Fefferman-type metrics and the projective geometry of sprays in two dimensions

M. Crampin
Department of Mathematical Physics and Astronomy,
Ghent University,
Krijgslaan 281, B–9000 Gent, Belgium
and D. J. Saunders
Department of Applied Mathematics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, UK

September 1, 2005

1 Introduction

There has recently been some interest in split-signature versions of Fefferman metrics ([11], see also [9, 10]). These are conformal families of metrics, of signature $(2, 2)$, defined on a 4-dimensional manifold; it is shown in [11] that each equivalence class of second-order ordinary differential equations (where two such equations are deemed equivalent if one is obtained from the other by a point transformation of variables) defines a conformal class of this type. This correspondence between differential equations and conformal geometries is one of several instances of similar phenomena, studied in [10], of which the association between the conformal geometry of space-time and the properties of a certain pair of partial differential equations which arises in the null surface formulation of general relativity [8] is a prime example.

The original Fefferman metrics [7] have Lorentz signature and are specified using a particular type of CR-structure, namely a real 3-dimensional manifold embedded in $\mathbf{C}^2$. The construction of the split-signature Fefferman-type metrics given in [11] is based on a close analogy between the Cartan geometry of a CR-structure and that of a second-order ordinary differential equation. The latter geometry was first described by Cartan himself in his projective connections paper of 1924 [2].

Our purpose in the present work is to relate the cited studies to some of the more classical results of path geometry, or in other words the projective geometry of sprays. Much of the development of path geometry was carried out in the early part of the last century, more or less independently of Cartan's approach: a major part of it may be found in [6, 15, 16, 17, 18]. The relationship between path geometry and Cartan connections has been established rather more recently, for instance, in [1, 4, 12], and we shall use some of the properties of this relationship to help us elicidate some of the properties of the split-signature metrics. We shall recall some of these properties, and also the relevant parts of the Fefferman construction from [11], in Section 2. In Section 3 we examine in some detail the construction of the conformal metrics in the flat case, using an approach which, while it is based on Cartan's version of connection theory, is closely related to the link with path geometry; and in the final section we exploit this link to discuss the construction in the general case.

2 Preliminaries

The Klein view of geometry is of a family of properties invariant under the action of a group. Given a Lie group G and a closed subgroup H, the Klein geometry on the homogeneous space G/H is given by the action of H on the fibres of the projection G $\to$ G/H; the Maurer-Cartan form of G captures the infinitesimal essence of this action. Such a geometry is 'flat', and forms the model for a 'curved' version called a Cartan geometry. For the latter, the projection G $\to$ G/H is replaced by a general principal H-bundle $P \to B$ where $\dim P = \dim \mathrm{G}$, and the link with the flat model is provided by a Cartan connection form ω (see [13]). Such a form is a $\mathfrak{g}$-valued form on P satisfying

- if $\mathfrak{h}$ is the Lie algebra of H and $X \in \mathfrak{h}$ then $\langle \tilde{X}, \omega \rangle = X$, where $\tilde{X}$ is the vertical vector field on P generated by X through the action of H;
- for each $p \in P$, $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism;
- for each $h \in \mathrm{H}$, $R_h^* \omega = \mathrm{ad}(h^{-1})\omega$.

The curvature of the connection is the $\mathfrak{g}$-valued 2-form $\Omega = d\omega + \omega \wedge \omega$. In the Klein case where $P = \mathrm{G}$, the Cartan connection form is taken to be the Maurer-Cartan form itself, and of course in that case the curvature vanishes.

A particular example of a Cartan connection form is the normal Cartan projective connection ω associated with a given second-order ordinary differential equation. This is an $\mathfrak{sl}(3)$-valued 1-form on a certain principal H-bundle $P \to \mathrm{P}TM$ over the projective tangent bundle to a 2-dimensional manifold M, where H is the subgoup of SL(3) consisting of matrices h of the form

$$h = \begin{pmatrix} h_0^0 & h_1^0 & h_2^0 \\ 0 & h_1^1 & h_2^1 \\ 0 & 0 & h_2^2 \end{pmatrix}$$

with $h_0^0 h_1^1 h_2^2 = 1$. We express the connection form as

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_1^0 & \omega_2^0 \\ \omega_0^1 & \omega_1^1 & \omega_2^1 \\ \omega_0^2 & \omega_1^2 & \omega_2^2 \end{pmatrix}.$$

We consider on P the symmetric covariant 2-tensor $\hat{g}_F = \sum_i \omega_i^2 \odot \omega_0^i$, $i = 0, 1, 2$; equivalently (using the fact that $\mathrm{tr}\,\omega = 0$) we could write $\hat{g}_F = \omega_0^1 \odot \omega_1^2 - \omega_1^1 \odot \omega_0^2$. This tensor will be the precursor of the Fefferman-type metrics.

From the properties of the Cartan connection form we see that, for any $X \in \mathfrak{h}$,

$$\hat{g}_F(\tilde{X}, \cdot) = X_0^1\omega_1^2 - X_1^1\omega_0^2 + X_1^2\omega_0^1 - X_0^2\omega_1^1 = -X_1^1\omega_0^2,$$

where the X_j^i are the entries in X considered as a matrix, since for any $X \in \mathfrak{h}$, $X_0^1 = X_0^2 = X_1^2 = 0$. Thus if $X_1^1 = 0$ then $\hat{g}_F(\tilde{X}, \cdot) = 0$.

Now the map $\mathrm{H} \to \mathbf{R}$ by $h \mapsto h_1^1$ is a homomorphism, so the set of h such that $h_1^1 = 1$ is a closed normal subgroup $\mathrm{K} \subset \mathrm{H}$ whose Lie algebra $\mathfrak{k}$ generates a distribution of vertical vector fields on P which are totally null with respect to $\hat{g}_F$. Furthermore, under the right action of $h \in \mathrm{H}$, $\hat{g}_F$ transforms to $h_0^0(h_2^2)^{-1}\hat{g}_F$, that is, it transforms conformally; and a fortiori $\hat{g}_F$ transforms conformally under the action of K. So, by factoring out by K, we obtain a line bundle L over $\mathrm{P}TM$ on which lives a conformal class of metrics. This construction is in direct analogy with one method of constructing the Fefferman class associated with a 3-dimensional CR struture, as is shown in [11], so the metrics are said to be of Fefferman type; for convenience we may call them the Fefferman metrics associated with the second-order differential equation. The metrics associated in this way with a second-order differential equation have signature $(2, 2)$, in contrast to those for a CR structure, which have standard Lorentz signature $(1, 3)$.

An explicit formula for a representative g of the Fefferman class associated with the second-order differential equation $y'' = Q(x, y, y')$ can be found in [10, 11]; it is

$$g = (dp - Qdx)dx - (dy - pdx)\left(dw + \tfrac{2}{3}Q_p dx + \tfrac{1}{6}Q_{pp}(dy - pdx)\right);$$

here (x, y, p) are affine coordinates on $\mathrm{P}TM$, identifying it locally with a 1-jet bundle, w is a fibre coordinate on L, and subscripts indicate partial derivatives with respect to the corresponding variable. Each metric g of the Fefferman class has the following properties [10, 11]:

- both the self-dual and the anti-self-dual parts of the Weyl tensor of g have Petrov type N;
- the self-dual and the anti-self-dual parts of the Weyl tensor are each expressible in terms of a single component;
- one of the principal null directions of g is a conformal Killing field for g, and is tangent to a shear-free but twisting null geodesic congruence on L.

We shall show that these properties (which, it is stated in [11], characterize this conformal class of metrics) have a simple interpretation in terms of path geometry.

The geometry of a single second-order ordinary differential equation under point transformations of coordinates is equivalent to the projective geometry of a 2-dimensional spray; the latter point of view seems to us to be preferable in that it is more directly tensorial in character. The analysis of the Klein geometry described above leads to a natural construction for the Fefferman class of metrics in terms of the projective geometry of 2-dimensional sprays, from which it is easy to see that the conformal invariants of the Fefferman class are just the projective invariants of the sprays. We therefore summarise the essential features of spray geometry that we shall need for this analysis.

A spray S on an m-dimensional manifold M is a second-order vector field on the slit tangent bundle $T°M$; taking coordinates x^a on M and fibre coordinates v^a on $T°M$ $(a = 1, 2, \ldots, m)$ we have

$$S = v^a \frac{\partial}{\partial x^a} - 2\Gamma^a \frac{\partial}{\partial v^a}$$

where the functions Γ^a are positively homogeneous of degree 2 in the fibre coordinates; when they are quadratic the spray is the geodesic spray of a symmetric affine connection, and is therefore said to be affine. As usual, we write $\Gamma^a_{bc} = \partial^2\Gamma^a/\partial v^b\,\partial v^c$; then Γ^a_{bc} is homogeneous of degree 0.

The geodesics of a spray are the projections onto M of its integral curves. There is a unique geodesic through a given point of M with a given initial tangent vector. In general the geodesic through x with initial tangent vector $-v$ does not coincide (as a point set) with the geodesic with initial tangent vector v; sprays for which the two geodesics do always coincide are said to be reversible, and we shall deal only with reversible sprays, since it is these that correspond to systems of second-order ordinary differential equations (this point is discussed in [4]).

Two sprays are said to be projectively equivalent if they differ by a multiple of the dilation field Δ on $T°M$; their geodesics on M differ only by a reparametrization. In any coordinate neighbourhood we can choose within a projective equivalence class a spray for which Γ^1 (say) vanishes; the integral curves of this spray are tangent to the local submanifold $v^1 = 1$, and are given by the solutions of the differential equations

$$\frac{d^2x^\alpha}{d(x^1)^2} = f^\alpha\left(x^1, x^\beta, \frac{dx^\gamma}{dx^1}\right), \qquad \alpha, \beta, \gamma = 2, 3, \ldots.$$

Conversely, given a system of second-order differential equations in the dependent variables x^α, with independent variable x^1, we can locally recover a spray by setting

$$\Gamma^1 = 0, \quad \Gamma^\alpha(x^i, v^i) = -\tfrac{1}{2}(v^1)^2 f^\alpha(x^i, v^\beta/v^1).$$

Under a coordinate transformation involving all of the coordinates x^a (a so-called point transformation) the spray corresponding to the new system of differential equations will be projectively equivalent to that corresponding to the original one. The invariants of the system of second-order ordinary differential equations under point transformations will be the projective invariants of the corresponding projective equivalence class of sprays.

Douglas [6] was the first to point out that the quantity

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{n+1}\left(\Gamma_c\delta^a_b + \Gamma_b\delta^a_c + B_{bc}v^a\right), \quad \Gamma_a = \Gamma^b_{ab} = \Gamma^b_{ba}, \quad B_{bc} = \frac{\partial\Gamma_b}{\partial v^c} = \frac{\partial\Gamma_c}{\partial v^b}$$

is projectively invariant; he called it the fundamental projective invariant, because all projective invariants are derived from it. The Π^a_{bc} are not in fact the components of a tensor, nor even of a connection, but nevertheless the quantities we are about to derive from them are tensorial. It is important to note that Π^a_{bc} is trace-free.

The first projective invariant is just

$$D^a_{bcd} = \frac{\partial\Pi^a_{bc}}{\partial v^d},$$

which is called the Douglas tensor. It is completely symmetric in its lower indices, trace-free, and satisfies $D^a_{bcd}v^d = 0$ by Euler's theorem on homogeneous functions. The vanishing of the Douglas tensor is the necessary and sufficient condition for the projective class to contain an affine spray.

As is so often happens, the two-dimensional case is special, so we restrict our discussion of projective invariants to this case. The first thing we can say is that in two dimensions the Douglas tensor can be expressed in terms of a single component (see for example [14]); in fact if we take a second-order differential equation $y'' = Q(x, y, y')$ to represent the projective class of sprays every component of the Douglas tensor is a multiple of Q_{pppp}, the fourth partial derivative of Q with respect to the derivative variable.

To derive a second projective invariant we first introduce a quantity modelled on the Riemann curvature:

$$\mathfrak{R}^a_{bcd} = H_c(\Pi^a_{bd}) - H_d(\Pi^a_{bc}) + \Pi^a_{ce}\Pi^e_{bd} - \Pi^a_{de}\Pi^e_{bc},$$

where

$$H_a = \frac{\partial}{\partial x^a} - \Pi^b_{ac}v^c\frac{\partial}{\partial v^b}.$$

It follows from the fact that Π^a_{bc} is trace-free that $\mathfrak{R}^a_{acd} = 0$. Since we are in two dimensions $\mathfrak{R}^a_{bcd}$ is determined by $\mathfrak{R}_{ab} = \mathfrak{R}^c_{acb}$, just as in two dimensions the Riemann curvature is determined by the Ricci tensor. In fact

$$\mathfrak{R}^1_{112} = \mathfrak{R}_{12}, \quad \mathfrak{R}^1_{212} = \mathfrak{R}_{22}, \quad \mathfrak{R}^2_{112} = -\mathfrak{R}_{11}, \quad \mathfrak{R}^2_{212} = -\mathfrak{R}_{21}.$$

It follows that $\mathfrak{R}_{21} = \mathfrak{R}_{12}$. If we set $\mathfrak{R} = \mathfrak{R}_{ab}v^av^b$ then

$$\frac{\partial\mathfrak{R}}{\partial v^a} = 2\mathfrak{R}_{ab}v^b, \quad \frac{\partial^2\mathfrak{R}}{\partial v^a\partial v^b} = \mathfrak{R}_{ab},$$

since $\mathfrak{R}_{ab}$ is homogeneous of degree 0. Thus $\mathfrak{R}^a_{bcd}$ is completely determined by $\mathfrak{R}$. We have

$$[H_1, H_2] = -\mathfrak{R}^a_{b12}v^b\frac{\partial}{\partial v^a} = -\mathfrak{R}^a_{12}\frac{\partial}{\partial v^a},$$

and using the expressions for $\mathfrak{R}^a_{b12}$ in terms of $\mathfrak{R}_{ab}$ we find that $\mathfrak{R}^1_{12} = \mathfrak{R}_{b2}v^b = \frac{1}{2}\mathfrak{R}_2$ and $\mathfrak{R}^2_{12} = -\mathfrak{R}_{b1}v^b = -\frac{1}{2}\mathfrak{R}_1$, where we have written $\mathfrak{R}_a$ for $\partial\mathfrak{R}/\partial v^a$.

In dimension 2 the second projective invariant is the Berwald invariant B, which is given by

$$B = \tfrac{1}{2}(H_1(\mathfrak{R}_2) - H_2(\mathfrak{R}_1)).$$

We return to the general case. We may represent an entire projective class of sprays on M by using a single spray on an $(m+1)$-dimensional manifold $\mathcal{V}M$, the volume bundle of M [4]. This is a principal $\mathbf{R}_+$ bundle $\mathcal{V}M \to M$ whose elements are $\pm$ equivalence classes of m-covectors on M. Taking coordinates (x^0, x^a), where for any $\theta \in \mathcal{V}M$

$$\theta = \{x^0(\theta)\}^{m+1}dx^1 \wedge \ldots \wedge dx^m,$$

and taking $\mathbf{R}_+$ to act by multiplying the fibre coordinate x^0, we obtain a fundamental vector field Υ given by $x^0\,\partial/\partial x^0$. The volume bundle also has a tautological $(m+1)$-density $d\Theta$ given by $|(m+1)(x^0)^m dx^0 \wedge dx^1 \wedge \ldots \wedge dx^m|$. The projective class of sprays containing S may be represented by the normal Berwald-Thomas-Whitehead spray, or *BTW*-spray, a second-order vector field on $T^\circ(\mathcal{V}M)$ given by

$$v^0 \frac{\partial}{\partial x^0} + v^a \frac{\partial}{\partial x^a} + \frac{\mathfrak{R}}{m-1} x^0 \frac{\partial}{\partial v^0} - \left(\Pi^a_{bc} v^b v^c + (x^0)^{-1} v^0 v^a\right) \frac{\partial}{\partial v^a}.$$

The normal *BTW*-spray may be defined invariantly [4], and is unique.

3 The flat case

It is tempting to draw the conclusion from the group-theoretic perspective that the Cartan geometry associated with the pair of groups $(\mathrm{SL}(3), \mathrm{K})$ is a subgeometry of the Cartan conformal geometry with signature $(2,2)$. This turns out not quite to be the case: the homogeneous space $\mathrm{SL}(3)/\mathrm{K}$ is not identical with the homogeneous space of the conformal geometry, but with an open submanifold of it. We shall explain this by considering in detail the Klein geometry in the two cases. The Klein geometry associated with a conformal structure is known as a Möbius geometry [13]; we begin by constructing a version of the Möbius geometry with signature $(2,2)$.

Consider the 6-dimensional vector space $V = \mathbf{R}^3 \oplus \mathbf{R}^{3*}$. If $\{e_i\}$ $(i = 0,1,2)$ is the standard basis of $\mathbf{R}^3$ and $\{\varepsilon^j\}$ is the dual basis of $\mathbf{R}^{3*}$ then we shall call $\{(e_i, 0), (0, \varepsilon^j)\}$ the standard basis of V. The canonical symmetric bilinear form q on V is defined by

$$q\left((x,\xi),(y,\eta)\right) = \langle x, \eta\rangle + \langle y, \xi\rangle$$

and has signature $(3,3)$. We denote by $\mathcal{N}$ the null cone of q, that is, the set of null lines through the origin in V; it is the quadric cone given by $\langle x, \xi\rangle = 0$. When deprived of its vertex this defines, by projection, a submanifold Q of $\mathrm{P}V$ which is a quadric.

Incidentally, there is another way of approaching the definition of Q, which is to observe that $\bigwedge^2 \mathbf{R}^4$ is a 6-dimensional real vector space, which can be identified with V in such a way that $\mathcal{N}$ defines the decomposable elements of $\bigwedge^2 \mathbf{R}^4$, and then Q becomes the Grassmanian of 2-planes in $\mathbf{R}^4$. We have discussed a generalization of the construction of the Fefferman metrics from the Grassmanian point of view in [5], so we will not pursue this aspect of the matter any further here.

Let $(x,\xi) \in \mathcal{N}$, $x \neq 0$, $\xi \neq 0$, and let (u,μ) be a tangent vector to $\mathcal{N}$ at (x,ξ). The condition for $(u,\mu) \in T_{(x,\xi)}\mathcal{N}$ is that $q\left((x,\xi),(u,\mu)\right) = 0$; the restriction of q to $T_{(x,\xi)}\mathcal{N}$ is therefore singular, as the tangent to the null line $t \mapsto (tx, t\xi)$ is orthogonal to every vector. In general, taking $(v,\nu) \in T_{(x,\xi)}\mathcal{N}$, $r, s \in \mathbf{R}$,

$$q\left((u,\mu) + r(x,\xi), (v,\nu) + s(x,\xi)\right) = q\left((u,\mu),(v,\nu)\right).$$

On the other hand, dilations along the ruling of $\mathcal{N}$, that is, along the null lines, effect scale transformations of q. Thus the restriction of q to $\mathcal{N} - \{0\}$ defines a conformal class of metrics (of signature $(2,2)$) on Q.

Our choice of a standard basis of V allows us to define an action of GL(6) and its subgroups on V, and thus an action of PGL(6) on PV. The subgroup of GL(6) containing matrices A satisfying $A^T\Sigma A = \Sigma$ is the orthogonal group O(q); the quotient of this group by non-zero multiples of the identity (here, of course, just plus or minus the identity) is PO(q), the projective orthogonal group. The matrix pairs in this group may have determinant either $+1$ or -1. This group acts by conformal transformations on the 4-dimensional quadric Q in PV.

There is also an action of GL(3) and its subgroups on V, and thus an action of PGL(3) = SL(3) on PV. The action is defined for each $a \in \mathrm{GL}(3)$ by setting

$$a \cdot (x, \xi) = (ax, \xi a^{-1})$$

where the action of a on both the column vector x and the row vector ξ is matrix multiplication. This action obviously preserves q, so we may identify GL(3) with a subgroup of O(q). The projected action of SL(3) is, of course, $a \cdot [x, \xi] = [ax, \xi a^{-1}]$. By projection, we may identify SL(3) with a subgroup of PO(Q), of the form

$$\pm \begin{pmatrix} a & 0 \\ 0 & (a^T)^{-1} \end{pmatrix}$$

(the transpose arises, of course, because here a^{-1} acts on the left on the column vector ξ^T rather than on the right on the row vector ξ). In this way SL(3) acts by conformal transformations on the 4-dimensional quadric Q. The action is not transitive, and there are three orbits:

$$\begin{aligned} Q_+ &= \{[x, 0] : x \in \mathbf{R}^3_0\} \\ Q_- &= \{[0, \xi] : \xi \in \mathbf{R}^{3*}_0\} \\ Q_0 &= \{[x, \xi] : x \in \mathbf{R}^3_0, \xi \in \mathbf{R}^{3*}_0, \langle x, \xi \rangle = 0\}, \end{aligned}$$

where the subscript 0 indicates that the origin has been deleted. It is clear that SL(3) is transitive on both Q_+ and Q_-. Take $[e_0, 0] \in Q_+$ and $[0, \varepsilon^2] \in Q_-$; then the stabiliser subgroups of these points are

$$\mathrm{stab}([e_0, 0]) = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \qquad \mathrm{stab}([0, \varepsilon^2]) = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

so that $Q_+ \cong \mathrm{P}^2 \cong Q_-$ where P^2 denotes 2-dimensional projective space.

We shall now check that SL(3) is transitive on Q_0. It will be useful here to introduce the cross product $x \times y$ of elements x and y of $\mathbf{R}^3$, defined as the element of $\mathbf{R}^{3*}$ such that $\langle z, x \times y \rangle = \Omega(x, y, z)$ where Ω is the standard volume element $\varepsilon^0 \wedge \varepsilon^1 \wedge \varepsilon^2$ on $\mathbf{R}^3$; that is to say, we identify $\Omega(x, y, z)$ with the scalar triple product of the vectors x, y and z. Then for any $x, y \in \mathbf{R}^3$, $\langle x, x \times y \rangle = 0$; moreover, if $\langle x, \xi \rangle = 0$ with $x \neq 0$ we can find $y \in \mathbf{R}^3$ such that $x \times y = \xi$. Write a matrix $a \in \mathrm{SL}(3)$ in column form as (a_0, a_1, a_2); then the rows of a^{-1} are $a_1 \times a_2$, $a_2 \times a_0$, $a_0 \times a_1$. If the columns of a are given by $a_0 = x$,

$a_1 = y$ where $x \times y = \xi$, and $a_2 = z$ such that $\langle z, x \times y\rangle = 1$, then $a \in \mathrm{SL}(3)$, $ae_0 = x$ and $\varepsilon^2 a^{-1} = \xi$.

To express Q_0 as a homogeneous space of SL(3) we note that

$$\mathrm{stab}([e_0, \varepsilon^2]) = \begin{pmatrix} * & * & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix};$$

that is, $Q_0 = \mathrm{SL}(3)/\mathrm{K}$. Since every point of Q_0 may be expressed as $[ae_0, \varepsilon^2 a^{-1}]$ for some $a \in \mathrm{SL}(3)$, the conformal class of metrics induced from q is, but for sign, the conformal class of the reduction of $\sum_i \omega_i^2 \odot \omega_0^i$ as described in Section 2, where ω is the Maurer-Cartan form of SL(3).

Now every element of Q can be written in the form $[x, x \times y]$ for some $x, y \in \mathbf{R}^3$. Clearly, if we replace x by λx ($\lambda \in \mathbf{R}_0$) and y by $y + \mu x$ ($\mu \in \mathbf{R}$) then we obtain the same element of Q. This suggests another way of representing Q_0, as follows. Let $W = \mathbf{R}_0^3 \oplus \mathbf{R}^3$, and define an equivalence relation on the elements $(x, y) \in W$ by setting $(\hat{x}, \hat{y}) \sim (x, y)$ if $\hat{x} = \lambda x$ ($\lambda \in \mathbf{R}_0$) and $\hat{y} = y + \mu x$ ($\mu \in \mathbf{R}$); write

$$\mathcal{T}\mathrm{P}^2 = \{([x], [\![y]\!]_x) : x \in \mathbf{R}_0^3, y \in \mathbf{R}^3\}$$

for the set of equivalence classes. Evidently $\mathcal{T}\mathrm{P}^2$ projects to P^2 by $([x], [\![y]\!]_x) \mapsto [x]$; the vector space structure on the second component of W induces a vector bundle structure on this projection.

We identify $\mathcal{T}\mathrm{P}^2$ with $Q_+ \cup Q_0$ in the following way. Define the map $\psi : \mathcal{T}\mathrm{P}^2 \to \mathrm{P}V$ by

$$\psi([x], [\![y]\!]_x) = [x, x \times y].$$

It is clear that ψ takes its values in $Q \subset \mathrm{P}V$; as $x \neq 0$ it follows that the image is contained in $Q_+ \cup Q_0$. Furthermore, $x \times y = 0$ if and only if y is a scalar multiple of x, or equivalently if and only if $[\![y]\!]_x = 0$. Thus ψ is a bijection of the slit vector bundle $\mathcal{T}^o\mathrm{P}^2$ with Q_0.

We can use ψ to induce a conformal class of metrics on $\mathcal{T}^o\mathrm{P}^2$ from that on Q_0, as follows. First, consider the map $\phi : W \to V$ by $\phi(x, y) = (x, x \times y)$, which maps W onto that part of the null-cone in V for which $x \neq 0$, say $\bar{\mathcal{N}}$. We obtain a symmetric twice-covariant tensor G on W by taking the pull-back to W by ϕ of the restriction of q to the tangent bundle to $\bar{\mathcal{N}}$ (here q is thought of as a metric on V). If we identify tangent spaces to vector spaces with the vector spaces themselves, we have

$$\phi_{*(x,y)}(u, v) = (u, u \times y + x \times v).$$

Notice that

$$q_{\phi(x,y)}\left((x, x \times y), \phi_{*(x,y)}(u, v)\right) = \langle x, u \times y\rangle + \langle u, x \times y\rangle = 0,$$

so $\phi_{*(x,y)}(u, v)$ is indeed tangent to $\bar{\mathcal{N}}$; and

$$q_{\phi(x,y)}\left(\phi_{*(x,y)}(u_1, v_1), \phi_{*(x,y)}(u_2, v_2)\right)$$

$$
\begin{aligned}
&= q_{\phi(x,y)}\left((u_1, u_1 \times y + x \times v_1), (u_2, u_2 \times y + x \times v_2)\right) \\
&= \langle u_1, u_2 \times y\rangle + \langle u_1, x \times v_2\rangle + \langle u_2, u_1 \times y\rangle + \langle u_2, x \times v_1\rangle \\
&= \Omega(x, v_2, u_1) + \Omega(x, v_1, u_2).
\end{aligned}
$$

Define the tensor G on W by

$$G_{(x,y)}\left((u_1, v_1), (u_2, v_2)\right) = \Omega(x, v_2, u_1) + \Omega(x, v_1, u_2),$$

and consider the 1-parameter groups $(x, y) \mapsto (e^t x, y)$ and $(x, y) \mapsto (x, y + tx)$ which generate the equivalence from which we obtain $\mathcal{T}^o\mathrm{P}^2$. It is clear that the generators of these 1-parameter groups, which as tangent vectors at (x, y) are respectively $(x, 0)$ and $(0, x)$, are both totally null; moreover, the 1-parameter groups themselves act as conformal transformations. Thus G induces a conformal class of metrics on $\mathcal{T}^o\mathrm{P}^2$, giving another representation of the Fefferman class in the flat model.

We can write

$$G = \varepsilon_{ijk}\, x^i dx^j dy^k$$

where (x^i, y^j) are the coordinates on W, $i, j = 0, 1, 2$. If we take affine coordinates (z^a, u^a), $a, b = 1, 2$, on $\mathcal{T}^o\mathrm{P}^2$ (for $x^0 \neq 0$) by setting

$$z^a = x^a/x^0, \quad u^a = (x^0 y^a - x^a y^0)/x^0,$$

it is easy to check that (x^a, u^a) is constant on any orbit of each of the one-parameter group actions. The Fefferman metrics on $\mathcal{T}^o\mathrm{P}^2$ are those conformal to

$$dz^1 du^2 - dz^2 du^1.$$

4 The general case

In the general case we have a projective equivalence class of reversible sprays on a 2-dimensional manifold M; we shall show how to construct the Fefferman class from these data.

The first task is to identify the 4-dimensional manifold on which the metrics of the class live; as in the flat case, it is formed from a vector bundle over M, of fibre dimension 2, which we denote by $\mathcal{T}M$. The 4-dimensional manifold we actually require is the slit bundle $\mathcal{T}^\circ M$ of $\mathcal{T}M$. The construction of the bundle $\mathcal{T}M$ follows the pattern of the construction of $\mathcal{T}\mathrm{P}^2$ in the flat case, with $T(\mathcal{V}M)$ playing the role in relation to M that $\mathbf{R}^3$ plays in relation to P^2.

For a vector field X on any manifold we denote by X^{C} and X^{V} the complete and vertical lifts of X to the tangent bundle. Then for any vector fields X, Y we have $[X^{\mathrm{V}}, Y^{\mathrm{V}}] = 0$; $[X^{\mathrm{C}}, Y^{\mathrm{V}}] = [X, Y]^{\mathrm{V}}$; $[X^{\mathrm{C}}, Y^{\mathrm{C}}] = [X, Y]^{\mathrm{C}}$; moreover, $[\Delta, X^{\mathrm{V}}] = -X^{\mathrm{V}}$, $[\Delta, X^{\mathrm{C}}] = 0$ where Δ is the Liouville field, the infinitesimal generator of dilations in the fibres; see for example [19].

For the fundamental vector field Υ on $\mathcal{V}M$ we obtain vector fields

$$\Upsilon^{\mathrm{C}} = x^0 \frac{\partial}{\partial x^0} + v^0 \frac{\partial}{\partial v^0}, \qquad \Upsilon^{\mathrm{V}} = x^0 \frac{\partial}{\partial v^0}$$

on $T(\mathcal{V}M)$. We set $\Upsilon^{\mathrm{H}} = \Upsilon^{\mathrm{C}} - \tilde{\Delta}$ where $\tilde{\Delta}$ is the Liouville field on $T(\mathcal{V}M)$ (the reason for the notation will become clear shortly); then

$$\Upsilon^{\mathrm{H}} = x^0 \frac{\partial}{\partial x^0} - v^a \frac{\partial}{\partial v^a}.$$

It is clear both from the original definition and the coordinate formulæ that $[\Upsilon^{\mathrm{H}}, \Upsilon^{\mathrm{V}}] = \Upsilon^{\mathrm{V}}$, so the distribution $\mathcal{D}$ spanned by Υ^{H} and Υ^{V} is integrable. In fact the 1-parameter groups corresponding to Υ^{H} and Υ^{V} respectively are

$$(x^0, x^a, v^0, v^a) \mapsto (e^t x^0, x^a, v^0, e^{-t} v^a) \quad \text{and} \quad (x^0, x^a, v^0, v^a) \mapsto (x^0, x^a, v^0 + t x^0, v^a).$$

Together they define a free effective action of the affine group $\mathrm{A}(1)$ on $T(\mathcal{V}M)$. We denote by $\mathcal{T}M$ the quotient of $T(\mathcal{V}M)$ by the action; the fibres of the principal bundle $T(\mathcal{V}M) \to \mathcal{T}M$ are the integral manifolds of $\mathcal{D}$. For coordinates on $\mathcal{T}M$ we can take (x^a, u^a) with $u^a = x^0 v^a$: this shows that when $\dim M = 2$, $\mathcal{T}M$ is the bundle of contravariant vector densities of weight $\frac{1}{3}$. In the flat case, when $M = \mathrm{P}^2$, Υ is the radial vector field on $\mathbf{R}^3$, and the construction essentially reproduces that of $\mathcal{T}\mathrm{P}^2$.

To study the general case further, we shall need to establish some facts about the horizontal distribution defined by a spray S. The horizontal distribution on $T^\circ M$ determined by S is spanned by horizontal lifts X^{H} of vector fields X on M, where

$$X^{\mathrm{H}} = \tfrac{1}{2}(X^{\mathrm{C}} + [X^{\mathrm{V}}, S]).$$

The horizontal distribution is homogeneous, in the sense that for any X on M, $[\Delta, X^{\mathrm{H}}] = 0$, so that X^{H} is homogeneous of degree zero.

For the normal BTW-spray $\tilde{S}$ on the volume bundle of M we have $[\Upsilon^{\mathrm{C}}, \tilde{S}] = 0$, $[\Upsilon^{\mathrm{V}}, \tilde{S}] = \Upsilon^{\mathrm{C}} - 2\tilde{\Delta}$; indeed, these conditions form part of the invariant definition of a BTW-spray [4]. From the latter condition we see that Υ^{H} as defined above is just the horizontal lift of Υ.

We now compute the brackets of Υ^{V} and Υ^{H} with arbitrary vertical and horizontal lifts. We have

$$\begin{aligned}
[\Upsilon^{\mathrm{V}}, X^{\mathrm{V}}] &= 0; \\
[\Upsilon^{\mathrm{V}}, X^{\mathrm{H}}] &= \tfrac{1}{2}([\Upsilon^{\mathrm{V}}, X^{\mathrm{C}}] + [\Upsilon^{\mathrm{V}}, [X^{\mathrm{V}}, \tilde{S}]]) \\
&= \tfrac{1}{2}([\Upsilon, X]^{\mathrm{V}} + [X^{\mathrm{V}}, [\Upsilon^{\mathrm{V}}, \tilde{S}]]) \\
&= \tfrac{1}{2}([\Upsilon, X]^{\mathrm{V}} + [X^{\mathrm{V}}, \Upsilon^{\mathrm{C}} - 2\tilde{\Delta}]) \\
&= \tfrac{1}{2}([\Upsilon, X]^{\mathrm{V}} - [\Upsilon, X]^{\mathrm{V}} - 2X^{\mathrm{V}}) = -X^{\mathrm{V}}; \\
[\Upsilon^{\mathrm{H}}, X^{\mathrm{V}}] &= [\Upsilon^{\mathrm{C}}, X^{\mathrm{V}}] - [\tilde{\Delta}, X^{\mathrm{V}}] = ([\Upsilon, X] + X)^{\mathrm{V}}; \\
[\Upsilon^{\mathrm{H}}, X^{\mathrm{H}}] &= [\Upsilon^{\mathrm{C}}, X^{\mathrm{H}}] - [\tilde{\Delta}, X^{\mathrm{H}}] = [\Upsilon^{\mathrm{C}}, X^{\mathrm{H}}] \\
&= \tfrac{1}{2}([\Upsilon^{\mathrm{C}}, X^{\mathrm{C}}] + [\Upsilon^{\mathrm{C}}, [X^{\mathrm{V}}, \tilde{S}]]) \\
&= \tfrac{1}{2}([\Upsilon, X]^{\mathrm{C}} + [[\Upsilon^{\mathrm{C}}, X^{\mathrm{V}}], \tilde{S}]) \\
&= \tfrac{1}{2}([\Upsilon, X]^{\mathrm{C}} + [[\Upsilon, X]^{\mathrm{V}}, \tilde{S}]) = [\Upsilon, X]^{\mathrm{H}}.
\end{aligned}$$

Note in particular that $[\Upsilon^{\mathrm{V}}, X^{\mathrm{H}}]$ and $[\Upsilon^{\mathrm{H}}, X^{\mathrm{V}}]$ are both vertical lifts, while $[\Upsilon^{\mathrm{H}}, X^{\mathrm{H}}]$ is a horizontal lift.

We now restrict attention to 2-dimensional spray manifolds. For any such manifold, we define a symmetric 2-tensor G on $T(\mathcal{V}M)$ by specifying its values on vertical and horizontal lifts, as follows (using the scalar triple product, as in the flat case):

$$G(X^{\mathrm{V}}, Y^{\mathrm{V}}) = G(X^{\mathrm{H}}, Y^{\mathrm{H}}) = 0; \quad G(X^{\mathrm{H}}, Y^{\mathrm{V}}) = d\Theta(\Upsilon, X, Y).$$

Note that

$$G(X^{\mathrm{H}}, Y^{\mathrm{V}}) + G(Y^{\mathrm{H}}, X^{\mathrm{V}}) = 0.$$

We may give an alternative definition of G as follows. First, we define for each point $u \in T(\mathcal{V}M)$ two linear maps $T_uT(\mathcal{V}M) \to T_{\tau(u)}\mathcal{V}M$, determined by the horizontal distribution corresponding to the BTW-spray, denoted by $\xi \mapsto \xi_{\mathrm{V}}$ and $\xi \mapsto \xi_{\mathrm{H}}$. The vertical subspace of $T_uT(\mathcal{V}M)$ is naturally identified with $T_{\tau(u)}\mathcal{V}M$, the horizontal subspace likewise. For $\xi \in T_uT(\mathcal{V}M)$, ξ_{V} and ξ_{H} are the vertical and horizontal components of ξ, considered as elements of $T_{\tau(u)}\mathcal{V}M$ using these identifications. Then for any $\xi, \eta \in T_uT(\mathcal{V}M)$,

$$G_u(\xi, \eta) = d\Theta_{\tau(u)}(\Upsilon, \xi_{\mathrm{H}}, \eta_{\mathrm{V}}) + d\Theta_{\tau(u)}(\Upsilon, \eta_{\mathrm{H}}, \xi_{\mathrm{V}}).$$

Evidently $G(\Upsilon^{\mathrm{V}}, \cdot) = 0$ and $G(\Upsilon^{\mathrm{H}}, \cdot) = 0$, so that the tangents to the fibres of the bundle $T(\mathcal{V}M) \to \mathcal{T}M$ are totally null. We therefore consider how G transforms under the actions of the group $\mathrm{A}(1)$ of the bundle, which we do by computing the Lie derivatives of G by the fundamental vector fields Υ^{V} and Υ^{H}.

Theorem 1 *The tensor G on $T(\mathcal{V}M)$ satisfies $\mathcal{L}_{\Upsilon^{\mathrm{V}}}G = 0$, $\mathcal{L}_{\Upsilon^{\mathrm{H}}}G = 2G$; it defines a conformal class of metrics on $\mathcal{T}^\circ M$.*

Proof It will be enough to calculate the Lie derivatives with arguments which are vertical or horizontal lifts.

For Υ^{V} we have $\Upsilon^{\mathrm{V}}(G(\cdot,\cdot)) = 0$ when the arguments are vertical or horizontal lifts, since then $G(\cdot,\cdot)$ is either zero or basic. Thus

$$(\mathcal{L}_{\Upsilon^{\mathrm{V}}}G)(X^{\mathrm{V}}, Y^{\mathrm{V}}) = -G([\Upsilon^{\mathrm{V}}, X^{\mathrm{V}}], Y^{\mathrm{V}}) - G(X^{\mathrm{V}}, [\Upsilon^{\mathrm{V}}, Y^{\mathrm{V}}]) = 0$$

because $[\Upsilon^{\mathrm{V}}, X^{\mathrm{V}}] = [\Upsilon^{\mathrm{V}}, Y^{\mathrm{V}}] = 0$;

$$(\mathcal{L}_{\Upsilon^{\mathrm{V}}}G)(X^{\mathrm{H}}, Y^{\mathrm{V}}) = -G([\Upsilon^{\mathrm{V}}, X^{\mathrm{H}}], Y^{\mathrm{V}}) - G(X^{\mathrm{H}}, [\Upsilon^{\mathrm{V}}, Y^{\mathrm{V}}]) = 0$$

because $[\Upsilon^{\mathrm{V}}, X^{\mathrm{H}}]$ is vertical; and

$$\begin{aligned}(\mathcal{L}_{\Upsilon^{\mathrm{V}}}G)(X^{\mathrm{H}}, Y^{\mathrm{H}}) &= -G([\Upsilon^{\mathrm{V}}, X^{\mathrm{H}}], Y^{\mathrm{H}}) - G(X^{\mathrm{H}}, [\Upsilon^{\mathrm{V}}, Y^{\mathrm{H}}]) \\ &= G(X^{\mathrm{V}}, Y^{\mathrm{H}}) + G(X^{\mathrm{H}}, Y^{\mathrm{V}}) = 0.\end{aligned}$$

Thus $\mathcal{L}_{\Upsilon^{\mathrm{V}}}G = 0$.

For Υ^{H} we have $\Upsilon^{\mathrm{H}}(G(\cdot,\cdot)) = 0$ when both arguments are vertical lifts or both are horizontal lifts; moreover $[\Upsilon^{\mathrm{H}}, X^{\mathrm{V}}]$ is a vertical lift and $[\Upsilon^{\mathrm{H}}, X^{\mathrm{H}}]$ is a horizontal lift; so

$$(\mathcal{L}_{\Upsilon^{\mathrm{H}}}G)(X^{\mathrm{V}}, Y^{\mathrm{V}}) = (\mathcal{L}_{\Upsilon^{\mathrm{H}}}G)(X^{\mathrm{H}}, Y^{\mathrm{H}}) = 0.$$

For the final part we need the fact that $\mathcal{L}_\Upsilon d\Theta = 3d\Theta$, which follows from a simple coordinate calculation (or the observation that $d\Theta$ is essentially homogeneous of degree 3 in x^0). Then

$$\begin{aligned}\Upsilon^{\mathrm{H}}(G(X^{\mathrm{H}},Y^{\mathrm{V}})) &= \Upsilon(d\Theta(\Upsilon,X,Y)) \\ &= 3d\Theta(\Upsilon,X,Y)+d\Theta(\Upsilon,[\Upsilon,X],Y)+d\Theta(\Upsilon,X,[\Upsilon,Y]),\end{aligned}$$

whereas

$$\begin{aligned}&G([\Upsilon^{\mathrm{H}},X^{\mathrm{H}}],Y^{\mathrm{V}})+G(X^{\mathrm{H}},[\Upsilon^{\mathrm{H}},Y^{\mathrm{V}}]) \\ &\quad= G([\Upsilon,X]^{\mathrm{H}},Y^{\mathrm{V}})+G(X^{\mathrm{H}},([\Upsilon,Y]+Y)^{\mathrm{V}}) \\ &\quad= d\Theta(\Upsilon,[\Upsilon,X],Y)+d\Theta(\Upsilon,X,[\Upsilon,Y])+d\Theta(\Upsilon,X,Y);\end{aligned}$$

thus

$$\begin{aligned}\mathcal{L}_{\Upsilon^{\mathrm{H}}}G(X^{\mathrm{H}},Y^{\mathrm{V}}) &= \Upsilon^{\mathrm{H}}(G(X^{\mathrm{H}},Y^{\mathrm{V}}))-G([\Upsilon^{\mathrm{H}},X^{\mathrm{H}}],Y^{\mathrm{V}})-G(X^{\mathrm{H}},[\Upsilon^{\mathrm{H}},Y^{\mathrm{V}}]) \\ &= 3d\Theta(\Upsilon,X,Y)+d\Theta(\Upsilon,[\Upsilon,X],Y)+d\Theta(\Upsilon,X,[\Upsilon,Y]) \\ &\quad -(d\Theta(\Upsilon,[\Upsilon,X],Y)+d\Theta(\Upsilon,X,[\Upsilon,Y])+d\Theta(\Upsilon,X,Y)) \\ &= 2d\Theta(\Upsilon,X,Y)=2G(X^{\mathrm{H}},Y^{\mathrm{V}}).\end{aligned}$$

That is to say, $\mathcal{L}_{\Upsilon^{\mathrm{H}}}G=2G$.

It follows from the fact that Υ^{V} and Υ^{H} are totally null that every vector field in the distribution spanned by Υ^{V} and Υ^{H} is an infinitesimal conformal transformation of G. Thus the conformal class of G passes to the quotient under the distribution spanned by Υ^{V} and Υ^{H}, to define a conformal class on $\mathcal{T}M$. ∎

Let us take coordinates (x^0,x^a) on $T(\mathcal{V}M)$ adapted to the structure, so that (x^a) may be regarded as coordinates on M. We denote by $\{H_0,H_a,V_0,V_a\}$ the horizontal and vertical lifts of the coordinate fields with respect to the normal BTW-spray, and by $\{dx^0,dx^a,\tilde{\pi}^0,\tilde{\pi}^a\}$ the dual basis of local 1-forms. Then $G=3(x^0)^3(dx^1\tilde{\pi}^2-dx^2\tilde{\pi}^1)$. A short calculation using the expression for the normal BTW-spray given earlier shows that

$$\tilde{\pi}^a=(x^0)^{-1}(x^0dv^a+v^adx^0+x^0\Pi^a_{bc}v^cdx^b+v^0dx^a);$$

then (ignoring a superfluous overall numerical factor)

$$\begin{aligned}G &= (x^0)^2\Big(dx^1(x^0dv^2+v^2dx^0+\Pi^2_{bc}x^0v^cdx^b) \\ &\qquad\qquad -dx^2(x^0dv^1+v^1dx^0+\Pi^1_{bc}x^0v^cdx^b)\Big) \\ &= (x^0)^2\Big(dx^1(du^2+\Pi^2_{bc}u^cdx^b)-dx^2(du^1+\Pi^1_{bc}u^cdx^b)\Big),\end{aligned}$$

where $u^a=x^0v^a$. Here $\Pi^a_{bc}=\Pi^a_{bc}(x,v)$; but Π^a_{bc} is positively homogeneous of degree 0 in the velocity variables, and is independent of v^0, so $\Pi^a_{bc}(x,v)=\Pi^a_{bc}(x,u)$. We may therefore take

$$dx^1(du^2+\Pi^2_{bc}u^cdx^b)-dx^2(du^1+\Pi^1_{bc}u^cdx^b)$$

as a representative of the Fefferman class on $\mathcal{T}M$, in the coordinates (x^a, u^a).

We now confirm that this is the same metric (up to a conformal factor) as that found in [10, 11] and quoted earlier. We set $x = x^1$, $y = x^2$, $t = u^1$, $p = u^2/u^1$: then

$$\begin{aligned} du^1 + \Pi^1_{ab}u^a dx^b &= dt + t(\Pi^1_{1a} + p\Pi^1_{2a})dx^a \\ du^2 + \Pi^2_{ab}u^a dx^b &= (pdt + tdp) + t(\Pi^2_{1a} + p\Pi^2_{2a})dx^a. \end{aligned}$$

We calculated the connection forms ω^i_j for the normal Cartan connection (in arbitrary dimension, in fact) in [4]; using the formulæ given there we find that for the equation $y'' = Q(x, y, y')$

$$\begin{aligned} \omega^1_0 &= dx \\ \omega^1_1 &= (\Pi^1_{1a} + p\Pi^1_{2a})dx^a = \tfrac{1}{3}Q_p dx + \tfrac{1}{6}Q_{pp}(dy - pdx) \\ \omega^2_0 &= dy - pdx \\ \omega^2_1 &= dp + (\Pi^2_{1a} - p\Pi^1_{1a} + p\Pi^2_{2a} - p^2\Pi^1_{2a})dx^a \\ &= (dp - Qdx) - \tfrac{1}{3}Q_p(dy - pdx). \end{aligned}$$

Thus

$$\begin{aligned} du^1 + \Pi^1_{ab}u^a dx^b &= dt + t\omega^1_1 \\ du^2 + \Pi^2_{ab}u^a dx^b &= (pdt + tdp) + t(\omega^2_1 + p\omega^1_1 - dp) = pdt + t(\omega^2_1 + p\omega^1_1). \end{aligned}$$

If we substitute into the expression for the representative Fefferman metric and ignore an overall factor t (using conformal equivalence) we obtain

$$\begin{aligned} & dx \odot (pt^{-1}dt + \omega^2_1 + p\omega^1_1) - dy \odot (t^{-1}dt + \omega^1_1) \\ &= \omega^1_0 \odot \omega^2_1 - \omega^2_0 \odot (t^{-1}dt + \omega^1_1) \\ &= (dp - Qdx)dx - (dy - pdx)\left(t^{-1}dt + \tfrac{2}{3}Q_p dx + \tfrac{1}{6}Q_{pp}(dy - pdx)\right), \end{aligned}$$

which agrees with the earlier formula (with $t = e^w$).

We next compute the connection, curvature and conformal curvature forms of the Levi-Civita connection of the representative Fefferman metric, using the 1-form basis $\{\theta^A\}$, $A = 1, 2, 3, 4$, with $\theta^a = dx^a$, $\theta^{a+2} = du^a + \Pi^a_{bc}u^c dx^b = \pi^a$. The dual basis of vector fields is

$$H_a = \frac{\partial}{\partial x^a} - \Pi^b_{ac}u^c\frac{\partial}{\partial u^b}, \quad \frac{\partial}{\partial u^a}.$$

The connection forms ϖ^A_B of the Levi-Civita connection relative to this basis must satisfy $\varpi^C_A\eta_{BC} + \varpi^C_B\eta_{AC} = 0$, with

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\varpi = \begin{pmatrix} \varpi^1_1 & \varpi^1_2 & \alpha & 0 \\ \varpi^2_1 & \varpi^2_2 & 0 & \alpha \\ \beta & 0 & -\varpi^2_2 & \varpi^1_2 \\ 0 & \beta & \varpi^2_1 & -\omega^1_1 \end{pmatrix};$$

the curvature must have the same structure. Using this form of the connection in the structure equations $d\theta^A + \varpi^A_B \wedge \theta^B = 0$ we find that

$$\varpi^a_b = \Pi^a_{bc} dx^c, \quad \alpha = 0, \quad \beta = \tfrac{1}{2}\mathfrak{R}_a dx^a.$$

Notice that ϖ^a_b is trace-free, so the bottom right 2×2 block of the connection form matrix is just ϖ^a_b again.

The curvature turns out to be (in block 2×2 form)

$$\begin{pmatrix} \mathfrak{R}^a_{b12} dx^1 \wedge dx^2 - D^a_{bcd} dx^c \wedge \pi^d & 0 \\ (B dx^1 \wedge dx^2)\delta^a_b & \mathfrak{R}^a_{b12} dx^1 \wedge dx^2 - D^a_{bcd} dx^c \wedge \pi^d \end{pmatrix}.$$

The only non-vanishing Ricci tensor components are

$$\begin{aligned} R_{11} &= 2\mathfrak{R}^2_{121} = 2\mathfrak{R}_{11}, & R_{12} &= 2\mathfrak{R}^1_{112} = 2\mathfrak{R}_{12}, \\ R_{21} &= 2\mathfrak{R}^2_{221} = 2\mathfrak{R}_{21}, & R_{22} &= 2\mathfrak{R}^1_{212} = 2\mathfrak{R}_{22}. \end{aligned}$$

The scalar curvature vanishes. It follows that the conformal curvature form is

$$\begin{pmatrix} -D^a_{bcd} dx^c \wedge \pi^d & 0 \\ (B dx^1 \wedge dx^2)\delta^a_b & -D^a_{bcd} dx^c \wedge \pi^d \end{pmatrix},$$

and is therefore determined completely by projective invariants of the sprays, namely the Douglas tensor D^a_{bcd} and the Berwald scalar B.

The Hodge star operation corresponding to the Fefferman metrics is

$$\begin{aligned} *(dx^1 \wedge dx^2) &= dx^1 \wedge dx^2, & *(\pi^1 \wedge \pi^2) &= \pi^1 \wedge \pi^2, \\ *(dx^1 \wedge \pi^1) &= -dx^1 \wedge \pi^1, & *(dx^2 \wedge \pi^2) &= -dx^2 \wedge \pi^2, \\ *(dx^1 \wedge \pi^2) &= -dx^2 \wedge \pi^1, & *(dx^2 \wedge \pi^1) &= -dx^1 \wedge \pi^2. \end{aligned}$$

Evidently the Berwald terms $(B dx^1 \wedge dx^2)\delta^a_b$ in the conformal curvature form are self-dual (with respect to the star operation), while the Douglas terms $-D^a_{bcd} dx^c \wedge \pi^d$ are anti-self-dual (since D^a_{bcd} is symmetric in its lower indices).

We can now confirm the properties of the Weyl (conformal curvature) tensor of the Fefferman metrics mentioned in Section 2.

Theorem 2 *For the representative Fefferman metric g given above*

- *both the self-dual and the anti-self-dual parts of the Weyl tensor of g have Petrov type N;*
- *the self-dual and the anti-self-dual parts of the Weyl tensor are each expressible in terms of a single component.*

Proof The Liouville field $\Delta = u^a\partial/\partial u^a$ on $\mathcal{T}^\circ M$ is evidently null. Clearly $\Delta \lrcorner B dx^1 \wedge dx^2 = 0$; moreover $\Delta \lrcorner D^a_{bcd} dx^c \wedge \pi^d = -(D^a_{bcd}u^d)dx^c = 0$, from which the first assertion follows. The self-dual part of the Weyl tensor is determined by the Berwald invariant B. As we pointed out earlier, the Douglas tensor in two dimensions is completely determined by a single component, so this applies to the anti-self-dual part of the Weyl tensor too. ∎

The corresponding normal conformal connection takes the simple form

$$\begin{pmatrix} 0 & -\mathfrak{R}_{ab}dx^b & 0 & 0 \\ dx^a & \Pi^a_{bc}dx^c & 0 & 0 \\ \phi^a & \frac{1}{2}(\mathfrak{R}_c dx^c)\delta^a_b & \Pi^a_{bc}dx^c & * \\ 0 & * & * & 0 \end{pmatrix},$$

where the stars are determined by the conditions that the matrix lies in $\mathfrak{so}(3,3)$. The 3×3 matrix in the upper left hand corner is effectively the normal Cartan projective connection (see [4]), while the bottom right hand corner is its dual (see [3]).

Finally, we consider the null geodesics of the Fefferman metrics.

Theorem 3 *Among the null geodesics of the Fefferman metrics are*

- *curves $t \mapsto (x^a(t), u^a(t))$ where $t \mapsto (x^a(t))$ is a geodesic of the projective class of sprays defining the Fefferman class, suitably parametrized, and $u^a(t) \propto \dot{x}^a(t)$;*
- *the straight lines in the fibres of $\mathcal{T}^\circ M$.*

In particular, Δ is a null geodesic field.

Proof We express the tangent vector of a curve $(x^a(t), u^a(t))$ in $\mathcal{T}^\circ M$ in terms of the basis vector fields H_a and $V_a = \partial/\partial u^a$ as $\dot{x}^a H_a + \xi^a V_a$, where $\xi^a = \dot{u}^a + \Pi^a_{bc}(t)\dot{x}^b u^c$. Then the geodesic equations are

$$\ddot{x}^a + \Pi^a_{bc}(t)\dot{x}^b\dot{x}^c = 0, \quad \dot{\xi}^a + \Pi^a_{bc}(t)\xi^b\dot{x}^c + \beta_b(t)\dot{x}^b\xi^a = 0.$$

Here $\Pi^a_{bc}(t)$ and $\beta_b(t)$ are the values of these quantities along the curve. Moreover, for the curve to be null we must have $\dot{x}^1\xi^2 = \dot{x}^2\xi^1$.

Consider first those geodesics for which u^a is a multiple of $\dot{x}^a$, say $u^a = k\dot{x}^a$, $k > 0$. For such a curve $\xi^a = k(\ddot{x}^a + \Pi^a_{bc}\dot{x}^b\dot{x}^c) + \dot{k}k\dot{x}^a$, where now $\Pi^a_{bc}(t) = \Pi^a_{bc}(x(t), k\dot{x}(t)) = \Pi^a_{bc}(x(t), \dot{x}(t))$. The first geodesic equation is $\ddot{x}^a + \Pi^a_{bc}(t)\dot{x}^b\dot{x}^c = 0$, again with $\Pi^a_{bc}(t) = \Pi^a_{bc}(x(t), \dot{x}(t))$; this is the equation for the geodesics of the projective class of sprays parametrized with what Douglas calls the projective parameter. If the first geodesic equation is satisfied then $\xi^a \propto \dot{x}^a$ and the curve is automatically null. The second geodesic equation specifies how k varies along the curve. Thus null geodesics of this form project onto the geodesics of the sprays in M, parametrized with the projective parameter.

On the other hand, vertical curves are clearly null. The first geodesic equation for any vertical curve is vacuous; we have $\xi^a = \dot{u}^a$, and the second geodesic equation is simply $\ddot{u}^a = 0$. Thus a vertical curve is geodesic if and only if it is a straight line in a fibre. In particular the dilation field Δ on $\mathcal{T}^\circ M$ is a null geodesic field (albeit not affinely parametrized). ∎

The null geodesic field Δ turns out to be shear-free and twisting. Moreover, it generates infinitesimal conformal (in fact homothetic) transformations, by homogeneity of the representative metric.

Acknowledgements

The first author is a Guest Professor at Ghent University, and a Visiting Senior Research Fellow of King's College, University of London: he is grateful to the Department of Mathematical Physics and Astronomy at Ghent and the Department of Mathematics at King's for their hospitality.

Address for correspondence

M. Crampin, 65 Mount Pleasant, Aspley Guise, Beds MK17 8JX, UK
Crampin@btinternet.com

References

[1] Bailey, T. N., Eastwood, M.G. and Gover, A.R: Thomas's structure bundle for conformal, projective and related structures *Rocky Mountain J. Math.* **24** (1994) 1191–1217

[2] Cartan, E: Sur les variétés à connexion projective *Bull. Soc. Math. France* **52** (1924) 205–241

[3] Crampin, M. and Saunders, D.J: Cartan's concept of duality for second-order ordinary differential equations *J. Geom. Phys.* **54** (2005) 146–172

[4] Crampin, M. and Saunders, D.J: On projective connections: the general case *Preprint* Ghent University (2004) (available at `http://maphyast.ugent.be` Theoretical Mechanics)

[5] Crampin, M. and Saunders, D.J: Path geometries and almost Grassmann structures *Preprint* Ghent University (2005) (available at `http://maphyast.ugent.be` Theoretical Mechanics)

[6] Douglas, J: The general geometry of paths *Ann. Math.* **29** (1928) 143–168

[7] Fefferman, C: Monge-Ampère equations, the Bergman kernel, and geometry of pseudo-convex domains *Ann. Math.* **103** (1976) 359–416

[8] Fritelli, S., Kozameh, C. and Newman, E.T: Differential geometry from differential equations *Comm. Math. Phys.* **223** (2001) 383–408

[9] Newman, E. T. and Nurowski, P: Projective connections associated with second-order ODEs *Classical Quantum Gravity* **20** (2003) 2325–2335

[10] Nurowski, P: Differential equations and conformal structures *Preprint* Warsaw University (2004) `arXiv:math.DG/0406400 v1`

[11] Nurowski, P. and Sparling, G. A. J: Three-dimensional Cauchy-Riemann structures and second-order ordinary differential equations *Classical Quantum Gravity* **20** (2003) 4995–5016

[12] Roberts, C: The projective connections of T. Y. Thomas and J. H. C. Whitehead applied to invariant connections *Diff. Geom. Appl.* **5** (1995) 237–255

[13] Sharpe, R. W: *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program* (Springer, 1997)

[14] Shen, Z: *Differential Geometry of Spray and Finsler Spaces* (Kluwer, 2001)

[15] Thomas, T. Y: On the projective and equi-projective geometries of paths *Proc. Nat. Acad. Sci.* **11** (1925) 199–203

[16] Thomas, T. Y: A projective theory of affinely connected manifolds *Math. Zeit.* **25** (1926) 723–733

[17] Weyl, H: Zur Infinitesimalgeometry; Einordnung der projektiven und der konformen Auffassung *Gött. Nach.* (1921) 99-112

[18] Whitehead, J. H. C: The representation of projective spaces *Ann. Math.* **32** (1931) 327–360

[19] Yano, K. and Ishihara, S: *Tangent and Cotangent Bundles* (Marcel Dekker, 1973)