition functions for the bundle $P \rightarrow M$, and thus define it implicitly. Let $(\omega_\beta^\alpha), (\hat{\omega}_\beta^\alpha)$ be gauged connection forms for a projective connection, in standard gauge with respect to two overlapping coordinate patches with coordinates (x^a) and (\hat{x}^a) . By considering the gauge transformation of $(\hat{\omega}_\beta^\alpha)$ to standard form with respect to the coordinates (x^a) we have $\omega_\beta^\alpha = \bar{h}_\gamma^\alpha \hat{\omega}_\delta^\gamma h_\beta^\delta + h_\gamma^\alpha dh_\beta^\gamma$ with

$$h_0^0 = \varepsilon_J |J|^{-1/(m+1)}, \quad h_b^a = \varepsilon_J |J|^{-1/(m+1)} J_b^a, \quad h_c^0 dx^c = \varepsilon_J d|J|^{-1/(m+1)}$$

where as before (J_b^a) is the Jacobian matrix of the coordinate transformation, J is the Jacobian determinant, and $\varepsilon_J = J/|J|$. That is,

$$\begin{aligned} h &= \begin{pmatrix} \varepsilon_J |J|^{-1/(m+1)} & -\frac{\varepsilon_J}{m+1} |J|^{-(m+2)/(m+1)} \frac{\partial |J|}{\partial x^b} \\ 0 & \varepsilon_J |J|^{-1/(m+1)} J_b^a \end{pmatrix} \\ &= \varepsilon_J |J|^{-1/(m+1)} \begin{pmatrix} 1 & -\frac{1}{m+1} \frac{\partial \log |J|}{\partial x^b} \\ 0 & J_b^a \end{pmatrix}. \end{aligned}$$

If m is even then this is a matrix in $\mathrm{SL}(m+1)$, but if m is odd then both $(m+1)$ -th roots must be taken and the result is a pair of matrices in $\mathrm{GL}(m+1)$ whose determinants are of absolute value 1. In either case we obtain an element of $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$.

Thus given a manifold M with a Cartan projective connection we have an open covering of M by coordinate neighbourhoods $\{U_\lambda\}$ and smooth maps $h_{\mu\lambda} : U_\lambda \cap U_\mu \rightarrow \mathrm{H}_{m+1}$ determined by the gauge transformation between the gauged connections in standard form on the two coordinate patches. The maps $h_{\mu\lambda}$ satisfy

$$h_{\nu\mu}h_{\mu\lambda} = h_{\nu\lambda} \quad \text{on} \quad U_\lambda \cap U_\mu \cap U_\nu;$$

this follows from their construction, but can also be established easily from the explicit formula. They are therefore transition functions in the definition of a principal H_{m+1} -bundle P ; then the connection form in standard gauge will be the pull-back by a suitable local section of a global Cartan connection form on P , and we regain the principal bundle definition of the projective connection.

The transition functions are derived from consideration of the left column of the gauged connection form alone. Using the transition functions and assuming that we have a globally defined Cartan connection form we can compute the coordinate transformation properties of the remaining entries in the gauged connection form. We find, in particular, that if we set $\omega_a^c = \omega_{ab}^c dx^b$ then

$$\hat{\omega}_{ab}^c = \bar{J}_a^d \bar{J}_b^e (J_f^c \omega_{de}^f - J_{de}^c) + \frac{1}{m+1} \frac{\partial \log |J|}{\partial x^d} (\bar{J}_a^d \delta_b^c + \bar{J}_b^d \delta_a^c),$$

that is, that the symmetric part of ω_{ab}^c ,

$$\omega_{(ab)}^c = \frac{1}{2} (\omega_{ab}^c + \omega_{ba}^c),$$

transforms as the fundamental descriptive invariant of a projective equivalence class. However, although $\omega_{ab}^a = 0$, it is not necessarily the case that $\omega_{(ab)}^a = 0$.

3.3 Curvature and torsion

The curvature of a Cartan projective connection is the $\mathfrak{sl}(m+1)$ -valued 2-form (Ω_β^α) where

$$\Omega_\beta^\alpha = d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma.$$

The vanishing of the curvature is the necessary and sufficient condition for the Cartan geometry to be locally diffeomorphic to \mathbb{P}^m , the Klein geometry on which it is modelled.

The torsion of the Cartan connection is the \mathbf{R}^m -valued 2-form (Ω_0^a) .

We can consider curvature and torsion in a local gauge; the definitions are formally the same. Under a change of gauge the curvature transforms by $\hat{\Omega}_\beta^\alpha = \bar{h}_\gamma^\alpha \Omega_\delta^\gamma h_\beta^\delta$; it follows that the torsion transforms by $\hat{\Omega}_0^a = h_0^0 \bar{h}_b^a \Omega_0^b$. Of particular interest are connections

with vanishing torsion. It is clear from the transformation rule that this is a gauge-independent property of a connection. If we take a connection in standard gauge, its torsion is just

$$\Omega_0^a = -\omega_{bc}^a dx^b \wedge dx^c;$$

so the connection has zero torsion if and only if ω_{bc}^a is symmetric in its lower indices.

3.4 Geodesics

The definition of a geodesic in a Cartan geometry depends on the notion of the development of a curve in M into a curve in the Klein geometry G/H on which the Cartan geometry is modelled. Let ω be the Cartan connection form, and κ a gauge. A curve $x(t)$ in M defines a curve X_κ in \mathfrak{g} by

$$X_\kappa(t) = \langle \dot{x}(t), \kappa^* \omega \rangle.$$

We assume that G is a matrix group, for simplicity. Let $g(t)$ be a curve in G which is a solution of the matrix differential equation $\dot{g} = gX_\kappa$, and set $\xi(t) = g(t)\xi_0$ where ξ_0 is the point in the homogeneous space of which H is the stabilizer. It is easy to see that, unlike $g(t)$, $\xi(t)$ is unchanged by a change of gauge: in fact $g(t)$ changes to $g(t)h(t)$ where $h(t)$ is a curve in H . Then $\xi(t)$ is a development of $x(t)$. It is clear that there is a development of a given curve in M through each point of G/H .

If the Klein geometry contains straight lines, a curve in M is called a geodesic of the Cartan geometry if all of its developments into G/H are straight lines.

Since projective space P^m contains straight lines, any Cartan projective geometry has geodesics. We will now find them. We take a connection form in standard gauge and write $X(t)$ for

$$\begin{pmatrix} 0 & \omega_{bc}^0 \dot{x}^c \\ \dot{x}^a & \omega_{bc}^a \dot{x}^c \end{pmatrix}.$$

Then any development $\xi(t)$ of $x(t)$ into P^m is given by $\xi(t) = g(t)\xi_0$ where $\xi_0 = [1, 0, \dots, 0]$ and $g(t)$ satisfies $\dot{g} = gX$. Now $\xi(t)$ is a curve in projective space P^m ; if we wish to consider the equation defining it as a vector equation we must introduce an arbitrary non-vanishing scalar factor, say $\phi(t)$. That is, the development of $x(t)$ is $[u(t)]$ where $u(t)$ is a curve in \mathbf{R}^{m+1} such that $u(t) = \phi(t)g(t)e_0$ where $e_0 = (1, 0, \dots, 0)$. Let us assume that the parametrization is chosen such that the straight line in P^m is given by $\ddot{u} = 0$. Then

$$0 = \frac{d^2}{dt^2}(\phi g)e_0 = (\ddot{\phi}g + 2\dot{\phi}\dot{g} + \phi\ddot{g})e_0 = g(\ddot{\phi}I + 2\dot{\phi}X + \phi(\dot{X} + X^2))e_0.$$

Thus $x(t)$ will be a geodesic if and only if there is some function $\phi(t)$ such that

$$(\ddot{\phi}I + 2\dot{\phi}X + \phi(\dot{X} + X^2))e_0 = 0.$$

This is equivalent to a pair of equations, one vector and one scalar:

$$\ddot{x}^c + \omega_{ab}^c \dot{x}^a \dot{x}^b = -2(\dot{\phi}/\phi)\dot{x}^c, \quad \ddot{\phi} + \phi \omega_{ab}^0 \dot{x}^a \dot{x}^b = 0.$$

From the first of these we see that a global Cartan projective connection determines a restricted path space whose paths are its geodesics; and conversely, given a restricted path space there is a global Cartan connection (in fact there are many) whose geodesics are its paths. For a torsion-free Cartan projective connection the ω_{ab}^c are the components of the fundamental descriptive invariant of the corresponding path space. In fact the geodesic equations are exactly the same as the equations for the geodesics of a TW -connection obtained earlier, with the substitutions of $\omega_{(ab)}^c$ for $\tilde{\Gamma}_{ab}^c$, $\omega_{(ab)}^0$ for α_{ab} , and ϕ for x^0 .

3.5 Normalizing the Cartan projective connection

One of the achievements of Cartan [2] was to show that, although many projective connections give rise to the same restricted path space, there is a distinguished torsion-free connection which can be specified uniquely by conditions on its curvature.

Assume that we are given a restricted path space, and a torsion-free Cartan projective connection adapted to it as just described. We will show how to determine the remaining elements of the Cartan connection by further conditions on the curvature, so as to fix them uniquely. These conditions will be specified in terms of the standard gauge, but will be gauge-independent, which is to say that if they hold in one gauge they hold in any; we can then be sure that a connection which satisfies the conditions and is uniquely determined by them will be globally defined.

By assumption, in standard gauge the gauged connection and curvature forms are given by

$$\begin{pmatrix} 0 & \omega_b^0 \\ dx^a & \Pi_{bc}^a dx^c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Omega_0^0 & \Omega_b^0 \\ 0 & \Omega_b^a \end{pmatrix}.$$

First,

$$\Omega_0^0 = -\omega_{bc}^0 dx^b \wedge dx^c$$

where of course $\omega_b^0 = \omega_{bc}^0 dx^c$; thus if we take ω_{bc}^0 to be symmetric we will have $\Omega_0^0 = 0$. Note that if the connection is torsion-free then Ω_0^0 is unchanged by a gauge transformation, so this property is gauge-independent for torsion-free connections. Then

$$\begin{aligned} \Omega_b^a &= d\omega_b^a + \omega_c^a \wedge \omega_b^c + \omega_0^a \wedge \omega_b^0 \\ &= \frac{1}{2} \left(\mathfrak{R}_{bcd}^a + \delta_c^a \omega_{bd}^0 - \delta_b^a \omega_{cd}^0 \right) dx^c \wedge dx^d \end{aligned}$$

where (using notation from a previous subsection) \mathfrak{R}_{bcd}^a is the curvature ‘tensor’ derived from the Π_{bc}^a . Thus if $\Omega_b^a = \frac{1}{2} \Omega_{bcd}^a dx^c \wedge dx^d$, with Ω_{bcd}^a skew in c and d ,

$$\Omega_{bcd}^a = \mathfrak{R}_{bcd}^a + \delta_c^a \omega_{bd}^0 - \delta_d^a \omega_{bc}^0.$$

We can make Ω_{bcd}^a trace-free ($\Omega_{bcd}^c = 0$) by choosing $(m-1)\omega_{bc}^0 = -\mathfrak{R}_{bc}$, in which case

$$\Omega_{bcd}^a = \mathfrak{R}_{bcd}^a - \frac{1}{m-1} (\mathfrak{R}_{bd}\delta_c^a - \mathfrak{R}_{bc}\delta_d^a) = P_{bcd}^a,$$

the projective curvature tensor. The condition that Ω_{bcd}^a be trace-free is gauge-independent.

The conditions that $\Omega_0^0 = 0$ and $\Omega_{bcd}^c = 0$ determine ω uniquely. That is to say, given a restricted path space, there is a unique globally defined torsion-free $\mathfrak{sl}(m+1)$ -valued Cartan projective connection form with the paths as its geodesics, whose curvature satisfies $\Omega_0^0 = 0$ and $\Omega_{bcd}^c = 0$. It is called the normal projective connection form, and in the standard gauge it is given by

$$\omega = \begin{pmatrix} 0 & -\frac{1}{m-1}\mathfrak{R}_{bc}dx^c \\ dx^a & \Pi_{bc}^a dx^c \end{pmatrix}$$

Note that

$$\begin{aligned} \Omega_b^0 &= -\frac{1}{m-1} \left(d(\mathfrak{R}_{bc}dx^c) + \mathfrak{R}_{cd}dx^d \wedge \Pi_{be}^c dx^e \right) \\ &= \frac{1}{m-1} \mathfrak{R}_{b[c|d]} dx^c \wedge dx^d, \end{aligned}$$

where the brackets in the suffix indicate skew-symmetrization and the solidus ‘covariant differentiation’ with respect to the fundamental invariant. The curvature of the normal projective connection is therefore

$$\Omega = \begin{pmatrix} 0 & \frac{1}{m-1} \mathfrak{R}_{b[c|d]} dx^c \wedge dx^d \\ 0 & \frac{1}{2} P_{bcd}^a dx^c \wedge dx^d \end{pmatrix}.$$

4 The Cartan bundle

In this section we will describe a canonical procedure, starting with a manifold M , for constructing a principal bundle $\mathcal{CM} \rightarrow M$ with structure group $H_{m+1} \subset \text{PGL}(m+1)$ where $m = \dim M$. This procedure does not require a connection (of any kind) for the construction of the principal bundle: it just uses geometric properties of the manifold M . Nevertheless, the bundle constructed in this way has the same transition functions as one built synthetically using the transformation properties of a Cartan projective connection obtained above. We emphasise that the construction works whether or not M is orientable, and whether m is even or odd.

4.1 The Cartan algebroid

In order to construct a principal bundle as the domain for a Cartan projective connection, we will first consider the problem from Cartan's point of view: that at each point of M there should be attached a projective space of the same dimension m . Of course there is already a projective space of dimension $m - 1$, namely the fibre of the projective tangent bundle PTM , but this is too small for our purposes. There are, however, projective spaces of dimension m attached to each point of the volume bundle \mathcal{VM} , and so we will describe a mechanism for transferring these consistently to M . This mechanism will initially work on the underlying vector spaces, and so it will create a vector bundle $\mathcal{WM} \rightarrow M$ which we will call the Cartan algebroid.

We start with the tangent bundle to the volume bundle, $\tau_{\mathcal{VM}} : T(\mathcal{VM}) \rightarrow \mathcal{VM}$. Let $\mu_{s*} : T(\mathcal{VM}) \rightarrow T(\mathcal{VM})$ be the derivative of the action μ_s on the fibres of $\nu : \mathcal{VM} \rightarrow M$, and let \mathcal{WM} be the space of orbits of μ_* ; then \mathcal{WM} is a manifold with coordinates (x^a, u^a, w) where $w = (x^0)^{-1}u^0$, that is, if $\xi \in T(\mathcal{VM})$ and $[\xi]$ is its μ_* -orbit,

$$w([\xi]) = \frac{u^0(\xi)}{x^0(\xi)}$$

(recall that $x^0 > 0$). It is clear that the action μ_* respects the fibration $\nu_* : T(\mathcal{VM}) \rightarrow TM$, so that \mathcal{WM} is fibred over M . If ρ, τ are the two projections from \mathcal{WM} to TM and M respectively, and if χ satisfies $\nu_* = \rho \circ \chi$, then we have the following diagram.

$$\begin{array}{ccccc} T(\mathcal{VM}) & \xrightarrow{\chi} & \mathcal{WM} & \xrightarrow{\rho} & TM \\ \tau_{\mathcal{VM}} \downarrow & & \downarrow \tau & & \downarrow \tau_M \\ \mathcal{VM} & \xrightarrow{\nu} & M & \xrightarrow{=} & M \end{array}$$

Furthermore, the action μ_{s*} is linear on the fibres of $\tau_{\mathcal{VM}}$, so $\tau : \mathcal{WM} \rightarrow M$ is a vector bundle, and the projection $\chi : T(\mathcal{VM}) \rightarrow \mathcal{WM}$ is linear on the fibres. In fact χ is a fibrewise isomorphism, as is evident from the coordinate representation

$$u^a \circ \chi = u^a, \quad w \circ \chi = (x^0)^{-1}u^0$$

of the isomorphism $T_{[\pm\theta]}(\mathcal{VM}) \rightarrow \mathcal{W}_{\nu[\pm\theta]}M$. Two other significant facts about \mathcal{WM} are worth mentioning. First, the fibres of $\chi : T(\mathcal{VM}) \rightarrow \mathcal{WM}$ are the integral curves of Υ^C , the complete lift of Υ to $T(\mathcal{VM})$. Second, we may identify $T(\mathcal{VM})$ with the pullback $\nu^*(\mathcal{WM})$ by the map $\xi \mapsto (\tau_{\mathcal{VM}}(\xi), \chi(\xi))$.

Using the first of these observations we note that since the spray \tilde{S} corresponding to a TW -connection satisfies the condition $\mathcal{L}_{\Upsilon^C}\tilde{S} = 0$, it projects to a vector field on \mathcal{WM} , say $\tilde{S}_{\mathcal{W}}$. We can use $\tilde{S}_{\mathcal{W}}$ to construct the projective equivalence class of affine sprays on TM in another way, as follows. The line bundle $\rho : \mathcal{WM} \rightarrow TM$ admits global sections. A section σ is linear (in the fibre coordinates of $TM \rightarrow M$) if $\sigma_*(\Delta) = \Delta_{\mathcal{W}} \circ \sigma$, where Δ is the Liouville field of TM and $\Delta_{\mathcal{W}}$ is that of the vector bundle $\tau : \mathcal{WM} \rightarrow M$. For any linear section σ , $\rho_*(\tilde{S}_{\mathcal{W}}|_{\sigma})$ is a spray on M . The difference between two linear sections is a linear function on TM , so the corresponding sprays are projectively equivalent. In terms of the construction given previously, a 1-form ϑ on \mathcal{VM} defines a linear function $\hat{\vartheta}$ on $T(\mathcal{VM})$; if $\mathcal{L}_{\Upsilon}\vartheta = 0$ then $\Upsilon^C(\hat{\vartheta}) = 0$, in which case ϑ determines a function on \mathcal{WM} ; the zero set of such a function defines a linear section σ of $\mathcal{WM} \rightarrow TM$, and the spray on TM determined by σ is the spray of the connection on M determined by ϑ .

We will denote the vector space of vector fields on \mathcal{VM} by $\mathfrak{X}(\mathcal{VM})$, and the subspace of vector fields projectable to sections of $\tau : \mathcal{WM} \rightarrow M$ by $\mathfrak{X}_M(\mathcal{VM})$. The latter is a proper subspace of the space of ‘projectable vector fields’ in the ordinary sense, that is those projectable to vector fields on M : for instance ∂_0 projects to a vector field on M (it is, indeed, vertical) but does not project to a section of τ . In fact $\mathfrak{X}_M(\mathcal{VM})$, although not a module over the ring of all functions on \mathcal{VM} , is a module over the sub-ring of functions constant on the fibres of ν . If $X \in \mathfrak{X}_M(\mathcal{VM})$ then X must satisfy

$$X_{\mu_s[\pm\theta]} = \mu_{s*}(X_{[\pm\theta]}),$$

and a local basis for the module is given by

$$\left\{ \Upsilon, \frac{\partial}{\partial x^a} \right\}.$$

The global condition for $X \in \mathfrak{X}_M(\mathcal{VM})$ is $[X, \Upsilon] = 0$, and the Jacobi identity then implies that $\mathfrak{X}_M(\mathcal{VM})$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{VM})$. We will denote the image sections of the local basis by $\{e_0, e_a\}$ (where of course e_0 , as the image of Υ , is defined globally).

It follows from the preceding remarks that the bundle $\tau : \mathcal{WM} \rightarrow M$ is a Lie algebroid, with base dimension m and fibre dimension $m+1$. If $\underline{\chi} : \mathfrak{X}_M(\mathcal{VM}) \rightarrow \text{sect}(\tau)$ denotes the induced map of sections (so that $\underline{\chi}(X)_{\nu[\pm\theta]} = \chi(X_{[\pm\theta]})$) then $\underline{\chi}$ is a module isomorphism, and so may be used to define a Lie bracket on sections of τ ; the map $\rho : \mathcal{WM} \rightarrow TM$ is the anchor map. We will call this bundle the Cartan algebroid of M . It contains no more information than the canonical tangent bundle algebroid because the global section e_0 is in its centre: $[e_0, e] = 0$ for any section e .

The quotient of the Cartan algebroid by the equivalence relation of non-zero multiplication in the fibres is the Cartan projective bundle $P\mathcal{WM}$; this is the projective bundle with m -dimensional fibres that we need.

We have proposed that the construction of the Cartan projective bundle $P\mathcal{WM}$ corresponds to Cartan’s notion of attaching a projective space to each point of the manifold M . To make this correspondence even clearer, we now point out just how firmly the

projective spaces are attached to M by our construction: the Cartan projective bundle is actually soldered to M . (If one is given a Cartan projective connection one can use it to define a soldering; but here we are dealing just with the Cartan projective bundle and make no appeal to the existence of a connection.)

The following definition is taken from Kobayashi [8]. A fibre bundle $B \rightarrow M$ with standard fibre F is soldered to M if the following conditions are satisfied:

- $\dim F = \dim M$;
- B admits a cross-section which will be identified with M ;
- let \tilde{TM} be the space of all tangent vectors to F_x (fibre over $x \in M$) for all $x \in M$: then TM is isomorphic to \tilde{TM} ; more precisely, there is a mapping σ of TM onto \tilde{TM} such that, for each x in M , σ is a non-singular linear mapping of T_xM onto the space of all tangent vectors to F_x at x .

We now show that PWM is soldered to M according to this definition.

The condition on the dimensions is clearly satisfied. We know that $PWM \rightarrow M$ admits a global section, namely $[e_0]$. Notice that the projection $\rho : WM \rightarrow TM$ maps the section e_0 of WM to the zero section of TM , and more generally that the kernel of ρ (as a vector bundle over M) is just the 1-dimensional sub-bundle of WM spanned by e_0 . Let $V_0(WM)$ be the restriction to the section e_0 of the vertical sub-bundle of $T(WM)$, and $V_0(TM)$ the restriction to the zero section of the vertical sub-bundle of TTM , which can of course be canonically identified with TM . Then ρ_* restricts to a linear map of $V_0(WM)$ onto $V_0(TM)$, which is just ρ in a different guise; its kernel is again spanned by e_0 , considered now as a section of $V_0(WM)$ via its vertical lift e_0^V . We will show that $V_0(PWM)$, the restriction to the section $[e_0]$ of the vertical sub-bundle of $T(PWM)$, is canonically isomorphic to $V_0(WM)/\langle e_0^V \rangle$, the quotient bundle of $V_0(WM)$ by the 1-dimensional sub-bundle spanned by e_0^V . It will follow that $V_0(PWM)$ is canonically isomorphic to $V_0(TM)$, and therefore to TM , as required.

This is simply a matter of identifying the tangent space to a projective space in an appropriate way. Let W be a vector space with distinguished non-zero element e , PW the corresponding projective space with distinguished point $[e]$, and $\pi : W \rightarrow PW$ the projection. Then $\pi_* : T_eW \rightarrow T_{[e]}(PW)$ is a surjective linear map whose kernel is the 1-dimensional subspace of T_eW which is the tangent space to the ray through e ; and T_eW is canonically isomorphic to W , with the tangent space to the ray through e corresponding to the 1-dimensional subspace of W spanned by e itself. Thus $T_{[e]}(PW)$ is isomorphic to the quotient space $W/\langle e \rangle$, and the result follows.

It is worth noticing that the soldering isomorphism is canonical only because there is a canonical way of choosing a representative of the projective point $[e_0]$, that is, because WM has a canonical global section e_0 .

4.2 The Cartan principal bundles

By a frame of a vector bundle we mean an ordered basis of a fibre. We define an equivalence relation on frames of the Cartan algebroid as follows. Let (ζ_α) and $(\bar{\zeta}_\alpha)$ be frames of \mathcal{WM} at some point $x \in M$; we let $(\bar{\zeta}_\alpha) \equiv (\zeta_\alpha)$ if there is a non-zero real number λ such that $\bar{\zeta}_\alpha = \lambda\zeta_\alpha$. The corresponding equivalence class will be denoted by $[\zeta_\alpha]$, and is a reference $(m+1)$ -simplex for the m -dimensional projective space PW_xM . The bundle containing all these equivalence classes at all points of M will be denoted by $\mathcal{S}_{\mathcal{W}}M$: it is a principal $\text{PGL}(m+1)$ -bundle over M . If the first element ζ_0 of such an equivalence class is a multiple of the global vector section e_0 then we will call it a Cartan simplex. We will let $\mathcal{CM} \subset \mathcal{S}_{\mathcal{W}}M$ be the bundle containing all the Cartan simplices, and call it the Cartan bundle: it is a principal H_{m+1} -bundle over M , and is a reduction of $\mathcal{S}_{\mathcal{W}}M$.

As Cartan says: ‘It is natural to take each point of the manifold to be one of the vertices of the frame attached at that point’; this corresponds precisely to restricting one’s attention to Cartan simplices, having first identified M with the global section $[e_0]$ of PWM as specified in the definition of soldering.

The Cartan projective bundle PWM is an associated bundle of the principal bundle \mathcal{CM} , using the representation of H_{m+1} as a group of automorphisms of the standard fibre P^m .

4.3 Transition functions

We will now calculate the transition functions for the Cartan bundle \mathcal{CM} relative to local trivializations of the form $[e_\alpha]$, where (e_α) is a local frame field for the Cartan algebroid \mathcal{WM} which is the image of the local frame field

$$\left(\Upsilon, \frac{\partial}{\partial x^a} \right), \quad \Upsilon = x^0 \frac{\partial}{\partial x^0}$$

on \mathcal{VM} , which in turn is an ordered local basis of the module $\mathfrak{X}_M(\mathcal{VM})$ of vector fields projectable to \mathcal{WM} . In fact if $(e_\alpha), (\hat{e}_\alpha)$ are two such local frame fields for \mathcal{WM} , corresponding to coordinates $(x^a), (\hat{x}^a)$ on overlapping coordinate patches U, \hat{U} on M , and we define a $\text{GL}(m+1)$ -valued function G on $U \cap \hat{U}$ by $e_\alpha = G_\alpha^\beta \hat{e}_\beta$, then the transition function for $U \cap \hat{U}$ is just $[G]$, the projection of G into $\text{PGL}(m+1)$.

We must first find the transformation law for the above local basis of $\mathfrak{X}_M(\mathcal{VM})$ with respect to coordinate transformations on M . Suppose that (U, x^a) and (\hat{U}, \hat{x}^a) are overlapping coordinate patches on M , and that $(\nu^{-1}(U), x^a)$ and $(\nu^{-1}(\hat{U}), \hat{x}^a)$ are the corresponding coordinate patches on \mathcal{VM} . Then from $\hat{x}^0 = |J|^{-1/(m+1)} x^0$ we obtain

$$\begin{aligned} \frac{\partial}{\partial x^a} &= J_a^b \frac{\partial}{\partial \hat{x}^b} + \frac{\partial \hat{x}^0}{\partial x^a} \frac{\partial}{\partial \hat{x}^0} = J_a^b \frac{\partial}{\partial \hat{x}^b} + \frac{1}{\hat{x}^0} \frac{\partial \hat{x}^0}{\partial x^a} \Upsilon \\ &= J_a^b \frac{\partial}{\partial \hat{x}^b} - \frac{1}{m+1} \frac{\partial \log |J|}{\partial x^a} \Upsilon. \end{aligned}$$

As we have noted before, Υ is a global vector field and is unchanged by the coordinate transformation. To obtain the corresponding transformation for the e_α we have merely to replace $\partial/\partial x^a$ by e_a and Υ by e_0 in these formulæ. Thus

$$G = \begin{pmatrix} 1 & -\frac{1}{m+1} \frac{\partial \log |J|}{\partial x^b} \\ 0 & J_b^a \end{pmatrix}.$$

So in fact G takes its values in the affine group $A(m)$ (in its standard representation in $GL(m+1)$).

The transition function for \mathcal{CM} with respect to local trivializations $[e_\alpha], [\hat{e}_\alpha]$, is just the projective equivalence class of G . We may represent this projective class by a single matrix with determinant 1 for m even, or by a pair of matrices with determinant ± 1 for m odd, as before. Note that $\det G = J$. When m is even we can form $(\det G)^{-1/(m+1)} = J^{-1/(m+1)}$ whatever the sign of $\det G$, and then $(\det G)^{-1/(m+1)}G$ is the unique member of the projective equivalence class of G whose determinant is 1. When m is odd, on the other hand, we must treat the cases $\det G > 0$ and $\det G < 0$ differently. In the first case we can form $(\det G)^{-1/(m+1)} = J^{-1/(m+1)}$, and then $(\det G)^{-1/(m+1)}G$ gives the two members of the projective equivalence class of G with determinant 1. In the second case we can form $(-\det G)^{-1/(m+1)} = (-J)^{-1/(m+1)}$, and then $(-\det G)^{-1/(m+1)}G$ gives the two members of the projective equivalence class of G with determinant -1 . These prescriptions can be combined in the single formula

$$[G] \equiv \varepsilon_J |J|^{-1/(m+1)} \begin{pmatrix} 1 & -\frac{1}{m+1} \frac{\partial \log |J|}{\partial x^b} \\ 0 & J_b^a \end{pmatrix},$$

as before, which shows that the transition functions for \mathcal{CM} corresponding to the given local trivializations take their values in H_{m+1} and are exactly the functions obtained from the consideration of Cartan projective connections in standard gauge in the previous section. We conclude that the principal H_{m+1} -bundle implicitly defined via Cartan projective connections is (up to equivalence) \mathcal{CM} .

5 Projective connections

We now show how the theory of projective connections of Thomas and Whitehead fits in with the theory of Cartan. We will do so by showing how to construct a torsion-free Cartan connection on the bundle $\mathcal{CM} \rightarrow M$ from any trace-free TW -connection $\tilde{\nabla}$ on \mathcal{VM} . This will be a global construction, in that we will end up with an $\mathfrak{sl}(m+1)$ -valued form on \mathcal{CM} , although we will need to use local gauges to compare the connections obtained.

We first state and prove in general terms a result which will be used in the construction.

Consider a manifold \mathcal{N} with connection ∇ , on which there is defined a 1-parameter group ϕ_t of affine transformations whose infinitesimal generator X satisfies $\nabla X = \text{id}$, such that \mathcal{N} is fibred over an m -dimensional manifold \mathcal{M} where the fibres are the orbits of ϕ_t .

Let \mathcal{FN} be the frame bundle of \mathcal{N} (that is, the bundle of frames of $T\mathcal{N}$). The group $\mathbf{R} \times \mathbf{R} \times \{\pm 1\}$ acts freely on \mathcal{FM} by $\psi_{(s,t,\varepsilon)} : (x, (e_\alpha)) \mapsto (\phi_s x, (\varepsilon e^t \phi_{s*} e_\alpha))$; note that this action commutes with the right action of $\text{GL}(m+1)$ on \mathcal{FN} . Let $\mathcal{S}_\psi \mathcal{M}$ be the quotient of \mathcal{FN} under the action; it is a principal fibre bundle over \mathcal{M} with group $\text{PGL}(m+1)$, and for any $a \in \text{GL}(m+1)$, $\pi \circ R_a = R_{o(a)} \circ \pi$ where $\pi : \mathcal{FN} \rightarrow \mathcal{S}_\psi \mathcal{M}$ and $o : \text{GL}(m+1) \rightarrow \text{PGL}(m+1)$ are the projections.

Introduce local coordinates $(x^\alpha, x_\beta^\alpha)$ on \mathcal{FN} , where for a frame (e_α) , $e_\alpha = x_\beta^\alpha \partial_\beta$. The vector field X has a complete lift X^C to \mathcal{FN} given by

$$X^C = X^\alpha \frac{\partial}{\partial x^\alpha} + x_\beta^\gamma \frac{\partial X^\alpha}{\partial x^\gamma} \frac{\partial}{\partial x_\beta^\alpha};$$

it is the infinitesimal generator of $\psi_{(s,0,1)}$. The generator of $\psi_{(0,t,1)}$, or in other words the fundamental vector field on the $\text{GL}(m+1)$ -bundle \mathcal{FN} corresponding to the unit matrix $I \in \mathfrak{gl}(m+1)$, is

$$I^\dagger = x_\beta^\alpha \frac{\partial}{\partial x_\beta^\alpha}.$$

The vector fields X^C and I^\dagger commute, and the leaves of the integrable distribution \mathcal{D} they define on \mathcal{FN} are just the orbits of the $\psi_{(s,t,1)}$ action. The distribution is invariant under $\psi_{(0,0,\pm 1)}$.

With respect to the connection ∇ , X has a horizontal lift X^H to \mathcal{FN} given by

$$X^H = X^\alpha \frac{\partial}{\partial x^\alpha} - x_\beta^\gamma \Gamma_{\gamma\delta}^\alpha X^\delta \frac{\partial}{\partial x_\beta^\alpha}.$$

Thus

$$X^C - X^H = x_\beta^\gamma \left(\frac{\partial X^\alpha}{\partial x^\gamma} + \Gamma_{\gamma\delta}^\alpha X^\delta \right) \frac{\partial}{\partial x_\beta^\alpha}.$$

For any vector field X and connection ∇ , the condition that $\nabla X = \text{id}$ is

$$\frac{\partial X^\alpha}{\partial x^\beta} + \Gamma_{\beta\gamma}^\alpha X^\gamma = \delta_\beta^\alpha.$$

Thus if $\nabla X = \text{id}$

$$X^C - X^H = I^\dagger$$

(indeed, the two statements are equivalent). This is the frame bundle version of a result proved earlier in the tangent bundle case.

We denote by ω the connection form on \mathcal{FN} corresponding to ∇ . The vector field X generates affine transformations, and so its complete lift X^C satisfies $\mathcal{L}_{X^C} \omega = 0$

([9] Chapter VI, Prop. 2.2). Moreover, $\mathcal{L}_{I^\dagger}\omega = [I, \omega] = 0$. We can write any vector field in \mathcal{D} in the form $Z = fX^C + gI^\dagger$, and $\omega(X^C) = \omega(I^\dagger) = I$. It follows that $\mathcal{L}_Z\omega = I(df + dg)$. Furthermore, ω is clearly invariant under the action of $\psi_{(0,0,\pm 1)}$. We can therefore define an $\mathfrak{sl}(m+1)$ -valued 1-form $\hat{\omega}$ on $\mathcal{S}_\psi\mathcal{M}$ as follows: for $q \in \mathcal{S}_\psi\mathcal{M}$, $v \in T_q(\mathcal{S}_\psi\mathcal{M})$,

$$\langle v, \hat{\omega}_q \rangle = \langle u, o_*\omega_p \rangle$$

for any $p \in \mathcal{FN}$ such that $\pi(p) = q$, and any $u \in T_p(\mathcal{FN})$ such that $\pi_*u = v$, where $o_* : \mathfrak{gl}(m+1) \rightarrow \mathfrak{sl}(m+1)$ is the homomorphism of Lie algebras induced by o . Since $o_*\omega_p(u)$ is not affected if different choices are made of p and u (so long as the same conditions are satisfied), $\hat{\omega}$ is well-defined. We have $\pi^*\hat{\omega} = o_*\omega$, and so for any $a \in \text{GL}(m+1)$,

$$\begin{aligned} \pi^*(R_{o(a)}^*\hat{\omega}) &= R_a^*(\pi^*\hat{\omega}) = R_a^*(o_*\omega) \\ &= o_*(R_a^*\omega) = o_*(\text{ad}(a^{-1})\omega) = \text{ad}(o(a)^{-1})o_*\omega \\ &= \pi^*(\text{ad}(o(a)^{-1})\hat{\omega}), \end{aligned}$$

and so since π is surjective, $R_{o(a)}^*\hat{\omega} = \text{ad}(o(a)^{-1})\hat{\omega}$. Moreover, for any $A \in \mathfrak{gl}(m+1)$, we have $\pi_*(A^\dagger) = (o_*A)^\dagger$, and therefore

$$\hat{\omega}((o_*A)^\dagger) = (\pi^*\hat{\omega})(A^\dagger) = o_*(\omega(A^\dagger)) = o_*A.$$

Thus $\hat{\omega}$ is the connection form of a connection on the principal $\text{PGL}(m+1)$ -bundle $\mathcal{S}_\psi\mathcal{M}$.

From the fact that $\nabla X = \text{id}$ it follows that a non-zero multiple of X can be parallel along a non-trivial curve in \mathcal{N} only if that curve is (up to reparametrization) an integral curve of X : for $\nabla_u(fX) = u(f)X + fu = 0$ only if u is proportional to X . The same holds for a frame any member of which is a multiple of X . We now interpret this observation in terms of the properties of the connection on \mathcal{FN} , using the fact that a horizontal curve in \mathcal{FN} can be thought of as a curve in \mathcal{N} (its projection) with a parallel frame along it. Let us denote by $\mathcal{F}_X\mathcal{N} \subset \mathcal{FN}$ the sub-bundle consisting of those frames whose first member is a multiple of X . Then at any point $p \in \mathcal{F}_X\mathcal{N}$, we have $H_p \cap T_p(\mathcal{F}_X\mathcal{N}) = \langle X_p^H \rangle$, that is, the horizontal subspace at p (the kernel of ω_p) intersects the tangent space to $\mathcal{F}_X\mathcal{N}$ at p in the 1-dimensional subspace spanned by the horizontal lift of X to p . When we pass to the quotient, at any point $q \in \pi(\mathcal{F}_X\mathcal{N})$ we have $\ker \hat{\omega}_q \cap T_q(\pi(\mathcal{F}_X\mathcal{N})) = \{0\}$.

We now return to the TW -connection, which we represent in the standard way as an Ehresmann connection $\tilde{\omega}$ on the frame bundle $\mathcal{F}(\mathcal{VM})$ of the volume bundle; we wish to construct from this a Cartan connection form on \mathcal{CM} . Since the TW -connection satisfies the conditions of the theorem above with respect to Υ , we can as a first step construct from it a related Ehresmann connection $\hat{\omega}$ on the simplex bundle $\mathcal{S}_{\mathcal{VM}}$.

The second and final step of our construction is to use this Ehresmann connection on $\mathcal{S}_{\mathcal{VM}}$ to define a Cartan connection on the sub-bundle $\mathcal{CM} \subset \mathcal{S}_{\mathcal{VM}}$. This sub-bundle has codimension m , and so the restriction ω of $\hat{\omega}$ to \mathcal{CM} will define a Cartan connection if the intersection (in $T(\mathcal{S}_{\mathcal{VM}})$) of $\ker \hat{\omega}$ and $T(\mathcal{CM})$ contains only zero vectors ([14], Proposition A.3.1; see also [10]). But \mathcal{CM} is the image in $\mathcal{S}_{\mathcal{VM}}$ of the sub-bundle of

$\mathcal{F}(\mathcal{VM})$ consisting of those frames with first element a multiple of Υ , so this follows from the fact that for the TW -connection $\tilde{\nabla}\Upsilon = \text{id}$.

We now follow the construction through using explicit representations of the connections in suitable gauges. Since by assumption the TW -connection is trace-free, in the gauge (∂_α) on \mathcal{VM} it will be given by the $\mathfrak{gl}(m+1)$ -valued form

$$\tilde{\omega}_{(\partial_\alpha)} = \begin{pmatrix} 0 & x^0 \alpha_{bc} dx^c \\ (x^0)^{-1} dx^a & \Pi_{bc}^a dx^c + \delta_b^a (x^0)^{-1} dx^0 \end{pmatrix}.$$

This gauge is not, however, a projectable gauge, in that the vector field ∂_0 is not projectable to a local section of the Cartan algebroid. In the projectable gauge (Υ, ∂_a) , we have

$$\tilde{\omega}_{(\Upsilon, \partial_a)} = \begin{pmatrix} (x^0)^{-1} dx^0 & \alpha_{bc} dx^c \\ dx^a & \Pi_{bc}^a dx^c + \delta_b^a (x^0)^{-1} dx^0 \end{pmatrix} = ((x^0)^{-1} dx^0)I + \begin{pmatrix} 0 & \alpha_{bc} dx^c \\ dx^a & \Pi_{bc}^a dx^c \end{pmatrix}.$$

The Ehresmann connection $\hat{\omega}$ on the simplex bundle $\mathcal{S}_{\mathcal{VM}}$, in the gauge $[e_\alpha]$, is

$$\hat{\omega}_{[e_\alpha]} = \begin{pmatrix} 0 & \alpha_{bc} dx^c \\ dx^a & \Pi_{bc}^a dx^c \end{pmatrix},$$

and this is also the connection form of the Cartan connection ω in the same gauge.

If $[\nabla]$ is a projective equivalence class of affine connections on M , and $\tilde{\nabla}$ any corresponding trace-free TW -connection, it is immediate from the entries in the matrix that the Cartan connection we obtain has the same unparametrized geodesics as the equivalence class $[\nabla]$. Furthermore, this procedure establishes a 1-1 correspondence between the trace-free TW -connections and the torsion-free Cartan connections satisfying $\Omega_0^0 = 0$ associated with any given restricted path space, in which the normal TW -connection corresponds to the normal projective Cartan connection.

6 Conclusion

The investigation of the relationship between the theories of Cartan and of Thomas and Whitehead which we have described in this paper is, as we have mentioned, the first part of a more general study of projective connections. The second part, which is contained in [4], deals with the following generalization.

In place of a class of projectively equivalent symmetric affine connections on a manifold, and their corresponding (quadratic) geodesic sprays on the tangent manifold, we consider instead a class of projectively equivalent sprays on the slit tangent manifold whose coefficients Γ^a are homogeneous of degree 2 rather than quadratic. These are the path spaces described by Douglas, without the prefix ‘restricted’ [5]. They do not in general arise from affine connections on the manifold; instead they are related to connections on a pull-back bundle, the Berwald connections.

The corresponding general second-order differential equation was also studied by Cartan [2] with an indication that his calculations could be extended to systems of differential equations, and hence to manifolds of dimension greater than 2. In modern terminology, Cartan's constructions also take place on a pull-back bundle.

In [4] we show that, by analogy with the affine case, each equivalence class of sprays gives rise to a unique Cartan connection on a sub-bundle of a suitable pull-back bundle and that, once again, the geodesics of the sprays (their base integral curves, unparametrized) are precisely the geodesics of the Cartan connection. We also show that, when the spray is actually quadratic, so that its Berwald connection is the pull-back of an affine connection on the manifold, then the Cartan connections constructed by the restricted theory and the general theory are essentially the same.

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