On the orthogonal separation of variables in the Hamilton-Jacobi equation for geodesics in a Riemannian manifold

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October 5, 2004

1 Introduction

The Hamilton-Jacobi equation for a Hamiltonian H on the cotangent bundle T^*M of some manifold M is the first-order partial differential equation

$$H\left(x^i, \frac{\partial S}{\partial x^j}\right) = E$$

for constant E, where the unknown S is a function on M. We may think of any function S on M as defining a section of $T^*M \to M$ by $x \mapsto dS|_x$; such a section defines a Lagrangian submanifold of T^*M with respect to the canonical symplectic form $d(\sum_i p_i dx^i) = \sum_i dp_i \wedge dx^i$; a solution of the Hamilton-Jacobi equation is thus a Lagrangian submanifold projecting onto M (or, in the local version, onto some open subset of it) on which H is constant. A foliation of T^*M by such submanifolds is a complete solution of the Hamilton-Jacobi equation.

I deal here with the Hamilton-Jacobi equation for the energy Hamiltonian of a Riemannian manifold, $H = \frac{1}{2}\sum_{i,j} g^{ij} p_i p_j$, whose Hamiltonian flow represents the geodesics of the Riemannian metric. The Hamilton-Jacobi equation

$$\sum_{i,j} g^{ij} \frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} = 2E$$

is the necessary and sufficient condition for the normals to the level sets of S to be tangent to a congruence of geodesics.

The question under investigation is whether the Hamilton-Jacobi equation is solvable by the method of separation of variables, that is, whether there are coordinates with respect to which the equation has complete solutions of the form

$$S(x^1, x^2, \ldots, x^n) = S_1(x^1) + S_2(x^2) + \cdots + S_n(x^n)$$

(the n parameters that should occur in a complete solution have been suppressed). When there are such coordinates the equation, or more idiomatically the Hamiltonian or even in the Riemannian case the metric, is said to be separable, and the coordinates with respect to which the equation is separable are called separation coordinates. I will deal only with the case in which the separation coordinates are orthogonal for the metric, so that when it is written in terms of them $g_{ij} = 0$ for $i \neq j$; this is the most interesting situation.

Finding conditions for a metric to be separable is an old problem. From among the many contributions to it I want to mention three.

Firstly, Levi-Civita formulated conditions for separability of a Hamiltonian in 1904. For the energy Hamiltonian of a Riemannian metric, his condition for separability in orthogonal coordinates is

$$\frac{\partial^2 g^{kk}}{\partial x^i \partial x^j} - g_{jj}\frac{\partial g^{jj}}{\partial x^i}\frac{\partial g^{kk}}{\partial x^j} - g_{ii}\frac{\partial g^{ii}}{\partial x^j}\frac{\partial g^{kk}}{\partial x^i} = 0,$$

$i, j = 1, 2, \ldots, n$, $i \neq j$; there is no sum.

In 1929 Eisenhart related the orthogonal separability of a Riemannian metric to the existence of first integrals of the Hamiltonian system quadratic in momenta. Recall that the coefficient tensor of a quadratic first integral is a valence 2 Killing tensor. His conditions are that there should exist $n-1$ independent quadratic first integrals, which are diagonal with respect to the orthogonal coordinates (that is, whose corresponding Killing tensors are all diagonal), and are such that the Killing tensors in type $(1,1)$ form have everywhere simple eigenvalues. The conditions for a diagonal tensor $\mathrm{diag}(a^{11}, a^{22}, \ldots, a^{nn})$ to be a Killing tensor (or equivalently for $\frac{1}{2}\sum_i a^{ii}(p_i)^2$ to be a first integral) are

$$\frac{\partial a^{kk}}{\partial x^j} g^{jj} = \frac{\partial g^{kk}}{\partial x^j} a^{jj}, \quad j, k = 1, 2, \ldots, n.$$

The integrability conditions of these equations are just the Levi-Civita conditions. Furthermore, the first integrals commute pairwise (that is, have vanishing Poisson brackets), so the system is completely integrable.

The Levi-Civita conditions may of course be written in covariant form, when they become

$$\frac{\partial^2 g_{kk}}{\partial x^i \partial x^j} - 2g^{kk}\frac{\partial g_{kk}}{\partial x^i}\frac{\partial g_{kk}}{\partial x^j} + g^{jj}\frac{\partial g_{jj}}{\partial x^i}\frac{\partial g_{kk}}{\partial x^j} + g^{ii}\frac{\partial g_{ii}}{\partial x^j}\frac{\partial g_{kk}}{\partial x^i} = 0,$$

Let us use this version to enquire whether there are any conformally flat metrics which are separable. Substituting $g_{ii} = \varphi$, $i = 1, 2, \ldots, n$, we see that φ must satisfy

$$\frac{\partial^2 \varphi}{\partial x^i \partial x^j} = 0, \quad i \neq j,$$

and therefore

$$\varphi = \varphi_1(x^1) + \varphi_2(x^2) + \ldots + \varphi_n(x^n).$$

Separable metrics of this class are called Liouville metrics.

The Levi-Civita conditions are evidently fundamental; however, they suffer from the disadvantage of being expressed in terms of orthogonal coordinates, so that while they provide a method of determining whether orthogonal coordinates, once found, are separation coordinates, they do not offer much in the way of direct help in determining whether a given metric, not presented in orthogonal form, is separable. The purpose of this paper is to describe some ways of reformulating the Levi-Civita conditions so as to escape from this difficulty. Two such methods are presented. In the first, described in Section 2, the conditions are expressed in terms of forms. The second, in Section 3, is tensorial.

In Section 3 the Einstein summation convention is used; in Section 2, on the other hand, all sums are indicated explicitly with the summation symbol.

2 The Levi-Civita conditions in terms of forms

The basic idea is to correlate coordinate systems with type $(1,1)$ tensor fields with certain properties; then the conditions for separability can be expressed in terms of the existence of tensors satisfying appropriate conditions rather than the existence of particular kinds of coordinates. But separation coordinates are determined only up to coordinate transformations of the form $y^1 = y^1(x^1)$, $y^2 = y^2(x^2)$, $\ldots$, $y^n = y^n(x^n)$; the correspondence between coordinates and tensors must take account of this.

2.1 Coordinates and tensors

I will say that coordinates (x^i), (y^i) such that $y^i = y^i(x^i)$ $(dy^i/dx^i \neq 0)$ are diagonally equivalent. We can associate with a diagonal equivalence class of coordinates all those local $(1,1)$ tensor fields J such that

1. J is diagonal with respect to any, and thus all, coordinates of the class;
2. at each point the eigenvalues of J are simple;
3. the Nijenhuis torsion of J vanishes.

By (1)

$$J = \sum_i \lambda_i \frac{\partial}{\partial x^i} \otimes dx^i$$

where the λ_i are the eigenfunctions of J; by (2), since the eigenvalues are simple

$$\lambda_i = \lambda_i(x^i).$$

If $\lambda_i' \neq 0$ for all i the eigenfunctions may be chosen as coordinates, when

$$J = \sum_i x^i \frac{\partial}{\partial x^i} \otimes dx^i.$$

Conversely, a type $(1,1)$ tensor J with zero torsion and pointwise simple eigenvalues defines a diagonal equivalence class of coordinates, namely those with respect to which it is diagonal; if moreover its eigenfunctions are functionally independent then those functions may be taken as coordinates and define the diagonal equivalence class.

2.2 Expressing the Levi-Civita conditions in terms of forms

Suppose given the following data:

- a metric g on M, and the coresponding Hamiltonian $H = \frac{1}{2}\sum_{i,j} g^{ij} p_i p_j$ on T^*M;
- orthogonal coordinates (x^i) for g, so that $H = \frac{1}{2}\sum_i g^{ii} p_i^2$;
- a type $(1,1)$ tensor J which is symmetric with respect to g, has pointwise simple eigenvalues and vanishing Nijenhuis torsion, so that it is associated with the diagonal equivalence class of orthogonal coordinates;
- $\tilde{J}$, the complete lift to T^*M.

Then

$$J = \sum_i \lambda_i \frac{\partial}{\partial x^i} \otimes dx^i, \quad \tilde{J} = \sum_i \lambda_i \left(\frac{\partial}{\partial x^i} \otimes dx^i + \frac{\partial}{\partial p_i} \otimes dp_i \right).$$

I consider the 1-forms $d_{\tilde{J}} H = \tilde{J}(dH)$, $d_{\tilde{J}^2} H, \ldots$. Note that I do not distinguish notationally between the action of a type $(1,1)$ tensor field on a vector field and on a 1-form. We have

$$d_{\tilde{J}^r} H = \tfrac{1}{2} \sum_{i,j} \frac{\partial g^{jj}}{\partial x^i} (\lambda_i)^r p_j^2 dx^i + \sum_i g^{ii} (\lambda_i)^r p_i dp_i$$

for $r = 0, 1, 2, \ldots, n-1$. Then

$$\begin{bmatrix} dH \\ d_{\tilde{J}}H \\ d_{\tilde{J}^2}H \\ \vdots \\ d_{\tilde{J}^{n-1}}H \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ (\lambda_1)^2 & (\lambda_2)^2 & \cdots & (\lambda_n)^2 \\ \vdots & \vdots & & \vdots \\ (\lambda_1)^{n-1} & (\lambda_2)^{n-1} & \cdots & (\lambda_n)^{n-1} \end{bmatrix} \begin{bmatrix} A_1 dx^1 + g^{11}p_1 dp_1 \\ A_2 dx^2 + g^{22}p_2 dp_2 \\ A_3 dx^3 + g^{33}p_3 dp_3 \\ \vdots \\ A_n dx^n + g^{nn}p_n dp_n \end{bmatrix},$$

where I have written A_i for $\frac{1}{2}\sum_j \dfrac{\partial g^{jj}}{\partial x^i} p_j^2$. The Vandermonde matrix, which I will denote by V, is non-singular since by assumption λ_1, λ_2, $\ldots$, λ_n are distinct. Thus

$$p_i dp_i = g_{ii} \sum_{j=1}^{n} \left(\bar{V}_i^j d_{\tilde{J}^{j-1}} H - \frac{1}{2} \frac{\partial g^{jj}}{\partial x^i} p_j^2 dx^i \right) \qquad V^{-1} = (\bar{V}_i^j).$$

It follows that the 1-forms $d_{\tilde{J}^r}H$, $r = 0, 1, 2, \ldots, n-1$, are linearly independent of each other and of the basic 1-forms dx^i, $i = 1, 2, \ldots, n$, except where any one of the p_i is zero.

Furthermore

$$\begin{aligned} dd_{\tilde{J}}H &= \frac{1}{2} \sum_{i,j,k} \frac{\partial^2 g^{ii}}{\partial x^j \partial x^k} \lambda_k p_i^2 dx^j \wedge dx^k + \sum_{i,j} \frac{\partial g^{ii}}{\partial x^j} \lambda_j p_i dp_i \wedge dx^j \\ &\quad - \sum_{i,j} \frac{\partial g^{ii}}{\partial x^j} \lambda_i p_i dp_i \wedge dx^j - \sum_i g^{ii} p_i dp_i \wedge d\lambda_i \\ &= \sum_{i,k} \bar{V}_i^k \left(\sum_j g_{ii} \frac{\partial g^{ii}}{\partial x^i} (\lambda_i - \lambda_j) dx^j - d\lambda_i \right) \wedge d_{\tilde{J}^{k-1}} H \\ &\quad + \frac{1}{4} \sum_{i,j,k} \left(\frac{\partial^2 g^{ii}}{\partial x^j \partial x^k} - g_{jj} \frac{\partial g^{jj}}{\partial x^k} \frac{\partial g^{ii}}{\partial x^j} - g_{kk} \frac{\partial g^{kk}}{\partial x^j} \frac{\partial g^{ii}}{\partial x^k} \right) \\ &\qquad \times (\lambda_k - \lambda_j) p_i^2 dx^j \wedge dx^k. \end{aligned}$$

Thus if the metric is separable in orthogonal coordinates then for every tensor J associated with the diagonal equivalence class of separation coordinates

$$dd_{\tilde{J}}H = \sum_{r=0}^{n-1} \alpha^{(r)} \wedge d_{\tilde{J}^r} H$$

where the $\alpha^{(r)}$ are 1-forms on M given explicitly by

$$\alpha^{(r)} = \sum_{i,j} \bar{V}_i^{r+1} \left(g_{ii} \frac{\partial g^{ii}}{\partial x^j} (\lambda_i - \lambda_j) dx^j + d\lambda_i \right).$$

Of course, a different choice of J leads to different $\alpha^{(r)}$.

Conversely, if there is a locally defined type $(1,1)$ tensor field on M which is symmetric with respect to g, has pointwise simple eigenvalues and vanishing Nijenhuis torsion, and there are 1-forms $\alpha^{(r)}$ on M, $r = 0, 1, \ldots, n-1$, such that

$$dd_{\tilde{J}}H = \sum_{r=0}^{n-1} \alpha^{(r)} \wedge d_{\tilde{J}^r}H,$$

then any coordinates with respect to which J is diagonal are orthogonal separation coordinates for g.

The relation

$$dd_{\tilde{J}}H = \sum_{r=0}^{n-1} \alpha^{(r)} \wedge d_{\tilde{J}^r}H$$

holds for all J associated with a diagonal equivalence class of coordinates if it holds for one. In particular, we can replace J in it by J^s for any s with $0 \leq s \leq n-1$. The typical term on the right-hand side will then involve $(\widetilde{J^s})^r = \tilde{J}^{sr}$; but this can be expressed in terms of the $\tilde{J}^t$, $t = 0, 1, \ldots, n-1$, by repeated use of the minimal polynomial of $\tilde{J}$ (which is of degree n). Thus if the relation above holds, then there are 1-forms $\alpha^{(r)}_{(s)}$ on M, $r, s = 0, 1, \ldots, n-1$, such that

$$dd_{\tilde{J}^s}H = \sum_{r=0}^{n-1} \alpha^{(r)}_{(s)} \wedge d_{\tilde{J}^r}H, \quad s = 0, 1, \ldots, n-1.$$

Of course $\alpha^{(r)}_{(0)} = 0$.

The additional relations, those with $s > 1$, can be derived in another way. It is easy to show that for any type $(1,1)$ tensor field N with vanishing Nijenhuis torsion and any 1-form θ, on any manifold,

$$d(N^2\theta)(X, Y) = d(N\theta)(NX, Y) + d(N\theta)(X, NY) - d\theta(NX, NY)$$

for any vector fields X, Y. If we take $N = \tilde{J}$ and $\theta = d_{\tilde{J}^s}H$ on T^*M we obtain

$$dd_{\tilde{J}^{s+2}}H(X, Y) = dd_{\tilde{J}^{s+1}}H(\tilde{J}X, Y) + dd_{\tilde{J}^{s+1}}H(X, \tilde{J}Y) - dd_{\tilde{J}^s}H(\tilde{J}X, \tilde{J}Y).$$

This is in effect a second-order recurrence relation for the 2-forms $dd_{\tilde{J}^s}H$, from which one can see that $dd_{\tilde{J}^{s+2}}H$ has the desired form if $dd_{\tilde{J}^{s+1}}H$ and $dd_{\tilde{J}^s}H$ do; the initial conditions are $ddH = 0$ and $dd_{\tilde{J}}H = \sum_{r=0}^{n-1} \alpha^{(r)} \wedge d_{\tilde{J}^r}H$. In fact one can write down a recurrence relation for the $\alpha^{(r)}_{(s)}$ from this, which is second-order in s, and which shows that the $\alpha^{(r)}_{(s)}$ with $s \geq 2$ are completely determined by the $\alpha^{(r)}$. Thus the additional relations in principle provide no new information; they can be useful even so, as will appear shortly.

2.3 Interpretation

The results above have a somewhat more sophisticated interpretation, as follows.

Assume that there is a type $(1,1)$ tensor J, symmetric with respect to g, whose eigenvalues are everywhere simple and whose Nijenhuis torsion vanishes, and 1-forms $\alpha^{(r)}_{(s)}$ on M, $r, s = 0, 1, \ldots, n-1$, such that

$$dd_{\tilde{J}^s} H = \sum_{r=0}^{n-1} \alpha^{(r)}_{(s)} \wedge d_{\tilde{J}^r} H.$$

Consider the distribution $\mathcal{D} = \langle dH, d_{\tilde{J}} H, \ldots, d_{\tilde{J}^{n-1}} H \rangle^\circ$, consisting of those vector fields on T^*M annihilated by the n 1-forms $d_{\tilde{J}^r} H$, $r = 0, 1, \ldots, n-1$. Where the $d_{\tilde{J}^r} H$ are linearly independent $\mathcal{D}$ is n-dimensional and transverse to the fibres of T^*M. I denote by $\omega = \sum_i dp_i \wedge dx^i$ the canonical symplectic 2-form on T^*M, and by $Y_{(r)}$ the vector fields on T^*M defined by $Y_{(r)} \lrcorner\, \omega = d_{\tilde{J}^r} H$. An easy calculation in any coordinates in the diagonal equivalence class determined by J (which will be orthogonal with respect to g) shows that $\mathcal{D} = \langle Y_{(0)}, Y_{(1)}, \ldots, Y_{(n-1)} \rangle$. Thus

$$\omega(Y_{(r)}, Y_{(s)}) = \langle Y_{(s)}, d_{\tilde{J}^r} H \rangle = 0,$$

which is to say that $\mathcal{D}$ is a Lagrangian distribution. It is evidently integrable (in the sense of Frobenius). Thus the leaves of $\mathcal{D}$ are Lagrangian submanifolds of T^*M. Any leaf of $\mathcal{D}$ can be locally expressed in the form $p_i = \partial S / \partial x^i$ for some local function S on M, and since $\langle \mathcal{D}, dH \rangle = 0$ the function S satisfies

$$H\left(x^i, \frac{\partial S}{\partial x^i}\right) = \text{constant}.$$

Thus $\mathcal{D}$ defines locally a complete solution of the Hamilton-Jacobi equation. Furthermore, $\mathcal{D}$ is invariant under $\tilde{J}$:

$$\langle \tilde{J}(\mathcal{D}), d_{\tilde{J}^r} H \rangle = \langle \mathcal{D}, d_{\tilde{J}^{r+1}} H \rangle = 0.$$

Now the tangent space to the submanifold $p_i = \partial S / \partial x^i$ is spanned by the vector fields

$$\frac{\partial}{\partial x^i} + \sum_j \frac{\partial^2 S}{\partial x^i \partial x^j} \frac{\partial}{\partial p_j}, \qquad i = 1, 2, \ldots, n.$$

With respect to coordinates in terms of which J is diagonal we have

$$\tilde{J}\left(\frac{\partial}{\partial x^i} + \sum_j \frac{\partial^2 S}{\partial x^i \partial x^j} \frac{\partial}{\partial p_j}\right) = \lambda_i \frac{\partial}{\partial x^i} + \sum_j \frac{\partial^2 S}{\partial x^i \partial x^j} \lambda_j \frac{\partial}{\partial p_j};$$

but then the invariance of $\mathcal{D}$ under $\tilde{J}$ requires that

$$\lambda_i \frac{\partial}{\partial x^i} + \sum_j \frac{\partial^2 S}{\partial x^i \partial x^j} \lambda_j \frac{\partial}{\partial p_j} = \lambda_i \left(\frac{\partial}{\partial x^i} + \sum_j \frac{\partial^2 S}{\partial x^i \partial x^j} \frac{\partial}{\partial p_j}\right),$$

so that

$$(\lambda_i - \lambda_j)\frac{\partial^2 S}{\partial x^i \partial x^j} = 0,$$

and therefore

$$\frac{\partial^2 S}{\partial x^i \partial x^j} = 0 \quad i \neq j,$$

which means that

$$S(x^1, x^2, \dots, x^n) = S_1(x^1) + S_2(x^2) + \cdots + S_n(x^n).$$

Thus the Hamilton-Jacobi equation separates in the coordinates with respect to which J is diagonal.

2.4 The Eisenhart conditions

The Eisenhart conditions may be recovered by pursuing this line of reasoning further. It will be convenient to use a matrix notation. I will denote by D the row vector of 1-forms $[dH, d_{\bar{J}}H, \dots, d_{\bar{J}^{n-1}}H]$ and by A the matrix of 1-forms $[\alpha^{(r)}_{(s)}]$; then $dD = -D \wedge A$ (where matrix multiplication as well as wedge product is implied on the right-hand side). By taking the exterior derivative we obtain $D \wedge (dA + A \wedge A) = 0$. Now the elements of $dA + A \wedge A$ are 2-forms on M, while those of D are 1-forms on T^*M independent of forms on M; it follows that $dA + A \wedge A = 0$. But this is the necessary and sufficient condition for the existence of a non-singular matrix B of functions on M such that $A = B^{-1}dB$. Since the first column of A is zero, without loss of generality we may take the first column of B to be $[1, 0, \dots, 0]^T$. Consider now the vector of 1-forms DB^{-1} on T^*M. We have

$$d(DB^{-1}) = -(D \wedge A)B^{-1} + D \wedge B^{-1}(dB)B^{-1} = 0;$$

there are therefore locally defined functions $H_{(0)}, H_{(1)}, \dots, H_{(n-1)}$ on T^*M, with $H_{(0)} = H$, such that $DB^{-1} = [dH_{(0)}, dH_{(1)}, \dots, dH_{(n-1)}]$.

Since the $dH_{(r)}$ are linear combinations of the $d_{\bar{J}^s}H$ they define the same distribution $\mathcal{D}$. Moreover, their Hamiltonian vector fields $X_{H_{(r)}}$ are (to within sign) the same linear combinations of the $Y_{(r)}$, and so they also span $\mathcal{D}$. But $\mathcal{D}$ is Lagrangian: thus

$$\omega(X_{H_{(r)}}, X_{H_{(s)}}) = \{H_{(r)}, H_{(s)}\} = 0.$$

That is to say, the functions $H_{(r)}$, $r = 1, \dots, n-1$ are pairwise commuting first integrals of the Hamiltonian system. It is easy to see that, apart from constants (which may be ignored) the $H_{(r)}$ are quadratic in momenta, and that when expressed in terms of coordinates with respect to which J is diagonal these quadratic functions are of diagonal type.

2.5 The companion relations

Define σ_r, $r = 0, 1, 2, \dots, n-1$, by

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \lambda^n + \sigma_{n-1}\lambda^{n-1} + \cdots + \sigma_1\lambda + \sigma_0,$$

so that the σ_r are, up to sign, the elementary symmetric polynomials in the λ_i.

The companion matrix C of the Vandermonde matrix is

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\sigma_0 & -\sigma_1 & -\sigma_2 & \cdots & -\sigma_{n-1} \end{bmatrix}.$$

Set $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$; then $CV = V\Lambda$, or $C = V\Lambda V^{-1}$.

The σ_r satisfy the relations

$$d_J\sigma_0 = -\sigma_0 d\sigma_{n-1}, \quad d_J\sigma_k = d\sigma_{k-1} - \sigma_k d\sigma_{n-1}, \quad k = 1, 2, \dots, n-1;$$

I call these companion relations. The proof goes as follows.

Since $d_J\lambda_i = \lambda_i d\lambda_i$,

$$d_J V = (dV)\Lambda, \quad d_J\Lambda = (d\Lambda)\Lambda.$$

From the defining relation $CV = V\Lambda$ we get

$$(d_J C)V = (-C(dV) + (dV)\Lambda + V(d\Lambda))\Lambda = (dC)V\Lambda,$$

so that

$$d_J C = (dC)V\Lambda V^{-1} = (dC)C.$$

This is a trivial identity for all but the last rows of the two matrices $d_J C$ and $(dC)C$; the equality of the last rows gives the companion relations.

It happens that the 1-forms $\alpha^{(r)}$ satisfy relations which are formally the same, namely

$$d_J\alpha^{(0)} = -\sigma_0 d\alpha^{(n-1)}, \quad d_J\alpha^{(k)} = d\alpha^{(k-1)} - \sigma_k d\alpha^{(n-1)};$$

again, I call these companion relations. Here of course d_J acts on 1-forms; recall that its action on forms is determined by two properties: d_J is a derivation with respect to the wedge product, and $d_J d = -d d_J$.

The companion relations have a simple expression in terms of the vector of 1-forms $a = [\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(n-1)}]$, namely $d_J a = (da)C$. For the proof, observe that in terms of separation coordinates $a = bV^{-1}$, where $b = [\beta_1, \beta_2, \dots, \beta_n]$ and

$$\beta_i = \sum_j g_{ii}\frac{\partial g^{ii}}{\partial x^j}(\lambda_i - \lambda_j)dx^j + d\lambda_i = \sum_j \frac{\partial \phi^i}{\partial x^j}(\lambda_i - \lambda_j)dx^j + d\lambda_i$$

with $\phi^i = \log(g^{ii})$. Now

$$d\beta_i = \tfrac{1}{2}\sum_{j,k}\frac{\partial^2\phi^i}{\partial x^j\partial x^k}(\lambda_j - \lambda_k)dx^j \wedge dx^k + \sum_j \frac{\partial\phi^i}{\partial x^j}d\lambda_i \wedge dx^j,$$

while

$$d_J\beta_i = \sum_{j,k}\frac{\partial^2\phi^i}{\partial x^j\partial x^k}\lambda_j(\lambda_i - \lambda_k)dx^j \wedge dx^k + \sum_j \frac{\partial\phi^i}{\partial x^j}\lambda_i d\lambda_i \wedge dx^j.$$

By skew-symmetrizing in the first sum we obtain

$$d_J\beta_i = \tfrac{1}{2}\sum_{j,k}\frac{\partial^2\phi^i}{\partial x^j\partial x^k}\lambda_i(\lambda_j - \lambda_k)dx^j \wedge dx^k + \sum_j \frac{\partial\phi^i}{\partial x^j}\lambda_i d\lambda_i \wedge dx^j.$$

Thus $d_J\beta_i = \lambda_i d\beta_i$, or equivalently $d_J b = (db)\Lambda$. Then from the relation $aV = b$ we have

$$(d_J a)V = (-a(dV) + db)\Lambda = (da)V\Lambda,$$

so that

$$d_J a = (da)V\Lambda V^{-1} = (da)C.$$

3 The Levi-Civita conditions in terms of tensors

I now work in terms of general coordinates (that is, there is no assumption that the coordinates are separation coordinates or that they are coordinates with respect to which J is diagonal).

The Einstein summation convention is in force in this section.

3.1 A special case

I consider first the special case in which $\alpha^{(r)} = 0$, $r \geq 1$, so that

$$dd_{\bar{J}}H = \alpha \wedge dH, \qquad \text{where } H = \tfrac{1}{2}g^{ij}p_ip_j, \quad J = J^i_j\frac{\partial}{\partial x^i}\otimes dx^j, \quad \alpha = \alpha_i dx^i.$$

I shall use 1-forms on T^*M adapted to the Levi-Civita connection, which are dx^i and π_i where

$$\pi_i = dp_i - \Gamma^k_{ij}p_k dx^j, \qquad d\pi_i = \tfrac{1}{2}R^l_{\ ijk}p_l dx^j \wedge dx^k + \Gamma^k_{ij}dx^j \wedge \pi_k.$$

The dual basis of vector fields is $\{X_i, \partial/\partial p_i\}$ where

$$X_i = \frac{\partial}{\partial x^i} + \Gamma^k_{ij}p_k\frac{\partial}{\partial p_j}.$$

We need an expression for the complete lift of J to T^*M; we have

$$\tilde{J} = J^i_j \left(X_i \otimes dx^j + \frac{\partial}{\partial p_j} \otimes \pi_i \right) + \left(J^k_{i|j} - J^k_{j|i} \right) p_k \frac{\partial}{\partial p_i} \otimes dx^j$$

where the solidus denotes a covariant derivative. Indices will be raised and lowered with the metric tensor in the usual way. Where I refer below to the eigenfunctions of a valence 2 tensor I mean the eigenfunctions of its type $(1,1)$ form.

I now derive the consequences of the relation $dd_{\tilde{J}}H = \alpha \wedge dH$. First, we have

$$dH = g^{ij} p_j \pi_i, \qquad d_{\tilde{J}}H = J^{ij} p_j \pi_i + g_{il} \left(J^{kj|l} - J^{kl|j} \right) p_j p_k dx^i.$$

It follows from the latter that

$$\begin{aligned} dd_{\tilde{J}}H &= -J^{ij} \pi_i \wedge \pi_j + g_{il} \left(-J^{jk|l} + J^{jl|k} + J^{kl|j} \right) p_k dx^i \wedge \pi_j \\ &\quad + \left(\tfrac{1}{2} J^{kl} R^m_{kij} + g_{jk} (J^{lm|k} - J^{lk|m})_{|i} \right) p_l p_m dx^i \wedge dx^j. \end{aligned}$$

On comparing coefficients of $\pi_i \wedge \pi_j$ on both sides of the equation $dd_{\tilde{J}}H = \alpha \wedge dH$ we find that $J_{ji} = J_{ij}$, since there are no such termsd on the right-hand side; that is to say, the symmetry of J with respect to g is actually a consequence of the equation, and does not have to be assumed independently. From the $dx^i \wedge \pi_j$ terms we obtain

$$-J_{jk|i} + J_{ij|k} + J_{ik|j} = \alpha_i g_{jk},$$

or equivalently

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik}).$$

It turns out that the remaining conditions are satisfied as a consequence of these (see [7]).

A symmetric tensor J_{ij} with pointwise simple eigenvalues such that

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$$

will be called a Benenti tensor. Tensors satisfying the differential condition, but not necessarily the condition on the eigenvalues, have been called variously special conformal Killing tensors, L-tensors, and in the case of flat space elliptic coordinates matrices and planar inertia tensors. The name Benenti tensor or Benenti system was introduced by Bolsinov and Matveev [3], in a somewhat different context, in acknowledgement of the fact that such tensors were first studied by Benenti. The name special conformal Killing tensor, which was used by Sarlet, Thompson and myself [7], refers to the fact that such a J is a conformal Killing tensor:

$$J_{jk|i} + J_{ki|j} + J_{ij|k} = \alpha_i g_{jk} + \alpha_j g_{ki} + \alpha_k g_{ij},$$

though of course not all conformal Killing tensors take the special form. Benenti himself uses the name L-tensor [2], in honour of Levi-Civita, and because, as he says, L is a letter adjacent to K (for Killing). The name elliptic coordinates matrix, used by Lundmark [8], alludes to the fact that such a tensor is a generator of elliptic separation coordinates; while the name planar inertia tensor, again coined by Benenti [1], refers to a way of interpreting such tensors, in flat space, in mechanical terms.

An important feature of any Benenti tensor is that its Nijenhuis torsion vanishes automatically, as a consequence of the defining equation. This is in fact a particular case of a rather more general result: if J is symmetric and satisfies

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i K_{jk} + \alpha_j K_{ik}),$$

where K is symmetric and satisfies $J_i^k K_{jk} = J_j^k K_{ik}$, and α is any 1-form on M, then J has zero torsion. The proof is a simple tensor calculation:

$$\begin{aligned}
&2(J_i^l J_{jk|l} - J_j^l J_{ik|l} + J_k^l (J_{il|j} - J_{jl|i})) \\
&= J_i^l(\alpha_j K_{kl} + \alpha_k K_{jl}) - J_j^l(\alpha_i K_{kl} + \alpha_k K_{il}) + J_k^l(\alpha_i K_{lj} - \alpha_j K_{li}) \\
&= \alpha_i(J_k^l K_{jl} - J_j^l K_{kl}) - \alpha_j(J_k^l K_{il} - J_i^l K_{kl}) + \alpha_k(J_i^l K_{jl} - J_j^l K_{il}) \\
&= 0.
\end{aligned}$$

We can conclude that if a metric g admits a Benenti tensor it is separable. If the eigenfunctions of the Benenti tensor are functionally independent then they are separation coordinates; otherwise we have to find coordinates with respect to which J is diagonal.

On taking the trace in the equation defining the Benenti tensor we find that $\alpha = d(\operatorname{tr} J) = -d\sigma_{n-1}$. The companion relations give $d_J\alpha = d\alpha = 0$ for $n > 2$; but $d\sigma_{n-1} = d_J \log|\sigma_0|$, and $d_J^2 = 0$ when J has zero torsion.

3.2 Properties of Benenti tensors

We here list a number of properties of Benenti tensors which have appeared in other publications.

The equation $J_{ij|k} = \frac{1}{2}(\alpha_i K_{jk} + \alpha_j K_{ik})$, $\alpha = d(\operatorname{tr} J)$, is linear, so the set of its solutions is a real vector space. The metric g_{ij} is trivially a solution.

A valence 2 conformal Killing tensor with pointwise simple eigenvalues whose Nijenhuis torsion vanishes is necessarily a Benenti tensor [7].

If J is a Benenti tensor in type $(1,1)$ form and A is its cofactor tensor, so that for J non-singular $A = (\det J)J^{-1}$, then A is a valence 2 Killing tensor. Moreover, when J is Benenti so is $J - \lambda I$ for any constant λ, so the cofactor tensor $A(\lambda)$ of $J - \lambda I$ is Killing.

But $A(\lambda)$ is polynomial of degree $(n-1)$ in λ,

$$A(\lambda)_{ij} = \sum_{m=1}^{n} A_{ij}^{(m)} \lambda^{n-m}.$$

The $A^{(m)}$ satisfy the Eisenhart conditions [4]. When a metric admits a Benenti tensor, the commuting first integrals promised in Eisenhart's theorem can be computed purely algebraically.

The integrability conditions of the equation $J_{ij|k} = \frac{1}{2}(\alpha_i K_{jk} + \alpha_j K_{ik})$ are

$$-J_i^m R_{mjkl} - J_j^m R_{mikl} = \tfrac{1}{2}(g_{ik}\alpha_{j|l} + g_{jk}\alpha_{i|l} - g_{il}\alpha_{j|k} - g_{jl}\alpha_{i|k}).$$

In coordinates with respect to which J is diagonal, for i, j, k distinct we have

$$(\lambda_i - \lambda_j) R_{ijik} = \tfrac{1}{2} g_{ii}\alpha_{j|k} = 0,$$

so the curvature is of so-called orthogonal type: $R_{ijik} = 0$, $j \neq k$ [4]. (This condition is of course void for $n = 2$.)

For $n \geq 3$, there is no Benenti tensor for a general Liouville metric. The condition $R_{ijik} = 0$ with $g_{ii} = \varphi_1(x^1) + \varphi_2(x^2) + \cdots + \varphi_n(x^n)$ is $\varphi_j' \varphi_k' = 0$.

Spaces of constant curvature do admit Benenti tensors. Indeed, if a Riemannian space admits two Benenti tensors which are independent in the sense that they have no non-trivial common invariant subspaces then it is a space of constant curvature [4].

The general solution of the equation $J_{ij|k} = \frac{1}{2}(\alpha_i K_{jk} + \alpha_j K_{ik})$ in Euclidean space is $J_{ij} = a x_i x_j + b_i x_j + b_j x_i + c_{ij}$ where a, b_i and c_{ij} are constants with $c_{ji} = c_{ij}$; the eigenfunctions define elliptic coordinates when $a \neq 0$ and parabolic coordinates when $a = 0$ [4, 8].

Tensors satisfying the equation $J_{ij|k} = \frac{1}{2}(\alpha_i K_{jk} + \alpha_j K_{ik})$ arise also in certain situations not directly concerned with separability: in the study of projectively equivalent metrics [3, 5]; and in relation to a certain class of completely integrable non-conservative Lagrangian systems [6, 8].

3.3 Another 'one-step' case

Ecouraged by the comparative wealth of information obtained in the previous two subsections, one naturally wonders whether other situations in which there is only one term on the right-hand side lead to interesting results.

The next case to try is evidently

$$dd_{\bar{J}}H = \beta \wedge d_{\bar{J}}H.$$

But using a result quoted earlier, we see that for any 1-form θ and non-singular tensor K with vanishing torsion

$$d(K^{-1}\theta)(X,Y) = d\theta(K^{-1}X,Y) + d\theta(X,K^{-1}Y) - d(K\theta)(K^{-1}X,K^{-1}Y),$$

from which it follows that

$$dd_{\tilde{J}^{-1}}H(X,Y) = -dd_{\tilde{J}}H(\tilde{J}^{-1}X,\tilde{J}^{-1}Y),$$

so that

$$dd_{\tilde{J}^{-1}}H = \alpha\wedge dH, \quad \alpha = -J^{-1}\beta.$$

But $\tilde{J}^{-1} = \widetilde{J^{-1}}$ when J has vanishing torsion, so J^{-1} is a Benenti tensor — and it defines the same diagonal equivalence class of coordinates as J does. So this case is not essentially different from the case discussed previously.

3.4 The Levi-Civita conditions in terms of tensors: the general case

I consider again the equation

$$dd_{\tilde{J}}H = \sum_{r=0}^{n-1}\alpha^{(r)}\wedge d_{\tilde{J}^r}H,$$

which I regard as an equation for J, given H; and as before I don't assume that J is symmetric or has zero torsion. I express this equation in tensorial form in arbitrary coordinates.

First of all

$$\tilde{J}^r = (J^r)^i_j\left(X_i\otimes dx^j + \frac{\partial}{\partial p_j}\otimes\pi_i\right) + \cdots\frac{\partial}{\partial p_i}\otimes dx^j;$$

in particular, there are no $X_i\otimes\pi_j$ terms. Thus

$$d_{\tilde{J}^r}H = (J^r)^{ij}p_j\pi_i + \cdots dx^i.$$

As before,

$$dd_{\tilde{J}}H = -J^{ij}\pi_i\wedge\pi_j + g_{il}\left(-J^{jk|l} + J^{jl|k} + J^{kl|j}\right)p_k dx^i\wedge\pi_j + \cdots dx^i\wedge dx^j.$$

By comparing $\pi^i\wedge\pi^j$ terms we obtain $J_{ji} = J_{ij}$. From the $\pi^i\wedge dx^j$ terms we find that

$$-J_{jk|i} + J_{ij|k} + J_{ik|j} = \sum_{r=0}^{n-1}\alpha_i^{(r)}(J^r)_{jk},$$

whence

$$J_{ij|k} = \tfrac{1}{2}\sum_{r=0}^{n-1}\left(\alpha_i^{(r)}(J^r)_{jk} + \alpha_j^{(r)}(J^r)_{ik}\right).$$

Since J is symmetric, so is J^r for every r. A simple extension of a result proved earlier shows that J must have zero torsion.

The $dx^i \wedge dx^j$ terms give extra conditions in this case: after some calculation they are found to be

$$\sum_{r=0}^{n-1}(\alpha^{(r)}_{i|j} - \alpha^{(r)}_{j|i})(J^r)_{kl} + \sum_{r=1}^{n-1}\sum_{s=0}^{n-1}\sum_{t=0}^{r-1}\left(\alpha^{(r)}_i (J^t\alpha^{(s)})_j - \alpha^{(r)}_j (J^t\alpha^{(s)})_i\right)(J^{r+s-t-1})_{kl} = 0,$$

or equivalently

$$\sum_{r=0}^{n-1} d\alpha^{(r)} \otimes J^r - \sum_{r=1}^{n-1}\sum_{s=0}^{n-1}\sum_{t=0}^{r-1} \alpha^{(r)} \wedge J^t\alpha^{(s)} \otimes J^{r+s-t-1} = 0.$$

The power $r+s-t-1$ may exceed $n-1$. Its maximum value is $2n-3$, when $r = s = n-1$, $t = 0$ — but then the form multiplying $J^{r+s-t-1}$ vanishes. Thus the effective maximum is $2n-4$, which exceeds $n-1$ for $n \geq 4$. For such n those terms $J^{r+s-t-1}$ with $r+s-t-1 > n-1$ can be expressed as sums of terms J^m with $0 \leq m \leq n-1$ by repeated use of the minimal polynomial of J. Thus the condition takes the form

$$\sum_{r=0}^{n-1}(d\alpha^{(r)} + \beta^{(r)}) \otimes J^r = 0$$

for certain 2-forms $\beta^{(r)}$. The tensors J^r for $0 \leq r \leq n-1$ are linearly independent, so this is equivalent to the n conditions

$$d\alpha^{(r)} + \beta^{(r)} = 0.$$

The 2-forms $\beta^{(r)}$ are linear combinations of forms $\alpha^{(r)} \wedge J^t\alpha^{(s)}$ with coefficients which involve the σ_r. In fact these conditions constitute the relevant part of the condition $dA + A \wedge A = 0$ obtained in the previous section.

Thus a necessary and sufficient condition for separability is that there exist 1-forms $\alpha^{(r)}$ and a type $(1,1)$ tensor field J on M, symmetric with respect to g, and with pointwise simple eigenvalues, such that

$$J_{ij|k} = \tfrac{1}{2}\sum_{r=0}^{n-1}\left(\alpha^{(r)}_i (J^r)_{jk} + \alpha^{(r)}_j (J^r)_{ik}\right)$$

and

$$\sum_{r=0}^{n-1} d\alpha^{(r)} \otimes J^r - \sum_{r=1}^{n-1}\sum_{s=0}^{n-1}\sum_{t=0}^{r-1} \alpha^{(r)} \wedge J^t\alpha^{(s)} \otimes J^{r+s-t-1} = 0.$$

It is a consequence of the equation for $J_{ij|k}$ that J has vanishing torsion. If the eigenfunctions of J are functionally independent they may be taken as coordinates and are separation coordinates; more generally, the separation coordinates are those with respect to which J is diagonal.

When this holds, the $\alpha^{(r)}$ satisfy the companion relations

$$d_J\alpha^{(0)} = -\sigma_0 d\alpha^{(n-1)}, \quad d_J\alpha^{(k)} = d\alpha^{(k-1)} - \sigma_k d\alpha^{(n-1)},$$

$k = 1, 2, \ldots, n-1$, where the σs are, to within sign, the elementary symmetric polynomials in the eigenfunctions of J.

Furthermore, by taking the trace of the equation for $J_{ij|k}$ we find that

$$\sum_{r=0}^{n-1} J^r\alpha^{(r)} = d(\mathrm{tr}\,J).$$

3.5 Back to the 'one-step' case

The conditions derived in the previous subsection are not very amenable to further analysis in general. They can however be useful in particular cases. To illustrate this remark, I will show that there are few separable systems with only one term on the right-hand side: in fact there are no separable systems satisfying

$$dd_{\bar{J}}H = \alpha \wedge d_{\bar{J}^r}H \qquad \text{with } r > 1$$

for which J has functionally independent eigenfunctions. The proof goes as follows.

The companion relations reduce in this case to $d\alpha = d_J\alpha = 0$.

The conditions

$$\sum_{r=0}^{n-1} d\alpha^{(r)} \otimes J^r - \sum_{r=1}^{n-1}\sum_{s=0}^{n-1}\sum_{t=0}^{r-1} \alpha^{(r)} \wedge J^t\alpha^{(s)} \otimes J^{r+s-t-1} = 0$$

reduce to

$$\sum_{t=0}^{r-1} \alpha \wedge J^t\alpha \otimes J^{2r-t-1} = 0.$$

Write ω_t for $\alpha \wedge J^t\alpha$. For any vectors u, v, w,

$$\sum_{t=0}^{r-1} \omega_t(u,v)J^{2r-t-1}(w) = J^r\left(\sum_{t=0}^{r-1}\omega_t(u,v)J^{r-t-1}(w)\right) = 0.$$

But J is non-singular almost everywhere, so

$$\sum_{t=0}^{r-1} \omega_t(u,v)J^{r-t-1} = 0.$$

The matrices J^{r-t-1}, $t = 0, 1, \ldots, r-1$, are linearly independent since $r - 1 < n - 1$, and so $\omega_t = 0$, $t = 0, 1, \ldots, r-1$. In particular, since $r > 1$, $\alpha \wedge J\alpha = 0$, so that α is an eigenform of J. Now $d\lambda_i$ is an eigenform corresponding to the eigenfunction λ_i. Then $J^r\alpha = \mu\lambda_i^r d\lambda_i$ for some i (no sum), while $d(\operatorname{tr} J) = \sum_j d\lambda_j$; so $J^r\alpha \neq d(\operatorname{tr} J)$. There is therefore no non-zero 1-form α satisfying the separability conditions.

Acknowlegements

I am a Guest Professor at Ghent University, and a Visiting Senior Research Fellow of King's College, University of London: I should like to express my gratitude to the Department of Mathematical Physics and Astronomy at Ghent and the Department of Mathematics at King's for their hospitality.

The work described here grew out of a collaboration with Willy Sarlet and Gerard Thompson. The paper, especially Section 2, has been strongly influenced by the ideas of Franco Magri on separability of Hamiltonian systems; I am indebted to him for many illuminating remarks in conversations and seminars.

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