The Hilbert-Carathéodory form for parametric multiple integral problems in the calculus of variations

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1 Introduction

We are concerned in this paper with variational problems relating to m-dimensional submanifolds of an N-dimensional manifold E (where $N = m + n$) which are 'parametric', to use the terminology of Giaquinta and Hildebrandt [7], for example. That is to say, the extremal submanifolds are to be considered as subsets of E (with suitable smoothness properties) in 'general position': that is, not *ab initio* as sections of E over some m-dimensional base. Such problems may be considered geometrical, in contrast to physical problems in which, usually, one deals with fields which are regarded as sections of some bundle over a base representing time, space or space-time.

We can illustrate the issues we want to examine by first discussing the case $m = 1$. In this case parametric or geometric problems lead naturally to Finsler geometry, whereas the physical theory is typified by Lagrangian dynamics.

We can compare and contrast the two as follows. In dynamics, the Lagrangian is often considered as a function $L(t, q^\alpha, \dot{q}^\alpha)$ where the n variables q^α (with $\alpha = 1, 2, \ldots, n$) represent position and where $\dot{q}^\alpha$ are the corresponding velocities; L must be of class C^2 at least, but otherwise is quite general. More properly, though, the Lagrangian should be considered as the 1-form $\lambda = L\,dt$, as the variational problem is to find extremal curves σ for the variational problem

$$\delta \int_\sigma \lambda = 0$$

parametrized by the time variable t. On the other hand, in Finsler geometry the Finsler function $L(q^A, \dot{q}^A)$ depends on the $(n+1)$ variables q^A (with $A = 0, 1, \ldots, n$) and the corresponding velocities $\dot{q}^A$ in such a way that L is positively homogeneous of degree 1 in the $\dot{q}^A$, so that solutions of the variational problem

$$\delta \int_\sigma L\, dt = 0$$

are curves σ invariant under any change of parameter $t' = t'(t)$ where $dt'/dt > 0$. In particular, therefore, the arena of Lagrangian dynamics is the first jet bundle $J^1\pi$ of a manifold E fibred by π over the real line ('time'): if E is a product manifold $\mathbf{R} \times Q$ and if L happens to be independent of the time t then we may also write $J^1\pi$ as a product $\mathbf{R} \times TQ$ and consider L to be defined on the tangent bundle TQ. On the other hand, the homogeneity property of the Finsler function demonstrates that the realm of Finsler geometry is (or should be) the projectivized tangent bundle PTE of the configuration manifold, or perhaps better its sphere bundle as described in the next section.

This distinction is perhaps brought out most clearly by comparing the geometrical problem of a geodesic in three-dimensional Euclidean space with the physical problem of a free particle moving in that same space. In the former, $E = \mathbf{R}^3$ with a Lagrangian $\sqrt{(\dot{q}^0)^2 + (\dot{q}^1)^2 + (\dot{q}^2)^2}$, whereas in the latter $E = \mathbf{R} \times Q = \mathbf{R} \times \mathbf{R}^3$ with Lagrangian $\frac{1}{2}((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2)$. The homogeneity of the former Lagrangian allows us to work on the five-dimensional manifold PTE, whereas for the latter we must use the six-dimensional manifold TQ.

There are two approaches to working on PTE. One can use local affine coordinates — which is pretty much the same as introducing jet bundle type coordinates. Alternatively one can work on the slit tangent bundle $T^\circ E$ but always with geometric objects which pass to the quotient; this is the approach advocated by Bao, Chern and Shen in [1]. We shall show how to extend this philosophy to multiple integral problems; but we shall also explain what our formulations look like in affine coordinates, and thereby relate them to more familiar objects.

One example in Finsler geometry of a geometric object defined initially on $T^\circ E$ which is nevertheless a globally defined object on PTE is the so-called Hilbert 1-form, given in tangent bundle coordinates $(q^A, \dot{q}^A)$ by

$$\frac{\partial L}{\partial \dot{q}^A} dq^A.$$

In [4] Chern describes the Hilbert form as 'a powerful piece of data'; and indeed it does play a fundamental role in Finsler geometry (though this is not always acknowledged). In fact its role is parallel to that of the (Poincaré-)Cartan 1-form

$$L\,dt + \frac{\partial L}{\partial q^\alpha}(dq^\alpha - \dot{q}^\alpha dt)$$

in the jet-bundle theory — to which indeed it is closely related. Note that the Hilbert form may be written as $S(dL)$ where S is the vertical endomorphism (or 'almost tangent structure') on TE: we shall describe this construction in more detail below.

A similar distinction to the one we have described appears also in the case of multiple integral problems. When considering field theories, a suitable framework is again that of jet bundles, where $\pi : E \to M$ is a fibration and the Lagrangian function L is defined on the first jet bundle $J^1\pi$. Taking fibred coordinates (x^i, v^α) on E ($i = 1, 2, \ldots, m$, $\alpha = 1, 2, \ldots, n$), the derivative coordinates on $J^1\pi$ are v^α_i. We also require a volume form on the base M which we assume takes the coordinate form $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m d^m x$, so that the Lagrangian m-form $\lambda = L\,d^m x$ defines a variational problem

$$\delta \int_\sigma \lambda = 0$$

for (local) sections σ of the fibration π. On the other hand, geometrical problems are dealt with more appropriately in a homogeneous framework, where we consider m-dimensional submanifolds $\sigma : M \to E$ giving extremals of the variational problem

$$\delta \int_\sigma L\,d^m x = 0$$

which are independent of the particular parametrization of $\sigma(M)$. The prime example of such a problem is the problem of finding minimal surfaces in Euclidean 3-diemnsional space, and its generalizations.

In the jet bundle framework, the Poincaré-Cartan form may be generalized in the passage from a 1-dimensional to an m-dimensional base in several different ways. All of these require the resulting semi-basic m-form Θ to be a 'Lepage equivalent' of the Lagrangian m-form, in that the difference $\lambda - \Theta$ must be a contact form, and that the differential $d\Theta$ must vanish when contracted with any vector field on $J^1\pi$ vertical over E. In cordinates, therefore, Θ must be of the form

$$\Theta = L\,d^m x + \frac{\partial L}{\partial v^\alpha_i}\omega^\alpha \wedge d^{m-1}x_i + \cdots$$

where $d^{m-1}x_i$ denotes the contraction $\partial/\partial x^i \lrcorner\, d^m x$ and ω^α represents the contact 1-form $dv^\alpha - v^\alpha_j dx^j$, and where the dots signify terms which are at least 2-contact.

We therefore have some freedom in the choice of these remaining terms, and there are at least three options having an invariant meaning.

1. We may choose the remaining terms to be zero. This choice, originally made by De Donder and Weyl (see [10]) allows a straightforward generalization of the Legendre transformation and the Hamiltonian formalism used in the 1-dimensional case, and is often known as the 'multisymplectic approach'. [8]

2. We may choose the coefficients of the remaining terms to involve higher derivatives of the Lagrangian function, giving the 'fundamental Lepage equivalent' [2, 9]

$$\Theta = \sum_{r=0}^{\min\{m,n\}} \frac{1}{(r!)^2} \frac{\partial^r L}{\partial v^{\alpha_1}_{i_1} \partial v^{\alpha_2}_{i_2} \ldots \partial v^{\alpha_r}_{i_r}} \omega^{\alpha_1} \wedge \omega^{\alpha_2} \wedge \ldots \wedge \omega^{\alpha_r} \wedge d^{m-r} x_{i_1 i_2 \ldots i_r}.$$

This choice has the property that Θ is closed precisely when the Euler-Lagrange equations are trivial.

3. We may look for a simple m-form, such as the Carathéodory form [3]

$$\Theta = L^{-(m-1)} \bigwedge_{i=1}^{m} \left(L\, dx^j + \frac{\partial L}{\partial v^\alpha_i} \omega^\alpha \right).$$

This has the advantage, as observed by Dedecker [6], that it is invariant under arbitrary changes of coordinates on E, not merely under changes respecting the fibration. Thus one is pretty-well forced to choose the Carathéodory form when dealing with parametric multiple integral problems, a point emphasized also by Rund [10].

In fact the Carathéodory form (in the jet-bundle formalism) is uniquely determined by the conditions that it be semi-basic, a Lepage equivalent of $L\, d^m x$, and simple. Consider an m-form with these properties: it can be written as the m-fold exterior product of m 1-forms ϑ^i such that

$$\vartheta^i = A^i_j dx^j + B^i_\alpha \omega^\alpha.$$

Now

$$\bigwedge_{i=1}^{m} \vartheta^i = (\det A) d^m x + B^j_\alpha C^i_j \omega^\alpha \wedge d^{m-1} x_i + \cdots$$

where if we regard $(A^i_j) = A$ as a matrix, C is the matrix of cofactors of A. For this form to be a Lepage equivalent of $L\, d^m x$ we require that

$$\det A = L \quad \text{and} \quad B^j_\alpha C^i_j = \frac{\partial L}{\partial v^\alpha_i}.$$

The cofactor property says that $AC = (\det A) I$, so on multiplying the second equation by A we obtain

$$(\det A) B^i_\alpha = A^i_j \frac{\partial L}{\partial v^\alpha_j}.$$

Thus

$$\vartheta^i = L^{-1} A^i_j \left(L\, dx^j + \frac{\partial L}{\partial v^\alpha_i} \omega^\alpha \right),$$

so that

$$\begin{aligned} \bigwedge_{i=1}^{m} \vartheta^i &= L^{-m} (\det A) \bigwedge_{i=1}^{m} \left(L\, dx^j + \frac{\partial L}{\partial v^\alpha_i} \omega^\alpha \right) \\ &= L^{-(m-1)} \bigwedge_{i=1}^{m} \left(L\, dx^j + \frac{\partial L}{\partial v^\alpha_i} \omega^\alpha \right). \end{aligned}$$

One of the main aims of this paper is to define, and discuss some properties of, an m-form which stands in the same relation to the Carathéodory form as the Hilbert form does to the Poincaré-Cartan form. We have coined the name 'Hilbert-Carathéodory form' for it. The second main aim is to show how a generalization of the vertical endomorphism S can be defined in an appropriate geometrical set-up; we use this object in the achievement of the first aim.

2 The 1-dimensional case

In order to carry out our task, it is helpful to summarize the case of 1-dimensional parametric variational problems using rather more detail than that given in the Introduction.

We start, therefore, with our configuration manifold E of dimension $n+1$, with (as before) coordinates q^A for $A = 0, 1, \ldots, n$; we shall assume that this manifold (and the others we consider) are of class C^∞, connected and paracompact. We let TE be the tangent bundle of E, and $T^\circ E$ the 'slit tangent bundle', that is TE excluding the zero section. The latter is clearly an open submanifold of the former, and coordinates $(q^A, \dot{q}^A)$ may be used on both.

The *dilation field* Δ on TE is the infinitesimal generator of dilations in the fibres of TE; in coordinates it is

$$\Delta = \dot{q}^a \frac{\partial}{\partial \dot{q}^a}.$$

We use the same symbol for the restriction of the dilation field to $T^\circ E$.

A function L on $T^\circ E$ is *homogeneous of degree* m if $L(\alpha\xi) = \alpha^m L(\xi)$ for any $\xi \in T^\circ E$ and any non-zero real number α; it is *positively homogeneous* if this condition need hold only for $\alpha > 0$. Any positively homogeneous function satisfies $\Delta L = mL$ by Euler's Theorem, and indeed the converse is also true.

We define two equivalence relations on elements of $T^\circ E$. In each case we say that two elements ξ_1, ξ_2 in the same fibre of $T^\circ E$ over E are equivalent if $\xi_1 = \alpha\xi_2$ for some $\alpha \in \mathbf{R}$, the difference being that we require $\alpha \neq 0$ for the first relation and $\alpha > 0$ for the second. The two quotient spaces are clearly manifolds, and indeed the two projections

$$\rho_0 \colon T^\circ E \to PTE, \qquad \rho_+ \colon T^\circ E \to \mathcal{S}E$$

define principal bundles, the base of the former being the projectivized tangent bundle and that of the latter being the sphere bundle. We note that PTE may also be considered as the manifold $\mathcal{G}E$ of contact elements in E (namely jets of immersions of $\mathbf{R}$ in E), and that correspondingly $\mathcal{S}E$ may be considered as the manifold of oriented contact elements. The tangent bundle projection $\tau : T^\circ E \to E$ induces projections

$$\tau_0 \colon PTE \to E, \qquad \tau_+ \colon \mathcal{S}E \to E,$$

and we say that differential forms on PTE or $\mathcal{S}E$ are *semi-basic* if they vanish when contracted with vector fields vertical over these projections. A form on either manifold is called a *contact form* if it is pulled back to zero by the prolongation of every immersion defining the manifold.

A *Lagrangian form* is a semi-basic 1-form λ on PTE, and an *oriented Lagrangian form* is a semi-basic form on $\mathcal{S}E$. Two such forms λ_1, λ_2 are *equivalent* if the difference $\lambda_1 - \lambda_2$ is a contact form.

Any oriented Lagrangian form λ on $\mathcal{S}E$ defines a *Lagrangian function* L on $T^\circ E$. If $\xi \in T^\circ E$ then $\rho_+(\xi) \in \mathcal{S}E$, and we may consider $\lambda_{\rho_+(\xi)}$ to be an element of T^*E (rather than of $T^*\mathcal{S}E$) because λ is semi-basic. So define

$$L(\xi) = \langle \xi, \lambda_{\rho_+(\xi)} \rangle;$$

as $\rho_+(\alpha\xi) = \rho_+(\xi)$ for any $\alpha > 0$ we have immediately that $L(\alpha\xi) = \alpha L(\xi)$, so that L is positively homogeneous of degree 1. We may check, using coordinates, that a contact form on $\mathcal{S}E$ defines the zero function on $T^\circ E$, so that equivalent oriented Lagrangian forms define the same Lagrangian function. We note also that two oriented Lagrangian forms differing by an exact form df, where f is necessarily (the pull-back of) a function defined on E, give rise to two Lagrangian functions differing by the total derivative $df/dt = \dot{q}^A \partial f / \partial q^A = \langle \mathbf{T}, df \rangle$ where $\mathbf{T}$ is the tautological vector field along the tangent bundle projection.

In a similar way, a Lagrangian form λ on PTE my be used to define a function L on $T^\circ E$, and we now find that L is homogeneous of degree 1. In fact this is not normally an appropriate property for the types of variational problem we are considering — for instance, in Finsler geometry we might choose to consider functions satisfying

$L(\alpha\xi) = |\alpha|L(\xi)$ instead. Nevertheless these more general Lagrangian forms do fit within the general scheme of our discussion. (An alternative approach would be to make use of the 'odd forms' of de Rham [5]; we shall not adopt that approach in the present work.)

We remark that the function L does not project to a function on PTE or $\mathcal{S}E$ as it is not constant on the fibres of ρ_0 or ρ_+, so the function/form ambiguity found in the situation of affine jet bundles does not arise here. The function L does, however, define a function-like object on the appropriate manifold, and this will be a section of a certain line bundle. Define the bundle

$$\nu^0 \colon D^0 E \to PTE$$

by, for each $w \in PTE$, letting $D^0_w E$ be the set of functions f on the fibre $\rho_0^{-1}(w)$ satisfying

$$f(\alpha\xi) = \alpha f(\xi)$$

for any non-zero α. We now define L^0 as the section of ν^0 given by

$$L^0(w)(\xi) = L(\xi)$$

for each element ξ of $T^\circ E$ satisfying $\rho_0(\xi) = w$. In fact $D^0 E$ is the line bundle associated with the principal bundle $T^\circ E \to PTE$ by the multiplicative action of the group of non-zero real numbers on $\mathbf{R}$.

In the same way, we define the bundle

$$\nu^+ \colon D^+ E \to \mathcal{S}E$$

by letting $D^+_w E$ be the set of functions f satisfying $f(\alpha\xi) = \alpha f(\xi)$ for $\alpha > 0$, and we define L^+ to be the section of ν^+ given by the similar formula $L^+(w)(\xi) = L(\xi)$.

To construct the vertical endomorphism S, we use the 'vertical lift' operation defined on any vector bundle. Given any vector bundle $\pi : H \to E$, a point $p \in E$ and a point $x \in H_p$, there is a canonical linear isomorphism between the fibre H_p and the space of vertical tangent vectors $V_x\pi$, so that $y \in H_p$ corresponds to $y^{\mathrm{v}} \in V_x\pi$. Taking $\tau : TE \to E$ as our vector bundle, we may consider the vertical lift as a fibre-linear map $S : \tau^* TE \to TTE$, and as $T^\circ E \subset TE$ is open we may restrict this map to $S : \tau^* T^\circ E \to TT^\circ E$. Writing the map in coordinates as a tensor field, we have

$$S = dq^A \otimes \frac{\partial}{\partial \dot{q}^A}.$$

We can now define the *Hilbert form* of L to be the 1-form $S^*(dL)$, where S^* denotes the dual action of S on 1-forms. In coordinates,

$$S^*(dL) = \frac{\partial L}{\partial \dot{q}^A} dq^A.$$

This semi-basic 1-form is projectable to PTE and to $\mathcal{S}E$, because the coefficient functions of its coordinate representation are homogeneous (or positively homogeneous) of degree zero. The projected form is a canonical representative of the equivalence class of Lagrangian forms defining the variational problem. If we consider the subset of the chart on $T^\circ E$ corresponding to $\dot{q}^0 = 1$ and put $t = q^0$ then when L is homogeneous we may write

$$S^*(dL) = L\,dt + \frac{\partial L}{\partial \dot{q}^\alpha}(dq^\alpha - \dot{q}^\alpha dt)$$

where the index i runs from 1 to n; the restriction of the projection ρ_0 to this chart is a diffeomorphism, and the image chart may also be considered as a chart on the affine jet bundle $J^1\pi$ for a (perhaps locally defined) fibration π of E. By considering instead $\dot{q}^0 = \pm 1$ we may apply the same argument to $\mathcal{S}E$.

3 The geometrical background for m-dimensional variational problems

Our approach to constructing a generalization of the Hilbert form to the case of m-dimensional variational problems is to start with a careful consideration of the geometrical framework in which the problems should be posed.

So let us first take a configuration manifold E with $\dim E = N = m + n$. The space corresponding to the slit tangent bundle in the 1-dimensional case will be $\mathcal{F}^{(m)}E$, the bundle of m-frames over E: a point of $\mathcal{F}^{(m)}E$ consists of a pair $(u, \{e_i\})$ where $\{e_i\}$, $i = 1, 2, \ldots, m$ is an ordered set of linearly independent elements of T_uE; coordinates on $\mathcal{F}^{(m)}E$ are (u^A, u^A_i),where u^A, $A = 1, 2, \ldots, N$ are coordinates on E and

$$e_i = u^A_i \frac{\partial}{\partial u^A}.$$

The space $\mathcal{F}^{(m)}E$ is an open submanifold of the Whitney sum bundle $\bigoplus^m TE$ of arbitrary (not necessarily linearly independent) ordered m-tuples of tangent vectors.

The space corresponding to the projectivized tangent bundle is $\mathcal{G}^{(m)}E$, the m-plane bundle of E or bundle of m-dimensional contact elements: a point of $\mathcal{G}^{(m)}E$ consists of a point $u \in E$ and an equivalence class of m-dimensional immersed submanifolds of E through u, where two submanifolds are equivalent if they have the same tangent m-plane at u. In other words, the fibre of $\mathcal{G}^{(m)}E$ over u is the set of m-dimensional subspaces of T_uE.

Now $GL(m)$ acts on $\mathcal{F}^{(m)}E$ by $(u, \{e_i\}) \mapsto (u, \{a_i^j e_j\})$ for $a = (a_i^j) \in GL(m)$. But $\{e_i\}$ and $\{f_i\}$ span the same m-dimensional subspace of $T_u E$ if and only if $f_i = a_i^j e_j$ for some $a \in GL(m)$. Thus $\mathcal{G}^{(m)}E$ is the quotient manifold of $\mathcal{F}^{(m)}E$ under the $GL(m)$ action; and $\mathcal{F}^{(m)}E$ is a principal $GL(m)$-bundle over $\mathcal{G}^{(m)}E$.

We shall, however, prefer to consider not the quotient of $\mathcal{F}^{(m)}E$ under the full $GL(m)$ action but its quotient under the action of $GL(m)^+$, the group of matrices of positive determinant; this is the component of the identity of $GL(m)$. The quotient manifold, which is a double cover of $\mathcal{G}^{(m)}E$, could be called the bundle of oriented contact elements, and we shall denote it by $\mathcal{S}^{(m)}E$.

One reason for wanting to work with $GL(m)^+$ is that we can reduce considerations of the action to those of the fundamental vector fields on $\mathcal{F}^{(m)}E$ generated by it. The space of fundamental vector fields has local bases (over $\mathbf{R}$) of the form

$$\Delta_i^j = u_i^A \frac{\partial}{\partial u_j^A}.$$

A *Lagrangian form* is an m-form λ on $\mathcal{S}^{(m)}E$, semi-basic over E. Two such forms λ_1, λ_2 are *equivalent* if the difference $\lambda_1 - \lambda_2$ is a contact form. (Here we need to distinguish between different orders of contact forms, and our requirement is that the difference $\lambda_1 - \lambda_2$ be *at least 1-contact* so that it is pulled back to zero by the prolongation of every immersion of $\mathbf{R}^m$ in E defining the sphere bundle $\mathcal{S}^{(m)}E$.)

Any Lagrangian form defines a *Lagrangian function* L on $\mathcal{F}^{(m)}E$. We remark that elements of $\mathcal{S}^{(m)}E$ are equivalence classes of elements of $\mathcal{F}^{(m)}E$, where two m-frames ξ and ζ are equivalent if

$$\xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_m = \alpha\, \zeta_1 \wedge \zeta_2 \wedge \ldots \wedge \zeta_m$$

for some $\alpha > 0$. So if $\xi \in \mathcal{F}^{(m)}E$ is an m-frame then its equivalence class $\rho_+(\xi)$ is an element of $\mathcal{S}^{(m)}E$, and we may consider $\lambda_{\rho_+(\xi)}$ to be an element of $\bigwedge^m T^*E$ (rather than of $\bigwedge^m T^*\mathcal{S}^{(m)}E$) because λ is semi-basic. So define

$$L(\xi) = \langle \xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_m, \lambda_{\rho_+(\xi)} \rangle = \langle \xi, \lambda_{\rho_+(\xi)} \rangle,$$

where the angle brackets denote the pairing of an m-vector and an m-form on E. As

$$\rho_+(a \cdot \xi) = \rho_+(\xi)$$

for any $a \in GL(m)^+$ we have immediately that

$$L(a \cdot \xi) = \det(a) L(\xi).$$

An m-form on $\mathcal{S}^{(m)}E$ that is at least 1-contact defines the zero function on $\mathcal{F}^{(m)}E$, so that equivalent Lagrangian forms define the same Lagrangian function. We note also that two oriented Lagrangian forms differing by an exact form $d\eta$, where η is the pull-back of an $(m-1)$-form defined on E, give rise to two Lagrangian functions differing by the 'total derivative' $\langle \mathbf{T}^m, d\eta \rangle$ where $\mathbf{T}^m$ is the tautological vector field along the frame bundle projection.

The function L does not project to a function on $\mathcal{S}^{(m)}E$ as it is not constant on the fibres of ρ_+. It does, however, define a function-like object on $\mathcal{S}^{(m)}E$, and this will be a section of a certain line bundle. Define the bundle

$$\nu : DE \to \mathcal{S}^{(m)}E$$

by letting $D_w E$ be the set of functions f on the fibre $\rho_+^{-1}(w)$ satisfying

$$f(a \cdot \xi) = \det(a) f(\xi)$$

for any $a \in GL(m)^+$. We now define $\widetilde{L}$ as the section of ν given by

$$\widetilde{L}(w)(\xi) = L(\xi)$$

for each element ξ of $\mathcal{F}^{(m)}E$ satisfying $\rho_+(\xi) = w$. In fact DE is the line bundle associated with the principal bundle $\mathcal{F}^{(m)}E \to \mathcal{S}^{(m)}E$ by the action of $GL(m)^+$ on $\mathbf{R}$ by $(a, z) \mapsto (\det a)z$.

Finally in this section we consider a generalisation of the vertical endomorphism to the m-dimensional case. As the Whitney sum bundle $\bigoplus^m TE$ is a vector bundle over E with projection τ^m, it gives rise to a vertical lift operator in the manner described in the previous section. We may therefore regard this as an operator acting on sections of $\tau^m : \bigoplus^m TE \to E$ along τ^m to give vertical vector fields on $\bigoplus^m TE$, and we may restrict this operator to the open submanifold $\mathcal{F}^{(m)}E \subset \bigoplus^m TE$ to give, as a result, vertical vector fields on $\mathcal{F}^{(m)}E$. This restriction is the operator we shall call S.

To see a coordinate expression for this S operator, let $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ be an element of $\bigoplus^m TE$ at the point $p \in E$. If we have $\xi_i = \xi_i^A \partial/\partial u^A$ then at any frame $\zeta \in \mathcal{F}^{(m)}E$ (at the same point p) we have

$$S_\zeta(\xi) = \xi_i^A \frac{\partial}{\partial u_i^A}.$$

4 The Hilbert-Carathéodory form for m-dimensional variational problems

An immersion $\sigma: \mathbf{R}^m \to E$ — that is, a parametrized submanifold of E — defines, at each point $\sigma(x)$, $x \in \mathbf{R}^m$, an m-frame

$$\left\{ \sigma_* \frac{\partial}{\partial x^i} \right\}$$

and therefore an immersion $\widehat{\sigma}: \mathbf{R}^m \to \mathcal{F}^{(m)}E$ given in coordinates by

$$(x^i) \mapsto \left(u^A(x), \frac{\partial u^A}{\partial x^i}(x) \right).$$

An m-dimensional submanifold W of E defines a covering submanifold $\widetilde{W}$ of $\mathcal{S}^{(m)}E$, which in turn is covered by the image of $\widehat{\sigma}$ for any σ whose image is in W.

Let L be a function on $\mathcal{F}^{(m)}E$. Then for any $\sigma: \mathbf{R}^m \to E$ we define the m-form $(\widehat{\sigma}^* L) d^m x$ on $\mathbf{R}^m$. This is to be thought of as the integrand of a variational problem. For a parametric problem it should not depend on the parametrization, provided the orientation is unchanged: that is, if we make an orientation-preserving parameter transformation $y^i = y^i(x)$ we should have

$$L \left(u^A(y(x)), \frac{\partial u^A}{\partial y^i}(y(x)) \right) d^m y^1 = L \left(u^A(x), \frac{\partial u^A}{\partial x^i}(x) \right) d^m x.$$

This requires that L satisfies

$$L(u^A, a_i^j u_j^A) = (\det a) L(u^A, u_i^A)$$

for all $a \in GL(m)^+$.

Now differentiate this equation partially with respect to a_i^j at the identity, to obtain

$$\Delta_i^j(L) = u_i^A \frac{\partial L}{\partial u_j^A} = \delta_i^j L,$$

which is the homogeneity condition that L must satisfy for parametric problems. In fact this is equivalent to the determinantal condition for $GL(m)^+$ (see also [10], in which it is shown that this is the sufficient as well as the necessary condition).

On differentiating the homogeneity condition again with respect to u_k^B we obtain

$$u_i^A \frac{\partial^2 L}{\partial u_j^A \partial u_k^B} = \delta_i^j \frac{\partial L}{\partial u_k^B} - \delta_i^k \frac{\partial L}{\partial u_j^B}.$$

Also, by differentiating the determinantal condition with respect to u_i^A we find that

$$\frac{\partial L}{\partial u_i^A}(u^B, a_j^k u_k^B) = (\det a)\bar{a}_j^i \frac{\partial L}{\partial u_j^A}(u^B, u_k^B)$$

where $\bar{a} = a^{-1}$.

Now $L\,d^m x$ is not a geometric object on any space which occurs naturally in this set-up. It would be nice to be able to formulate the variational integral in terms of a geometric object on one of the spaces defined above. The following simple m-form does the trick:

$$\Theta = L^{-(m-1)} \bigwedge_{i=1}^{m} \frac{\partial L}{\partial u_i^A} du^A.$$

We shall show that:

1. Θ is well-defined as an m-form on $\mathcal{F}^{(m)}E$.
2. Θ passes to the quotient to define a Lagrangian form $\tilde{\Theta}$ on $\mathcal{S}^{(m)}E$; the Lagrangian function associated with $\tilde{\Theta}$ is L itself.
3. For any σ parametrizing a submanifold of E, $\widehat{\sigma}^*\Theta = (\widehat{\sigma}^* L)d^m x$.
4. If we choose coordinates u^A such that $u^i = x^i$, $i = 1, 2, \ldots, m$, and we set $u^{m+\alpha} = v^\alpha$, $\alpha = 1, 2, \ldots, n$, and if we define $\check{L}$ by

   $$\check{L}(x^i, v^\alpha, v_i^\alpha) = L(x^i, v^\alpha, \delta_i^j, v_i^\alpha),$$

   then

   $$\Theta = \check{L}^{-(m-1)} \bigwedge_{i=1}^{m} \left(\check{L} dx^i + \frac{\partial \check{L}}{\partial v_i^\alpha}(dv^\alpha - v_j^\alpha dx^j) \right).$$

This is the Carathéodory form of $\check{L}$, which is a Lepage equivalent of $\check{L}\,d^m x$. In the case $m = 1$, Θ is the Hilbert 1-form of Finsler geometry. In general, we call Θ the Hilbert-Carathéodory form. The corresponding form $\tilde{\Theta}$ on $\mathcal{S}^{(m)}E$is a canonical representative of the equivalence class of Lagrangian forms defining the variational problem.

Proofs of 1-4:
1. We shall demonstrate this by using the generalization of the vertical endomorphism S. As S is a linear map between sections of vector bundles, it has a dual operator S^*, so that $S^*(dL)$ is a section of the dual bundle $\bigoplus^m T^*E \to E$ along $\mathcal{F}^{(m)}E \to E$. To obtain from this a semi-basic m-form on $\mathcal{F}^{(m)}E$, we use the natural map $\wedge : \bigoplus^m T^*E \to \bigwedge^m T^*E$, given by

$$\wedge(\alpha^1, \alpha^2, \ldots, \alpha^m) = \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^m;$$

clearly $\wedge$ takes its values in the submanifold of $\bigwedge^m T^*E$ containing the decomposable m-covectors. We use the same notation for the map acting on sections along $\mathcal{F}^{(m)}E \to E$, and so finally we can write the Hilbert-Carathéodory form as

$$\Theta = L^{-(m-1)}\left(\wedge(S^*(dL))\right).$$

2. Set

$$\theta^i = \frac{\partial L}{\partial u^A_i} du^A.$$

Under the action of $a \in GL(m)^+$, $\theta^i \mapsto (\det a)\bar{a}^i_j\theta^j$, so

$$\bigwedge_{i=1}^m \theta^i \mapsto (\det a)^m \bigwedge_{i=1}^m \bar{a}^i_j\theta^j = (\det a)^{m-1}\bigwedge_{i=1}^m \theta^i;$$

but $L \mapsto (\det a)L$, so Θ is invariant. Since it is semi-basic, it passes to the quotient. (Alternatively, one can calculate the Lie derivative with respect to Δ^j_i.) Moreover,

$$\det\left\langle u^A_i \frac{\partial}{\partial u^A}, \theta^j \right\rangle = \det(\delta^j_i L) = L^m,$$

so the Lagrangian function of $\tilde{\Theta}$ is L.
3. If σ is given by $u^A = u^A(x)$ then

$$du^A = \frac{\partial u^A}{\partial x^i} dx^i = u^A_i dx^i,$$

so

$$\theta^i = u^A_j \frac{\partial L}{\partial u^A_i} dx^j = \delta^i_j L\, dx^j = L\, dx^i,$$

and $\Theta = L\, d^m x$.
4. With these coordinates the homogeneity condition gives

$$u^k_j \frac{\partial L}{\partial u^k_i} + v^\alpha_j \frac{\partial L}{\partial v^\alpha_i} = \delta^i_j L;$$

so when we restrict to $u^k_j = \delta^k_j$ we obtain

$$\frac{\partial L}{\partial u^k_i} = \delta^i_j \check{L} - v^\alpha_j \frac{\partial \check{L}}{\partial v^\alpha_i}$$

and so

$$\theta^i = \frac{\partial L}{\partial u^j_i} dx^j + \frac{\partial \check{L}}{\partial v^\alpha_i} dv^\alpha = \check{L}\, dx^i + \frac{\partial \check{L}}{\partial v^\alpha_i}\left(dv^\alpha - v^\alpha_j dx^j\right)$$

as required.

5 The Euler-Lagrange equations

We now show how to derive the Euler-Lagrange equations using the Hilbert-Carathéodory form Θ. We do so for the sake of completeness, and because the calculations reveal some useful properties of Θ.

A region R of a submanifold W of E, with boundary B, defines an integral I by

$$I = \int_{\widetilde{R}} \tilde{\Theta}|_{\widetilde{W}},$$

where $\widetilde{R}$ is the subset of $\widetilde{W}$ sitting over R. If σ is any parametrization of W then $I = \int_R \widehat{\sigma}^*\Theta$ (where Θ denotes the form on $\mathcal{F}^{(m)}E$ in the second version but its projection into $\mathcal{S}^{(m)}E$ in the first).

Consider a variation in the following way. Let V be a vector field on E. It has a natural lift to $\mathcal{F}^{(m)}E$, denoted by $\widehat{V}$, given by the induced action of its flow on m-frames: in coordinates

$$\widehat{V} = V^A \frac{\partial}{\partial u^A} + u_i^B \frac{\partial V^A}{\partial u^B} \frac{\partial}{\partial u_i^A}.$$

Moreover, $\widehat{V}$ passes to the quotient to define a vector field $\widetilde{V}$ on $\mathcal{S}^{(m)}E$, as follows from the definition, and equally from the fact that $\widehat{V}$ commutes with Δ_j^i. Now let the flow of V transform W: the resulting rate of change of I is given by

$$\dot{I} = \int_{\widetilde{R}} \left(\mathcal{L}_{\widetilde{V}}\tilde{\Theta}\right)\Big|_{\widetilde{W}} = \int_{\widetilde{R}} \left(\widetilde{V} \lrcorner\, d\tilde{\Theta}\right)\Big|_{\widetilde{W}} + \int_{\widetilde{B}} \left(\widetilde{V} \lrcorner\, \tilde{\Theta}\right)\Big|_{\widetilde{W}}.$$

Then W is an extremal for the fixed boundary problem if and only if

$$\left(\widetilde{V} \lrcorner\, \tilde{\Theta}\right)\Big|_{\widetilde{W}} = 0 \quad \text{for all } V.$$

That is to say,

$$\widehat{\sigma}^* \left(\widehat{V} \lrcorner\, \Theta\right) = 0 \quad \text{for all } V,$$

for any parametrization σ of W. We shall show that for any σ,

$$\widehat{\sigma}^* \left(\frac{\partial}{\partial u_i^A} \lrcorner\, d\Theta\right) = 0, \quad \text{while } \widehat{\sigma}^* \left(\frac{\partial}{\partial u^A} \lrcorner\, d\Theta\right) = \left(\frac{\partial L}{\partial u^A} - \frac{d}{dx^i}\left(\frac{\partial L}{\partial u_i^A}\right)\right) d^m x.$$

The first of these implies that for any vector field V which is vertical with respect tot he projection $\mathcal{F}^{(m)}E \to E$, $V \lrcorner\, d\Theta$ is a contact form.

Set $X_i = u_i^A \partial/\partial u^A$ (X_i is the ith component of the tautological frame field); notice that $\langle X_i, \theta^j \rangle = \delta_i^j L$, and therefore

$$X_i \lrcorner\, \Theta = (-1)^{i-1} L^{-(m-2)} \theta^1 \wedge \cdots \widehat{\theta^i} \cdots \wedge \theta^m.$$

Thus

$$
\begin{aligned}
d\Theta &= L^{-1}\left(-(m-1)dL\wedge\Theta + d\theta^i\wedge(X_i\lrcorner\,\Theta)\right)\\
&= L^{-1}\left(\left(u_i^A d\left(\frac{\partial L}{\partial u_i^A}\right) - (m-1)dL\right)\wedge\Theta - d\left(\frac{\partial L}{\partial u_i^A}\right)\wedge\left(X_i\lrcorner\,(du^A\wedge\Theta)\right)\right).
\end{aligned}
$$

But $u_i^A(\partial L/\partial u_i^A) = mL$, and so

$$
Ld\Theta = \frac{\partial L}{\partial u^A}du^A\wedge\Theta - d\left(\frac{\partial L}{\partial u_i^A}\right)\wedge(X_i\lrcorner\,(du^A\wedge\Theta)).
$$

Now $\widehat{\sigma}_*(\partial/\partial x^i)$ differs from X_i by a vertical vector field, so

$$
\widehat{\sigma}^*(X_i\lrcorner\,\Theta) = L\frac{\partial}{\partial x^i}\lrcorner\, d^m x = Ld^{m-1}x_i.
$$

Moreover, for any semi-basic form ω such that $\widehat{\sigma}^*\omega = 0$, and in particular for any semi-basic $(m+1)$-form ω, $\widehat{\sigma}^*(X_i\lrcorner\,\omega) = 0$; that is to say, if ω is semi-basic and contact then $X_i\lrcorner\,\omega$ is contact. We shall write $\omega_1 \cong \omega_2$ if $\omega_1 - \omega_2$ is contact. Now

$$
X_i\lrcorner\,(du^A\wedge\Theta) = u_i^A\Theta - du^A\wedge(X_i\lrcorner\,\Theta),
$$

and therefore

$$
du^A\wedge(X_i\lrcorner\,\Theta) \cong u_i^A\Theta.
$$

It follows from the formula for $d\Theta$ that

$$
L\left(\frac{\partial}{\partial u_i^A}\lrcorner\, d\Theta\right) = -\frac{\partial^2 L}{\partial u_i^A\partial u_j^B}X_j\lrcorner\,(du^B\wedge\Theta)\cong 0.
$$

Next, we have

$$
\begin{aligned}
&L\left(\frac{\partial}{\partial u^A}\lrcorner\, d\Theta\right)\\
&= \frac{\partial L}{\partial u^A}\Theta - \frac{\partial L}{\partial u^B}du^B\wedge\left(\frac{\partial}{\partial u^A}\lrcorner\,\Theta\right) - \frac{\partial^2 L}{\partial u^A\partial u_i^B}X_i\lrcorner\,(du^A\wedge\Theta)\\
&\quad - d\left(\frac{\partial L}{\partial u_i^B}\right)\wedge\left(X_i\lrcorner\,\left(\frac{\partial}{\partial u^A}\lrcorner\,(du^B\wedge\Theta)\right)\right)\\
&\cong \frac{\partial L}{\partial u^A}\Theta - d\left(\frac{\partial L}{\partial u_i^A}\right)\wedge(X_i\lrcorner\,\Theta)\\
&\quad - \frac{\partial L}{\partial u^B}du^B\wedge\left(\frac{\partial}{\partial u^A}\lrcorner\,\Theta\right) + d\left(\frac{\partial L}{\partial u_i^B}\right)\wedge\left(X_i\lrcorner\,\left(du^B\wedge\left(\frac{\partial}{\partial u^A}\lrcorner\,\Theta\right)\right)\right).
\end{aligned}
$$

When we pull back with $\widehat{\sigma}$ the first two terms will give the Euler-Lagrange equations; we must therefore show that the last two terms are collectively contact. Now

$$\frac{\partial}{\partial u^A} \lrcorner\, \Theta = L^{-1} \frac{\partial L}{\partial u_i^A} X_i \lrcorner\, \Theta,$$

so that

$$du^B \wedge \left(\frac{\partial}{\partial u^A} \lrcorner\, \Theta \right) = L^{-1} \frac{\partial L}{\partial u_i^A} (u_i^B \Theta - X_i \lrcorner\, (du^B \wedge \Theta)) \cong L^{-1} \frac{\partial L}{\partial u_i^A} u_i^B \Theta.$$

It follows that

$$\begin{aligned}
& -\frac{\partial L}{\partial u^B} du^B \wedge \left(\frac{\partial}{\partial u^A} \lrcorner\, \Theta \right) + d \left(\frac{\partial L}{\partial u_i^B} \right) \wedge \left(X_i \lrcorner \left(du^B \wedge \left(\frac{\partial}{\partial u^A} \lrcorner\, \Theta \right) \right) \right) \\
\cong\ & L^{-1} \frac{\partial L}{\partial u_i^A} \left(-u_i^B \frac{\partial L}{\partial u^B} \Theta + u_i^B d \left(\frac{\partial L}{\partial u_j^B} \right) \wedge (X_j \lrcorner\, \Theta) \right) \\
\cong\ & L^{-1} \frac{\partial L}{\partial u_i^A} \left(-u_i^B \frac{\partial L}{\partial u^B} \Theta + dL \wedge (X_i \lrcorner\, \Theta) - \frac{\partial L}{\partial u_j^B} du_i^B \wedge (X_j \lrcorner\, \Theta) \right) \\
\cong\ & L^{-1} \frac{\partial L}{\partial u_i^A} \frac{\partial L}{\partial u_j^B} (du_j^B \wedge (X_i \lrcorner\, \Theta) - du_i^B \wedge (X_j \lrcorner\, \Theta)).
\end{aligned}$$

But for any σ,

$$\begin{aligned}
\widehat{\sigma}^*(du_i^B \wedge (X_j \lrcorner\, \Theta)) &= L \frac{\partial^2 u^B}{\partial x^i \partial x^k} dx^k \wedge d^{m-1} x_j = L \frac{\partial^2 u^B}{\partial x^i \partial x^j} d^m x \\
&= \widehat{\sigma}^*(du_j^B \wedge (X_i \lrcorner\, \Theta));
\end{aligned}$$

so we are left with

$$\begin{aligned}
\widehat{\sigma}^* \left(\frac{\partial}{\partial u^A} \lrcorner\, d\Theta \right) &= \frac{\partial L}{\partial u^A} d^m x - d \left(\frac{\partial L}{\partial u_i^A} \right) \wedge d^{m-1} x_i \\
&= \left(\frac{\partial L}{\partial u^A} - \frac{d}{dx^i} \left(\frac{\partial L}{\partial u_i^A} \right) \right) d^m x.
\end{aligned}$$

In the course of this argument we have shown that for any vector field V on $\mathcal{F}^{(m)}E$ which is vertical with respect to the projection $\mathcal{F}^{(m)}E \to E$, $V \lrcorner\, \Theta \cong 0$. Moreover, it is clear from the formula for $d\Theta$ and the fact that $X_i \lrcorner\, \omega \cong 0$ for any semi-basic form ω such that $\omega \cong 0$ that $X_i \lrcorner\, \Theta \cong 0$; this is equivalent to the well-known fact that

$$\left(\frac{\partial L}{\partial u^A} - \frac{d}{dx^i} \left(\frac{\partial L}{\partial u_i^A} \right) \right) \frac{\partial u^A}{\partial x^j} = 0$$

identically on any submanifold (see, for example, [10]). Thus to obtain those of the Euler-Lagrange equations that are non-trivial we need consider only variations which are transverse to the submanifold, and it is immaterial how they are lifted from E to $\mathcal{F}^{(m)}E$.

6 Minimal submanifolds

The classical problem of finding a minimal surface in Euclidean 3-space is a good example for illustrating our results: indeed, this problem, and its generalizations (which we will discuss below), should probably be regarded as the paradigms of parametric multiple integral problems in the calculus of variations.

We can see how the minimal surface problem fits into the framework we have described as follows. We have $M = \mathbf{R}^2$ and $E = \mathbf{R}^3$ with the standard Euclidean structure. We take coordinates $u^A = (u, v, w)$ on $\mathbf{R}^3$, so that coordinates on $\mathcal{F}^{(2)}\mathbf{R}^3$ are (u^A, u_1^A, u_2^A). The Lagrangian function for this problem is

$$\begin{aligned} L &= \sqrt{\det(g_{ij})} \\ &= \sqrt{(v_1 w_2 - v_2 w_1)^2 + (w_1 u_2 - w_2 u_1)^2 + (u_1 v_2 - u_2 v_1)^2} \end{aligned}$$

where g_{ij} is the pull-back by σ of the metric on E. The Lagrangian satisfies the homogeneity condition $\Delta^i_j(L) = \delta^i_j L$, and so we can derive the Hilbert-Carathéodory form: we find that

$$\Theta = \frac{(v_1 w_2 - v_2 w_1) dv \wedge dw + (w_1 u_2 - w_2 u_1) dw \wedge du + (u_1 v_2 - u_2 v_1) du \wedge dv}{\sqrt{(v_1 w_2 - v_2 w_1)^2 + (w_1 u_2 - w_2 u_1)^2 + (u_1 v_2 - u_2 v_1)^2}}.$$

This 2-form is projectable to the sphere bundle $\mathcal{S}^{(2)}\mathbf{R}^3$, and the resulting form $\tilde{\Theta}$ is a canonical representative of the equivalence class of Lagrangian forms on the sphere bundle whose variations define the minimal surface problem. Formally, this expression for Θ is just the area (volume) form induced on a surface in terms of the unit normal field N obtained from the vector product of the x^i-coordinate tangent vectors, and the Euclidean volume 3-form vol, namely $N \lrcorner \, \mathrm{vol}$. Moreover, in order to obtain the (single) Euler-Lagrange equation for this problem we need only consider $\widehat{\sigma}^*(N \lrcorner \, d\Theta)$ where N is the normal field considered as a vector field along the projection $\mathcal{F}^{(2)}\mathbf{R}^3 \to \mathbf{R}^3$, and it is not difficult to see from this that $\widehat{\sigma}^*(N \lrcorner \, d\Theta) = -HN \lrcorner \, \mathrm{vol}$ where H is the mean curvature of the surface.

With these observations for guidance, we now consider the more general problem of m-dimensional minimal submanifolds of an N-dimensional Riemannian space, which we assume to be orienteable (and oriented).

Let E be a Riemannian space with metric g. The Lagrangian function for the m-dimensional minimal submanifold problem is

$$L(u^A, u_i^A) = \sqrt{\det(g_i j)} = \sqrt{\det(g_{AB} u_i^A u_j^B)};$$

it clearly satisfies $L(a \cdot \xi) = \det(a) L(\xi)$; moreover, for any map σ parametrizing a submanifold of E, $(\widehat{\sigma}^* L) d^m x$ is the volume form induced on the submanifold.

For any $w \in \mathcal{S}^{(m)}E$, we say that $N \in T_w \mathcal{S}^{(m)}E$ is a *normal* vector if $\rho_{+*}N$ is perpendicular with respect to $g_{\rho_+(w)}$ to the subspace of $T_{\rho_+(w)}E$ spanned by one, and therefore every, m-frame in the equivalence class w. Of course this makes every ρ_+-vertical vector normal, but we shall be interested only in normal vectors with non-vanishing projections. This definition may be made to apply to $\mathcal{F}^{(m)}E$ with obvious modifications.

Let $w \in \mathcal{S}^{(m)}E$, and let ξ be an m-frame representative of w which is orthonormal. Let N_1, N_2, ..., N_n be n normal vectors at w such that

$$\mathrm{vol}(\rho_{+*}(N_1), \rho_{+*}(N_2), \ldots, \rho_{+*}(N_n), \xi_1, \xi_2, \ldots, \xi_m) = 1,$$

where vol is the Riemannian volume form. Then if $\nu = N_1 \wedge N_2 \wedge \ldots \wedge N_n$, the m-form $\nu \lrcorner \rho_+^* \,\mathrm{vol}$ is well-defined, that is, independent of which orthonormal m-frame we choose to represent w, and which normal vectors we choose, provided they satisfy the given condition. We claim that $\nu \lrcorner \rho_+^* \,\mathrm{vol} = \tilde{\Theta}$. Now

$$\frac{\partial L}{\partial u_i^A} = L^{-1} g^{ij} g_{AB} u_j^B,$$

where (g^{ij}) is the inverse of the matrix g_{ij} (which is non-singular on the m-frame bundle). It follows that $\langle N, \theta^i \rangle = 0$ for any normal vector on $\mathcal{F}^{(m)}E$, whence $N \lrcorner \Theta = 0$, and so

$$N \lrcorner \tilde{\Theta} = 0 = N \lrcorner (\nu \lrcorner \rho_+^* \,\mathrm{vol})$$

for any normal vector N on $\mathcal{S}^{(m)}E$. At any frame, the tautological vectors X_i satisfy $\langle X_i, \theta^j \rangle = \delta_i^j L$, and if the frame is orthonormal then $L = 1$; thus for any vectors Ξ_i at w whose projections into E form an orthonormal m-frame ξ in the equivalence class w,

$$\begin{aligned} \tilde{\Theta}_w(\Xi_1, \Xi_2, \ldots, \Xi_m) &= \Theta_\xi(X_1, X_2, \ldots, X_m) = 1 \\ &= \mathrm{vol}_{\rho_+(w)}(\rho_{+*}\nu, \xi_1, \xi_2, \ldots, \xi_m) = (\nu \lrcorner \rho_+^* \,\mathrm{vol})_w(\Xi_1, \Xi_2, \ldots, \Xi_m). \end{aligned}$$

Since both forms are semi-basic, it follows that they are equal.

We next review the concept of mean curvature, in a form suitable for our purposes. Let $\mathcal{D}$ be an m-dimensional distribution on E, and N any unit vector field perpendicular to $\mathcal{D}$, so that $g(N_x, \mathcal{D}_x) = 0$. In fact in all of the following discussion the geometrical objects involved need only be locally defined; we shall not bother to mention this repeatedly. Let P be the operator of perpendicular projection onto $\mathcal{D}$, considered as a type $(1,1)$ tensor field on E. The *Weingarten map* associated with

$\mathcal{D}$ and N is the map $\mathcal{D} \to \mathcal{D}$ by $Z \mapsto -P(\nabla_Z N))$, $Z \in \mathcal{D}$, where ∇ is the covariant derivative operator of the Levi-Civita connection on E. The *mean curvature* of $\mathcal{D}$ in the direction N, $H_N(\mathcal{D})$, is the trace of the Weingarten map. Let $\{X_i\}$ be an orthonormal m-frame field tangent to $\mathcal{D}$, and $\{N_\alpha\}$, $\alpha = 1, 2, \ldots, n$, an orthonormal basis of vector fields normal to $\mathcal{D}$ such that $\{N_1, N_2, \ldots, N_n, X_1, X_2, \ldots, X_m\}$ is positively oriented; let $\nu = N_1 \wedge N_2 \wedge \ldots, \wedge N_n$ and set $\Omega = \nu \lrcorner \operatorname{vol}$. Then $N \lrcorner \Omega = 0$ for any vector field N normal to $\mathcal{D}$, and $\Omega(X_1, X_2, \ldots, X_m) = 1$. It follows that

$$d\Omega(N, X_1, X_2, \ldots, X_m) = \sum_{k=1}^{m} (-1)^k \Omega([N, X_k], X_1, \ldots \widehat{X_k} \ldots, X_m),$$

since $N(\Omega(X_1, X_2, \ldots, X_m)) = 0$, and in every other term N appears as an argument of Ω. The only component of $[N, X_k]$ which makes a non-zero contribution to the sum on the right-hand side is its X_k-component, which is $g([N, X_k], X_k)$; in fact

$$(N \lrcorner d\Omega)(X_1, X_2, \ldots, X_m) = -\sum_{k=1}^{m} g([N, X_k], X_k).$$

But $g([N, X_k], X_k) = -g(\nabla_{X_k} N, X_k)$ since $g(X_k, X_k) = 1$, and so

$$N \lrcorner d\Omega = -H_N(\mathcal{D})\Omega.$$

In particular, if W is an m-dimensional integral manifold of $\mathcal{D}$ then $H_N(W) = H_N(\mathcal{D})|_W$ is the mean curvature of W in the normal direction N, and $\Omega|W = \operatorname{vol}_W$, the volume form induced on W by Riemannian structure of E. Note that we do not intend to imply here that $\mathcal{D}$ is integrable in the sense of Frobenius. In fact, given any point x of any m-dimensional submanifold W of E, there is a neighbourhood U of x in E and an m-dimensional distribution $\mathcal{D}$ on U such that $W \cap U$ is an integral manifold of $\mathcal{D}$; the strategy of our calculation is to embed W in a distribution in this way.

An m-dimensional distribution $\mathcal{D}$ on E, together with choice of orientation of tangent m-frames, determines a section γ of $\mathcal{S}^{(m)}E \to E$ such that $\gamma^*\tilde{\Theta} = \Omega$, and $\gamma_* N$ is a normal field on the image of γ. Thus

$$\gamma^*(\gamma_* N \lrcorner d\tilde{\Theta}) = N \lrcorner d(\gamma^*\tilde{\Theta}) = N \lrcorner d\Omega = -H_N(\mathcal{D})\Omega.$$

In particular, if W is a submanifold of E, parametrized by σ, and $\mathcal{D}$ is any distribution of which W is an integral manifold, then

$$H_N(W)\sigma^* \operatorname{vol}_W = -(\gamma \circ \sigma)^*(\gamma_* N \lrcorner d\tilde{\Theta}).$$

But $\rho_+ \circ \widehat{\sigma} = \gamma \circ \sigma$, whence

$$H_N(W)\sigma^* \operatorname{vol}_W = -\widehat{\sigma}^* \rho_+^*(\gamma_* N \lrcorner d\tilde{\Theta}).$$

We can clearly find a vector field N' on $\mathcal{F}^{(m)}E$ such that $\rho_{+*}N' - \gamma_* N$ is vertical with respect to $\mathcal{S}^{(m)}E \to E$, whence

$$H_N(W)\sigma^* \operatorname{vol}_W = -\widehat{\sigma}^*(N' \lrcorner\, d(\rho_+^* \tilde{\Theta})) = -\widehat{\sigma}^*(N' \lrcorner\, d\Theta).$$

It follows that the Euler-Lagrange equations for this problem state that the mean curvature of W in every normal direction should vanish.

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