The Hilbert-Carathéodory and Poincaré-Cartan forms for higher-order multiple-integral variational problems

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September 19, 2002

Abstract. We consider higher-order homogeneous multiple-integral variational problems defined in the context of m-frame bundles, and construct an m-form (the Hilbert-Carathéodory form) with the same extremals as the Lagrangian. We show that for second-order problems, and for problems where $m = 1$, this m-form is projectable to the corresponding sphere bundle. Where a second-order problem of this kind has been obtained from a problem on an affine jet bundle, the projected m-form is a generalization of the Carathéodory form used in the study of first-order problems.

Keywords: calculus of variations; multiple-integral problems; higher-order problems; homogeneous Lagrangian; Carathéodory form; Poincaré-Cartan form; Lepage equivalent
Subject classification: 53C60

1 Introduction

Problems in the calculus of variations fall into two classes, which are distinguished by the way the differential equations determining their solution manifolds (extremals) behave under change of parametrization. One of the simplest examples of this distinction is the difference between the geodesic equations in Finsler geometry and the equations of motion of a free particle in classical Lagrangian mechanics. In the latter case, where the parameter is the time, the equations of motion are invariant in form only under affine transformations of the parameter; in the former the equations are form-invariant under any smooth, sense-preserving reparametrization. It might be said that in the Lagrangian formulation of mechanics the parametrization of the extremals is determined by the problem, whereas in Finsler geometry the extremals are just oriented paths (1-dimensional submanifolds), with no preferred parametrization. These properties of extremals reflect properties of the Lagrangian functions from which they are derived: in particular, the parameter-independence of the extremals in the Finsler case is a consequence of the fact that the Lagrangian (the Finsler function) is positively homogeneous.

The concept of homogeneity of a Lagrangian can be generalized so that it applies to the most general type of variational problem, in which there are arbitrarily many (say m) independent variables (so that the variational integral is in general a multiple integral), and the Lagrangian depends on formal derivatives of the dependent variables of arbitrary finite order (say k). The cases discussed in the opening paragraph are the simplest, since they deal with single-integral problems ($m = 1$) of first order ($k = 1$). We have discussed the properties of homogeneous Lagrangians, and their relationship with non-homogeneous ones, for the case of first-order multiple-integral problems ($k = 1$, m arbitrary) in an earlier paper [3]. In the present work we extend our results to higher-order problems, so that in this paper both m and k are arbitrary.

For kth-order problems in which the Lagrangian is non-homogeneous the geometrical context is normally that of a bundle of k-jets of sections of a fibred manifold $\pi: E \to M$. In such a context, one of the most important geometrical objects associated with variational problems is the Poincaré-Cartan form. This is a differential form on the jet bundle, with the same extremals as the Lagrangian, and with the additional property that the extremals may be calculated using arbitrary variations rather than those prolonged from E. The form is, in fact, a Lepage equivalent of the Lagrangian; see for example [1, 8, 12].

The Poincaré-Cartan form was defined originally in the context of first-order single-integral problems, but the definition was extended without too much difficulty to higher-order single-integral problems, and to first order multiple-integral problems. It was therefore natural to expect that a further extension, to higher-order multiple-integral problems, would soon become available. Unfortunately, though, attempts to find such an extension encountered serious obstacles. A straightforward development using coordinate formulæ [6] turned out not to be invariant under a general change of coordinates for Lagrangians of third-order and above, and therefore did not represent a global geometric object [4]. Other approaches used sequences of affine bundles [9], sheaf theory [13] and non-holonomic jets [14, 18], although these constructions were not unique except in the second-order case where, for example, semi-holonomic jets could be employed to advantage [20]. A different technique used supplementary data, in the shape of a connection, to construct a suitable form [5, 7]. Indeed, theoretical studies have shown that there is no unique form for multiple-integral problems of third-order and above in the absence of additional data [10].

A separate observation regarding multiple-integral problems is that, even in the first-order case, the (well-defined) Poincaré-Cartan form is no longer a unique Lepage equivalent of the Lagrangian, as an arbitrary 2-contact form may be added without affecting its properties. In fact there are several other invariant choices, and one of particular interest is the Carathéodory form [2]: this is a simple (decomposable) form, and it has the important additional property that it is invariant under changes of coordinates which mix up the base coordinates and the fibre coordinates.

In our previous work [3] we have shown how the Carathéodory form may be constructed geometrically by using a somewhat different approach. There, we considered a generalization of the homogeneous formulation of Finsler geometry to first-order multiple-integral problems defined on a bundle of frames $\mathcal{F}^1_{(m)}E$, and we defined an m-form which we called the Hilbert-Carathéodory form. This is a form on the frame bundle, which has the same extremals as the Lagrangian from which it is derived, and which is invariant under the action of a group of transformations generated by reparametrizations of m-dimensional submanifolds of the base manifold E. We showed that if this homogeneous problem was derived from a variational problem on a jet bundle, then the projection of our Hilbert-Carathéodory form was just the Carathéodory form. In the present paper, we extend this approach to higher-order problems. We have two main aims: to lay the foundations of a general theory of the calculus of variations for multiple-integral problems of arbitrary order in which the Lagrangian is homogeneous, by generalizing the definition of the Hilbert-Carathéodory form; and to cast new light on the difficulties inherent in the choice of a Poincaré-Cartan form which we have described above.

We shall show that there is a well-defined method of constructing from a Lagrangian of any order both a (singular) coframe (ϑ^i) (an ordered set of m 1-forms on the appropriate m-frame bundle), and an m-form Θ, each of which may be used to generate the Euler-Lagrange equations when the Lagrangian is homogeneous. We call Θ the Hilbert-Carathéodory form for the variational problem. In the second-order case, Θ is again projectable, and so we are able to present the second-order Carathéodory form: in fibred coordinates (x^i, v^α) it is, as perhaps might be expected,

$$\frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \left(L\, dx^i + \left(\frac{\partial L}{\partial v_i^\alpha} - \frac{d}{dx^j} \frac{1}{\#(ij)} \frac{\partial L}{\partial v_{ij}^\alpha} \right) \theta^\alpha + \frac{1}{\#(ij)} \frac{\partial L}{\partial v_{ij}^\alpha} \theta_j^\alpha \right)$$

(where $\#(ij)$ equals 1 if $i = j$ and equals 2 otherwise, and where θ^α, θ^α_j are the contact forms). Of course if this object were given simply by a coordinate formula then its invariance under coordinate changes would be far from obvious, and indeed the corresponding third-order formula is not invariant; however our construction is geometrical, and so invariance is immediate. On the other hand, we find that the form defined on the frame bundle for Lagrangians of order greater than two is in general no longer projectable in this way. These conclusions apply to multiple-integral problems; in the single-integral case the frame bundle is simply a non-singular tangent bundle, and the m-form, which now coincides with the coframe, is a generalization of the Hilbert form from Finsler geometry and is always projectable. We shall show that the Hilbert-Carathéodory form we define is uniquely determined by certain natural properties, and we therefore conclude that the fact that difficulties arise when one attempts to define a Poincaré-Cartan form for multiple-integral problems of order greater than two can be explained by the fact that the Hilbert-Carathéodory form, though well-defined and unique, is not projectable in these cases.

In the Lagrangian formulation of mechanics (which gives rise to a class of single integral first-order problems) the Lagrangian is usually assumed to be regular, in the sense that its Hessian with respect to the velocity variables should be non-singular; a Finsler function, on the other hand, cannot be regular in this sense, as a consequence of its homogeneity. This is an important aspect of the difference between the two types of variational problem which we described in the opening paragraph, when $m = 1$ and $k = 1$. However, it is not easy to extend this idea to the general case considered in this paper, because the notion of regularity becomes less tractable; it is perhaps fortunate, therefore, that no concept of regularity plays any part in the theory developed below.

In Sections 2 and 3 of this paper we recall some of the properties of the frame bundle context in which we work, and establish several multi-index formulæ we shall need. In Section 4 we discuss the homogeneity of Lagrangian functions. In Section 5 we describe the construction of the Hilbert-Carathéodory form and in Section 6 we examine its invariance under reparametrization, and hence its projectability to the appropriate quotient bundle. Although indices and multi-indices appear in these latter two sections, we wish to emphasize that they frequently refer not to coordinates but to components of a frame, and so the objects labelled by them are nevertheless invariant and globally-defined. Finally in Section 7 we give coordinate formulæ for our constructions, and in particular obtain the formula given above for the second-order Carathéodory form.

2 The geometry of the frame bundle

In this paper we shall be concerned with a smooth (C^∞) connected manifold E of finite dimension N, and with its infinite-order m-frame bundle $\mathcal{F}^\infty_{(m)}E$; we shall devote most of this section to defining, and describing some relevant properties of, the latter manifold.

In [3] we defined a (first-order) m-frame on a manifold E as an ordered linearly independent set of m tangent vectors at a point of E, and regarded the bundle of m-frames as an open submanifold of the Whitney sum $\bigoplus^m TE$; the bundle of 1-frames (the case $m = 1$) is just the slit tangent bundle $TE-$zero section. However, an alternative description involves the use of 1-jets at the origin $0 \in \mathbb{R}^m$ of smooth maps $\mathbb{R}^m \to E$ (of course, such a map need be defined only on a neighbourhood of $0 \in \mathbb{R}^m$); an m-frame is the 1-jet of an immersion. The bundle of 1-jets at 0 of smooth maps $\mathbb{R}^m \to E$ is called the bundle of first-order m-velocities, and is denoted by $T^1_{(m)}E$. When dealing with higher-order problems we could replace $T^1_{(m)}E$ by $T^k_{(m)}E$, the bundle of kth-order m-velocities [11], or k-jets at 0 of smooth maps into E from a neighbourhood of the origin $0 \in \mathbb{R}^m$, for some suitable k; the kth order m-frame bundle is the submanifold of $T^k_{(m)}E$ consisting of jets of immersions. In fact for many of our constructions the distinction between m-frames and m-velocities will not be important.

For convenience, however, we will use infinite jets rather than finite ones: that is, we will work with $T^\infty_{(m)}E$, the collection of infinite jets of smooth maps into E from a neighbourhood of the origin $0 \in \mathbb{R}^m$, evaluated of course at 0. This is an infinite-dimensional Fréchet manifold, but the difficulties sometimes associated with such manifolds will not concern us: all our calculations could be repeated, at the cost of dealing with more complicated formulæ, by using objects defined on finite-order jet bundles or along the projection maps between them. In fact we will treat $T^\infty_{(m)}E$ merely as the projective limit of the sequence of finite-order m-velocity bundles and projections, and similarly for geometric objects defined on it. An infinite order m-frame is the infinite jet of an immersion, and the bundle of infinite order m-frames is denoted by $\mathcal{F}^\infty_{(m)}E$.

As we have just noted, $T^\infty_{(m)}E$ is the collection of jets at 0 of smooth maps $\sigma\colon \mathbb{R}^m \to E$. There is no loss of generality in restricting ourselves to jets at 0, since the jet of σ at $x \in \mathbb{R}^m$ is the jet at 0 of the translate $\sigma \circ \tau_x$ of σ (where $\tau_x\colon \mathbb{R}^m \to \mathbb{R}^m$ is the translation map $\tau_x(y) = y + x$).

We shall also be concerned with the infinite-order Grassmannian bundle over E, which is the quotient of $\mathcal{F}^\infty_{(m)}E$ under the equivalence relation in which two points of $\mathcal{F}^\infty_{(m)}E$ are equivalent when the maps defining them are related by a diffeomorphism of $\mathbb{R}^m$ preserving the origin. If we consider only orientation-preserving diffeomorphisms then we obtain the infinite-order sphere bundle. We shall, below, look in more detail at the Lie aglebra of fundamental vector fields associated with the projection from the frame bundle to the sphere bundle.

After we have defined the Hilbert-Carathéodory form we shall examine its projection to the Carathéodory form, and so we shall have to consider the relationship of our constructions to those on bundles of jets $J^\infty\pi$ of sections of a fibration $\pi: E \to M$. We note here that if E admits such a fibration then we may consider the corresponding (affine) jet bundle $J^\infty\pi$ as an open submanifold of either the Grassmannian bundle or the sphere bundle.

Returning to the frame bundle, we shall use indices and multi-indices both to indicate the components of a frame, where the objects concerned are defined globally, and to represent coordinates that are defined only locally. The summation convention will mostly be in abeyance; the exceptions are that summation over repeated indices A, which are the coordinate indices on the base manifold E, will be understood, and that occasionally we shall state explicitly that the convention should apply to a particular formula. We shall use $I, J, \ldots \in \mathbb{N}^m$ to represent multi-indices with the usual addition and partial ordering; I_k will be the kth component of I, and 1_j will be the multi-index with 1 in the jth position and 0 elsewhere. We shall write $|I| = I_1 + I_2 + \cdots + I_m$ for the length of I, and $I! = I_1!I_2!\cdots I_m!$ for its factorial. If $I \leq J$ then $J - I$ will be the multi-index whose kth component is $J_k - I_k$.

The ratio $|I|!/I!$, often called the weight of I, will appear frequently in formulæ: it is used when converting a sum over a multi-index into a sum over the corresponding individual indices, so that if some object $\psi(I)$ depends upon the multi-index I then

$$\sum_{|I|=r} \frac{|I|!}{I!}\psi(I) = \sum_{i_1,i_2,\ldots,i_r} \psi(1_{i_1} + 1_{i_2} + \ldots + 1_{i_r}),$$

and consequently

$$\sum_{|J|=r+1} \frac{|J|!}{J!}\psi(J) = \sum_i \sum_{|I|=r} \frac{|I|!}{I!}\psi(I + 1_i).$$

For any variables $x^1, x^2, \ldots, x^m$, we write x^I for the monomial $(x^1)^{I_1}(x^2)^{I_2}\cdots(x^m)^{I_m}$. The following version of the binomial theorem holds: if $z^i = x^i + y^i$,

$$z^I = \sum_{J\leq I} \frac{I!}{J!(I-J)!} x^J y^{I-J};$$

as a particular case we have

$$\sum_{J\le I}\frac{(-1)^{|J|}I!}{J!(I-J)!}=0.$$

The power notation extends to any m objects which have a commutative multiplication rule defined on them (pairwise commuting operators, for example). In particular, if we set $D_i=\partial/\partial x^i$, we have

$$D_I=\frac{\partial^{|I|}}{(\partial x^1)^{I_1}(\partial x^2)^{I_2}\cdots(\partial x^m)^{I_m}};$$

then D_{1_i} is just D_i, and we shall use the two forms interchangeably.

We define coordinates on $T^\infty_{(m)}E$ as follows. For any point in the bundle, let $\sigma\colon\mathbb{R}^m\to E$ be any smooth map of which the point is the ∞-jet at 0; we write $j^\infty\sigma$ for the jet, the fact that it is evaluated at the origin being understood. Choose local coordinates u^A on E, so that σ is represented by functions σ^A, and set

$$u^A_I=D_I\sigma^A|_0.$$

For a frame we require that the matrix (u^A_i) has rank m, where $u^A_i=u^A_I$ for $I=1_i$.

Now $D_Ix^I|_0=I!$, so the Taylor series of σ^A about 0 is

$$\sum_I\frac{1}{I!}u^A_Ix^I.$$

Moreover, $D_Ix^J=0$ unless $J\ge I$, in which case

$$D_Ix^J=\frac{J!}{(J-I)!}x^{J-I}.$$

Any smooth map $\sigma\colon\mathbb{R}^m\to E$ lifts to a map $\widehat{\sigma}\colon\mathbb{R}^m\to T^\infty_{(m)}E$, given by $\widehat{\sigma}(x)=j^\infty(\sigma\circ\tau_x)$; in terms of the given coordinates on $T^\infty_{(m)}E$ we have

$$\widehat{\sigma}^A_I(x)=\sum_{J\ge I}\frac{1}{(J-I)!}u^A_Jx^{J-I}=\sum_K\frac{1}{K!}u^A_{I+K}x^K;$$

the u^A_J here are the coordinates of $j^\infty\sigma$. We shall use these lifts to characterize certain differential forms defined on $T^\infty_{(m)}E$: we shall say that a form θ is a contact form if $\widehat{\sigma}^*\theta=0$ for every map $\sigma\colon\mathbb{R}^m\to E$.

Note that a vector field on $T^\infty_{(m)}E$ may be expressed in these coordinates as a formal infinite sum

$$\sum_J X^A_J\frac{\partial}{\partial u^A_J};$$

on the other hand, in the corresponding expression for a 1-form

$$\sum_J \eta_A^J du_J^A,$$

the sum is finite at each point of $T^\infty_{(m)}E$ although globally the number of non-zero terms may be unbounded. Contracting a vector field with a 1-form therefore gives rise to a well-defined finite sum at each point (see, for example, [19] Chapter 7).

In the remainder of this section we shall define some families of geometrical objects on $T^\infty_{(m)}E$ that will be used extensively in the rest of the paper. The first of these will be a generalization of the vertical endomorphism defined on tangent bundles. Other ways of defining similar objects have been given in [15, 18, 21].

We first define a vertical lift operation, which takes a first-order m-velocity ζ at a point $j^\infty\sigma \in T^\infty_{(m)}E$ and constructs from it a vector at the same point which is vertical over E. The construction is carried out as follows. We take a map $\chi\colon \mathbb{R}^m \times \mathbb{R}^m \to E$ with the following two properties: if we write $\chi_y(x)$ for $\chi(x,y)$ then $j^\infty(\chi_0) = j^\infty\sigma$; and the map $\gamma\colon \mathbb{R}^m \to T^\infty_{(m)}E$ given by $\gamma(y) = j^\infty(\chi_y)$ satisfies $j^1\gamma = \zeta$. We say that χ is a representative of ζ; we shall show how to find a representative of any m-velocity shortly.

Given a representative χ of ζ, we define a new map $\overline{\chi}\colon \mathbb{R}^m \times \mathbb{R} \to E$ by $\overline{\chi}(x,t) = \chi(x,tx)$. If we now write $\overline{\chi}_t(x)$ for $\overline{\chi}(x,t)$, then $j^\infty(\overline{\chi}_0) = j^\infty\sigma$, and the map $\overline{\gamma}\colon \mathbb{R} \to T^\infty_{(m)}E$ given by $\overline{\gamma}(t) = j^\infty(\overline{\chi}_t)$ is a curve in $T^\infty_{(m)}E$ whose derivative at zero is a tangent vector denoted ζ^{v}, the vertical lift of the m-velocity ζ. It is clear from checking the equality of derivatives that applying the same procedure to any other representative χ' of ζ will produce the same result ζ^{v}, so that taking the vertical lift is a well-defined operation.

We now carry out this construction in coordinates. Suppose that the coordinates of $j^\infty\sigma$ are (u_I^A), and that the frame ζ is given by

$$\zeta_i = \sum_J \zeta^A_{J,i} \frac{\partial}{\partial u_J^A}.$$

If we set

$$\chi(x,y) = \sum_I \frac{1}{I!}(u_I^A + \zeta^A_{I,i} y^i) x^I$$

then χ is a representative of ζ. We have

$$\overline{\chi}(x,t) = \sum_I \frac{1}{I!}(u_I^A + t\zeta^A_{I,i} x^i) x^I,$$

whence

$$\overline{\gamma}^A_{I+1_i}(t) = u^A_{I+1_i} + t(I_i+1)\zeta^A_{I,i}, \quad \overline{\gamma}^A_J(t) = u^A_J \text{ if } J_i = 0,$$

so that

$$\zeta^{\mathrm{v}} = \sum_{I,i}(I_i + 1)\zeta^A_{I,i}\frac{\partial}{\partial u^A_{I+1_i}}.$$

Note that ζ^{v} depends linearly on $\zeta \in T^1_{(m)}(T^\infty_{(m)}E)$.

Define the linear map $S\colon T^1_{(m)}(T^\infty_{(m)}E) \to T(T^\infty_{(m)}E)$ by $S(\zeta) = \zeta^{\mathrm{v}}$, and then let S^i be the restriction of S to the i-th summand, that is, S acting on m-velocities such that $\zeta_j = 0$, $j \neq i$. We may think of each S^i as an endomorphism of $T(T^\infty_{(m)}E)$ (or, equivalently, of $T^*(T^\infty_{(m)}E)$), and in coordinates

$$S^i = \sum_I (I_i + 1)\frac{\partial}{\partial u^A_{I+1_i}} \otimes du^A_I = \sum_I \frac{(I+1_i)!}{I!}\frac{\partial}{\partial u^A_{I+1_i}} \otimes du^A_I.$$

It is easy to see that

$$S^i \circ S^j = \sum_I \frac{(I+1_i+1_j)!}{I!}\frac{\partial}{\partial u^A_{I+1_i+1_j}} \otimes du^A_I = S^j \circ S^i.$$

Since S^i and S^j commute, S^I is defined for any non-zero multi-index I, and clearly

$$S^I = \sum_J \frac{(I+J)!}{J!}\frac{\partial}{\partial u^A_{I+J}} \otimes du^A_J.$$

The next family of objects to be defined on $T^\infty_{(m)}E$ will be a family of vector fields. The ith total derivative operator Δ_i is the vector field on $T^\infty_{(m)}E$ whose value at $j^\infty\sigma$ is the image under $\widehat{\sigma}_*$ of $\partial/\partial x^i|_0$; this image does not depend on the choice of map σ representing the m-velocity $j^\infty\sigma$, as may be seen from the coordinate representation. If the coordinates of $j^\infty\sigma$ are (u^A_I), then

$$\left.\frac{\partial\widehat{\sigma}^A_I}{\partial x^i}\right|_0 = u^A_{I+1_i},$$

so

$$\Delta_i = \sum_I u^A_{I+1_i}\frac{\partial}{\partial u^A_I}.$$

It is easy to see that the vector fields Δ_i pairwise commute, and therefore so do the corresponding Lie derivatives, which we denote by d_i. Thus d_I is defined for all non-zero multi-indices I.

The total derivative operators can be used to define prolongations of vector fields from E to $T^\infty_{(m)}E$. For any vector field X on E, there is a unique vector field $\widehat{X}$ on

$T^\infty_{(m)}E$, the prolongation of X, which projects onto X and commutes with Δ_i for each i: it is given by

$$\widehat{X} = \sum_I d_I(X^A)\frac{\partial}{\partial u^A_I}, \quad \text{where } X = X^A\frac{\partial}{\partial u^A}.$$

There is an important relationship between the total derivatives and the vertical endomorphism operators, where the latter are regarded as operators on differential forms. It is straightforward to check (for example, from the coordinate representation) that $[S^i, d_j] = \delta^i_j$, and we can then obtain the following result.

Lemma 2.1 *For any $n \geq 1$,*

$$\sum_{|I|=n} \frac{1}{I!}S^i d_I S^I = \sum_{|J|=n-1} \frac{1}{J!}d_J S^{J+1_i} + \sum_{|K|=n} \frac{1}{K!}d_K S^{K+1_i}.$$

Proof This is certainly true when $n = 1$, as it is just

$$(S^i d_j)S^j = (\delta^i_j + d_j S^i)S^j = S^i + d_j S^i S^j.$$

If we suppose it to be true for a particular value of n, we find that

$$\begin{aligned}
\sum_{|I|=n+1} \frac{1}{I!}S^i d_I S^I &= \frac{1}{n+1}\sum_j \sum_{|J|=n} \frac{1}{I!}S^i d_{J+1_j} S^{J+1_j} \\
&= \frac{1}{n+1}\sum_j \sum_{|J|=n} \frac{1}{I!}(\delta^i_j + d_j S^i) d_J S^{J+1_j} \\
&= \frac{1}{n+1}\left(\sum_{|J|=n} \frac{1}{I!}d_J S^{J+1_i} + \sum_j \sum_{|J|=n} \frac{1}{I!}d_j S^i d_J S^{J+1_j}\right).
\end{aligned}$$

Using the induction hypothesis, the second term above can be rearranged as follows:

$$\begin{aligned}
&\sum_j \sum_{|I|=n} \frac{1}{I!}d_j S^i d_I S^{I+1_j} \\
&= \sum_j d_j \left(\sum_{|J|=n-1} \frac{1}{J!}d_J S^{J+1_i} + \sum_{|K|=n} \frac{1}{K!}d_K S^{K+1_i}\right) S^j \\
&= \sum_j \sum_{|J|=n-1} \frac{1}{J!}d_{J+1_j} S^{J+1_i+1_j} + \sum_j \sum_{|K|=n} \frac{1}{K!}d_{K+1_j} S^{K+1_i+1_j} \\
&= n\sum_{|J|=n} \frac{1}{J!}d_J S^{J+1_i} + (n+1)\sum_{|K|=n+1} \frac{1}{K!}d_K S^{K+1_i}.
\end{aligned}$$

On combining this with the first term we complete the induction step. ∎

3 Reparametrizations

If $\sigma\colon \mathbb{R}^m \to E$ is a smooth map and $\phi\colon \mathbb{R}^m \to \mathbb{R}^m$ is a diffeomorphism, then we regard $\sigma \circ \phi$ as a reparametrization of σ, by ϕ. We consider in particular reparametrizations ϕ that leave the origin of $\mathbb{R}^m$ fixed. Any such reparametrization induces a transformation Φ of $T^\infty_{(m)}E$, fibred over the identity of E, by

$$\Phi(j^\infty\sigma) = j^\infty(\sigma \circ \phi);$$

such a Φ maps $\mathcal{F}^\infty_{(m)}E$ to itself.

Note that Φ depends only on the infinite jet of ϕ at 0. If we endow the space of infinite jets of local diffeomorphisms about 0 of $\mathbb{R}^m$ leaving 0 fixed, with the multiplication operation induced from composition of the mappings, it becomes a group, called the (infinite) jet group $G^\infty_{(m)}$. Here again we regard the infinite-order object as the projective limit of finite-order ones; each finite-order jet group $G^k_{(m)}$ is a Lie group, whose Lie algebra can be identified with the vector space of kth-order polynomial vector fields on $\mathbb{R}^m$ vanishing at the origin, with the bracket operation being the usual bracket of vector fields followed by truncation at order k (see [11] section 13). We can therefore regard the vector space of formal power series vector fields on $\mathbb{R}^m$ vanishing at the origin as playing the rôle of Lie algebra of $G^\infty_{(m)}$.

The formula $\Phi(j^\infty\sigma) = j^\infty(\sigma \circ \phi)$ defines a right action of $G^\infty_{(m)}$ on $T^\infty_{(m)}E$ (and $\mathcal{F}^\infty_{(m)}E$). We seek the corresponding fundamental vector fields. It will be enough to consider the effects of 1-parameter families $\phi^I_i(t)$ of reparametrizations of the form

$$x^i \mapsto x^i + tx^I, \quad x^j \mapsto x^j \; j \neq i,$$

for some fixed index i and multi-index I. Then if $\sigma^A(x) = \sum_J (1/J!) u^A_J x^J$,

$$(\sigma \circ \phi^I_i(t))^A(x) = \sum_J \frac{1}{J!} u^A_J (x^J + tJ_i x^{I+J-1_i} + O(t^2));$$

although x^{I+J-1_i} will not be defined if $I_i = J_i = 0$, the presence of the coefficient J_i makes this unimportant (terms of this kind will occur frequently in calculations below). If we differentiate this K times and put $x = 0$ we obtain

$$\Phi^I_i(t)^A_K(j^\infty\sigma) = \begin{cases} u^A_K + t\dfrac{(K_i - I_i + 1)K!}{(K - I + 1_i)!} u^A_{K-I+1_i} + O(t^2) & \text{if } K \geq I, \\ u^A_K + O(t^2) & \text{else.} \end{cases}$$

The expression for $K \geq I$ can equivalently be written

$$\Phi^I_i(t)^A_{I+J}(j^\infty\sigma) = u^A_{I+J} + t\frac{(I+J)!}{J!} u^A_{J+1_i} + O(t^2).$$

Now differentiate with respect to t and set $t=0$, to obtain the vector field

$$\Delta_i^I = \sum_J \frac{(I+J)!}{J!} u^A_{J+1_i} \frac{\partial}{\partial u^A_{I+J}}.$$

We can regard these vector fields, for $|I|>0$, as the infinitesimal generators of the action of reparametrizations on $T^\infty_{(m)}E$. If we take $|I|=0$ we recover the total derivative operators Δ_i. Note that

$$\Delta_i^I = S^I \Delta_i.$$

A short computation leads to the following formula for the Lie derivative of S^J by Δ_i^I:

$$\mathcal{L}_{\Delta_i^I} S^J = -J_i S^{I+J-1_i}.$$

This formula will play an important role in the calculations below.

It can be shown that

$$[\Delta_i^I, \Delta_j^J] = I_j \Delta_i^{I+J-1_j} - J_i \Delta_j^{I+J-1_i},$$

including the cases in which one or both of I and J is zero. The Δ_i^I, for all I, span a Lie algebra $\mathfrak{h}$; those for which $|I|>0$ form a subalgebra $\mathfrak{h}^+$. In fact $\mathfrak{h}$ is anti-isomorphic to the Lie algebra of vector fields on $\mathbb{R}^m$ whose coefficients are formal power series in the x^i; the subalgebra $\mathfrak{h}^+$ corresponds to the algebra of such vector fields which are zero at the origin.

The particular case of the bracket formula in which $I=1_k$, $J=1_l$ gives

$$[\Delta_i^k, \Delta_j^l] = \delta_j^k \Delta_i^l - \delta_i^l \Delta_j^k,$$

writing Δ_i^k for $\Delta_i^{1_k}$; the Δ_i^j span a Lie algebra $\mathfrak{h}^1$ which is isomorphic to the corresponding algebra in the 1-jet case described in [3].

The important case of this bracket formula, from our point of view, is the one in which J is zero:

$$[\Delta_i^I, \Delta_j] = I_j \Delta_i^{I-1_j}.$$

It follows from this, for example, that for $|I|>1$

$$\mathcal{L}_{\Delta_i^I} \circ d_j = d_j \circ \mathcal{L}_{\Delta_i^I} + I_j \mathcal{L}_{\Delta_i^{I-1_j}}.$$

We shall be concerned later with calculating the Lie derivatives with respect to Δ_i^I of 1-forms of the form $d_J\alpha$ for some 1-form α; we will want to express the answer in terms of 1-forms of the same form. The formula above solves the simplest case of this problem; repeated use of it suggests the following result for the general case:

$$\mathcal{L}_{\Delta_i^I} \circ d_J = \sum_{K \leq I,J} a_K(I,J) d_{J-K} \circ \mathcal{L}_{\Delta_i^{I-K}}$$

for some coefficients $a_K(I,J)$. The values of the coefficients are given in the following Lemma, in a slightly more general context.

Lemma 3.1 *Let δ_X, for any vector field X, be a derivation of $\bigwedge T^*\left(T^\infty_{(m)}E\right)$ such that $[\mathcal{L}_X, \delta_Y] = \delta_{[X,Y]}$. Then*

$$\delta_{\Delta_i^I} \circ d_J = \sum_{K \leq I,J} \frac{I!J!}{K!(I-K)!(J-K)!} d_{J-K} \circ \delta_{\Delta_i^{I-K}}.$$

Proof by induction on the length of J. The formula is trivially true for $|J| = 0$, and follows from the bracket relation $[\Delta_i^I, \Delta_j] = I_j \Delta_i^{I-1_j}$ when $|J| = 1$. Suppose it is true for all J of a given length; then

$$\delta_{\Delta_i^I} \circ d_{J+1_j} = \sum_{K \leq I,J} \frac{I!J!}{K!(I-K)!(J-K)!} d_{J-K} \circ \delta_{\Delta_i^{I-K}} \circ d_j,$$

composing on the right with d_j. That is,

$$\delta_{\Delta_i^I} \circ d_{J+1_j} = \\ \sum_{K \leq I,J} \frac{I!J!}{K!(I-K)!(J-K)!} \left(d_{J-K+1_j} \circ \delta_{\Delta_i^{I-K}} + (I-K)_j d_{J-K} \circ \delta_{\Delta_i^{I-K-1_j}} \right).$$

The coefficient of $d_{J+1_j-K} \circ \delta_{\Delta_i^{I-K}}$ in this expression, for $K \leq I$, $K \leq J$, $K_j \neq 0$, is

$$\begin{aligned} \frac{I!J!}{K!(I-K)!(J-K)!} &+ \frac{I!J!(I_j - K_j + 1)}{(K-1_j)!(I-K+1_j)!(J-K+1_j)!} \\ &= \frac{I!J!(J_j - K_j + 1 + K_j)}{K!(I-K)!(J-K+1_j)!} \\ &= \frac{I!(J+1_j)!}{K!(I-K)!(J+1_j-K)!}. \end{aligned}$$

If $K \leq I$, $K \leq J$, $K_j = 0$ the coefficient is just $I!J!/K!(I-K)!(J-K)!$: but in this case $(J+1_j)!/(J+1_j-K)! = J!/(J-K)!$, and so

$$\frac{I!J!}{K!(I-K)!(J-K)!} = \frac{I!(J+1_j)!}{K!(I-K)!(J+1_j-K)!}.$$

We must also compare the ranges of summation, $K \leq I$, $K \leq J$ and $K \leq I$, $K \leq J + 1_j$. In the former case we have $K_i \leq \min(I_i, J_i)$, in the latter $K_i \leq \min(I_i, (J+1_j)_i)$; the ranges will be the same if $I_j \leq J_j$, whereas if $I_j > J_j$ we have in the second range of summation an additional set of multi-indices with $K_j = J_j + 1$. But when $I_j > J_j$ the term $(I-K)_j d_{J-K} \circ \delta_{\Delta_i^{I-K-1_j}}$ makes further contributions to $\delta_{\Delta_i^I} \circ d_{J+1_j}$ not so far counted, namely those with $K_j = J_j$; these contributions are

$$
\begin{aligned}
&\sum_{\substack{K \leq I, J \\ K_j = J_j}} \frac{I! J! (I_j - J_j)}{K!(I-K)!(J-K)!} d_{J-K} \circ \delta_{\Delta_i^{I-K-1_j}} \\
&\qquad = \sum_{\substack{K \leq I, J+1_j \\ K_j = J_j + 1}} \frac{I! J! (I_j - J_j)}{(K-1_j)!(I-K+1_j)!(J-K+1_j)!} d_{J-K+1_j} \circ \delta_{\Delta_i^{I-K}} \\
&\qquad = \sum_{\substack{K \leq I, J+1_j \\ K_j = J_j + 1}} \frac{I! J! (J_j+1)(I_j - J_j)}{K!(I_j - J_j)(I-K)!(J-K+1_j)!} d_{J-K+1_j} \circ \delta_{\Delta_i^{I-K}} \\
&\qquad = \sum_{\substack{K \leq I, J+1_j \\ K_j = J_j + 1}} \frac{I!(J_j+1)!}{K!(I-K)!(J+1_j-K)!} d_{J+1_j-K} \circ \delta_{\Delta_i^{I-K}}.
\end{aligned}
$$

It follows that

$$
\delta_{\Delta_i^I} \circ d_{J+1_j} = \sum_{K \leq I, J+1_j} \frac{I!(J_j+1)!}{K!(I-K)!(J+1_j-K)!} d_{J+1_j-K} \circ \delta_{\Delta_i^{I-K}},
$$

as required. ∎

4 Homogeneous Lagrangians

In the remainder of this paper we shall be concerned with variational problems in the context just described, and so we shall work, not with the geometrical objects we have specified on the bundle $T^\infty_{(m)}E$ of m-velocities, but with their restrictions to the open sub-bundle $\mathcal{F}^\infty_{(m)}E$ of m-frames.

Let L be a smooth function of globally finite order on $\mathcal{F}^\infty_{(m)}E$: that is, L is the pull-back of a function defined on some finite-order frame bundle $\mathcal{F}^n_{(m)}E$. We shall call the smallest such n the order of L; the function itself is, of course, the Lagrangian of the problem. Some of our constructions will apply to any such function, but many will be valid only for Lagrangians that are nowhere zero and are homogeneous in the following sense.

Definition 4.1 *A Lagrangian L on $\mathcal{F}^\infty_{(m)}E$ will be called* homogeneous *if*

- $\Delta^j_i(L) = \delta^j_i L$;
- $\Delta^J_i(L) = 0$ *for* $|J| \geq 2$.

When $m = 1$ these conditions for a Lagrangian to be homogeneous are called the Zermelo conditions: see [16].

The variational problem associated with L is that of finding extremals $\sigma: C \to E$ of the integral

$$\int_C (\widehat{\sigma}^* L) d^m x$$

where C is a compact connected m-dimensional submanifold of $\mathbb{R}^m$ with boundary ∂C and $d^m x$ is the canonical volume form, subject to variations that leave $\sigma(\partial C)$ fixed. If L is homogeneous then such extremals will be independent of parametrization, in that if ϕ is an orientation-preserving diffeomorphism of $\mathbb{R}^m$ then $\sigma \circ \phi: \phi^{-1}C \to E$ will be an extremal exactly when σ is. We shall now justify this claim. In fact we shall show that the homogeneity of the Lagrangian is the necessary and sufficient condition for the integral above to be invariant under any orientation-preserving diffeomorphism; that is to say, if ϕ is any diffeomorphism such that $\det \phi'(x) > 0$ for all $x \in \mathbb{R}^m$, where ϕ' is the Jacobian matrix of ϕ, then for any C,

$$\int_{\phi^{-1}C} ((\widehat{\sigma \circ \phi})^* L) d^m x = \int_C (\widehat{\sigma}^* L) d^m x.$$

This generalizes the corresponding result for the first-order case, which is Theorem 1 of Chapter 4, Section 5 of [17].

Theorem 4.2 *The necessary and sufficient condition for every integral $\int_C (\widehat{\sigma}^* L) d^m x$ to be invariant under any orientation-preserving reparametrization is that L be homogeneous.*

Proof By the change of variables formula,

$$\int_{\phi^{-1}C} ((\widehat{\sigma \circ \phi})^* L) d^m x = \int_C \phi^{-1*}(((\widehat{\sigma \circ \phi})^* L) d^m x);$$

thus the condition for the integral to be invariant, assuming that ϕ is orientation-preserving, is that

$$((\widehat{\sigma \circ \phi}) \circ \phi^{-1})^* L = (\det \phi' \circ \phi^{-1}) \widehat{\sigma}^* L.$$

So far we have not assumed that ϕ leaves the origin of $\mathbb{R}^m$ invariant. Note however that for any $x \in \mathbb{R}^m$,

$$(\widehat{\sigma \circ \phi})(\phi^{-1}(x)) = j^\infty(\sigma \circ \phi \circ \tau_{\phi^{-1}(x)}),$$

and

$$\sigma \circ \phi \circ \tau_{\phi^{-1}(x)} = (\sigma \circ \tau_x) \circ (\tau_{-x} \circ \phi \circ \tau_{\phi^{-1}(x)});$$

but $\phi_x = \tau_{-x} \circ \phi \circ \tau_{\phi^{-1}(x)}$ is a diffeomorphism which fixes the origin, and $\phi'_x(0) = \phi'(\phi^{-1}(x))$. It follows that in the condition $((\widehat{\sigma \circ \phi}) \circ \phi^{-1})^*(L) = (\det \phi' \circ \phi^{-1})\hat{\sigma}^* L$ we may take ϕ to be a diffeomorphism which fixes the origin. Thus the necessary and sufficient condition for the invariance of the integral is that

$$\Phi^* L = (\det \phi'(0)) L$$

for every diffeomorphism ϕ which leaves the origin fixed.

In particular, therefore,

$$\Phi_i^I(t)^* L = (\det \phi_i^I(t)'(0)) L,$$

from which it follows, on differentiating with respect to t and setting $t = 0$, that L must be homogeneous according to the definition above.

We will now prove the converse. Let $X = \sum_i X^i D_i$ be a complete vector field on $\mathbb{R}^m$ which vanishes at the origin, and set $X_I^i = D_I X^i|_0$; let $\phi = \phi_1$ where ϕ_t is the flow of X, so that ϕ is an orientation-preserving diffeomorphism of $\mathbb{R}^m$ fixing the origin. Let Φ be the transformation of $\mathcal{F}^\infty_{(m)}E$ generated by the diffeomorphism ϕ, and Φ_t that corresponding to ϕ_t. Define a function $L(t)$ by $L(t) = L(\Phi_t(j^\infty\sigma))$ for any point $j^\infty\sigma \in \mathcal{F}^\infty_{(m)}E$ (this point remains fixed throughout the argument). Then

$$\frac{dL}{dt}(t) = \sum_{i,I} X_I^i \Delta_i^I(L)(\Phi_t(j^\infty\sigma)) = \left(\sum_{i,j} X_j^i \delta_i^j\right) L(t) = \left(\sum_i X_i^i\right) L(t).$$

On the other hand,

$$\frac{d\phi_t}{dt} = X \circ \phi_t,$$

so that

$$\frac{d\phi'_t(0)}{dt} = X' \phi'_t(0)$$

where X' is the matrix whose components are X_j^i (and $\phi'_t(0)$ is the Jacobian matrix of ϕ_t at the origin of $\mathbb{R}^m$). But

$$\frac{d}{dt} \det \phi'_t(0) = \det \phi'_t(0) \operatorname{tr}\left(\frac{d\phi'_t(0)}{dt} \phi'_t(0)^{-1}\right) = \det \phi'_t(0) \operatorname{tr} X'.$$

It follows that

$$\frac{d}{dt}\left(\frac{L(t)}{\det \phi_t'(0)}\right) = 0,$$

and therefore that

$$\frac{\Phi^* L}{\det \phi'(0)} = L.$$

This proves the invariance of the integral under diffeomorphisms which can be obtained by exponentiating vector fields. Now for any k the (finite order) jet group $G^k_{(m)}$ is the semi-direct product of $GL(m)$ and the subgroup B_1 of $G^k_{(m)}$ consisting of the jets of diffeomorphisms with Jacobian at 0 equal to the identity; and furthermore, the exponential map is a diffeomorphism of the Lie algebra of B_1 with B_1 (see [11] subsection 13.4). So if ϕ is any diffeomorphism of $\mathbb{R}^m$ leaving the origin fixed, at every finite order there is a jet-equivalent diffeomorphism ψ which is a finite product of exponentials, and is such that $\psi'(0) = \phi'(0)$. For such ψ we have $\Psi^* L = (\det \psi'(0))L$. But since L is of globally finite order, this means that $\Phi^* L = (\det \phi'(0))L$. ∎

5 Hilbert forms, the Hilbert coframe and the Hilbert-Carathéodory form

Our first main definition in this section will be of a singular coframe associated with any Lagrangian function.

Definition 5.1 *The* Hilbert coframe *of L is the m-coframe $(\vartheta^1, \ldots, \vartheta^m)$ on $\mathcal{F}^\infty_{(m)}E$ given by*

$$\vartheta^i = \sum_I \frac{(-1)^{|I|}}{(|I|+1)I!} d_I(S^{I+1_i} dL);$$

the individual 1-forms ϑ^i will be called Hilbert forms.

We see from this definition that the Hilbert forms of L are of globally finite order. If the order of L (and hence of dL) is n then the form $S^{I+1_i} dL$ vanishes whenever $|I + 1_i| > n$, and the Lie derivative d_I will increase the order by no more than $|I|$, so the order of ϑ^i will be no more than $2n - 1$.

The main reason for our definition of ϑ^i is that the difference between dL and the total divergence of the Hilbert coframe is semi-basic over the manifold E and, as we shall see later, incorporates the Euler-Lagrange equations for L.

Theorem 5.2 *The 1-form* $\varepsilon = dL - \sum_{i=1}^m d_i\vartheta^i$ *satisfies* $S^j(\varepsilon) = 0$ *for* $j = 1, \ldots, m$ *and is hence semi-basic over* E.

Proof From the definition of ϑ^i we have

$$\begin{aligned}\sum_{i=1}^m d_i\vartheta^i &= \sum_{i=1}^m \sum_I \frac{(-1)^{|I|}}{(|I|+1)I!} d_{I+1_i}(S^{I+1_i}dL) \\ &= \sum_{|J|>0} \frac{(-1)^{|J|-1}}{J!} d_J(S^J dL)\end{aligned}$$

so that

$$\varepsilon = \sum_J \frac{(-1)^{|J|}}{J!} d_J(S^J dL);$$

the result then follows immediately from Lemma 2.1. ∎

We shall call the 1-form ε the Euler-Lagrange form of L, and the formula $\varepsilon = dL - \sum_{i=1}^m d_i\vartheta^i$ the integration by parts formula. The reason for the terminology is this. Any vector field X on E defined in a neighbourhood of the image of σ generates a variation of σ; the corresponding rate of variation of the integral $\int_C (\widehat{\sigma}^* L) d^m x$ is $\int_C \widehat{\sigma}^*(\mathcal{L}_{\widehat{X}} L) d^m x = \int_C \widehat{\sigma}^* \langle \widehat{X}, dL\rangle d^m x$, and by Theorem 5.2

$$\langle \widehat{X}, dL\rangle = \langle \widehat{X}, \varepsilon\rangle + \sum_{i=1}^m \langle \widehat{X}, d_i\vartheta^i\rangle.$$

But prolongations commute with total derivative operators, so $\sum_{i=1}^m \langle \widehat{X}, d_i\vartheta^i\rangle = \sum_{i=1}^m d_i\langle \widehat{X}, \vartheta^i\rangle$. This leads to the fundamental variation formula

$$\int_C \widehat{\sigma}^* \langle \widehat{X}, dL\rangle d^m x = \int_C \widehat{\sigma}^* \langle X, \varepsilon\rangle d^m x + \sum_{i=1}^m \int_C \frac{\partial \varphi^i}{\partial x^i} d^m x,$$

where $\varphi^i = \widehat{\sigma}^* \langle \widehat{X}, \vartheta^i\rangle$, and the hat has been dropped from X in the first term because ε is semi-basic. This formula is the generalization of the familiar variation formula in first-order, single-integral variational problems, obtained by integration by parts. The fact that the Euler-Lagrange form is uniquely determined by the conditions that ε is semi-basic and that $dL - \varepsilon$ is of the form $\sum d_i\vartheta^i$ for some ϑ^i is shown in [21]; what is of importance here is the specific choice of the ϑ^i, especially when L is homogeneous.

We consider only homogeneous Lagrangian functions from now on. In this case we have several additional properties. The first is that the module spanned by the Hilbert forms is closed under the action of the Lie algebra $\mathfrak{h}^1$.

Proposition 5.3 *If L is homogeneous then, for any i, j and k,*

$$\mathcal{L}_{\Delta^j_k}\vartheta^i = \delta^j_k\vartheta^i - \delta^i_k\vartheta^j.$$

Proof We recall from Lemma 3.1 that

$$[\mathcal{L}_{\Delta^j_k}, d_I] = I_j d_{I+1_k-1_j}.$$

It follows that

$$\mathcal{L}_{\Delta^j_k}\vartheta^i = \sum_I \frac{(-1)^{|I|}}{(|I|+1)I!}\left(d_I(\mathcal{L}_{\Delta^j_k}(S^{I+1_i}dL)) + I_j d_{I+1_k-1_j}(S^{I+1_i}dL)\right).$$

Now

$$\begin{aligned}\mathcal{L}_{\Delta^j_k}(S^{I+1_i}dL) &= -(I+1_i)_k S^{I-1_k+1_i+1_j}dL + S^{I+1_i}d(\Delta^j_k L)\\ &= -(I_k+\delta^i_k)S^{I-1_k+1_i+1_j}dL + \delta^j_k S^{I+1_i}dL;\end{aligned}$$

the first term makes sense because the coefficient is zero unless either $k = i$ or $I_k \neq 0$ (or both). Thus

$$\begin{aligned}&\mathcal{L}_{\Delta^j_k}\vartheta^i\\ &= \delta^j_k\vartheta^i + \sum_I \frac{(-1)^{|I|}}{(|I|+1)I!}\left(-(I_k+\delta^i_k)d_I(S^{I-1_k+1_i+1_j}dL) + I_j d_{I+1_k-1_j}(S^{I+1_i}dL)\right).\end{aligned}$$

Now consider the second term in the sum, that is,

$$\sum_I \frac{(-1)^{|I|}I_j}{(|I|+1)I!}d_{I+1_k-1_j}(S^{I+1_i}dL).$$

We can rewrite it, first by replacing I by $I+1_j$ to obtain

$$\sum_I \frac{(-1)^{|I|+1}}{(|I|+2)I!}d_{I+1_k}(S^{I+1_i+1_j}dL),$$

and then by replacing I by $I-1_k$ in this to obtain

$$\sum_I \frac{(-1)^{|I|}I_k}{(|I|+1)I!}d_I(S^{I-1_k+1_i+1_j}dL)$$

(in the first step we use the fact that $(I_j+1)/(I+1_j)! = 1/I!$; in the second, similarly, the fact that $1/(I-1_k)! = I_k/I!$). This gives

$$\mathcal{L}_{\Delta^j_k}\vartheta^i = \delta^j_k\vartheta^i - \delta^i_k\sum_I \frac{(-1)^{|I|}}{(|I|+1)I!}d_I(S^{I-1_k+1_i+1_j}dL).$$

But if $i = k$ the sum is ϑ^j, while if $i \neq k$ it doesn't matter what it is: so we have

$$\mathcal{L}_{\Delta^j_k} \vartheta^i = \delta^j_k \vartheta^i - \delta^i_k \vartheta^j$$

as claimed. ∎

We may use this property to establish a uniqueness result.

Proposition 5.4 *Suppose that, for a homogeneous Lagrangian L, we consider 1-forms ϑ^i of the form*

$$\vartheta^i = \sum_I \frac{(-1)^{|I|}}{I!} \alpha^i_I d_I (S^{I+1_i} dL)$$

for some constant coefficients α^i_I. Then there is a unique choice of α^i_I such that

$$dL - \sum_i d_i \vartheta^i = \sum_J \frac{(-1)^{|J|}}{J!} d_J (S^J dL) = \varepsilon$$

and

$$\mathcal{L}_{\Delta^j_k} \vartheta^i = \delta^j_k \vartheta^i - \delta^i_k \vartheta^j;$$

this choice is

$$\alpha^i_I = \frac{1}{(|I| + 1)} \quad \textit{(for all } i\textit{).}$$

Proof We have

$$\sum_i d_i \vartheta^i = \sum_i \sum_I \frac{(-1)^{|I|}}{I!} \alpha^i_I d_{I+1_i} (S^{I+1_i} dL);$$

for this to equal $dL - \varepsilon$ we must have, for each $J \neq 0$,

$$\sum_{\substack{I,i \\ I+1_i = J}} \frac{\alpha^i_I}{I!} = \frac{1}{J!},$$

or equivalently

$$\sum_{\substack{I,i \\ I+1_i = J}} J_i \alpha^i_I = 1.$$

Now by performing the Lie derivative calculation as in in the proof of Proposition 5.3 we have

$$\begin{aligned} &\mathcal{L}_{\Delta^j_k} \vartheta^i - (\delta^j_k \vartheta^i - \delta^i_k \vartheta^j) \\ &\quad = - \sum_I \frac{(-1)^{|I|}}{I!} \alpha^i_I \left(I_k d_I (S^{I-1_k+1_i+1_j} dL) - I_j d_{I+1_k-1_j} (S^{I+1_i} dL) \right). \end{aligned}$$

We may replace I by $I-1_k$ in the first part of the sum, and by $I-1_j$ in the second; then the condition that $\mathcal{L}_{\Delta^j_k}\vartheta^i = \delta^j_k\theta^i - \delta^i_k\theta^j$ becomes

$$\sum_I (\alpha^i_{I+1_j} - \alpha^i_{I+1_k})\frac{(-1)^{|I|}}{I!}d_{I+1_k}(S^{I+1_i+1_j}dL) = 0.$$

A sufficient condition for this equality to hold is that $\alpha^i_{I+1_j} = \alpha^i_{I+1_k}$ for all I; in fact this condition is also necessary. To see this, we argue by induction on the length of I, using the fact that the coefficients α^i_I are constants. So suppose the condition holds whenever $|I| < n$. Let L be a homogeneous Lagrangian of global order $n+2$; then the only non-zero terms in the sum are of the form

$$(\alpha^i_{I+1_j} - \alpha^i_{I+1_k})\frac{(-1)^{|I|}}{I!}d_{I+1_k}\left(\frac{\partial L}{\partial u^A_{I+1_i+1_j}}du^A\right)$$

with $|I| = n$. The coefficient of $du^A_{I+1_k}$ in this expression is

$$\frac{(-1)^{|I|}}{I!}(\alpha^i_{I+1_j} - \alpha^i_{I+1_k})\frac{\partial L}{\partial u^A_{I+1_i+1_j}},$$

and this must vanish. But for any A, I, i and j, and any point $u \in E$, we can find a point in the fibre over u and a homogeneous Lagrangian L such that $\partial L/\partial u^A_{I+1_i+1_j}$ is not zero at that point; thus we must have $\alpha^i_{I+1_j} = \alpha^i_{I+1_k}$. A similar argument serves in the case $I = 0$.

A consequence of this condition on the coefficients is that, for each fixed i, α^i_I is the same for all I of the same length: any such I may be transformed into any other such I by a succession of operations in which 1 is subtracted from one non-zero component and 1 is added to another component. So α^i_I may be replaced by α^i_n where $n = |I|$. Then α^i_n must satisfy the condition

$$\sum_i J_i\alpha^i_n = 1$$

for any J with $|J| = n+1$. In particular, we can take J to have zeros everywhere except in the ith position; for this J, $J_i = |J| = n+1$, and the sum reduces to a single term, so that

$$(n+1)\alpha^i_n = 1,$$

as required. ∎

Another property of the Hilbert forms for a homogeneous Lagrangian will be useful later on.

Lemma 5.5 *If L is homogeneous then $\langle \Delta_j, \vartheta^i \rangle = \delta^i_j L$.*

Proof Since $[\Delta_i, \Delta_j] = 0$ we have, for any form α,

$$\Delta_j \lrcorner\, d_i \alpha = d_i(\Delta_j \lrcorner\, \alpha),$$

and so more generally $\Delta_j \lrcorner\, (d_I \alpha) = d_I(\Delta_j \lrcorner\, \alpha)$. Thus for any $I \neq 0$,

$$\langle \Delta_j, d_I(S^{I+1_i} dL) \rangle = d_I \langle \Delta_j, S^{I+1_i} dL \rangle = d_I \langle S^{I+1_i} \Delta_j, dL \rangle = d_I \langle \Delta_j^{I+1_i}, dL \rangle = 0,$$

while

$$\langle \Delta_j, S^i dL) \rangle = \langle \Delta^i_j, dL \rangle = \delta^i_j L.$$

Thus

$$\langle \Delta_j, \vartheta^i \rangle = \sum_I \frac{(-1)^{|I|}}{(|I|+1)I!} \langle \Delta_j, d_I(S^{I+1_i} dL) \rangle = \delta^i_j L,$$

as asserted. ∎

Corollary 5.6 *For any map $\sigma \colon \mathbb{R}^m \to E$, $\hat{\sigma}^* \vartheta^i = (\hat{\sigma}^* L) dx^i$.* ∎

Lemma 5.7 *The Euler-Lagrange form ε associated with a homogeneous Lagrangian L is a contact form.*

Proof We have to show that for any map $\sigma \colon \mathbb{R}^m \to E$, $\hat{\sigma}^* \varepsilon = 0$. But from the definition of the operators d_i,

$$\hat{\sigma}^* \left(\sum_i d_i \vartheta^i \right) = \sum_i \mathcal{L}_{\partial/\partial x^i}((\hat{\sigma}^* L) dx^i) = \sum_i \frac{\partial(\hat{\sigma}^* L)}{\partial x^i} dx^i = \hat{\sigma}^* dL.$$

∎

Lemma 5.5 has another useful corollary.

Corollary 5.8 *For any homogeneous function L on $\mathcal{F}^\infty_{(m)} E$, $mL = \sum_{i=1}^m \langle \Delta_i, \vartheta^i \rangle$, and so*

$$m\, dL = \sum_{i=1}^m d\langle \Delta_i, \vartheta^i \rangle = \sum_{i=1}^m d_i \vartheta^i - \sum_{i=1}^m \Delta_i \lrcorner\, d\vartheta^i.$$

∎

We shall take advantage of this property of the Hilbert forms in the particular case when the homogeneous function L never vanishes.

Definition 5.9 *If L is homogeneous and never vanishes then the* Hilbert-Carathéodory form *of L is the m-form Θ on $\mathcal{F}^\infty_{(m)}E$ given by*

$$\Theta = L^{-(m-1)} \bigwedge_{i=1}^{m} \vartheta^i .$$

Proposition 5.10 *The exterior derivative of the Hilbert-Carathéodory form satisfies*

$$L\, d\Theta = \varepsilon \wedge \Theta + \sum_{i=1}^{m} \Delta_i \lrcorner (d\vartheta^i \wedge \Theta).$$

Proof Note first that, from Lemma 5.5, we have

$$\Delta_i \lrcorner\, \Theta = L^{-(m-2)} \bigwedge_{j \neq i} \vartheta^j .$$

Consequently

$$\begin{aligned}
L\, d\Theta &= -(m-1)L^{-(m-1)} dL \wedge \bigwedge_i \vartheta^i + L^{-(m-2)} \sum_i \left(d\vartheta^i \wedge \bigwedge_{j\neq i} \vartheta^j \right) \\
&= dL \wedge \Theta - \left(\sum_i d_i \vartheta^i - \sum_i \Delta_i \lrcorner\, d\vartheta^i \right) \wedge \Theta + \sum_i d\vartheta^i \wedge (\Delta_i \lrcorner\, \Theta) \\
&= \left(dL - \sum_i d_i \vartheta^i \right) \wedge \Theta + \sum_i \Delta_i \lrcorner (d\vartheta^i \wedge \Theta) \\
&= \varepsilon \wedge \Theta + \sum_i \Delta_i \lrcorner (d\vartheta^i \wedge \Theta),
\end{aligned}$$

as asserted. ∎

We can now see that the Hilbert-Carathéodory form Θ is an m-form on the infinite frame bundle having the same extremals as L; thus by using the Hilbert-Carathéodory form we can formulate the variational problem for a homogeneous Lagrangian in terms of a geometrical object that lives on $\mathcal{F}^\infty_{(m)}E$, rather than in terms of a mixture of objects from $\mathcal{F}^\infty_{(m)}E$ and $\mathbb{R}^m$.

Theorem 5.11 *If C is a compact connected m-dimensional submanifold of $\mathbb{R}^m$ and $\sigma\colon C \to E$ then $\widehat{\sigma}^*\Theta = (\widehat{\sigma}^* L) d^m x$; furthermore σ satisfies*

$$\int_C \widehat{\sigma}^* (\mathcal{L}_{\widehat{X}} L) d^m x = 0$$

for every vector field X on E with $X_{\sigma(\partial C)} = 0$ exactly when it satisfies

$$\int_C \widehat{\sigma}^*(\mathcal{L}_Y \Theta) = 0$$

for every vector field Y on $\mathcal{F}^\infty_{(m)}E$ which is projectable to E, such that $Y_{\widehat{\sigma}(\partial C)} = 0$.

Proof First, from Lemma 5.7 we have $\widehat{\sigma}^*\Theta = (\widehat{\sigma}^* L) d^m x$. Now consider the variational problem for Θ. Its extremals satisfy $\int_C \widehat{\sigma}^*(\mathcal{L}_{\widehat{X}}\Theta) = 0$. From Proposition 5.10 we have

$$\mathcal{L}_{\widehat{X}}\Theta = L^{-1}\left(\langle \widehat{X}, \varepsilon \rangle \Theta - \varepsilon \wedge (\widehat{X} \lrcorner \Theta) - \sum_i \Delta_i \lrcorner (\widehat{X} \lrcorner (d\vartheta^i \wedge \Theta)) \right) + d(\widehat{X} \lrcorner \Theta).$$

Now ε is contact; moreover, $\Delta_i \lrcorner (\widehat{X} \lrcorner (d\vartheta^i \wedge \Theta))$ is contact for each i because Δ_i is $\widehat{\sigma}$-related to $\partial/\partial x^i$ and $\widehat{X} \lrcorner (d\vartheta^i \wedge \Theta)$ is an $(m+1)$-form. Thus

$$\widehat{\sigma}^*(\mathcal{L}_{\widehat{X}}\Theta) = \widehat{\sigma}^* \langle X, \varepsilon \rangle d^m x + d(\widehat{\sigma}^*(\widehat{X} \lrcorner \Theta)).$$

So Θ has the same extremals as L; and the variation integral for Θ, for variations leaving the boundary of C fixed, depends only on the value of X on the image of σ, so that we may replace the prolongation $\widehat{X}$ by an arbitrary projectable vector field Y. ∎

6 The invariance of the Hilbert-Carathéodory form

In this section we retain the assumption that the Lagrangian function L is homogeneous. We show first that the Hilbert forms vanish when contracted with elements of $\mathfrak{h}^+$, the Lie algebra of infinitesimal generators of reparametrizations.

Proposition 6.1 *For any vector field Δ^J_j with $|J| > 0$,*

$$\langle \Delta^J_j, \vartheta^i \rangle = 0.$$

Proof We observe first that

$$\begin{aligned} \langle \Delta^J_j, \vartheta^i \rangle &= \sum_I \frac{(-1)^{|I|}}{|I + 1_i| I!} \langle \Delta^J_j, d_I(S^{I+1_i} dL) \rangle \\ &= \sum_I \frac{(-1)^{|I|}}{|I + 1_i| I!} \sum_{K \leq I, J} \frac{I! J!}{K!(I-K)!(J-K)!} d_{I-K} \langle \Delta^{J-K}_j, S^{I+1_i} dL \rangle \end{aligned}$$

using Lemma 3.1. But from the relationship $\Delta_j^J = S^J\Delta_j$ we have

$$\langle \Delta_j^{J-K}, S^{I+1_i}dL\rangle = \langle \Delta_j^{I+J-K+1_i}, dL\rangle = \Delta_j^{I+J-K+1_i}(L),$$

and this vanishes whenever $|I+J-K+1_i| \geq 2$, so it must always vanish when $K \leq I$ and $|J| > 0$. ∎

It follows from this that the Hilbert-Carathéodory form Θ also vanishes when contracted with a vector field in $\mathfrak{h}^+$.

To determine the projectability of Θ to the sphere bundle we must now examine its invariance under the Lie derivative action of these vector fields. We have seen in Proposition 5.3 that

$$\mathcal{L}_{\Delta_k^j}\vartheta^i = \delta_k^j\vartheta^i - \delta_k^i\vartheta^j,$$

and we may use this to show that Θ is invariant under the Lie derivative action of the subalgebra $\mathfrak{h}^1$.

Proposition 6.2 *For any vector field Δ_k^j, $\mathcal{L}_{\Delta_k^j}\Theta = 0$.*

Proof This follows immediately from the definition $\Theta = L^{-(m-1)}\bigwedge\vartheta^i$, Proposition 5.3, and the fact that $\Delta_k^j(L) = \delta_k^j L$. ∎

We now wish to consider whether $\mathcal{L}_{\Delta_i^I}\Theta = 0$ for $|I| > 1$. By using Lemma 3.1 again we have

$$\begin{aligned}
\mathcal{L}_{\Delta_i^I}\vartheta^j &= \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!}\left(\sum_{K\leq I,J}\frac{I!J!}{K!(I-K)!(J-K)!}d_{J-K}\left(\mathcal{L}_{\Delta_i^{I-K}}(S^{J+1_j}dL)\right)\right) \\
&= \sum_J\sum_{K\leq I,J}\frac{(-1)^{|J|}I!}{(|J|+1)K!(I-K)!(J-K)!}d_{J-K}\left(\mathcal{L}_{\Delta_i^{I-K}}(S^{J+1_j}dL)\right).
\end{aligned}$$

Now reverse the order of summation and replace J by $J+K$ to obtain

$$\mathcal{L}_{\Delta_i^I}\vartheta^j = \sum_{K\leq I}\sum_J \frac{(-1)^{|J|+|K|}I!}{(|J|+|K|+1)J!K!(I-K)!}d_J\left(\mathcal{L}_{\Delta_i^{I-K}}(S^{J+K+1_j}dL)\right).$$

The term with $K = I$ needs to be treated separately from the rest in the sum, so we write

$$\mathcal{L}_{\Delta_i^I}\theta_j \;=\; \sum_{K<I}\sum_J \frac{(-1)^{|J|+|K|}I!}{(|J|+|K|+1)J!K!(I-K)!}d_J\left(\mathcal{L}_{\Delta_i^{I-K}}(S^{J+K+1_j}dL)\right)$$

$$
\begin{aligned}
&+\sum_J \frac{(-1)^{|I|+|J|}}{(|I|+|J|+1)J!} d_{J+1_i}(S^{I+J+1_j}dL)\\
= \quad & -\sum_{K<I}\sum_J \frac{(-1)^{|J|+|K|}I!(J+K+1_j)_i}{(|J|+|K|+1)J!K!(I-K)!} d_J(S^{I+J-1_i+1_j}dL)\\
&+\sum_{K<I}\sum_J \frac{(-1)^{|J|+|K|}I!}{(|J|+|K|+1)J!K!(I-K)!} d_J(S^{J+K+1_j}d(\Delta_i^{I-K}L))\\
&-\sum_J \frac{(-1)^{|I|+|J|}J_i}{(|I|+|J|)J!} d_J(S^{I+J-1_i+1_j}dL).
\end{aligned}
$$

We have replaced J by $J-1_i$ in the last term. In the first term, we have to be sure that the sum doesn't contain any terms for which $I+J-1_i+1_j$ doesn't make sense. The only question is what happens if $i \neq j$ and $I_i = 0$; but then since $K < I$ in the sum, $K_i = 0$, whence $(J+K+1_j)_i = J_i$, so we have a non-zero coefficient only if $J_i \neq 0$ and all is well. The middle term is non-zero only if $I_i \neq 0$, and then only the term with $K = I - 1_i$ remains; for this term $I!/K!(I-K)! = I_i$, so we are left with

$$
\begin{aligned}
\mathcal{L}_{\Delta_i^I}\vartheta^j \quad = \quad & -\sum_J \frac{(-1)^{|J|}}{J!}\left(\sum_{K<I}\frac{(-1)^{|K|}(J+K+1_j)_i I!}{(|J|+|K|+1)K!(I-K)!}\right) d_J(S^{I+J-1_i+1_j}dL)\\
&-(-1)^{|I|}\sum_J \frac{(-1)^{|J|}(I_i+J_i)}{J!(|I|+|J|)} d_J(S^{I+J-1_i+1_j}dL).
\end{aligned}
$$

The next task is to evaluate the sum

$$
\sum_{K<I}\frac{(-1)^{|K|}(J+K+1_j)_i I!}{(|J|+|K|+1)K!(I-K)!}.
$$

We shall in fact evaluate

$$
\sum_{K\le I}\frac{(-1)^{|K|}(J+K+1_j)_i I!}{(|J|+|K|+1)K!(I-K)!};
$$

note that

$$
\begin{aligned}
\sum_{K<I}\frac{(-1)^{|K|}(J+K+1_j)_i I!}{(|J|+|K|+1)K!(I-K)!} \quad = \quad & \sum_{K\le I}\frac{(-1)^{|K|}(J+K+1_j)_i I!}{(|J|+|K|+1)K!(I-K)!}\\
&-\frac{(-1)^{|I|}(I_i+J_i+\delta_i^j)}{|I|+|J|+1}.
\end{aligned}
$$

By the multi-index binomial theorem

$$
\sum_{K\le I}\frac{I!}{K!(I-K)!}x^{J+K+1_j} = x^{J+1_j}(x+1)^I,
$$

where $x = (x^1, x^2, \ldots, x^n)$ and 1 here stands for $(1, 1, \ldots, 1)$. Partially differentate this with respect to x^i to obtain

$$\sum_{K \leq I} \frac{(J + K + 1_j)_i I!}{K!(I - K)!} x^{J+K-1_i+1_j} = (J + 1_j)_i x^{J-1_i+1_j}(x + 1)^I + I_i x^{J+1_j}(x + 1)^{I-1_i}.$$

Diagonalize, that is, set $x^i = z$ for $i = 1, 2, \ldots, n$, to obtain

$$\sum_{K \leq I} \frac{(J + K + 1_j)_i I!}{K!(I - K)!} z^{|J|+|K|} = (J + 1_j)_i z^{|J|}(z + 1)^{|I|} + I_i z^{|J|+1}(z + 1)^{|I|-1}$$

(where 1 has now reverted to its usual meaning). Now integrate from 0 to -1:

$$\begin{aligned} &\sum_{K \leq I} \frac{(-1)^{|J|+|K|+1}(J + K + 1_j)_i I!}{(|J| + |K| + 1)K!(I - K)!} \\ &= \frac{(-1)^{|J|+1}(J_i + \delta_i^j)|I|!|J|!}{(|I| + |J| + 1)!} + \frac{(-1)^{|J|} I_i(|I| - 1)!(|J| + 1)!}{(|I| + |J| + 1)!}, \end{aligned}$$

using the fact that

$$\int_0^{-1} z^r (z + 1)^s dz = (-1)^{r+1} \frac{r!s!}{(r + s + 1)!},$$

a formula obtained by repeated integration by parts. Thus

$$\begin{aligned} &\sum_{K<I} \frac{(-1)^{|J|+|K|+1}(J + K + 1_j)_i I!}{(|J| + |K| + 1)K!(I - K)!} \\ &= \frac{(J_i + \delta_i^j)|I|!|J|! - I_i(|I| - 1)!(|J| + 1)!}{(|I| + |J| + 1)!} - \frac{(-1)^{|I|}(I_i + J_i + \delta_i^j)}{|I| + |J| + 1} \\ &= \frac{J_i|I|!|J|! - I_i(|I| - 1)!(|J| + 1)!}{(|I| + |J| + 1)!} - \frac{(-1)^{|I|}(I_i + J_i)}{|I| + |J| + 1} \\ &\quad + \delta_i^j \left(\frac{|I|!|J|!}{(|I| + |J| + 1)!} - \frac{(-1)^{|I|}}{|I| + |J| + 1} \right). \end{aligned}$$

So finally

$$\mathcal{L}_{\Delta_i^I} \vartheta^j = -\sum_J \frac{(-1)^{|J|}}{J!} C_J d_J(S^{I+J-1_i+1_j} dL)$$

where the coefficient C_J is given by

$$\begin{aligned} C_J &= J_i \left(\frac{|I|!|J|! + (-1)^{|I|}(|I| + |J| - 1)!}{(|I| + |J| + 1)!} \right) \\ &\quad - I_i \left(\frac{(|I| - 1)!(|J| + 1)! - (-1)^{|I|}(|I| + |J| - 1)!}{(|I| + |J| + 1)!} \right) \\ &\quad + \delta_i^j \left(\frac{|I|!|J|! - (-1)^{|I|}(|I| + |J|)!}{(|I| + |J| + 1)!} \right). \end{aligned}$$

We remark that, in the case $|I| = 1$, the term multiplying J_i is zero, and in fact

$$C_J = (-I_i + \delta_i^j)\frac{(|J|! + (|J|+1)!)}{(|J|+2)!};$$

but

$$\frac{(|J|! + (|J|+1)!)}{(|J|+2)!} = \frac{|J|!(|J|+2)}{(|J|+2)!} = \frac{1}{|J|+1},$$

from which one can again obtain the formula

$$\mathcal{L}_{\Delta_i^k}\vartheta^j = \delta_i^k\vartheta^j - \delta_i^j\vartheta^k.$$

The new results from our calculation are these.

Theorem 6.3 *If L is a second-order homogeneous Lagrangian function then the Hilbert-Carathéodory form Θ is invariant under the action of the reparametrization algebra, and so is projectable to the sphere bundle.*

Proof We know that Θ is invariant under $\mathfrak{h}^1$, so we need consider only what happens when $|I| = 2$. The only contribution to $\mathcal{L}_{\Delta_i^I}\vartheta^j$ comes from $J = 0$, and it is easy to see that when $J = 0$ and $|I|$ is even, $C_0 = 0$. ∎

Theorem 6.4 *If $m = 1$ and L is a homogeneous Lagrangian function of arbitrary globally finite order then the Hilbert-Carathéodory form Θ (which in this case is the same as the single Hilbert form ϑ) is invariant under the action of the reparametrization algebra, and so is projectable to the sphere bundle.*

Proof Putting $r = |I|$ and $s = |J|$ we have, in this case,

$$\begin{aligned} C_s &= s\left(\frac{r!s! + (-1)^r(r+s-1)!}{(r+s+1)!}\right) \\ &\quad - r\left(\frac{(r-1)!(s+1)! - (-1)^r(r+s-1)!}{(r+s+1)!}\right) \\ &\quad + \left(\frac{r!s! - (-1)^r(r+s)!}{(r+s+1)!}\right) \\ &= 0. \end{aligned}$$

∎

Finally in this section we shall see how a homogeneous variational problem on a frame bundle may be derived from an inhomogeneous problem on a sphere bundle (and, therefore, from a variational problem on an affine jet bundle). We shall denote the sphere bundle by $\mathcal{S}^\infty_{(m)}E$ and the projection $\mathcal{F}^\infty_{(m)}E \to \mathcal{S}^\infty_{(m)}E$ by ρ; given a Lagrangian m-form λ of globally finite order on $\mathcal{S}^\infty_{(m)}E$ we may construct a function L on $\mathcal{F}^\infty_{(m)}E$ by taking the contraction $\langle \mathbf{\Delta}, \rho^*\lambda\rangle$ where $\mathbf{\Delta} = (\Delta_1, \Delta_2, \ldots, \Delta_m)$ is the tautological m-frame field on $\mathcal{F}^\infty_{(m)}E$. (A Lagrangian form is of order r if it is the pull-back of a form defined on $\mathcal{S}^r_{(m)}E$ and semi-basic over $\mathcal{S}^{r-1}_{(m)}E$.)

A Lagrangian function on $\mathcal{F}^\infty_{(m)}E$ constructed in this way will automatically be homogeneous. For

$$\Delta^I_i(L) = \left(\sum_{j=1}^m \Delta_m \lrcorner \ldots \lrcorner [\Delta^I_i, \Delta_j] \lrcorner \ldots \lrcorner \Delta_1 \lrcorner \rho^*\lambda\right) + \langle \mathbf{\Delta}, \mathcal{L}_{\Delta^I_i}(\rho^*\lambda)\rangle;$$

but $\mathcal{L}_{\Delta^I_i}(\rho^*\lambda) = 0$, and from $[\Delta^I_i, \Delta_j] = I_j \Delta^{I-1_j}_i$ and the fact that $I_j\Delta^{I-1_j}_i \lrcorner \rho^*\lambda = 0$ unless $I = 1_j$ we obtain

$$\Delta^j_i(L) = \delta^j_i L, \qquad \Delta^I_i(L) = 0 \text{ for } |I| \geq 2$$

as required. We can therefore obtain the following results.

Theorem 6.5 *If λ is a Lagrangian m-form of order 2 on $\mathcal{S}^\infty_{(m)}E$ then the Hilbert-Carathéodory form Θ for the Lagrangian function $L = \langle \mathbf{\Delta}, \rho^*\lambda\rangle$ is projectable to an m-form on $\mathcal{S}^\infty_{(m)}E$.*

Proof As λ is the pull-back of a form defined on $\mathcal{S}^2_{(m)}E$ and semi-basic over $\mathcal{S}^1_{(m)}E$, the contraction $L = \langle \mathbf{\Delta}, \rho^*\lambda\rangle$ is the pull-back of a function defined on $\mathcal{F}^2_{(m)}E$. ∎

Theorem 6.6 *If λ is a Lagrangian 1-form of globally finite order on $\mathcal{S}^\infty_{(1)}E$ then the Hilbert form ϑ for the Lagrangian function $L = \langle \Delta, \rho^*\lambda\rangle$ is projectable to a 1-form on $\mathcal{S}^\infty_{(1)}E$.* ∎

Now suppose that E has a fibration π over some orientable m-dimensional manifold M with volume form ω. Consider a second-order Lagrangian form $\lambda = \widetilde{L}\omega$ on $J^\infty\pi$; if we think of this as being defined on an open submanifold of $\mathcal{S}^\infty_{(m)}E$ then the corresponding Lagrangian function L is defined on an open submanifold of $\mathcal{F}^\infty_{(m)}E$ and is clearly second-order. (The most general second-order Lagrangian form λ differs from $\widetilde{L}\omega$ by a contact form, and so this difference vanishes on contraction with $\mathbf{\Delta}$.) We may therefore construct the Hilbert-Carathéodory form Θ and its projection $\widetilde{\Theta}$ on $J^\infty\pi$; we call this projection the Carathéodory form of $\widetilde{L}\omega$. It is a globally-defined third-order m-form, and from the properties of the Hilbert-Carathéodory form it is a Lepage equivalent of λ. If $m = 1$ then we can do this for a Lagrangian form of arbitrary (but globally-finite) order, obtaining the Poincaré-Cartan form.

7 Coordinate formulæ

In this section we shall examine the coordinate representation of the Hilbert-Carathéodory form and the Euler-Lagrange form, using the coordinates u^A_I on $\mathcal{F}^\infty_{(m)}E$ derived from the coordinates u^A on E. For any set $j_1, \ldots, j_r$ of indices in the range $1 \leq j_k \leq m$, we shall use the notation $\#(j_1 \ldots j_r)$ to denote the number of distinct sequences of length r that may be obtained from the numbers $j_1, \ldots, j_r$, so that if we define the multi-index J by $J = 1_{j_1} + \ldots + 1_{j_r}$ then

$$\#(j_1 \ldots j_r) = \frac{|J|!}{J!}.$$

Proposition 7.1 *The coordinate representation of* Θ *is*

$$\frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \sum_J \sum_K (-1)^{|J|} \frac{(J+K+1_i)!|J|!|K|!}{(|J|+|K|+1)!J!K!} d_J \left(\frac{\partial L}{\partial u^A_{J+K+1_i}} \right) du^A_K$$

in multi-index notation, and

$$\frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \left(\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r}{\#(j_1 \ldots j_r)} d_{k_1} \cdots d_{k_s} \left(\frac{\partial L}{\partial u^A_{i j_1 \ldots j_r k_1 \ldots k_s}} \right) du^A_{j_1 \ldots j_r} \right)$$

(using the summation convention) in ordinary index notation.

Proof We shall establish a suitable formula for the Hilbert forms ϑ^i, from which the first formula for Θ will follow. The second formula can then be derived by observing that the sums over the multi-indices J and K correspond to the implicit sums over the individual indices $j_1, \ldots, j_r$ and $k_1, \ldots, k_s$ when we incorporate the numerical factors $|J|!/J!$ and $|K|!/K!$.

Now

$$S^I dL = \sum_G \frac{(I+G)!}{G!} \left(\frac{\partial L}{\partial u^A_{I+G}} \right) du^A_G,$$

so that

$$\begin{aligned}
&\sum_I \frac{(-1)^{|I|}}{(|I|+1)I!} d_I \left(S^{I+1_i} dL \right) \\
&= \sum_I \frac{(-1)^{|I|}}{(|I|+1)!} d_I \left(\sum_G \frac{(I+G+1_i)!}{G!} \left(\frac{\partial L}{\partial u^A_{I+G+1_i}} \right) du^A_G \right) \\
&= \sum_{I,G} \frac{(-1)^{|I|}(I+G+1_i)!}{(|I|+1)I!G!} \left(\sum_{H\le I} \frac{I!}{H!(I-H)!} d_{I-H} \left(\frac{\partial L}{\partial u^A_{I+G+1_i}} \right) du^A_{G+H} \right) \\
&= \sum_{J,K} \frac{(-1)^{|J|}(J+K+1_i)!}{J!K!} \\
&\qquad \times \sum_{H\le K} \frac{(-1)^{|H|}K!}{(|H|+|J|+1)H!(K-H)!} d_J \left(\frac{\partial L}{\partial u^A_{J+K+1_i}} \right) du^A_K,
\end{aligned}$$

where we have written $I - H = J$, $G + H = K$. We shall evaluate the second sum in this expression. The multi-index binomial theorem for $x = (x^i)$ gives

$$\sum_{H\le K} \frac{K!}{H!(K-H)!} x^{H+J} = x^J (x+1)^K.$$

Diagonalize, by setting $x^i = z$ for all i, to obtain

$$\sum_{H\le K} \frac{K!}{H!(K-H)!} z^{|H|+|J|} = z^{|J|}(z+1)^{|K|}.$$

Integrate from $z = 0$ to -1 to obtain

$$\sum_{H\le K} \frac{(-1)^{|H|+|J|+1}K!}{(|H|+|J|+1)H!(K-H)!} = \frac{(-1)^{|J|+1}|J|!|K|!}{(|J|+|K|+1)!}$$

(the right-hand side is obtained by repeated partial integration), so that

$$\sum_{H\le K} \frac{(-1)^{|H|}K!}{(|H|+|J|+1)H!(K-H)!} = \frac{|J|!|K|!}{(|J|+|K|+1)!}$$

as required. ∎

Proposition 7.2 *The coordinate representation of ε is*

$$\sum_I (-1)^{|J|} d_J \left(\frac{\partial L}{\partial u^A_J} \right) du^A$$

in multi-index notation, and

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{\#(j_1 \dots j_r)} d_{j_1 \dots j_r} \left(\frac{\partial L}{\partial u^A_{j_1 \dots j_r}} \right) du^A$$

(using the summation convention) in ordinary index notation.

Proof Once again we prove only the multi-index result, as the other follows directly. We have

$$\begin{aligned}
\sum_I \frac{(-1)^{|I|}}{I!} d_I(S^I dL) &= \sum_{I,G} \sum_{H \le I} \frac{(-1)^{|I|}(I+G)!}{G!H!(I-H)!} d_{I-H} \left(\frac{\partial L}{\partial u^A_{I+G}} \right) du^A_{G+H} \\
&= \sum_{J,K} \frac{(-1)^{|J|}(J+K)!}{J!K!} \left(\sum_{H \le K} \frac{(-1)^{|H|}K!}{H!(K-H)!} \right) d_J \left(\frac{\partial L}{\partial u^A_{J+K}} \right) du^A_K .
\end{aligned}$$

But for $|K| > 0$,

$$\sum_{H \le K} \frac{(-1)^{|H|}K!}{H!(K-H)!} = 0,$$

by the multi-index binomial theorem. Thus

$$\sum_I \frac{(-1)^{|I|}}{I!} d_I(S^I dL) = \sum_J (-1)^{|J|} d_J \left(\frac{\partial L}{\partial u^A_J} \right) du^A$$

as asserted. ∎

Finally, suppose that E has a fibration π over some orientable m-dimensional manifold M with volume form ω. Let x^i $(1 \le i \le m)$ be coordinates on M satisfying $\omega = d^m x$, and let v^α $(1 \le \alpha \le n)$ be fibre coordinates on E, so that $N = m + n$. We take coordinates u^A related to the adapted coordinates (x^i, v^α) by

$$u^1 = x^1, \dots, u^m = x^m, u^{m+1} = v^1, \dots, u^{m+n} = v^n.$$

Suppose we have a homogeneous second-order Lagrangian function L defined on an open submanifold of $\mathcal{F}^\infty_{(m)}E$, and suppose the function $\widetilde{L}$ on $J^\infty\pi$ is related to L in these coordinates by

$$\widetilde{L}(x^i, v^\alpha, v^\alpha_i, v^\alpha_{ij}) = L(x^i, v^\alpha, \delta^j_i, v^\alpha_i, 0, v^\alpha_{ij});$$

we may check that $L = \langle \mathbf{\Delta}, \rho^*(L\omega) \rangle$. If we construct the Hilbert-Carathéodory form Θ on the open submanifold of $\mathcal{F}^\infty_{(m)}E$ and the corresponding Carathéodory form $\widetilde{\Theta}$ on $J^\infty\pi$, we find after some calculation, using the facts that $\Delta^i_j(L) = \delta^i_j L$ and $\Delta^i_{jk}(L) = 0$, that

$$\widetilde{\Theta} = \frac{1}{\widetilde{L}^{m-1}} \bigwedge_{i=1}^{m} \left(\widetilde{L}\, dx^i + \left(\frac{\partial \widetilde{L}}{\partial v^\alpha_i} - \frac{d}{dx^j} \frac{1}{\#(ij)} \frac{\partial \widetilde{L}}{\partial v^\alpha_{ij}} \right) \theta^\alpha + \frac{1}{\#(ij)} \frac{\partial \widetilde{L}}{\partial v^\alpha_{ij}} \theta^\alpha_j \right)$$

as described in the introduction.

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