Affine and projective transformations of Berwald connections

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Abstract

We discuss the infinitesimal affine transformations of the Berwald connection of a spray, and the relation between the projective transformations of a spray and the affine transformations of its Berwald-Thomas-Whitehead connection.

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1 Introduction

In the context of differential geometry, an affine transformation of a manifold with a connection is a diffeomorphism that preserves parallel transport, while a projective transformation is a diffeomorphism that maps each geodesic to a geodesic as a path (that is, as a point set but possibly with a sense-preserving reparametrization). A vector field whose flow consists of affine transformations is called an infinitesimal affine transformation; likewise, one whose flow consists of projective transformations is an infinitesimal projective transformation.

Of course the infinitesimal affine transformations of $\mathbf{R}^n$ (with the standard flat connection) are the affine vector fields in the usual sense. The infinitesimal projective transformations of real projective space P^m can be realised as equivalence classes of linear vector fields on $\mathbf{R}_0^{m+1}$, where the subscript indicates the exclusion of the origin, modulo constant multiples of the dilation field. The linear vector fields are just the infinitesimal affine transformations of $\mathbf{R}_0^{m+1}$ which commute with the dilation field, while the constant multiples of the dilation field itself are the infinitesimal affine transformations which are tangent to the fibration $\mathbf{R}_0^{m+1} \to \mathrm{P}^m$.

Our ultimate aim in this paper is to show that this relation between projective transformations of an m-dimensional manifold and affine transformations of another manifold of dimension $m+1$, fibred over the first, holds in very general circumstances, perhaps the most general circumstances in which the concepts make sense. We have shown in [3, 7] that associated with an m-dimensional manifold M there is a canonically defined $(m+1)$-manifold $\mathcal{V}M$, fibred over M, which stands in the same relation to M as $\mathbf{R}_0^{m+1}$ does to P^m. Furthermore, if M is equipped with a spray S, and the corresponding Berwald connection, then there is a uniquely-defined spray $\tilde{S}$ on $\mathcal{V}M$, with corresponding Berwald connection, which encodes all the projective information about S (and therefore about its projective equivalence class). We called $\tilde{S}$ the Berwald-Thomas-Whitehead spray, or BTW-spray, of (the projective equivalence class of) S, and its Berwald connection the Berwald-Thomas-Whitehead connection, or BTW-connection; the terminology is supposed to suggest (what is in fact the case) that this is a generalization of a construction which in effect was first given, for affine connections, by T.Y. Thomas [10, 11] and by Whitehead [12]. We will show here that infinitesimal projective transformations of S correspond to infinitesimal affine transformations of the BTW-connection in exactly the way we described above for projective space.

In order to achieve this objective we must define, and develop the properties of, infinitesimal affine transformations of a Berwald connection. The transformation theory of sprays and Berwald connections was in vogue towards the middle of the last century — Chapter VIII of Yano's book 'The Theory of Lie Derivatives and its Applications' [13] gives an excellent survey of the state of the art in 1957 — but then went out of fashion; the subject has been taken up again very recently by Lovas [5]. The definition of an infinitesimal affine transformation of a Berwald connection is not entirely straightforward, because a Berwald connection is defined on a pull-back bundle (a pull-back of a tangent bundle in fact). We feel that the concept of the Lie derivative of a section of such a pull-back bundle has not received the careful geometrical consideration that it deserves. We will attempt to remedy this defect below; in doing so we will introduce a broadening of the concept of infinitesimal affine transformation, compared with that of the authors mentioned above, which has some interesting and useful features.

We discuss the definition of the Lie derivative of a section of a pull-back of a tangent bundle in Section 2. In Section 3 we review the basic theory of sprays and their Berwald connections, and we define the infinitesimal affine transformations of a Berwald connection and discuss their properties in Section 4. Sections 5 and 6 are devoted to the theory of infinitesimal projective transformations of sprays. In Section 7 we describe the construction of the BTW-connection associated with a projective equivalence class of sprays. Sections 8 and 9 are devoted to the main results, relating the projective transformations of a spray with the affine transformations of the BTW-connection of its projective equivalence class. The final section contains an example.

We use the Einstein summation convention.

2 Transformations of pull-back bundles

As a preliminary to our consideration of Berwald connections and their transformations, we make some general observations regarding the pull-backs of tangent bundles, leading to a discussion of Lie derivatives of sections of such a bundle.

Let $\pi : E \to B$ be a bundle, $\tau_B : TB \to B$ the tangent bundle of the base manifold B, and $\pi^*(TB) \to E$ the pullback of TB to E. Let $\varphi : E \to E$ be a fibred map, fibred over $f : B \to B$. Then we can lift φ to a map $\tilde{\varphi} : \pi^*(TB) \to \pi^*(TB)$, fibred over φ, as follows: for $p \in E$, $u \in TB$ with $\pi(p) = \tau_B(u)$, set

$$\tilde{\varphi}(p, u) = (\varphi(p), f_*(u)),$$

where $f_* : TB \to TB$ is the tangent map to f. We have $\pi \circ \varphi = \tau_B \circ f_* = f$, so $\tilde{\varphi}$ is well-defined. In coordinates (x^a, y^A, u^a), with $\varphi(x, y) = (f^a(x), \varphi^A(x, y))$, we have

$$\tilde{\varphi}(x, y, u) = \left(f^a(x), \varphi^A(x, y), \frac{\partial f^a}{\partial x^b} u^b \right).$$

For two such maps φ_1 and φ_2 we have $\widetilde{\varphi_1 \circ \varphi_2} = \tilde{\varphi}_1 \circ \tilde{\varphi}_2$.

A projectable vector field Z on E lifts to a vector field $\tilde{Z}$ on $\pi^*(TB)$ by applying the lifting process to its flow. If

$$Z = X^a(x) \frac{\partial}{\partial x^a} + Z^A(x, y) \frac{\partial}{\partial y^A},$$

so that

$$\pi_* Z = X^a \frac{\partial}{\partial x^a} = X \text{ say},$$

then

$$\tilde{Z} = X^a(x) \frac{\partial}{\partial x^\alpha} + Z^A(x, y) \frac{\partial}{\partial y^A} + u^b \frac{\partial X^a}{\partial x^b} \frac{\partial}{\partial u^a}.$$

Furthermore, $\widetilde{[Z_1, Z_2]} = [\tilde{Z}_1, \tilde{Z}_2]$.

We shall refer to sections of $\pi^*(TB) \to E$ as vector sections for brevity; in other terms, a vector section is a vector field along π. Thus for example if Y is any vector field on E then $\pi_* Y$ is a vector section.

Given any vector section ξ and any fibred diffeomorphism $\varphi : E \to E$ we can define a new vector section $\varphi_\sharp \xi$ by $\varphi_\sharp \xi(p) = \tilde{\varphi}(\xi(\varphi^{-1}(p)))$. We can thus define a Lie derivative operator of projectable vector fields on vector sections: for any projectable vector field Z and vector section ξ set

$$\mathcal{L}_Z \xi = \frac{d}{dt} (\psi_{(-t)\sharp} \xi)_{t=0},$$

where ψ_t is the flow of Z. If we think of ξ as a derivation from functions on B to functions on E, we have for a function F on B

$$\mathcal{L}_Z\xi(F) = Z(\xi(F)) - \xi(X(F)).$$

In coordinates

$$\mathcal{L}_Z\left(\xi^a\frac{\partial}{\partial x^a}\right) = \left(X^b\frac{\partial \xi^a}{\partial x^b} + Z^A\frac{\partial \xi^a}{\partial y^A} - \xi^b\frac{\partial X^a}{\partial x^b}\right)\frac{\partial}{\partial x^a}.$$

For any vector field Y on E we have

$$\mathcal{L}_Z(\pi_* Y) = \pi_*[Z, Y];$$

changing Y by the addition of a vertical vector field leaves this equality intact because Z is assumed to be projectable.

Any vector field on B defines a vector section by composition with the projection $E \to B$; we will call such a vector section basic, and use the same symbol for both vector field and basic vector section. If ξ is a basic vector section then $\mathcal{L}_Z\xi$ is the basic vector section $[X, \xi]$, where the bracket is the bracket of vector fields on B.

3 Sprays and Berwald connections

From now on, we focus our attention on a manifold M and its slit tangent bundle $\tau_M^\circ : T^\circ M \to M$; vector sections will be sections of the pull-back bundle $\tau_M^{\circ *}(TM) \to T^\circ M$. We use coordinates (x^a, u^a) on $T^\circ M$.

A connection of Berwald type on M is a covariant differentiation operator on sections of $\tau_M^{\circ *}(TM) \to T^\circ M$, determined by a horizontal distribution on $T^\circ M$. Suppose that the horizontal distribution is spanned by vector fields H_a where

$$H_a = \frac{\partial}{\partial x^a} - \Gamma_a^b\frac{\partial}{\partial u^b}.$$

Then the connection is given by

$$\nabla_{H_a}\left(\frac{\partial}{\partial x^b}\right) = \Gamma_{ab}^c\frac{\partial}{\partial x^c}, \quad \nabla_{V_a}\left(\frac{\partial}{\partial x^b}\right) = 0,$$

where $V_a = \partial/\partial u^a$ and

$$\Gamma_{ab}^c = \frac{\partial \Gamma_a^c}{\partial u^b};$$

it is not necessarily the case that $\Gamma_{ba}^c = \Gamma_{ab}^c$.

It is a defining property of a connection of Berwald type that it reduces to the canonical complete parallelism on the fibres of $\tau_M^{\circ *}(TM)$, that is, for any basic vector section ξ and any vertical vector field V, $\nabla_V\xi = 0$.

A connection of Berwald type is said to be homogeneous if the Γ^a_{bc} are homogeneous of degree 0 in the u^a.

A spray on a manifold M is a vector field S on $T^\circ M$ of the form

$$S = u^a \frac{\partial}{\partial x^a} - 2\Gamma^a \frac{\partial}{\partial u^a}$$

where the coefficients Γ^a are homogeneous of degree 2 in the u^a. Associated with a spray is a horizontal distribution with

$$\Gamma^a_b = \frac{\partial \Gamma^a}{\partial u^b},$$

and in turn a connection of Berwald type, which is called the Berwald connection of the spray. The Berwald connection of a spray is homogeneous, as is its horizontal distribution; moreover for a Berwald connection we have $\Gamma^c_{ba} = \Gamma^c_{ab}$.

These matters are discussed in greater detail in, for example, [1, 9].

4 Affine transformations of Berwald connections

Suppose that M is equipped with a spray S and the corresponding Berwald connection ∇. Using the connection one defines the covariant derivative of a vector section along a curve in $T^\circ M$ in the usual way; a vector section is parallel along a curve if its covariant derivative is zero.

A fibred diffeomorphism of $T^\circ M$ which preserves parallelism is called an affine transformation. That is to say, a fibred diffeomorphism $\varphi : T^\circ M \to T^\circ M$ is an affine transformation if for every curve σ in $T^\circ M$ and every parallel vector section ξ along σ, $\varphi_\sharp \xi$ (which is a vector section along the image curve $\varphi \circ \sigma$) is again parallel.

For any frame $\{e_a\}$ at $\sigma(0)$ (that is, any basis of $T_{\tau^\circ_M(\sigma(0))}M$) there is a unique parallel frame $\{E_a(t)\}$ along σ such that $E_a(0) = e_a$. For any vector section ξ along σ we may write $\xi = \xi^a E_a$, and then $\nabla_{\dot\sigma}\xi = \dot\xi^a E_a$. If $\varphi : T^\circ M \to T^\circ M$ is affine, then $\varphi_\sharp E_a$ is parallel along $\varphi \circ \sigma$, and so

$$\varphi_\sharp(\nabla_{\dot\sigma}\xi) = \dot\xi^a \varphi_\sharp E_a = \nabla_{\varphi_*\dot\sigma}\varphi_\sharp\xi.$$

It follows that a fibred diffeomorphism $\varphi : T^\circ M \to T^\circ M$ is an affine transformation if and only if

$$\varphi_\sharp(\nabla_Y \xi) = \nabla_{\varphi_* Y}\varphi_\sharp\xi$$

for every vector field Y on $T^\circ M$ and every vector section ξ.

A complete projectable vector field Z on $T^\circ M$ is an infinitesimal affine transformation if its one-parameter group consists of affine transformations. The necessary and sufficient condition for Z to be an infinitesimal affine transformation is that

$$\mathcal{L}_Z(\nabla_Y \xi) = \nabla_{[Z,Y]}\xi + \nabla_Y \mathcal{L}_Z \xi$$

for every vector field Y on $T^\circ M$ and every vector section ξ, or

$$[\mathcal{L}_Z, \nabla_Y] = \nabla_{[Z,Y]}$$

for every vector field Y on $T^\circ M$. More generally, any vector field (complete or not) which satisfies this condition is called an infinitesimal affine transformation. Roughly speaking, an infinitesimal affine transformation is a vector field whose flow consists of affine transformations, as we said in the Introduction.

Note that we require only that Z is projectable. Other authors [5, 13] have always taken an infinitesimal affine transformation to be the complete lift of a vector field X on M. This is of course the most interesting case, but our generalization has its points of interest too. If $Z = X^{\mathrm{C}}$ is an infinitesimal affine transformation we will call X (a vector field on M) a strict infinitesimal affine transformation.

We will for the moment continue with projectable vector fields, but we will find it convenient to write them in the form $Z = X^{\mathrm{C}} + V$, where V is of course vertical.

We now give the coordinate conditions for a projectable vector field Z to be an infinitesimal affine transformation of a Berwald connection. We first show that for any projectable vector field Z and any vertical vector field W on $T^\circ M$, $[\mathcal{L}_Z, \nabla_W] = \nabla_{[Z,W]}$, so that this part of the condition for Z to be an infinitesimal affine transformation is automatically satisfied. It will be enough to prove that $\mathcal{L}_Z(\nabla_W \xi) - \nabla_W \mathcal{L}_Z \xi = \nabla_{[Z,W]}\xi$ whenever ξ is basic and W is vertical. But since the connection is of Berwald type $\nabla_W \xi = 0$; $\mathcal{L}_Z \xi$ is basic so $\nabla_W \mathcal{L}_Z \xi = 0$; and $[Z, W]$ is vertical so $\nabla_{[Z,W]}\xi = 0$.

We evaluate the condition $\mathcal{L}_Z(\nabla_Y \xi) = \nabla_{[Z,Y]}\xi + \nabla_Y \mathcal{L}_Z \xi$ with $\xi = \partial/\partial x^b$, $Y = H_a$ and

$$Z = X^a \frac{\partial}{\partial x^a} + \left(u^b \frac{\partial X^a}{\partial x^b} + V^a \right) \frac{\partial}{\partial u^a}.$$

We find that

$$\mathcal{L}_Z \left(\frac{\partial}{\partial x^b} \right) = -\frac{\partial X^c}{\partial x^b} \frac{\partial}{\partial x^c}, \quad [Z, H_a] = -\frac{\partial X^c}{\partial x^a} H_c \pmod{V_b}.$$

The condition with $Y = H_a$ is

$$\frac{\partial^2 X^c}{\partial x^a \partial x^b} + \frac{\partial X^d}{\partial x^a}\Gamma^c_{db} + \frac{\partial X^d}{\partial x^b}\Gamma^c_{ad} - \frac{\partial X^c}{\partial x^d}\Gamma^d_{ab} + Z(\Gamma^c_{ab}) = 0.$$

This condition can equivalently be expressed in terms of Berwald covariant derivatives, as follows:

$$\nabla_{H_a}\nabla_{H_b} X^c + X^d R^c_{bda} + (\nabla_S X^d + V^d) B^c_{abd} = 0,$$

where R^d_{cab} is the Riemann curvature of the Berwald connection,

$$R^d_{cab}\frac{\partial}{\partial x^d} = \left(\nabla_{H_a}\nabla_{H_b} - \nabla_{H_b}\nabla_{H_a} - \nabla_{[H_a,H_b]} \right) \frac{\partial}{\partial x^c},$$

B^d_{abc} its Berwald curvature,

$$B^d_{abc}\frac{\partial}{\partial x^d} = \left(\nabla_{V_a}\nabla_{H_b} - \nabla_{H_b}\nabla_{V_a} - \nabla_{[V_a,H_b]}\right)\frac{\partial}{\partial x^c} = \frac{\partial\Gamma^d_{bc}}{\partial u^a}\frac{\partial}{\partial x^d};$$

and $\nabla_S X^d$ is the dth component of $\nabla_S X$, the covariant derivative of the basic vector section X with respect to the spray $S = u^a H_a$.

Notice that the necessary and sufficient condition for a vertical vector field to be an infinitesimal affine transformation is that $V^d B^c_{abd} = 0$. In particular, since the connection is homogeneous the Liouville field is an infinitesimal affine transformation. Since the vanishing of the Berwald curvature is the necessary and sufficient condition for a spray to be affine, we have the following amusing result: a necessary and sufficient condition for a spray to be affine is that every vertical vector field is an infinitesimal affine transformation of its Berwald connection.

On the other hand, the condition for a vector field X on M to be a strict infinitesimal affine transformation is that

$$\frac{\partial^2 X^c}{\partial x^a \partial x^b} + \frac{\partial X^d}{\partial x^a}\Gamma^c_{db} + \frac{\partial X^d}{\partial x^b}\Gamma^c_{ad} - \frac{\partial X^c}{\partial x^d}\Gamma^d_{ab} + X^{\mathrm{C}}(\Gamma^c_{ab}) = 0,$$

or in terms of covariant derivatives

$$\nabla_{H_a}\nabla_{H_b}X^c + X^d R^c_{bda} + \nabla_S X^d B^c_{abd} = 0,$$

the condition given by Yano [13]. In particular, if the spray in question is actually affine then this condition (without the term involving B^c_{abd}) is just the condition for X to be an infinitesimal affine transformation of the corresponding affine connection.

In this latter special case we know that X is an infinitesimal affine transformation if and only if $[X^{\mathrm{C}}, S] = 0$. We now consider the same question in the general case, where the spray need not be affine, and where $Z = X^{\mathrm{C}} + V$. Now for any complete lift X^{C} and spray S, $[X^{\mathrm{C}}, S]$ is vertical, and for any vertical vector field V,

$$[V, S] = V^a\frac{\partial}{\partial x^a} \pmod{V_b}.$$

Thus in order that $[Z, S] = 0$ we must have $V = 0$, so that in fact Z must be a complete lift; when that is the case then the remaining part of the condition becomes

$$\tfrac{1}{2}u^a u^b\frac{\partial^2 X^c}{\partial x^a \partial x^b} - \frac{\partial X^c}{\partial x^d}\Gamma^d + X^d\frac{\partial\Gamma^c}{\partial x^d} + u^e\frac{\partial X^d}{\partial x^e}\frac{\partial\Gamma^c}{\partial u^d} = 0.$$

This is equivalent to the condition given above for X to be strictly affine: differentiating the second equation twice, with respect to u^a and u^b, gives the first; contracting the first with u^a and u^b gives second by homogeneity.

Much of the theory above could be applied without much change to connections of Berwald type, for what it would be worth.

5 Projectively equivalent sprays and projective transformations

For an extensive discussion of the projective differential geometry of sprays see [2, 3]; also, Shen's book [8] is a standard modern reference.

Two sprays are projectively equivalent if they have the same geodesics, considered as oriented paths (point sets traversed in a given sense). A necessary and sufficient condition for S, $\hat{S}$ to be projectively equivalent is $S - \hat{S} = \alpha\Delta$ for some function α on $T^\circ M$, positively homogeneous of degree 1.

A vector field X on M is an infinitesimal projective transformation of a spray S if the flow of X^{C} maps S to a projectively equivalent spray. The condition for this is that

$$[X^{\mathrm{C}}, S] = F\Delta$$

for some function F on $T^\circ M$, positively homogeneous of degree 1. Such a vector field will then be an infinitesimal projective transformation of each spray in the projective equivalence class. In coordinates, the condition for X to be an infinitesimal projective transformation is

$$\tfrac{1}{2}u^b u^c \frac{\partial^2 X^a}{\partial x^b \partial x^c} + u^c \frac{\partial X^b}{\partial x^c}\Gamma^a_b - \frac{\partial X^a}{\partial x^b}\Gamma^b + X^b \frac{\partial \Gamma^a}{\partial x^b} = -\tfrac{1}{2}F u^a.$$

By differentiating with respect to u^b and taking the trace over a and b, using the homogeneity of F, one obtains an expression for F involving $m = \dim M$:

$$-\tfrac{1}{2}(m+1)F = u^c \frac{\partial(\operatorname{div} X)}{\partial x^c} + u^d \frac{\partial X^c}{\partial x^d}\Gamma_c + X^c \frac{\partial \Gamma}{\partial x^c},$$

where $\Gamma = \Gamma^a_a$, $\Gamma_c = \partial\Gamma/\partial u^c$, and $\operatorname{div} X = \partial X^a/\partial x^a$.

Using this result, the conditions for X to be an infinitesimal projective transformation can be written in a convenient form in terms of the quantities

$$\Pi^a = \Gamma^a - \frac{1}{m+1}\Gamma u^a, \qquad \Pi^a_b = \frac{\partial \Pi^a}{\partial u^b}, \qquad \Pi^a_{bc} = \frac{\partial^2 \Pi^a}{\partial u^b \partial u^c}.$$

The Π^a_{bc} are called by Douglas [4] the fundamental invariants of the projective class; we will use this term indiscriminately for any of the above quantities; they are after all essentially equivalent, since $\Pi^a = \frac{1}{2}\Pi^a_b u^b = \frac{1}{2}\Pi^a_{bc}u^b u^c$, by homogeneity. It is an important property of the fundamental invariants that $\Pi^a_a = 0$. The conditions for X to be an infinitesimal projective transformation become

$$\tfrac{1}{2}u^b u^c \frac{\partial^2 X^a}{\partial x^b \partial x^c} + u^c \frac{\partial X^b}{\partial x^c}\Pi^a_b - \frac{\partial X^a}{\partial x^b}\Pi^b + X^b \frac{\partial \Pi^a}{\partial x^b} = \frac{1}{m+1} u^b \frac{\partial(\operatorname{div} X)}{\partial x^b} u^a,$$

or equivalently

$$\frac{\partial^2 X^a}{\partial x^b \partial x^c} + \frac{\partial X^d}{\partial x^b}\Pi^a_{cd} + \frac{\partial X^d}{\partial x^c}\Pi^a_{bd} - \frac{\partial X^a}{\partial x^d}\Pi^d_{bc} + X^d \frac{\partial \Pi^a_{bc}}{\partial x^d} + u^e \frac{\partial X^d}{\partial x^e}\frac{\partial \Pi^a_{bc}}{\partial u^d}$$
$$= \frac{1}{m+1}\left(\frac{\partial(\operatorname{div} X)}{\partial x^b}\delta^a_c + \frac{\partial(\operatorname{div} X)}{\partial x^c}\delta^a_b\right)$$

6 Projective transformations of projective space

Before proceeding with the general discussion, we briefly examine the special case of projective space itself. For a fuller account see [2].

Points in m-dimensional projective space P^m are equivalence classes of points (rays) in $\mathbf{R}^{m+1}_0$; the projective structure on P^m may be considered as arising from the standard flat connection on $\mathbf{R}^{m+1}_0$. We denote by Υ the radial vector field on $\mathbf{R}^{m+1}_0$. Each vector field on P^m may be represented by an equivalence class of vector fields on $\mathbf{R}^{m+1}_0$ invariant under dilations, where two such vector fields X, $\hat{X}$ are equivalent if their difference $\hat{X} - X$ is a multiple of Υ. Each equivalence class (whose elements satisfy $[\Upsilon, X] = 0$) has a canonical representative with vanishing divergence (with respect to the standard volume on $\mathbf{R}^{m+1}_0$), using the fact that $\operatorname{div}\Upsilon = m+1$, so that $\operatorname{div}\hat{X} = 0$ where

$$\hat{X} = X - \frac{\operatorname{div} X}{m+1}\Upsilon.$$

Similarly, a diffeomorphism of P^m may be represented as an equivalence class of projectable diffeomorphisms of $\mathbf{R}^{m+1}_0$, where two such diffeomorphisms ϕ, $\hat{\phi}$ are equivalent if $\hat{\phi} = r_f \circ \phi$ where r_f is the radial diffeomorphism of $\mathbf{R}^{m+1}_0$ defined by $r_f(x) = f(x)x$ for a non-vanishing function f on $\mathbf{R}^{m+1}_0$.

Now specialise to projective diffeomorphisms of P^m, which preserve the geodesics. These diffeomorphisms may be represented by equivalence classes of linear transformations of $\mathbf{R}^{m+1}$ (restricted to $\mathbf{R}^{m+1}_0$), where two such linear transformations g, $\hat{g}$ are equivalent if $\hat{g} = r_\lambda \circ g$ where r_λ is the radial diffeomorphism of $\mathbf{R}^{m+1}_0$ defined by $r_\lambda(x) = \lambda x$ and where, necessarily, λ is a non-zero number rather than a non-zero function. In other words, we quotient the group $\mathrm{GL}(m+1)$ of non-singular matrices by the subgroup of non-zero multiples of the identity, giving $\mathrm{PGL}(m+1)$. When m is even, there is a unique representative of the equivalence class which is volume-preserving, so that we represent $\mathrm{PGL}(m+1)$ by $\mathrm{SL}(m+1)$. When m is odd, there are two representatives, which are either both volume-preserving or both volume-reversing. We note that general affine transformations of $\mathbf{R}^{m+1}$ do not restrict to $\mathbf{R}^{m+1}_0$ unless they are linear.

An infinitesimal projective transformation of P^m is a vector field on P^m whose flow consists of projective diffeomorphisms. Thus we may represent such a vector field as an equivalence class of linear vector fields on $\mathbf{R}^{m+1}_0$, differing by constant multiples of the radial vector field Υ. We note that general infinitesimal affine transformations of $\mathbf{R}^{m+1}$

restrict to the open submanifold $\mathbf{R}_0^{m+1}$, but they do not project to P^m unless they are linear. Nevertheless, the pure infinitesimal translations of $\mathbf{R}_0^{m+1}$, that is, the constant vector fields, do have a rôle to play in projective space. Take a constant vector field X. Let $p \in \mathrm{P}^m$ be the ray in $\mathbf{R}_0^{m+1}$ given by the direction of X. At each point of any other ray in $\mathbf{R}_0^{m+1}$, the vectors defined by X project to vectors on P^m in the same direction but of differing magnitudes. Then X defines a line element field on $\mathrm{P}^m - \{p\}$ whose integral manifolds are the lines in P^m passing through the point p.

7 The BTW-connection of a projective class of sprays

We now return to the general case of a projective equivalence class of sprays on a manifold M of dimension m. The case of projective space itself suggests that there should be another manifold corresponding to $\mathbf{R}_0^{m+1}$, of dimension $m+1$, carrying a connection which gives rise to the class of sprays. The construction of such a manifold was carried out by Roberts [6], based on the ideas of Thomas and Whitehead [10, 11, 12]; a modified version of Roberts's construction was used in our earlier work [2, 3]. We call the new manifold the volume bundle $\nu : \mathcal{V}M \to M$. An element of $\mathcal{V}M$ is an unsigned, non-vanishing m-covector $\pm\theta$, where $\theta \in \bigwedge^m T^*M$. If v is the usual volume coordinate (that is, $\theta = v(\theta) dx^1 \wedge \ldots \wedge dx^m$) then we take, as fibre coordinate on $\mathcal{V}M$, $x^0 = |v|^{1/(m+1)}$; in this way, $\mathcal{V}M \to M$ becomes a principal $\mathbf{R}_+$-bundle. We shall adopt the convention for labelling coordinates that Latin indices $(x^a) = (x^1, \ldots, x^m)$ will denote the coordinates on M, whereas Greek indices $(x^\alpha) = (x^0, x^1, \ldots, x^m)$ will denote coordinates on $\mathcal{V}M$. In these coordinates, the fundamental vector field of the principal bundle structure is $\Upsilon = x^0 \partial/\partial x^0$. There is also a canonical odd scalar density $|d\Theta|$ on $\mathcal{V}M$, obtained from the tautological m-form Θ on $\bigwedge^m T^*M$; in coordinates, $|d\Theta| = \pm(m+1)(x^0)^m dx^0 \wedge \ldots \wedge dx^m$.

Dealing with the case of affine connections and sprays, Roberts [6], in our modified interpretation, defines a Thomas-Whitehead connection (a TW-connection) as a symmetric affine connection $\tilde\nabla$ on $\mathcal{V}M$, satisfying the conditions that $\tilde\nabla\Upsilon = \mathrm{id}$ and that $\tilde\nabla$ is invariant under the $\mathbf{R}_+$-action on the fibres of $\mathcal{V}M$; the latter condition may be restated in infinitesimal form as that Υ is an infinitesimal affine transformation of $\tilde\nabla$, so that

$$[\Upsilon, \tilde\nabla_X Y] = \tilde\nabla_{[\Upsilon, X]} Y + \tilde\nabla_X [\Upsilon, Y]$$

for all vector fields X, Y on $\mathcal{V}M$. In coordinates, a TW-connection $\tilde\nabla$ has connection coefficients $\tilde\Gamma^\gamma_{\alpha\beta}$ satisfying

$$\tilde\Gamma^\gamma_{00} = \tilde\Gamma^0_{a0} = 0, \qquad \tilde\Gamma^c_{a0} = (x^0)^{-1}\delta^c_a$$

and

$$\Upsilon(\tilde\Gamma^c_{ab}) = 0, \qquad \Upsilon(\tilde\Gamma^0_{ab}) = \tilde\Gamma^0_{ab}.$$

Roberts goes on to show that any Ehresmann connection form ϑ on the principal bundle $\nu : \mathcal{V}M \to M$ gives rise to a symmetric affine connection ∇ on M, and that different connection forms give rise to projectively-related affine connections. If the connection form is

$$\vartheta = (x^0)^{-1} dx^0 + \vartheta_a dx^a$$

then the related affine connection ∇ on M has connection coefficients

$$\Gamma^c_{ab} = \tilde{\Gamma}^c_{ab} - \vartheta_a \delta^c_b - \vartheta_b \delta^c_a.$$

We may describe this construction of ∇ in an alternative way when the connection form ϑ is exact and arises from a global section $\sigma : M \to \mathcal{V}M$ of the volume bundle, so that if $\sigma^0 = x^0 \circ \sigma$ then

$$\vartheta = (x^0)^{-1} dx^0 + (\sigma^0)^{-1} \frac{\partial \sigma^0}{\partial x^a} dx^a.$$

To use the alternative method, first consider the tangent map $\sigma_* : TM \to T(\mathcal{V}M)$; this is a section of $\nu_* : T(\mathcal{V}M) \to TM$. Write $\dot{\sigma}^0 = u^0 \circ \sigma_*$, so that

$$\dot{\sigma}^0 = u^a \frac{\partial \sigma^0}{\partial x^a};$$

then $\dot{\sigma}^0$, as a function on TM, is the total time derivative of σ^0. The image of the tangent map σ_* is a submanifold of $T(\mathcal{V}M)$, defined locally by the equations $x^0 = \sigma^0$, $u^0 = \dot{\sigma}^0$.

Now consider the geodesic spray of the TW-connection. This is the vector field

$$\tilde{S} = u^\gamma \frac{\partial}{\partial x^\gamma} - 2\tilde{\Gamma}^\gamma \frac{\partial}{\partial u^\gamma}$$

on $T(\mathcal{V}M)$, where $2\tilde{\Gamma}^\gamma = \tilde{\Gamma}^\gamma_{\alpha\beta} u^\alpha u^\beta$; thus

$$\tilde{S} = u^\gamma \frac{\partial}{\partial x^\gamma} - \left(\tilde{\Gamma}^c_{ab} u^a u^b + (x^0)^{-1} u^0 u^c \right) \frac{\partial}{\partial u^c} - \tilde{\Gamma}^0_{ab} u^a u^b \frac{\partial}{\partial u^0}.$$

At each point $v \in \operatorname{Im}(\sigma_*) \subset T(\mathcal{V}M)$, $\tilde{\Gamma}_v \in T_v T(\mathcal{V}M)$, so define the vector field S on TM by $S_{\nu_*(v)} = \nu_{**}(\tilde{S}_v)$; this is well-defined as $\nu_*|_{\operatorname{Im}(\sigma_*)}$ is an injection. Here, ν_{**} is the tangent map of ν_*, not of its restriction to the image of σ_*, so that its domain is the whole of $T_v T(\mathcal{V}M)$. In coordinates,

$$S = u^c \frac{\partial}{\partial x^c} - \left(\tilde{\Gamma}^c_{ab} u^a u^b + (\sigma^0)^{-1} \frac{\partial \sigma^0}{\partial x^a} u^a u^c \right) \frac{\partial}{\partial u^c}$$

so that S is the geodesic spray of the affine connection ∇.

We now reproduce this approach for a general spray, again working locally in coordinates initially (though we will consider the global aspects of the construction in due course). We consider the complete and vertical lifts of Υ to $T^\circ(\mathcal{V}M)$, Υ^{C} and Υ^{V}. Both of course are projectable to $\mathcal{V}M$: in coordinates

$$\Upsilon^{\mathrm{C}} = x^0 \frac{\partial}{\partial x^0} + u^0 \frac{\partial}{\partial u^0}, \qquad \Upsilon^{\mathrm{V}} = x^0 \frac{\partial}{\partial u^0}.$$

A Berwald-Thomas-Whitehead connection (or, more briefly, a *BTW*-connection) is a Berwald connection on $\mathcal{V}M$ with covariant derivative $\tilde{\nabla}$ (operating on sections of the pull-back bundle $\tau^{\circ *}_{\mathcal{V}M}(T(\mathcal{V}M)) \to T^\circ(\mathcal{V}M))$, corresponding to a spray $\tilde{S}$ on $T^\circ(\mathcal{V}M)$, which satisfies the following conditions:

1. as a basic vector section Υ satisfies $\tilde{\nabla}\Upsilon = (\tau^\circ_{\mathcal{V}M})_*$ — or, more explicitly, $\tilde{\nabla}_Y \Upsilon = (\tau^\circ_{\mathcal{V}M})_* Y$ for any vector field Y on $T^\circ(\mathcal{V}M)$, where $(\tau^\circ_{\mathcal{V}M})_* Y$ is the corresponding vector section;
2. both Υ^{C} and Υ^{V} are infinitesimal affine transformations of $\tilde{\nabla}$.

In coordinates, the connection coefficients $\tilde{\Gamma}^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\beta\alpha}$ of a *BTW*-connection (which of course are defined on $T^\circ(\mathcal{V}M)$ rather than on $\mathcal{V}M$) must satisfy

$$\tilde{\Gamma}^\gamma_{00} = \tilde{\Gamma}^0_{a0} = 0, \qquad \tilde{\Gamma}^c_{a0} = (x^0)^{-1}\delta^c_a$$

from the first condition,

$$\Upsilon^{\mathrm{C}}(\tilde{\Gamma}^c_{ab}) = 0, \qquad \Upsilon^{\mathrm{C}}(\tilde{\Gamma}^0_{ab}) = \tilde{\Gamma}^0_{ab}$$

because Υ^{C} is affine (that is, Υ is strictly affine), and

$$\Upsilon^{\mathrm{V}}(\tilde{\Gamma}^\gamma_{\alpha\beta}) = 0$$

because Υ^{V} is affine. Thus in particular $\tilde{\Gamma}^c_{ab}$ is projectable to $T^\circ M$, as is $(x^0)^{-1}\tilde{\Gamma}^0_{ab}$. As before,

$$\tilde{S} = u^\gamma \frac{\partial}{\partial x^\gamma} - \left(\tilde{\Gamma}^c_{ab} u^a u^b + (x^0)^{-1} u^0 u^c\right) \frac{\partial}{\partial u^c} - \tilde{\Gamma}^0_{ab} u^a u^b \frac{\partial}{\partial u^0}.$$

In [3] we define a *BTW*-spray to be a spray $\tilde{S}$ on $T^\circ(\mathcal{V}M)$ satisfying $\mathcal{L}_{\Upsilon^{\mathrm{C}}}\tilde{S} = 0$ and $\mathcal{L}_{\Upsilon^{\mathrm{V}}}\tilde{S} = \Upsilon^{\mathrm{C}} - 2\tilde{\Delta}$, where $\tilde{\Delta}$ is the dilation field on $T^\circ(\mathcal{V}M)$; the coordinate expression above shows that the spray of a *BTW*-connection is a *BTW*-spray.

Now let $\sigma : T^\circ M \to T^\circ(\mathcal{V}M)$ be a section of $\nu_* : T^\circ(\mathcal{V}M) \to T^\circ M$ which projects to a section $\bar{\sigma} : M \to \mathcal{V}M$, and which has the property that $\sigma_*(\Delta) = \tilde{\Delta} \circ \sigma$. Write $\sigma^0 = u^0 \circ \sigma$ and $\bar{\sigma}^0 = x^0 \circ \sigma$; then $\Delta(\sigma^0) = \sigma^0$. Now at each point $v \in \mathrm{Im}(\sigma) \subset T^\circ(\mathcal{V}M)$, $\tilde{S}_v \in T_v T^\circ(\mathcal{V}M)$, so define the vector field S on $T^\circ M$ by $S_{\nu_*(v)} = \nu_{**}(\tilde{S}_v)$; this is well-defined as $\nu_*|_{\mathrm{Im}(\sigma)}$ is an injection. In coordinates,

$$S = u^c \frac{\partial}{\partial x^c} - \left(\tilde{\Gamma}^c_{ab} u^a u^b + (\bar{\sigma}^0)^{-1}\sigma^0 u^c\right) \frac{\partial}{\partial u^c}$$

so that S is a spray on $T^\circ M$; a different choice of section σ will give a projectively-related spray. Thus any *BTW*-connection on $\mathcal{V}M$ defines a projective equivalence class of sprays on M. The fundamental invariants Π^a of this projective equivalence class depend just on the $\tilde{\Gamma}^a_{bc}$, and are given by

$$\Pi^a = \tilde{\Gamma}^a - \frac{1}{m+1}\tilde{\Gamma} u^a,$$

where $\tilde{\Gamma}^a = \frac{1}{2}\tilde{\Gamma}^a_{bc}u^b u^c$ and $\tilde{\Gamma} = \tilde{\Gamma}^a_a = \tilde{\Gamma}^a_{ba}u^b$.

Next, we observe that for any BTW-connection

$$\tilde{\nabla}|d\Theta| = -(\tilde{\Gamma}^a_{ba}dx^b) \otimes |d\Theta|.$$

Thus if $\tilde{\nabla}|d\Theta| = 0$ then $\tilde{\Gamma}^a_{ba} = 0$, so that $\tilde{\Gamma} = 0$ and $\tilde{\Gamma}^a = \Pi^a$; the coefficients $\tilde{\Gamma}^a_{bc}$ of the BTW-connection in this case are just the fundamental invariants of the corresponding projective equivalence class of sprays on M.

Finally, we find that the Ricci tensor $\tilde{R}_{\alpha\beta} = \tilde{R}^{\gamma}_{\alpha\gamma\beta}$ of such a BTW-connection (where $\tilde{R}^{\delta}_{\alpha\beta\gamma}$ is its Riemann tensor) is given by

$$\tilde{R}_{\alpha 0} = \tilde{R}_{0\beta} = 0, \quad \tilde{R}_{ab} = \mathfrak{R}_{ab} + (m-1)(x^0)^{-1}\tilde{\Gamma}^0_{ab},$$

where $\mathfrak{R}_{ab}$ is the Ricci 'tensor' formed from the Π^a_{bc}. Thus if we take

$$\tilde{\Gamma}^0_{ab} = -\frac{1}{m-1}x^0\mathfrak{R}_{ab}$$

then the BTW-connection will have vanishing Ricci tensor.

Suppose now that we are given a projective equivalence class of sprays on M. Then in any coordinate patch on $\mathcal{V}M$ with coordinates $(x^0, x^1, \ldots, x^m)$, where the x^a are coordinates on M and x^0 the fibre coordinate defined earlier, there is a unique Berwald connection defined on the patch which satisfies the conditions

1. $\tilde{\nabla}\Upsilon = (\tau^{\circ}_{\mathcal{V}M})_*$;
2. Υ^{C} and Υ^{V} are infinitesimal affine transformations of $\tilde{\nabla}$;
3. $\tilde{\nabla}|d\Theta| = 0$;
4. the Ricci tensor of $\tilde{\nabla}$ vanishes;
5. the projective equivalence class of sprays it induces on M is the given one.

But since the local Berwald connection is uniquely determined, and the properties that determine it are coordinate-independent, it must be the local representation of a global Berwald connection with the same properties. We call this BTW-connection the normal BTW-connection for the projective class. The corresponding spray, given in coordinates by

$$\tilde{S} = u^{\gamma}\frac{\partial}{\partial x^{\gamma}} - 2(\Pi^c_{ab}u^a u^b + (x^0)^{-1}u^0 u^c)\frac{\partial}{\partial u^c} + \frac{1}{m-1}x^0\mathfrak{R}_{ab}u^a u^b\frac{\partial}{\partial u^0},$$

is called the normal BTW-spray of the projective class.

In [3] we derived the results above by a different method, based on the properties of sprays rather than connections. The argument above is closer to the one given for the affine case by Roberts [6].

The curvatures of the normal BTW-connection connection can be expressed in terms of projective invariants of the corresponding projective class of sprays on M. For the Riemann curvature $\tilde{R}^{\alpha}_{\beta\gamma\delta}$ we have

$$\tilde{R}^a_{bcd} = P^a_{bcd}, \qquad \tilde{R}^0_{bcd} = \frac{1}{m-1} x^0 \left(\mathfrak{R}_{bc|d} - \mathfrak{R}_{bd|c} \right),$$

and for the Berwald curvature $\tilde{B}^{\alpha}_{\beta\gamma\delta}$

$$\tilde{B}^a_{bcd} = D^a_{bcd}, \qquad \tilde{B}^0_{bcd} = -\frac{1}{m-1} x^0 \frac{\partial \mathfrak{R}_{bc}}{\partial u^d},$$

all other components being zero in each case. Here P^a_{bcd} is the projective curvature tensor and D^a_{bcd} the Douglas tensor of the projective class, and $\mathfrak{R}_{bc|d}$ is the 'covariant derivative' of $\mathfrak{R}_{bc}$, computed using the fundamental invariants Π^a_{bc} as connection coefficients. The vanishing of the Douglas tensor is the necessary and sufficient condition for the projective class to be affine, that is to contain an affine spray; the Π^a_{bc} are then independent of the u^d, and consequently so are the $\mathfrak{R}_{bc}$. When $D^a_{bcd} = 0$, and for $m \geq 3$, the vanishing of the projective curvature tensor is the necessary and sufficient condition for the projective class to be flat, or rectifiable, that is, to contain a spray for which $\Gamma^a = 0$ in some coordinates. When $m = 2$, however, P^a_{bcd} is identically zero, and the condition for projective flatness is that $\mathfrak{R}_{bc|d} = \mathfrak{R}_{bd|c}$. That is to say, a necessary and sufficient condition for a spray on M to be projectively flat (that is, projectively equivalent to the flat spray) is that its BTW-connection should be flat.

8 Affine transformations of a BTW-connection

The BTW-connection contains all the projective information about a spray on M (or equally its projective equivalence class) in the form of a single Berwald connection on $\mathcal{V}M$; it is therefore to be expected that the projective transformations of a spray on M should be correlated with the affine transformations of the corresponding BTW-connection. We now examine the extent to which this is the case.

Let us take a spray S on M and consider the conditions on the coefficients of the complete lift X^{C} of a vector field X on $\mathcal{V}M$, $X = X^{\alpha}\partial/\partial x^{\alpha}$, for it to be an infinitesimal affine transformation of the corresponding BTW-connection (that is, for X to be a strict infinitesimal affine transformation): these are

$$\frac{\partial^2 X^{\alpha}}{\partial x^{\beta} \partial x^{\gamma}} + \frac{\partial X^{\delta}}{\partial x^{\beta}} \tilde{\Gamma}^{\alpha}_{\delta\gamma} + \frac{\partial X^{\delta}}{\partial x^{\gamma}} \tilde{\Gamma}^{\alpha}_{\delta\beta} - \frac{\partial X^{\alpha}}{\partial x^{\delta}} \tilde{\Gamma}^{\delta}_{\beta\gamma} + X^{\mathrm{C}}(\tilde{\Gamma}^{\alpha}_{\beta\gamma}) = 0.$$

We will examine the equations obtained by taking various values for α, β and γ.

First of all, $(\alpha, \beta, \gamma) = (0, 0, 0)$ gives

$$\frac{\partial^2 X^0}{\partial (x^0)^2} = 0,$$

while $(\alpha, \beta, \gamma) = (a, 0, 0)$ gives

$$\frac{\partial^2 X^a}{\partial (x^0)^2} + \left(\frac{2}{x^0}\right) \frac{\partial X^a}{\partial x^0} = 0.$$

Thus

$$X^0 = A^0 x^0 + B^0, \quad X^a = A^a + (x^0)^{-1} B^a$$

for certain functions A^α, B^α on M. Thus

$$X = \left(A^0 x^0 \frac{\partial}{\partial x^0} + A^a \frac{\partial}{\partial x^a}\right) + \left(B^0 \frac{\partial}{\partial x^0} + (x^0)^{-1} B^a \frac{\partial}{\partial x^a}\right) = A + B,$$

say. Now $[\Upsilon, X] = -B$, so B is a well-defined vector field on $\mathcal{V}M$, and so is $A = X - B$. Moreover, the strict infinitesimal affine transformations of a Berwald connection form a Lie algebra [13], and by definition of the BTW-connection Υ is strict affine, so B is strict affine, and so is A.

We denote by $\mathfrak{a}$ the Lie algebra of strict infinitesimal affine trasformations of the normal BTW-connection; it is a Lie algebra of vector fields on $\mathcal{V}M$. As we have seen, any element X of $\mathfrak{a}$ can be expressed uniquely in the form $X = A+B$, with $A, B \in \mathfrak{a}$, where $[\Upsilon, A] = 0$ and $[\Upsilon, B] = -B$. Denote by $\mathfrak{A}$ the subspace of $\mathfrak{a}$ consisting of elements A satisfying $[\Upsilon, A] = 0$, and $\mathfrak{B}$ the subspace of elements B satisfying $[\Upsilon, B] = -B$; then $\mathfrak{a} = \mathfrak{A} \oplus \mathfrak{B}$ as vector spaces. Now for $A_1, A_2 \in \mathfrak{A}$, $[\Upsilon, [A_1, A_2]] = 0$ from the Jacobi identity, so $\mathfrak{A}$ is a subalgebra. For $B_1, B_2 \in \mathfrak{B}$, $[\Upsilon, [B_1, B_2]] = -2[B_1, B_2]$ so we must have $[B_1, B_2] = 0$, that is, $\mathfrak{B}$ is Abelian. Finally, for $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, $[\Upsilon, [A, B]] = -[A, B]$, so that $[\mathfrak{A}, \mathfrak{B}] \in \mathfrak{B}$. Thus $\mathfrak{B}$ is an Abelian ideal, and $\mathfrak{a}/\mathfrak{B} \cong \mathfrak{A}$.

The structure of $\mathfrak{a}$ we have just obtained is reflected in the fact that the remaining equations for an infinitesimal affine transformation separate out into two sets, one involving only the unknowns A^α, the other involving only the unknowns B^α. For the rest of this section we consider the elements of the subalgebra $\mathfrak{A}$, that is, we take $B = 0$ in the equations. We denote by $\bar{A}$ the projection of A to M, so that $\bar{A} = A^a \partial/\partial x^a$, and by $\bar{A}^{\mathrm{C}}$ the complete lift of $\bar{A}$ to $T^\circ M$.

The remaining cases of the affine transformation conditions with $B = 0$ yield two equations for the A^α: with $(\alpha, \beta, \gamma) = (0, b, c)$ we obtain

$$\frac{\partial^2 A^0}{\partial x^b \partial x^c} - \frac{\partial A^0}{\partial x^d} \Pi^d_{bc} - \frac{1}{m-1} \left(\frac{\partial A^d}{\partial x^b} \mathfrak{R}_{dc} + \frac{\partial A^d}{\partial x^c} \mathfrak{R}_{db} + \bar{A}^{\mathrm{C}}(\mathfrak{R}_{bc})\right) = 0,$$

and with $(\alpha, \beta, \gamma) = (a, b, c)$

$$\frac{\partial^2 A^a}{\partial x^b \partial x^c} + \frac{\partial A^d}{\partial x^b} \Pi^a_{dc} + \frac{\partial A^d}{\partial x^c} \Pi^a_{db} - \frac{\partial A^a}{\partial x^d} \Pi^d_{bc} + \bar{A}^{\mathrm{C}}(\Pi^a_{bc}) + \frac{\partial A^0}{\partial x^b} \delta^a_c + \frac{\partial A^0}{\partial x^c} \delta^a_b = 0.$$

By taking the trace over a and c in the second of these we obtain

$$\frac{\partial A^0}{\partial x^b} = -\frac{1}{m+1} \frac{\partial (\operatorname{div} \bar{A})}{\partial x^b};$$

so that

$$A^0 = -\frac{\operatorname{div}\bar{A}}{m+1} + k$$

where k is a constant. Taking $k = 1$, $A^a = 0$ gives Υ, which we know to be affine. On the other hand, with $k = 0$ the equation above with $(\alpha, \beta, \gamma) = (a, b, c)$ is just the condition, expressed in terms of the fundamental invariants, for the vector field $\bar{A}$ on M to be a projective transformation.

On substitution for A^0 the equation with $(\alpha, \beta, \gamma) = (0, b, c)$ becomes

$$\frac{m-1}{m+1}\left(\frac{\partial^2(\operatorname{div}A)}{\partial x^b \partial x^c} - \frac{\partial(\operatorname{div}A)}{\partial x^d}\Pi^d_{bc}\right) + \left(\frac{\partial A^d}{\partial x^b}\mathfrak{R}_{dc} + \frac{\partial A^d}{\partial x^c}\mathfrak{R}_{db} + \bar{A}^{\mathrm{C}}(\mathfrak{R}_{bc})\right) = 0.$$

As it happens, this holds as a consequence of the fact that $\bar{A}$ is a projective transformation, as we now show. First of all, the second bracket on the left-hand side can be written

$$\frac{\partial A^d}{\partial x^b}\mathfrak{R}_{dc} + \frac{\partial A^d}{\partial x^c}\mathfrak{R}_{db} + \bar{A}^{\mathrm{C}}(\mathfrak{R}_{bc}) = \tfrac{1}{2}\frac{\partial^2(\bar{A}^{\mathrm{C}}(\mathfrak{R}))}{\partial u^b \partial u^c},$$

where $\mathfrak{R} = \mathfrak{R}_{bc}u^b u^c$. Next, we take the condition for $\bar{A}$ to define a projective transformation in the form which is homogeneous of degree 2 in the u^a:

$$\tfrac{1}{2}u^b u^c \frac{\partial^2 A^a}{\partial x^b \partial x^c} + u^c \frac{\partial A^b}{\partial x^c}\Pi^a_b - \frac{\partial A^a}{\partial x^b}\Pi^b + A^b \frac{\partial \Pi^a}{\partial x^b} = \frac{1}{m+1}u^b \frac{\partial(\operatorname{div}\bar{A})}{\partial x^b}u^a;$$

on differentiating this with respect to x^a and taking a trace we obtain

$$\begin{aligned}
&\tfrac{1}{2}u^b u^c \frac{\partial^2(\operatorname{div}\bar{A})}{\partial x^b \partial x^c} + u^c \frac{\partial^2 A^b}{\partial x^a \partial x^c}\Pi^a_b + u^c \frac{\partial A^b}{\partial x^c}\frac{\partial \Pi^a_b}{\partial x^a} - \frac{\partial(\operatorname{div}\bar{A})}{\partial x^b}\Pi^b + A^b \frac{\partial^2 \Pi^a}{\partial x^a \partial x^b} \\
&\quad = \frac{1}{m+1}u^b u^c \frac{\partial^2(\operatorname{div}\bar{A})}{\partial x^b \partial x^c}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{m-1}{m+1}\left(\tfrac{1}{2}u^b u^c \frac{\partial^2(\operatorname{div}\bar{A})}{\partial x^b \partial x^c} - \Pi^b \frac{\partial(\operatorname{div}\bar{A})}{\partial x^b}\right) \\
&\quad + \left(-\frac{2}{m+1}\frac{\partial(\operatorname{div}\bar{A})}{\partial x^b}\Pi^b + u^c \frac{\partial^2 A^b}{\partial x^a \partial x^c}\Pi^a_b + u^c \frac{\partial A^b}{\partial x^c}\frac{\partial \Pi^a_b}{\partial x^a} + A^b \frac{\partial^2 \Pi^a}{\partial x^a \partial x^b}\right) = 0.
\end{aligned}$$

If we differentiate the projective transformation condition with respect to u^b and contract the result with Π^b_a we find that

$$u^c \frac{\partial^2 A^a}{\partial x^b \partial x^c}\Pi^b_a + u^d \frac{\partial A^c}{\partial x^d}\Pi^a_{bc}\Pi^b_a + A^c \frac{\partial \Pi^a_b}{\partial x^c}\Pi^b_a = \frac{2}{m+1}\frac{\partial(\operatorname{div}\bar{A})}{\partial x^b}\Pi^b,$$

since Π^a_b is trace-free and $u^a \Pi^b_a = 2\Pi^b$ by homogeneity. Thus

$$\begin{aligned}
&\frac{m-1}{m+1}\left(\tfrac{1}{2}u^b u^c \frac{\partial^2(\operatorname{div}\bar{A})}{\partial x^b \partial x^c} - \Pi^b \frac{\partial(\operatorname{div}\bar{A})}{\partial x^b}\right) \\
&\quad + \left(A^b\left(\frac{\partial^2 \Pi^a}{\partial x^a \partial x^b} - \frac{\partial \Pi^a_c}{\partial x^b}\Pi^c_a\right) + u^c \frac{\partial A^b}{\partial x^c}\left(\frac{\partial \Pi^a_b}{\partial x^a} - \Pi^a_{bd}\Pi^d_a\right)\right) = 0.
\end{aligned}$$

But the terms in the second line are just $\frac{1}{2}\bar{A}^{\mathrm{C}}(\mathfrak{R})$, so if this equation is twice differentiated, with respect to u^b and u^c, the required equation is obtained. Thus if $A \in \mathfrak{A}$ then A is projectable to M and $\nu_* A = \bar{A}$ is an infinitesimal projective transformation of S.

Now any diffeomorphism ϕ of M induces a transformation of m-covectors by $\theta \mapsto (\phi^{-1})^*\theta$, and $(\phi^{-1})^*\theta = 0$ if and only if $\theta = 0$. Thus ϕ lifts to a diffeomorphism $\hat{\phi}$ of $\mathcal{V}M$, fibred over ϕ. Similarly, any vector field X on M lifts to a vector field $\hat{X}$ on $\mathcal{V}M$, which projects onto X; in fact

$$\hat{X} = X^a \frac{\partial}{\partial x^a} - \frac{\operatorname{div} X}{m+1}\Upsilon.$$

Alternatively we can note that $\mathcal{L}_{\hat{X}}|d\Theta| = 0$, and define $\hat{X}$ as the unique vector field on $\mathcal{V}M$ which projects onto X and satisfies this condition. Clearly $[\Upsilon, \hat{X}] = 0$. So if a vector field X on M is an infinitesimal projective transformation of S and we set $A = \hat{X}$ then $A \in \mathfrak{A}$.

Let $\mathfrak{p}$ denote the Lie algebra of projective transformations of S. The map $\mathfrak{p} \to \mathfrak{A}$ by $X \to \hat{X}$ is injective, and we see that $\mathfrak{A} \cong \mathfrak{p} \oplus \langle \Upsilon \rangle$.

9 Translations

We now turn to consideration of $\mathfrak{B}$. When $(\alpha, \beta, \gamma) = (0, b, 0)$ the conditions for an infinitesimal affine transformation with $A = 0$ are

$$\frac{\partial B^0}{\partial x^b} - \frac{1}{m-1}\mathfrak{R}_{bc}B^c = 0,$$

while taking $(\alpha, \beta, \gamma) = (a, b, 0)$ gives

$$\frac{\partial B^a}{\partial x^b} + \Pi^a_{bc}B^c + B^0\delta^a_b = 0.$$

These two equations, together with the explicit form of the vector field B, say that as a basic vector section B satisfies

$$\tilde{\nabla} B = 0.$$

The further conditions on B are obtained from the general condition for a strict infinitesimal affine transformation X in the form

$$\tilde{\nabla}_{\tilde{H}_\alpha}\tilde{\nabla}_{\tilde{H}_\beta}X^\gamma + X^\delta \tilde{R}^\gamma_{\beta\delta\alpha} + \tilde{\nabla}_{\tilde{S}}X^\delta \tilde{B}^\gamma_{\alpha\beta\delta} = 0;$$

this reduces when $X = B$ with $\tilde{\nabla} B = 0$ to $B^\delta \tilde{R}^\gamma_{\beta\delta\alpha} = 0$, or

$$B^d P^c_{bda} = 0, \qquad B^d(\mathfrak{R}_{bc|d} - \mathfrak{R}_{bd|c}) = 0,$$

using the expressions for the components of $\tilde{R}^\gamma_{\beta\delta\alpha}$ given earlier.

To understand the significance of these infinitesimal affine transformations we look once again at the case of projective space. The BTW-spray should be the standard flat spray on $\mathbf{R}_0^{m+1}$; however, in the coordinates x^α not all of the connection coefficients vanish (we have $\tilde{\Gamma}^a_{0b} = \tilde{\Gamma}^a_{b0} = (x^0)^{-1}\delta^a_b$), which indicates that these are not Cartesian coordinates. If we take the x^a to be affine coordinates on P^m, related to Cartesian coordinates y^α by $x^a = y^a/y^0$, then the construction of the volume bundle gives $x^0 = y^0$, so Cartesians are given in terms of the x^α by

$$y^0 = x^0, \quad y^a = x^0 x^a.$$

The connection coefficients do indeed vanish in these coordinates, as can most easily be seen by writing down the geodesic equations for the BTW-spray, which in the original coordinates are

$$\ddot{x}^0 = 0, \quad \ddot{x}^a + 2(x^0)^{-1}\dot{x}^0\dot{x}^a = 0;$$

these give $\ddot{y}^\alpha = 0$ since

$$\ddot{y}^a = \ddot{x}^0 x^a + 2\dot{x}^0\dot{x}^a + x^0\ddot{x}^a.$$

The transformation laws for the coordinate vector fields will be useful: they are

$$\frac{\partial}{\partial y^0} = \frac{\partial}{\partial x^0} - \frac{x^a}{x^0}\frac{\partial}{\partial x^a}, \quad \frac{\partial}{\partial y^a} = \frac{1}{x^0}\frac{\partial}{\partial x^a};$$

of course

$$\Upsilon = x^0\frac{\partial}{\partial x^0} = y^0\frac{\partial}{\partial y^0} + y^a\frac{\partial}{\partial y^a} = y^\alpha\frac{\partial}{\partial y^\alpha}.$$

In terms of the x^α the equations for the affine transformations of the BTW-connection are

$$\frac{\partial B^a}{\partial x^b} + B^0\delta^a_b = 0, \quad \frac{\partial B^0}{\partial x^b} = 0$$

for elements of $\mathfrak{B}$ (there are no further conditions on B in this case), and

$$\frac{\partial^2 A^a}{\partial x^b \partial x^c} + \frac{\partial A^0}{\partial x^b}\delta^a_c + \frac{\partial A^0}{\partial x^c}\delta^a_b = 0, \quad \frac{\partial^2 A^0}{\partial x^b\partial x^c} = 0$$

for the elements of $\mathfrak{A}$. Thus B^0 is constant, and $B^a = -B^0x^a + C^a$ where C^a is constant. The corresponding vector field on $\mathbf{R}^{m+1}$ is

$$X = B^0\left(\frac{\partial}{\partial x^0} - \left(\frac{x^a}{x^0}\right)\frac{\partial}{\partial x^a}\right) + C^a\frac{1}{x^0}\frac{\partial}{\partial x^a} = B^0\frac{\partial}{\partial y^0} + C^a\frac{\partial}{\partial y^a};$$

so these are the translations. For the A^α we have

$$A^0 = A^0_a x^a + K^0, \quad A^a = -A^0_b x^b x^a + K^a_b x^b + K^a,$$

where all the coefficients are constants, and so, finally, the infinitesimal affine transformation is

$$\begin{aligned} X &= x^0A^0\frac{\partial}{\partial x^0} + A^a\frac{\partial}{\partial x^a} \\ &= A^0_b x^b\left(x^0\frac{\partial}{\partial x^0} - x^a\frac{\partial}{\partial x^a}\right) + K^0x^0\frac{\partial}{\partial x^0} + (K^a_b x^b + K^a)\frac{\partial}{\partial x^a} \\ &= A^0_b y^b\frac{\partial}{\partial y^0} + K^0y^\alpha\frac{\partial}{\partial y^\alpha} + (K^a_b y^b + K^a y^0)\frac{\partial}{\partial y^a}. \end{aligned}$$

Thus in this case $\mathfrak{A}$ consists of the linear vector fields on $\mathbf{R}_0^{m+1}$ and $\mathfrak{B}$ the translations.

In the general case, therefore, it is natural to regard the elements of $\mathfrak{B}$ as generalized translations.

The strict infinitesimal affine transformations of a Berwald connection form a Lie algebra of maximum dimension $n(n+1)$ where n is the dimension of the base manifold [13]; projective transformations likewise, of maximum dimension $n(n+2)$. The maximum dimension is attained in each case only for the flat structure. In our situation we have an algebra of affine transformations $\mathfrak{a}$ on $\mathcal{V}M$, with $n = m+1$, and one of projective transformations $\mathfrak{p}$ on M, with $n = m$. Furthermore, since the elements of $\mathfrak{B}$ satisfy $\tilde{\nabla} B = 0$, the maximum dimension of $\mathfrak{B}$ is the fibre dimension of the bundle $\tau^{\circ *}_{\mathcal{V}M}(T(\mathcal{V}M)) \to T^\circ(\mathcal{V}M)$, which is $m+1$.

Now $\dim(\mathfrak{a}) = \dim(\mathfrak{A}) + \dim(\mathfrak{B}) = \dim(\mathfrak{p}) + 1 + \dim(\mathfrak{B}$; as a matter of simple arithmetic, if $\mathfrak{p}$ and $\mathfrak{B}$ both attain their maximum dimensions then $\dim(\mathfrak{a}) = m(m+2)+1+m+1 = (m+1)(m+2)$ is maximum also.

In fact if $\dim(\mathfrak{B}) = \dim(\mathcal{V}M)$ then the spray S on M is projectively flat. A *BTW*-connection, considered simply as a connection on the bundle $\tau^{\circ *}_{\mathcal{V}M}(T(\mathcal{V}M)) \to T^\circ(\mathcal{V}M)$, gives rise to a horizontal distribution on the total space $\tau^{\circ *}_{\mathcal{V}M}(T(\mathcal{V}M))$; an integral section of such a connection is a vector section ξ whose image is an integral manifold of the distribution. For a connection given by a covariant derivative $\tilde{\nabla}$, the integral sections are those satisfying $\tilde{\nabla}\xi = 0$. Such a section must evidently be basic. Now any element of $\mathfrak{B}$, a vector field on $\mathcal{V}M$, gives rise to a basic vector section which is an integral section of the *BTW*-connection. If $\dim(\mathfrak{B}) = m+1$ then at each point of $T^\circ(\mathcal{V}M)$ there are $m+1$ linearly independent integral sections, so that there is an integral section through each point of $\tau^{\circ *}_{\mathcal{V}M}(T(\mathcal{V}M))$. Thus, by Frobenius's Theorem, the *BTW*-connection has vanishing curvature; any spray on $T^\circ M$ in the associated projective class must therefore be projectively flat.

It is even enough to assume that $\mathfrak{B}$ contains m linearly independent vector fields everywhere transverse to the fibre of $\nu : \mathcal{V}M \to M$, to draw the same conclusion. If this condition holds, then at any point in M we can choose the values of the B^a arbitrarily. It follows from the fact that $\tilde{\nabla} B = 0$ for each $B \in \mathfrak{B}$ that $D^a_{bcd} B^c = 0$, where D^a_{bcd} is the Douglas tensor of the spray S; this is an integrability condition for $\tilde{\nabla} B = 0$, as can be seen most easily by differentiating the equation

$$\frac{\partial B^a}{\partial x^b} + \Pi^a_{bc} B^c + B^0 \delta^a_b = 0$$

with respect to u^d and using the fact that $\partial \Pi^a_{bc}/\partial u^d = D^a_{bcd}$. Thus since the values of the B^c may be chosen arbitrarily, $D^a_{bcd} = 0$, and the spray S is projectively affine. But as we noted earlier, we have $B^d P^c_{bda} = 0$ and $B^d(\mathfrak{R}_{bc|d} - \mathfrak{R}_{bd|c}) = 0$. Thus if $m \geq 3$ then $P^c_{bda} = 0$, the other condition is satisfied as a consequence, and the spray S on M is projectively flat. If $m = 2$ on the other hand then P^c_{bda} is identically zero, $\mathfrak{R}_{bc|d} - \mathfrak{R}_{bd|c} = 0$, and again the spray S on M is projectively flat.

10 Example: Shen's circles

The following nice example of a non-affine spray is due to Shen [8].

Consider the spray on $\mathbf{R}^2$ (coordinates (x, y), velocity components (u, v)):

$$S = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \sqrt{u^2+v^2}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right).$$

Its base integral curves are circles of unit radius, traversed counter-clockwise. (Notice that u^2+v^2 is a constant on any integral curve; if the value of the constant is $k > 0$ then with an appropriate choice of origin for t we have $\dot{x} = k\sin(kt)$, $\dot{y} = k\cos(kt)$, so that $x = -\cos(kt) + a$, $y = \sin(kt) + b$.)

Orientation-preserving Euclidean motions are clearly projective transformations of S; we show that there are no others. The equations for an infinitesimal projective tranformation

$$Z = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}$$

are

$$\begin{aligned}
&\sqrt{u^2+v^2}\left(u^2\frac{\partial^2 X}{\partial x^2} + 2uv\frac{\partial^2 X}{\partial x\partial y} + v^2\frac{\partial^2 X}{\partial y^2} + Fu\right)\\
&\qquad = -v^3\frac{\partial X}{\partial x} + u(u^2+2v^2)\left(\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x}\right) + v(u^2+2v^2)\frac{\partial Y}{\partial y}\\
&\sqrt{u^2+v^2}\left(u^2\frac{\partial^2 Y}{\partial x^2} + 2uv\frac{\partial^2 Y}{\partial x\partial y} + v^2\frac{\partial^2 Y}{\partial y^2} + Fv\right)\\
&\qquad = -u(2u^2+v^2)\frac{\partial X}{\partial x} - v(2u^2+v^2)\left(\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x}\right) + u^3\frac{\partial Y}{\partial y}.
\end{aligned}$$

for some function F positively homogeneous of degree 1 in u and v. These hold for all $(u, v) \neq (0, 0)$. In particular the first holds for $u = 0$, $v \neq 0$, whence

$$|v|v^2\frac{\partial^2 X}{\partial y^2} = -v^3\left(\frac{\partial X}{\partial x} - 2\frac{\partial Y}{\partial y}\right),$$

from which it follows that

$$\frac{\partial^2 X}{\partial y^2} = 0, \quad \frac{\partial X}{\partial x} = 2\frac{\partial Y}{\partial y}.$$

Similarly, from the other equation,

$$\frac{\partial^2 Y}{\partial x^2} = 0, \quad 2\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}.$$

Thus

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} = 0,$$

and

$$X = a + \alpha y, \quad Y = b + \beta x.$$

On substituting these back into the original equations we find that

$$F = \frac{(\alpha+\beta)(u^2+2v^2)}{\sqrt{u^2+v^2}} = -\frac{(\alpha+\beta)(2u^2+v^2)}{\sqrt{u^2+v^2}},$$

whence $\alpha + \beta = 0$, and $F = 0$. Then

$$Z = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

This is an infinitesimal generator of orientation-preserving Euclidean motions. Notice that no reflection can be projective, since reflections reverse the sense in which the circles are traversed. Thus the projective transformations are the orientation-preserving Euclidean motions, and all projective transformations are affine.

Among the equations for generalized translations we find

$$\begin{aligned}
\frac{\partial B^1}{\partial x} - \frac{v^3}{2r^3}B^1 - \frac{u^3}{2r^3}B^2 + B^0 &= 0\\
\frac{\partial B^1}{\partial y} - \frac{u^3}{2r^3}B^1 - \frac{v(3u^2+2v^2)}{2r^3}B^2 &= 0\\
\frac{\partial B^2}{\partial x} + \frac{u(2u^2+3v^2)}{2r^3}B^1 + \frac{v^3}{2r^3}B^2 &= 0\\
\frac{\partial B^2}{\partial y} + \frac{v^3}{2r^3}B^1 + \frac{u^3}{2r^3}B^2 + B^0 &= 0,
\end{aligned}$$

where $r = \sqrt{u^2+v^2}$. On changing the signs of u and v we see that the terms in each equation involving these variables, and those independent of them, must vanish separately. Thus in particular

$$\begin{aligned}
\frac{u^3}{2r^3}B^1 + \frac{v(3u^2+2v^2)}{2r^3}B^2 &= 0\\
\frac{u(2u^2+3v^2)}{2r^3}B^1 + \frac{v^3}{2r^3}B^2 &= 0;
\end{aligned}$$

but the matrix of coefficients is non-singular (except on a curve in each fibre), and therefore $B^1 = B^2 = 0$, whence $B^0 = 0$. So there are no non-zero generalized translations in this case. Thus the strict infinitesimal affine transformations of the BTW-connection of this spray are the vector fields

$$Z = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + kz\frac{\partial}{\partial z}$$

on $\mathcal{V}\mathbf{R}^2$, where z is the fibre coordinate.

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