

Some remarks on the Finslerian version of Hilbert's fourth problem

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Abstract

The Finslerian version of Hilbert's fourth problem is the problem of finding projective Finsler functions. Álvarez Paiva (J. Diff. Geom. **69** (2005) 353–378) has shown that projective absolutely homogeneous Finsler functions correspond to symplectic structures on the space of oriented lines in \mathbf{R}^n with certain properties. I give new and direct proofs of his main results, and show how they are related to the more classical formulations of the problem due to Hamel and Rapcsák.

1 Introduction

From the point of view of Finsler geometry, Hilbert's fourth problem is usually regarded as the problem of finding projective Finsler functions, that is, Finsler functions on $T^\circ\mathbf{R}^n$ (the tangent bundle of \mathbf{R}^n with zero section removed) whose geodesics, as point sets, are straight lines. As initially formulated, the problem was to find metrics (in the topological sense) on \mathbf{R}^n with the property that the shortest curve joining two points is the straight line segment between them. The Finslerian version is more specific in that differentiability properties are assumed, but also more general in that Finsler functions do not define genuine metrics. A general Finsler function, one which is merely positively homogeneous of degree one in the velocity variables, defines a distance function which has two of the properties of a metric (it is positive and satisfies the triangle inequality) but lacks the third, symmetry. For the latter property to hold the Finsler function must be absolutely

homogeneous. The strict Finslerian version of Hilbert's fourth problem is to find projective absolutely homogeneous Finsler functions. This paper deals with both the strict and the more general forms of the problem.

There are in fact many projective Finsler functions (see for example [6, 7] and references therein), so that 'finding' them, at least in the sense of listing them, becomes rather a tall order. In fact this paper is concerned with ways of characterizing projective Finsler spaces, or to be more precise with two apparently rather dissimilar approaches to the problem of doing so; indeed one of its aims is to reconcile these approaches.

The first approach, which might be called classical, is the reformulation of Hilbert's fourth problem by Hamel in the early 20th century, and the related work of Rapcsák. Hamel's conditions will be rederived below, but for some background and a more extensive discussion with references see [8].

Much more recently, a new approach to the problem using symplectic geometry and Crofton formulae has been developed by Álvarez Paiva [1]. Álvarez Paiva deals entirely with the strict version of the problem. One aim of the present paper is to show that most of Álvarez Paiva's results can be derived by rather more elementary methods than he uses. Of course one pays a price in loss of elegance; on the other hand, one gains some different insights, and in particular one sees that there is a close link between Álvarez Paiva's characterization of projective Finsler spaces, in the case of absolute homogeneity, and that of Hamel.

One unfortunate but unavoidable feature of the approach adopted here is that the requirement of a Finsler function that it be strongly convex has to be treated separately from the rest of the problem. Moreover, it turns out to be more convenient to deal directly with the Finsler function than with its energy, whereas in most treatments the condition for strong convexity is stated in terms of the energy. I begin therefore, in Section 2, with a general discussion of strong convexity adapted to the needs of the paper; some of the contents of this section are, I believe, new, and interesting in their own right.

In Section 3 I discuss Rapcsák's and Hamel's contributions to the problem, and in Section 4 I give a restatement of Hamel's conditions in terms of the existence on $T^*\mathbf{R}^n$ of a 2-form with certain properties. This is a half-way stage to the formulation of the problem in terms of symplectic geometry, which will be found in Section 5.

2 Strong convexity

A Finsler function F on a slit tangent bundle $T^\circ M$ is required to be strongly convex. The condition for strong convexity is usually given in terms of the Hessian of the energy $E = \frac{1}{2}F^2$ ('Hessian' will always mean 'Hessian with respect to the natural fibre coordinates'); it is that for each $(x, y) \in T^\circ M$ the symmetric bilinear form on $T_x M$ whose components are

$$g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j} = F \frac{\partial^2 F}{\partial y^i \partial y^j} + \frac{\partial F}{\partial y^i} \frac{\partial F}{\partial y^j}$$

is positive definite. For the purposes of this article, however, it will be more useful to state the condition directly in terms of the Hessian of F . (Of course the Hessian of F is, apart from a factor of F , the angular metric; but this identification does not seem to be particularly helpful here.)

One preliminary observation is necessary. As is pointed out in [2] for example, from this conventional definition it follows that if a function F on $T^\circ M$ is positively homogeneous and strongly convex then it is never vanishing, so that when defining what it is for a function to be a Finsler function it is enough to require that the function is nonnegative. In the following discussion this point has to be treated with a certain amount of care.

Since F is positively homogeneous

$$y^i \frac{\partial^2 F}{\partial y^i \partial y^j} = 0.$$

I will say that the Hessian of F is positive semidefinite at (x, y) if for all $u \in T_x M$,

$$\frac{\partial^2 F}{\partial y^i \partial y^j} u^i u^j \geq 0, \quad \frac{\partial^2 F}{\partial y^i \partial y^j} u^i u^j = 0 \text{ if and only if } u^i = \lambda y^i$$

for some scalar λ . Similar terminology will be used for certain other bilinear forms that occur later, but always with the understanding that at (x, y) it is y that is the 'null' vector.

Lemma 1. *If F is positively homogeneous and nonnegative then F is strongly convex at (x, y) if and only if $F(x, y) > 0$ and the Hessian of F is positive semidefinite at (x, y) .*

Proof. Suppose that F is strongly convex. Then from the formula above for g_{ij} in terms of F we have

$$g_{ij} y^j = F(x, y) \frac{\partial F}{\partial y^i}, \quad g_{ij} y^i y^j = F(x, y)^2;$$

from the latter we see that $F(x, y)$ is positive. Then for any $u \in T_x M$ we may set

$$u^i - \frac{1}{F(x, y)} \left(u^k \frac{\partial F}{\partial y^k} \right) y^i = v^i;$$

v can be thought of as the component of u tangent to the level set of F in which (x, y) lies. It is easy to see that

$$\frac{\partial^2 F}{\partial y^i \partial y^j} u^i u^j = \frac{g_{ij} v^i v^j}{F(x, y)}.$$

So the left-hand side is nonnegative, and is zero only if $v = 0$, in which case u is a scalar multiple of y .

Conversely, if $F(x, y) > 0$ and the Hessian of F is positive semidefinite at (x, y) , then for any u

$$g_{ij} u^i u^j = F(x, y) \frac{\partial^2 F}{\partial y^i \partial y^j} u^i u^j + \left(u^k \frac{\partial F}{\partial y^k} \right)^2 \geq 0.$$

Moreover, $g_{ij} u^i u^j = 0$ if and only if both terms on the right-hand side are zero individually. Then $u^i = \lambda y^i$ from the first, and then

$$0 = \lambda y^k \frac{\partial F}{\partial y^k} = \lambda F(x, y),$$

so $\lambda = 0$. So F is strongly convex at (x, y) . □

In general one cannot deduce from positive-semidefiniteness of the Hessian of F that F is nonvanishing. The following simple example is quite instructive. The most obvious projective Finsler function is the Euclidean length function, $F(x, y) = |y| = \sqrt{\delta_{ij} y^i y^j}$. Then

$$\frac{\partial^2 F}{\partial y^i \partial y^j} = \frac{1}{|y|^3} (|y|^2 \delta_{ij} - y_i y_j)$$

where $y_i = \delta_{ij} y^j$. Consider now $\hat{F}(x, y) = |y| + \alpha_i y^i$, where α is any constant covector. The Hessian of \hat{F} is evidently identical to the Hessian of F . Whether such a function \hat{F} is a Finsler function or not depends on $|\alpha|$: we must have $|\alpha| < 1$ for it to be a Finsler function; if $|\alpha| \geq 1$ there will be values of y for which $\hat{F}(y) = 0$. That is to say, one cannot tell in general from considerations of the Hessian alone whether or not \hat{F} is nonvanishing. It is worth remarking that the Euclidean length function is uniquely distinguished in this class of positively homogeneous functions by the fact that it is absolutely homogeneous; and it of course is nonvanishing. I will return to this point at the end of the section.

Returning to the general case, we can evidently regard the Hessian of F at (x, y) as defining a symmetric bilinear form on $T_x M / \langle y \rangle$, which is positive definite if and only if the Hessian itself is positive semidefinite. More generally, if F is positively homogeneous at (x, y) its Hessian defines a symmetric bilinear form on $T_x M / \langle y \rangle$, which I call the reduced Hessian (and again the same terminology will be used without comment in other situations later on). I will be interested below only in such functions F for which this form is nonsingular: a positively homogeneous function whose reduced Hessian is everywhere nonsingular but not necessarily positive definite will be called a pseudo-Finsler function. I will refer to the signature of the reduced Hessian of a pseudo-Finsler function F as the signature of F . The signature of F at (x, y) is also the signature of the restriction of the Hessian of F to any subspace of $T_x M$ which is complementary to $\langle y \rangle$. For $F(x, y) > 0$, one such subspace is the tangent space to the level set of F in which (x, y) lies.

The following result is due to Lovas [5]. Lovas's proof uses g_{ij} ; here, in keeping with my earlier remarks, I prove the result using only the Hessian of F .

Lemma 2. *A pseudo-Finsler function which takes only positive values is a Finsler function.*

Proof. I show that at any $x \in M$ there is a point of $T_x^\circ M$, the tangent space at x with origin deleted, at which the Hessian of the pseudo-Finsler function F is positive semidefinite. Then since the signature of F cannot change without the reduced Hessian becoming singular, F must be positive semidefinite all over $T_x^\circ M$.

The argument takes place entirely within $T_x^\circ M$ so I will ignore the fact that F depends on x and regard it as a function just on $T_x^\circ M$. I work in coordinates, which is to say that I identify $T_x^\circ M$ with $\mathbf{R}^n - \{0\}$, and I equip the latter space with the Euclidean metric.

Consider the level set Σ of F of value 1. It cannot contain any critical points of F , since $y^i \partial F / \partial y^i = 1$ on Σ . It is therefore a submanifold of $T_x^\circ M$ of codimension 1, and at each $y \in T_x^\circ M$ it is transverse to the ray $\{\lambda y : \lambda > 0\}$. Thus Σ is topologically a sphere, and in particular is compact. The function on Σ which maps each y to its Euclidean length $|y|$ achieves its maximum value. At a maximum, say y_0 , we have

$$\frac{\partial F}{\partial y^i}(y_0) = \delta_{ij} \frac{y_0^j}{|y_0|^2},$$

by the method of undetermined multipliers. Now choose any $u \in T_{y_0} \Sigma$, and let $c(t)$ be a curve in Σ with $c(0) = y_0$, $\dot{c}(0) = u$. Then

$$u^i \frac{\partial F}{\partial y^i}(y_0) = 0, \quad \ddot{c}^i(0) \frac{\partial F}{\partial y^i}(y_0) + u^i u^j \frac{\partial^2 F}{\partial y^i \partial y^j}(y_0) = 0.$$

From the first of these we obtain $u \cdot y_0 = 0$. Now $|c(t)|$ has a maximum at $t = 0$. Thus

$$\begin{aligned} 0 &\geq \frac{d^2}{dt^2} (|c(t)|)_{t=0} = \frac{1}{|y_0|} (\ddot{c}(0) \cdot y_0 + |\dot{c}(0)|^2) \\ &= -|y_0| u^i u^j \frac{\partial^2 F}{\partial y^i \partial y^j} (y_0) + \frac{|u|^2}{|y_0|}. \end{aligned}$$

It follows that for every nonzero $u \in T_{y_0} \Sigma$,

$$u^i u^j \frac{\partial^2 F}{\partial y^i \partial y^j} (y_0) \geq \left(\frac{|u|}{|y_0|} \right)^2 > 0.$$

That is to say, the restriction of the Hessian of F to $T_{y_0} \Sigma$ is positive definite. \square

I pointed out earlier that one cannot in general tell from consideration of the Hessian of F alone whether or not F is nonvanishing, even when the Hessian is positive semidefinite. However, if F is absolutely homogeneous (so that $F(x, -y) = F(x, y)$) it is possible to prove that when its Hessian is positive semidefinite it is nonvanishing, and in fact necessarily everywhere positive.

Lemma 3. *Suppose that the function F on $T^\circ M$ is absolutely homogeneous and its Hessian is positive semidefinite everywhere. Then F is everywhere positive, and so is a Finsler function.*

Proof. The key point about absolute homogeneity in this context is that if $F(x, z) = 0$ for some $(x, z) \in T^\circ M$ then $F(x, \lambda z) = 0$ for *all* nonzero scalars λ . Again, I restrict my attention to $T_x^\circ M$ for arbitrary x , and drop explicit mention of x in formulae.

The first point to establish is that F cannot be everywhere negative on $T_x^\circ M$. To do this I assume that it is everywhere negative, and argue as in Lemma 2, but with respect to the level set Σ of value -1 . As before, the Euclidean length function achieves its maximum on Σ , at y_0 say; but this time we have

$$\frac{\partial F}{\partial y^i} (y_0) = -\delta_{ij} \frac{y_0^j}{|y_0|^2}.$$

But then the condition that $|c(t)|$ has a maximum along the curve $c(t)$ at $t = 0$ reads

$$\begin{aligned} 0 &\geq \frac{d^2}{dt^2} (|c(t)|)_{t=0} = \frac{1}{|y_0|} (\ddot{c}(0) \cdot y_0 + |\dot{c}(0)|^2) \\ &= |y_0| u^i u^j \frac{\partial^2 F}{\partial y^i \partial y^j} (y_0) + \frac{|u|^2}{|y_0|}, \end{aligned}$$

which is a contradiction.

Thus $T_x^\circ M$ must contain points where F is nonnegative. I next show that it must contain a point where F is positive.

The zero set of F in $T_x^\circ M$ is evidently closed. On the other hand, F cannot vanish on an open subset of $T_x^\circ M$ and still have positive semidefinite Hessian. So the zero set of F in $T_x^\circ M$ is closed without interior points, and its complement (where F is nonzero) is open dense.

The following argument is based on the proof of the so-called fundamental inequality due to Bao et al., [2] page 9. Let y be any point of $T_x^\circ M$. For any u ,

$$F(y + u) = F(y) + u^i \frac{\partial F}{\partial y^i}(y) + \frac{1}{2} u^i u^j \frac{\partial^2 F}{\partial y^i \partial y^j}(y) + \epsilon u$$

for some ϵ , $0 \leq \epsilon \leq 1$, by the second mean-value theorem applied to the function $t \mapsto F(y + tu)$. Suppose that $F(y) = 0$. Then for any u (if y is a critical point of F), or for any u that is tangent to the level set Σ of F through y (if not), the second term on the right-hand side is zero. The third term is nonnegative by assumption, and indeed positive if we ensure that u is not a scalar multiple of y . Then if $F(y) = 0$, we have $F(y + u) > 0$ for such u .

Next, from the same formula but now with $F(y) > 0$ it follows that at all points on the tangent hyperplane to Σ at y the value of F is positive. Now if F has a zero, at z say, then F vanishes on the whole line $t \mapsto tz$ (excluding the origin); such a line therefore cannot intersect the tangent hyperplane. Thus at each point y where $F(y) > 0$ the line $t \mapsto y + tz$ lies in the tangent hyperplane to the level set of F through y . That is,

$$z^i \frac{\partial F}{\partial y^i}(y) = 0$$

for all y where $F(y) > 0$. But the set of points y where $F(y) > 0$ is open, so the relation above holds on an open set. We may therefore differentiate with respect to y^j to obtain

$$z^i \frac{\partial^2 F}{\partial y^i \partial y^j}(y) = 0$$

(z^i is constant). Clearly z is not a scalar multiple of y (because $F(z) = 0$ while $F(y) > 0$). But this contradicts the assumed positive-semidefiniteness of F . There are therefore no points z where $F(z) = 0$. It follows that F is everywhere positive. \square

3 Rapcsák's and Hamel's equations

Rapcsák's equations are conditions for the geodesic spray of a Finsler function to be projectively equivalent to a given spray (see for example [6] Chapter 12). They can be derived rather simply as follows. Let F be an arbitrary Finsler function, and consider the following version of the Euler-Lagrange equations in which F is taken as the Lagrangian:

$$S \left(\frac{\partial F}{\partial y^i} \right) - \frac{\partial F}{\partial x^i} = 0,$$

where S is assumed to be a spray. Then since

$$u^j \frac{\partial^2 F}{\partial y^i \partial y^j} = 0$$

if and only if u is a scalar multiple of y , S is determined up to the addition of a multiple of the Liouville field C . That is to say, the Euler-Lagrange equations (for the Finsler function rather than the energy), together with the assumption that S is a spray, determine a projective equivalence class of sprays; this class includes the canonical spray of F , and thus consists of all those sprays projectively equivalent to it. Thus (taking F to be given) in order for a spray S to be projectively equivalent to the canonical spray of F it is necessary and sufficient that it satisfies the above Euler-Lagrange equations. For much the same reasons (but now fixing S and regarding F as the unknown), a Finsler function F has the property that its canonical spray is projectively related to S if and only if F satisfies these equations. This is the essential content of Rapcsák's equations.

Consider in particular a Finsler function F on $T^o\mathbf{R}^n$ (one could take F to be defined just on the slit tangent bundle of some open subset of \mathbf{R}^n , but I leave this possibility to be understood). Then F has the property that its canonical spray is projectively related to the standard flat spray S , given by $y^i \partial / \partial x^i$ in rectilinear coordinates, if and only if

$$y^j \frac{\partial^2 F}{\partial x^j \partial y^i} - \frac{\partial F}{\partial x^i} = 0.$$

These are Rapcsák's equations applied to the case of a projective Finsler function; they are also one form of Hamel's equations. On differentiating again with respect to y^j we obtain

$$y^k \frac{\partial^3 F}{\partial x^k \partial y^i \partial y^j} + \frac{\partial^2 F}{\partial x^j \partial y^i} - \frac{\partial^2 F}{\partial x^i \partial y^j} = 0.$$

The part of this identity skew in i and j leads to the other Hamel equations, namely

$$\frac{\partial^2 F}{\partial x^j \partial y^i} = \frac{\partial^2 F}{\partial x^i \partial y^j};$$

these are easily seen to be equivalent to the first ones, assuming that F is positively homogeneous. The part of the identity symmetric in i and j says that the Hessian of F is invariant under S .

I have assumed in the discussion above that F is a Finsler function. Though we require F to be positively homogeneous, in fact it is enough that its reduced Hessian is nonsingular; so the results hold for a pseudo-Finsler function.

I summarize the discussion in the following proposition (which is of course well-known: see for example [6] Corollary 12.2.10 and [8] Corollary 8.1 for other versions).

Proposition 1. *A pseudo-Finsler function F on $T^o\mathbf{R}^n$ is projective if and only if it satisfies either of the following equivalent conditions (in rectilinear coordinates):*

$$y^j \frac{\partial^2 F}{\partial x^j \partial y^i} - \frac{\partial F}{\partial x^i} = 0; \quad \frac{\partial^2 F}{\partial x^j \partial y^i} = \frac{\partial^2 F}{\partial x^i \partial y^j}.$$

Further interesting consequences can be drawn from the Hamel conditions. It follows from the second version of these conditions that there is a function f such that

$$\frac{\partial F}{\partial y^i} = \frac{\partial f}{\partial x^i}.$$

Indeed, one can write down an explicit formula for f by adapting the usual formula for a homotopy operator for the exterior derivative acting on 1-forms:

$$f(x, y) = \int_{t=0}^1 x^i \frac{\partial F}{\partial y^i}(tx, y) dt;$$

the fact that f satisfies the required relation is a straightforward calculation using the Hamel conditions. The point of giving this formula is that it shows that f may be chosen to be positively homogeneous of degree zero in y . Addition to this f of any function of y alone will give a new function satisfying the given relation, but not necessarily one which is homogeneous.

Now from the defining relation above it follows that

$$y^i \frac{\partial f}{\partial x^i} = S(f) = y^i \frac{\partial F}{\partial y^i} = F,$$

where (here and below) S is the standard flat spray. This observation may be expressed in another form. Consider, for fixed x_0 and y_0 , the straight line $c(t) = x_0 + ty_0$. For this curve

$$F(c(t), \dot{c}(t)) = \frac{d}{dt}(f(c(t), \dot{c}(t))).$$

Thus the length of the line segment with $0 \leq t \leq 1$ as measured using the Finsler function F is

$$\int_{t=0}^1 F(c(t), \dot{c}(t)) dt = f(x_0 + y_0, y_0) - f(x_0, y_0).$$

That is, f determines the Finslerian distance function d_F by

$$d_F(x_1, x_2) = f(x_2, x_2 - x_1) - f(x_1, x_2 - x_1).$$

Of course, addition of a function of y alone to f has no effect on this formula. In general d_F will not be symmetric; but if F is absolutely homogeneous then (appealing again to the homotopy formula) we can choose f to satisfy $f(x, -y) = -f(x, y)$, and then

$$\begin{aligned} d_F(x_2, x_1) &= f(x_1, x_1 - x_2) - f(x_2, x_1 - x_2) \\ &= f(x_2, x_2 - x_1) - f(x_1, x_2 - x_1) = d_F(x_1, x_2). \end{aligned}$$

It is worth noting explicitly that

$$\frac{\partial^2 f}{\partial x^j \partial y^i} = \frac{\partial^2 F}{\partial y^i \partial y^j} = \frac{\partial^2 f}{\partial x^i \partial y^j}.$$

Furthermore,

$$y^j \frac{\partial^2 f}{\partial x^j \partial y^i} = 0 = S \left(\frac{\partial f}{\partial y^i} \right);$$

and it is easy to see that, conversely, if $S(\partial f / \partial y^i) = 0$ then

$$\frac{\partial^2 f}{\partial x^j \partial y^i} = \frac{\partial^2 f}{\partial x^i \partial y^j}.$$

Conversely, given a function f with such properties, we can find a projective Finsler function.

Proposition 2. *Let f be a function on $T^\circ \mathbf{R}^n$ which is positively homogeneous of degree zero in y and satisfies*

$$\frac{\partial^2 f}{\partial x^j \partial y^i} = \frac{\partial^2 f}{\partial x^i \partial y^j},$$

where the reduced version of the symmetric bilinear form so defined is nonsingular: then $S(f)$ is a projective pseudo-Finsler function. If in addition the symmetric bilinear form is positive semidefinite and $S(f) > 0$ then $S(f)$ is a projective Finsler function.

Proof. Set

$$F = S(f) = y^i \frac{\partial f}{\partial x^i}.$$

Then F is positively homogeneous of degree 1; furthermore

$$\frac{\partial F}{\partial y^i} = \frac{\partial f}{\partial x^i} + y^j \frac{\partial^2 f}{\partial x^j \partial y^i} = \frac{\partial f}{\partial x^i} + y^j \frac{\partial^2 f}{\partial x^i \partial y^j} = \frac{\partial f}{\partial x^i},$$

and so

$$\frac{\partial^2 F}{\partial x^j \partial y^i} = \frac{\partial^2 F}{\partial x^i \partial y^j} \quad \text{and} \quad \frac{\partial^2 F}{\partial y^i \partial y^j} = \frac{\partial^2 f}{\partial x^i \partial y^j}.$$

□

This result is essentially equivalent to Proposition 8.1 of [8].

4 The Hilbert forms of a projective Finsler function

I now consider the Hilbert 1-form of a projective Finsler function F ,

$$\theta = \frac{\partial F}{\partial y^i} dx^i,$$

and the Hilbert 2-form $d\theta$. From general considerations the Hilbert 2-form has the following properties:

1. $d\theta$ is singular, and its characteristic distribution is spanned by any spray S projectively equivalent to the canonical spray of F , and the Liouville field C ; this distribution contains the whole projective equivalence class of S , and is integrable by homogeneity;
2. since $d\theta$ is evidently closed, its Lie derivative by any vector field in its characteristic distribution is zero;
3. $d\theta(V_1, V_2) = 0$ for any pair of vertical vectors V_1, V_2 .

These results hold for any Finsler function; but it is quite interesting to see how they work out in the case of interest. So suppose that F is a projective Finsler function, and therefore satisfies the Hamel conditions stated in Proposition 1. Now consider the Hilbert forms of F . First of all,

$$d\theta = \frac{\partial^2 F}{\partial x^i \partial y^j} dx^i \wedge dx^j + \frac{\partial^2 F}{\partial y^i \partial y^j} dy^i \wedge dy^j;$$

but the first term is zero since its coefficient is symmetric in i and j . Thus

$$d\theta = \frac{\partial^2 F}{\partial y^i \partial y^j} dy^i \wedge dx^j.$$

Item 3 above follows immediately. We have

$$C \lrcorner d\theta = y^k \frac{\partial}{\partial y^k} \lrcorner \frac{\partial^2 F}{\partial y^i \partial y^j} dy^i \wedge dx^j = y^i \frac{\partial^2 F}{\partial y^i \partial y^j} dx^j,$$

while

$$S \lrcorner d\theta = y^k \frac{\partial}{\partial x^k} \lrcorner \frac{\partial^2 F}{\partial y^i \partial y^j} dy^i \wedge dx^j = -y^j \frac{\partial^2 F}{\partial y^i \partial y^j} dy^i,$$

where again S denotes the standard flat spray $y^i \partial / \partial x^i$; both are zero by homogeneity, whence item 1. Item 2 is a direct consequence, but can also be derived independently. In fact $\mathcal{L}_C \theta = 0$ by homogeneity, while

$$\mathcal{L}_S \theta = y^j \frac{\partial^2 F}{\partial x^j \partial y^i} dx^i + \frac{\partial F}{\partial y^i} dy^i = \frac{\partial F}{\partial x^i} dx^i + \frac{\partial F}{\partial y^i} dy^i = dF.$$

Recall that for any projective Finsler function F we can find a function f , positively homogeneous of degree 0, such that

$$\frac{\partial f}{\partial x^i} = \frac{\partial F}{\partial y^i}.$$

The Hilbert 1-form can be expressed in terms of f as $\theta = (\partial f / \partial x^i) dx^i$, so that

$$d\theta = d \left(\frac{\partial f}{\partial x^i} dx^i \right) = d \left(\frac{\partial f}{\partial x^i} \right) \wedge dx^i.$$

On the other hand, $\theta = df - (\partial f / \partial y^i) dy^i$, so that also

$$d\theta = -d \left(\frac{\partial f}{\partial y^i} dy^i \right) = dy^i \wedge d \left(\frac{\partial f}{\partial y^i} \right);$$

this will turn out to be the more significant formula of the two.

I now prove a partial converse to the statements above about the Hilbert 2-form of a projective Finsler function. This result in effect restates Hamel's conditions in terms of the properties of a 2-form on $T^\circ \mathbf{R}^n$.

Proposition 3. *Let Ω be a closed 2-form on $T^\circ \mathbf{R}^n$, whose characteristic distribution is 2-dimensional and is spanned by S , the standard flat spray, and C , the Liouville field. Suppose further that $\Omega = \Omega_{ij} dy^i \wedge dx^j$ in rectilinear coordinates, where Ω_{ij} is symmetric in its indices. Then Ω is the Hilbert 2-form of a projective pseudo-Finsler function F on $T^\circ \mathbf{R}^n$.*

Proof. The condition for the characteristic distribution of Ω to be spanned by S and C is that $\Omega_{ij}u^j = 0$ if and only if u is a scalar multiple of y .

The closure of Ω is equivalent to the conditions

$$\frac{\partial \Omega_{ij}}{\partial y^k} = \frac{\partial \Omega_{ik}}{\partial y^j}, \quad \frac{\partial \Omega_{ij}}{\partial x^k} = \frac{\partial \Omega_{ik}}{\partial x^j}$$

on its coefficients. From the first, there are functions ϕ_i , globally defined for $n > 2$, such that

$$\Omega_{ij} = \frac{\partial \phi_i}{\partial y^j} = \frac{\partial \phi_j}{\partial y^i}$$

(using symmetry). Since $\Omega_{ij}y^j = 0$,

$$y^j \frac{\partial \phi_i}{\partial y^j} = 0.$$

Set $\phi = \phi_i y^i$: then

$$\frac{\partial \phi}{\partial y^i} = \phi_i + y^j \frac{\partial \phi_j}{\partial y^i} = \phi_i + y^j \frac{\partial \phi_i}{\partial y^j} = \phi_i,$$

and therefore

$$\Omega_{ij} = \frac{\partial^2 \phi}{\partial y^i \partial y^j}.$$

From the second closure condition

$$\frac{\partial^3 \phi}{\partial y^i \partial y^j \partial x^k} = \frac{\partial^3 \phi}{\partial y^i \partial y^k \partial x^j},$$

so that

$$\frac{\partial^2 \phi}{\partial x^j \partial y^k} - \frac{\partial^2 \phi}{\partial x^k \partial y^j} = \psi_{jk}(x),$$

where ψ_{jk} , which is independent of the y^i , is skew in its indices. Now

$$\begin{aligned} & \frac{\partial \psi_{jk}}{\partial x^i} + \frac{\partial \psi_{ki}}{\partial x^j} + \frac{\partial \psi_{ij}}{\partial x^k} \\ &= \frac{\partial^3 \phi}{\partial x^i \partial x^j \partial y^k} - \frac{\partial^3 \phi}{\partial x^i \partial x^k \partial y^j} + \frac{\partial^3 \phi}{\partial x^j \partial x^k \partial y^i} - \frac{\partial^3 \phi}{\partial x^j \partial x^i \partial y^k} \\ &+ \frac{\partial^3 \phi}{\partial x^k \partial x^i \partial y^j} - \frac{\partial^3 \phi}{\partial x^k \partial x^j \partial y^i} \\ &= 0. \end{aligned}$$

There are therefore functions $\chi_i(x)$, again globally defined, such that

$$\psi_{ij} = \frac{\partial \chi_i}{\partial x^j} - \frac{\partial \chi_j}{\partial x^i}.$$

Now set

$$F = \phi + \chi_i y^i.$$

Then

$$y^i \frac{\partial F}{\partial y^i} = y^i \frac{\partial \phi}{\partial y^i} + y^i \chi_i = y^i \phi_i + y^i \chi_i = \phi + \chi_i y^i = F,$$

so F is positively homogeneous of degree one in the y^i . Moreover,

$$\frac{\partial^2 F}{\partial y^i \partial y^j} = \frac{\partial^2 \phi}{\partial y^i \partial y^j} = \Omega_{ij};$$

and

$$\begin{aligned} & \frac{\partial^2 F}{\partial x^i \partial y^j} - \frac{\partial^2 F}{\partial x^j \partial y^i} \\ &= \frac{\partial^2 \phi}{\partial x^i \partial y^j} + \frac{\partial \chi_j}{\partial x^i} - \frac{\partial^2 \phi}{\partial x^j \partial y^i} - \frac{\partial \chi_i}{\partial x^j} = \frac{\partial^2 \phi}{\partial x^i \partial y^j} - \frac{\partial^2 \phi}{\partial x^j \partial y^i} - \psi_{ij} = 0. \end{aligned}$$

Thus F satisfies the Hamel conditions, and its reduced Hessian is nonsingular. Moreover, Ω is the exterior derivative of the Hilbert 1-form of F . \square

If one can find a pseudo-Finsler function F which is nonvanishing, then if F is everywhere positive it is a Finsler function, by Lemma 2. If F is everywhere negative then one can simply replace Ω by $-\Omega$ and start again.

Corollary 1. *Suppose that there is a pseudo-Finsler function F for Ω which is everywhere positive. Then F is a projective Finsler function.*

Notice that according to Proposition 3, F is determined up to the addition of a total derivative, that is, a term of the form $(\partial \chi / \partial x^i) y^i$ where χ is any function on \mathbf{R}^n .

If we start with a Finsler function which is absolutely homogeneous then $d\theta$ changes sign under reflection; that is to say, if ρ is the reflection map, $\rho(x, y) = (x, -y)$, and $\rho^* F = F$ then $\rho^* d\theta = -d\theta$ (indeed, $\rho^* \theta = -\theta$). Conversely, suppose that Ω satisfies the hypotheses of Proposition 3 and in addition $\rho^* \Omega = -\Omega$, or equivalently $\Omega_{ij}(x, -y) = \Omega_{ij}(x, y)$. Then if F is a pseudo-Finsler function for Ω , so is $\bar{F} = \rho^* F$, and so is $\frac{1}{2}(F + \bar{F})$: the latter is absolutely homogeneous. Moreover, the absolutely homogeneous solution is unique: for any two solutions differ by a total derivative; but such a term is linear in y , and therefore changes sign under ρ ; so distinct solutions cannot both be absolutely homogeneous.

In these circumstances we can also deduce that a pseudo-Finsler function is a Finsler function by applying Lemma 3.

Corollary 2. *Suppose that in addition to satisfying the hypotheses of Proposition 3, Ω changes sign under reflection, and (Ω_{ij}) is positive semidefinite. Then the corresponding absolutely homogeneous pseudo-Finsler function F is a projective Finsler function.*

5 Path space and symplectic structure

Recall that the Hilbert 2-form of a projective Finsler function F (indeed any Finsler function) has for its characteristic distribution the span of any geodesic spray S of F and the Liouville field C . The distribution $\langle C, S \rangle$ is integrable, so we can (at least locally) take the quotient by its leaves. The result is a manifold of dimension $2n - 2$, each of whose points represents an unparametrized geodesic of F : it is the path space Γ . It follows from its other properties (as set out in Section 4) that $d\theta$ defines a 2-form ω on Γ which is closed and nonsingular, so is symplectic. Moreover, the set of all geodesic paths through any fixed point x_0 determines an $(n - 1)$ -dimensional submanifold of Γ which is Lagrangian. This construction is discussed at length in [3], as well as in [1].

To give a bit more detail in the projective case: the flow of the flat spray S on $T^\circ\mathbf{R}^n$ is just $(x^i, y^i) \mapsto (x^i + ty^i, y^i)$, while that of C is $(x^i, y^i) \mapsto (x^i, e^s y^i)$. In fact we have a left action of the affine group of the line by $(x^i, y^i) \mapsto (x^i + ty^i, e^s y^i)$; the path space Γ , that is, the space of oriented straight lines in \mathbf{R}^n , is the quotient of $T^\circ\mathbf{R}^n$ under this action (notice that the zero section of $T\mathbf{R}^n$ is pointwise fixed under the action of the affine group, so must be cut out before taking the quotient). Let $\pi : T^\circ\mathbf{R}^n \rightarrow \Gamma$ be the projection. Now $d\theta$ is invariant under the group action, and so passes to the quotient to define a 2-form on Γ , that is, a 2-form ω such that $\pi^*\omega = d\theta$. Evidently $\pi^*d\omega = 0$; but since π is surjective it follows that ω is closed. Moreover, since we have quotiented out the characteristic distribution of $d\theta$, ω is nonsingular. Thus ω is a symplectic 2-form. The form ω has one further important property: since $d\theta$ vanishes when restricted to any fibre of $T^\circ\mathbf{R}^n$, ω vanishes when restricted to the image of any fibre. The image of $T^\circ_{x_0}\mathbf{R}^n$ in Γ is an $(n - 1)$ -dimensional submanifold, which consists of all the lines through x_0 . Thus ω has the property that each submanifold of Γ consisting of all the lines through a given point of \mathbf{R}^n is a Lagrangian submanifold.

One concept of a ‘solution’ to Hilbert’s fourth problem, due to Álvarez Paiva [1], is a symplectic form on the path space such that lines through any point correspond to Lagrangian submanifolds, together with some condition ensuring strong convexity. His argument is indirect, involving as it does so-called Crofton formulas. However, one can work more directly, as I will show below.

I first examine the symplectic structure obtained from a projective Finsler function a little more closely; in fact the following comments apply equally to a projective pseudo-Finsler function, except for those that concern positive definiteness.

I will define certain local coordinates on path space Γ . These are modelled partly on the coordinates often used for real projective space. It is important to note however that Γ consists of *oriented* lines, so that the same line (as a point set) traversed in opposite directions determines two points of Γ . The map which takes each point of Γ to the direction of the corresponding oriented line defines a fibration of Γ over an $(n-1)$ -sphere. Without the insistence on oriented lines the base would indeed be a projective space. In fact, by taking the base to be a metric sphere S_{n-1} (with respect to the Euclidean metric) one can identify Γ with TS_{n-1} (see [4] for example); but I do not use this identification here.

We can cover Γ by $2n$ open sets U_k^\pm , where k is an integer, $1 \leq k \leq n$, and U_k^+ consists of those lines whose directions y satisfy $y^k > 0$, U_k^- those whose directions y satisfy $y^k < 0$. For coordinates on U_k^+ we take

$$(\xi^1, \xi^2, \dots, \xi^{k-1}, \xi^{k+1}, \dots, \xi^n, \eta^1, \eta^2, \dots, \eta^{k-1}, \eta^{k+1}, \dots, \eta^n),$$

where the η^i are the components of the direction vector of the line normalized with $y^k = 1$, and the ξ^i are the coordinates of the point where the line meets the hyperplane $x^k = 0$. The coordinates on U_k^- are similarly defined, except that the normalized direction vector has $y^k = -1$. (The numbering of the coordinates is somewhat unconventional, but this will not cause any problems.) The coordinate transformation between, for example, U_n^\pm and U_{n-1}^\pm is given by

$$\hat{\xi}^\alpha = (\xi^\alpha \eta^{n-1} - \xi^{n-1} \eta^\alpha) / \eta^{n-1}, \quad \hat{\xi}^n = -\epsilon(\xi^{n-1} / \eta^{n-1}),$$

and

$$\hat{\eta}^\alpha = \delta(\eta^\alpha / \eta^{n-1}), \quad \hat{\eta}^n = \delta\epsilon(1 / \eta^{n-1})$$

where $(\hat{\xi}^\alpha, \hat{\xi}^n, \hat{\eta}^\beta, \hat{\eta}^n)$, $1 \leq \alpha, \beta \leq n-2$, are the coordinates of a point in $U_{n-1}^\pm \cap U_n^\pm$ with respect to U_{n-1}^\pm , (ξ^a, η^b) , $1 \leq a, b \leq n-1$, the coordinates of the same point with respect to U_n^\pm ; $\delta = +1$ on U_{n-1}^+ , $\delta = -1$ on U_{n-1}^- , and ϵ is similarly defined for U_n^\pm . (To clarify the notation: U_k^\pm here stands for either U_k^+ or U_k^- , so that for example $U_{n-1}^\pm \cap U_n^\pm$ stands for any one of four different sets, and four coordinate transformations are being dealt with simultaneously, distinguished by the values of δ and ϵ .) Similar formulae hold on the other intersections of coordinate patches.

On U_n^\pm , say, the projection π has the coordinate representation $\pi(x, y) = (\xi^a, \eta^b)$ where

$$\xi^a = (x^a y^n - x^n y^a) / y^n, \quad \eta^a = y^a / |y^n|,$$

and similarly for the other coordinate patches.

Now suppose given a projective Finsler function F . On U_n^\pm the homogeneity condition may be written

$$\frac{\partial F}{\partial y^n} = \frac{1}{y^n} F - \frac{y^a}{y^n} \frac{\partial F}{\partial y^a}.$$

Thus

$$\frac{\partial^2 F}{\partial y^a \partial y^n} = -\frac{y^b}{y^n} \frac{\partial^2 F}{\partial y^a \partial y^b}, \quad \frac{\partial^2 F}{(\partial y^n)^2} = \frac{y^a y^b}{y^n y^n} \frac{\partial^2 F}{\partial y^a \partial y^b}.$$

Now consider the Hilbert 2-form $d\theta$. On $\pi^{-1}(U_n^\pm)$ we have $y^a = \epsilon \eta^a y^n$, $x^a = \xi^a + \epsilon \eta^a x^n$, whence

$$dy^a - \epsilon \eta^a dy^n = \epsilon y^n d\eta^a, \quad dx^a - \epsilon \eta^a dx^n = d\xi^a + \epsilon x^n d\eta^a.$$

Now

$$\begin{aligned} d\theta &= \frac{\partial^2 F}{\partial y^a \partial y^b} (dy^a \wedge dx^b - \epsilon \eta^a dy^n \wedge dx^b - \epsilon \eta^b dy^a \wedge dx^n + \eta^a \eta^b dy^n \wedge dx^n) \\ &= \frac{\partial^2 F}{\partial y^a \partial y^b} (dy^a - \epsilon \eta^a dy^n) \wedge (dx^b - \epsilon \eta^b dx^n) \\ &= \epsilon y^n \frac{\partial^2 F}{\partial y^a \partial y^b} d\eta^a \wedge (d\xi^b + \epsilon x^n d\eta^b) = \epsilon y^n \frac{\partial^2 F}{\partial y^a \partial y^b} d\eta^a \wedge d\xi^b \end{aligned}$$

using the symmetry of the coefficients. By the general theory, or an easy calculation, these must be functions on the appropriate coordinate neighbourhoods of Γ . Let me denote by F^\pm the restriction of F to $y^n = \pm 1$. Then on U_n^\pm , $F(y^i) = \epsilon y^n F^\pm(\eta^a)$, whence easily

$$\epsilon y^n \frac{\partial^2 F}{\partial y^a \partial y^b}(y^i) = \frac{\partial^2 F^\pm}{\partial \eta^a \partial \eta^b}(\eta^c).$$

Like each component of the Hessian of F , the right-hand side is invariant under the flow of the flat spray S . So for each a, b the right-hand side is a function on U_n^\pm . Furthermore, from the earlier calculations, for any v^i

$$\frac{\partial^2 F}{\partial y^i \partial y^j} v^i v^j = \frac{\partial^2 F}{\partial y^a \partial y^b} \left(v^a - \frac{y^a}{y^n} v^n \right) \left(v^b - \frac{y^b}{y^n} v^n \right).$$

By assumption, the left-hand side is nonnegative, and zero only if v is a scalar multiple of y . Thus all of three of the bilinear forms whose components are

$$\frac{\partial^2 F}{\partial y^a \partial y^b} \quad \text{and} \quad \frac{\partial^2 F^\pm}{\partial \eta^a \partial \eta^b}$$

must be positive definite (note that $\epsilon y^n = |y^n| > 0$). So the 2-form ω induced on Γ by $d\theta$ is given in U_n^\pm by

$$\omega = \frac{\partial^2 F^\pm}{\partial \eta^a \partial \eta^b} d\eta^a \wedge d\xi^b,$$

where the coefficients are the components of a positive-definite bilinear form. Similar representations hold on the other coordinate patches.

The reflection map on $T^\circ \mathbf{R}^n$ induces a map of Γ , also denoted by ρ , which sends each line to the same line (as a point set) traversed in the opposite direction. For its coordinate representation, we note that ρ maps U_k^+ to U_k^- and vice versa, and in terms of the coordinates on those two sets it is represented by $(\xi, \eta) \mapsto (\xi, -\eta)$. If F is absolutely homogeneous then $F^-(-\eta) = F^+(\eta)$, and so $\rho^*\omega = -\omega$.

Finally, let us consider a function f , positively homogeneous of degree zero, such that

$$\frac{\partial f}{\partial x^i} = \frac{\partial F}{\partial y^i}.$$

We saw earlier that the Hilbert 1-form of F is given by

$$\theta = df - \frac{\partial f}{\partial y^i} dy^i,$$

so that

$$d\theta = -d\left(\frac{\partial f}{\partial y^i} dy^i\right).$$

Let me set $(\partial f / \partial y^i) dy^i = \phi$. The homogeneity condition on f gives

$$y^i \frac{\partial f}{\partial y^i} = 0, \quad \frac{\partial f}{\partial y^i} + y^j \frac{\partial^2 f}{\partial y^i \partial y^j} = 0.$$

Now

$$\begin{aligned} S \lrcorner \phi &= 0; \\ C \lrcorner \phi &= y^i \frac{\partial f}{\partial y^i} = 0; \\ \mathcal{L}_S \phi &= y^j \frac{\partial^2 f}{\partial x^j \partial y^i} dy^i = y^j \frac{\partial^2 F}{\partial y^j \partial y^i} dy^i = 0; \\ \mathcal{L}_C \phi &= \left(y^j \frac{\partial^2 f}{\partial y^i \partial y^j} + \frac{\partial f}{\partial y^i} \right) dy^i = 0. \end{aligned}$$

Thus ϕ passes to the quotient Γ , unlike θ , and defines there a 1-form, say φ . We have $\pi^*(d\varphi) = d\phi = -d\theta = -\pi^*\omega$; but π is surjective, so $\omega = -d\varphi$. Thus ω is exact.

It is easy to see, by a calculation similar to the one leading to the coordinate formula for ω , that the coordinate representation of φ on U_n^\pm is

$$\varphi = \frac{\partial f^\pm}{\partial \eta^a} d\eta^a$$

where f^\pm is the restriction of f to $y^n = \pm 1$.

I now begin the proof of a converse to these properties of ω , that is, the demonstration that a suitable symplectic form on path space determines a projective Finsler function.

Lemma 4. *Let ω be a 2-form on Γ which vanishes on each submanifold of Γ consisting of all the lines through a point of \mathbf{R}^n . Then on U_n^\pm , ω takes the form $\omega = B_{ab}d\eta^a \wedge d\xi^b$, where $B_{ba} = B_{ab}$ (and similarly on the other coordinate patches).*

Proof. For $x_0 \in \mathbf{R}^n$, the submanifold of Γ consisting of the lines through x_0 is $\pi(T_{x_0}^o \mathbf{R}^n)$, the image of the fibre $T_{x_0}^o \mathbf{R}^n$ by the projection π . Now $\pi(T_{x_0}^o \mathbf{R}^n)$ consists of points (ξ^a, η^a) with

$$\xi^a = (x_0^a y^n - x_0^n y^a)/y^n, \quad \eta^a = (y^a/|y^n|),$$

with x_0^i fixed, y^i varying. On eliminating the y^i we find that $\pi(T_{x_0}^o \mathbf{R}^n)$ is given by

$$\xi^a + \epsilon x_0^n \eta^a = x_0^a.$$

Notice that for any point $(\xi^a, \eta^a) \in \Gamma$ and any value of $t \in \mathbf{R}$ we can find x_0^a such that $\pi(T_{x_0}^o \mathbf{R}^n)$ passes through (ξ^a, η^a) and $x_0^n = t$. Now let

$$\omega = A_{ab}d\xi^a \wedge d\xi^b + B_{ab}d\eta^a \wedge d\xi^b + C_{ab}d\eta^a \wedge d\eta^b,$$

where A and C are skew in their indices. Choose any point of Γ , and take an arbitrary real number t . Take the corresponding point $(x_0^i) \in \mathbf{R}^n$ such that $\pi(T_{x_0}^o \mathbf{R}^n)$ passes through the chosen point of Γ , and $x_0^n = t$. On $\pi(T_{x_0}^o \mathbf{R}^n)$ we have $d\xi^a = -\epsilon t d\eta^a$, and so the restriction of ω to that submanifold is

$$(t^2 A_{ab} + \epsilon t B_{ab} + C_{ab})d\eta^a \wedge d\eta^b.$$

By assumption, this must be zero. But t may be chosen arbitrarily, and A and C are skew; thus $A_{ab} = C_{ab} = 0$, $B_{ba} = B_{ab}$, and

$$\omega = B_{ab}d\eta^a \wedge d\xi^b.$$

□

If ω is symplectic then (B_{ab}) must be nonsingular. A symplectic, or even nonsingular, 2-form with the local representation described in the lemma has a well-defined signature.

Lemma 5. *If ω takes the form given in Lemma 4 in each coordinate patch, where each (B_{ab}) is everywhere nonsingular, then all of the bilinear forms (B_{ab}) have the same signature.*

The common signature is called the signature of ω .

Proof. It is enough to consider the effects of the coordinate transformation between U_n^\pm and U_{n-1}^\pm . A short calculation leads to the following transformation rule for the coefficients B_{ab} :

$$\begin{aligned}\hat{B}_{\alpha\beta} &= \delta\eta^{n-1}B_{\alpha\beta} \\ \hat{B}_{\alpha n} &= -\delta\epsilon\left((\eta^{n-1})^2B_{\alpha(n-1)} - \eta^{n-1}\eta^\beta B_{\alpha\beta}\right) \\ \hat{B}_{nn} &= \delta\left((\eta^{n-1})^3B_{(n-1)(n-1)} + 2(\eta^{n-1})^2\eta^\alpha B_{\alpha(n-1)} + \eta^{n-1}\eta^\alpha\eta^\beta B_{\alpha\beta}\right).\end{aligned}$$

This can be written as a matrix formula $\hat{B} = \delta\eta^{n-1}J^TBJ$, where the Jacobian J is given by

$$J_\alpha^\beta = \delta_\alpha^\beta, \quad J_\alpha^{(n-1)} = 0, \quad J_n^a = -\epsilon\eta^a.$$

(It is worth noticing that since the determinant of J is $-\epsilon\eta^{n-1}$, which by assumption is nonzero, J is nonsingular, and so \hat{B} is nonsingular if B is.) But since $\delta\eta^{n-1} = |\eta^{n-1}|$ is positive on the intersection of coordinate patches $U_{n-1}^\pm \cap U_n^\pm$, we see that B and \hat{B} have the same signature.

Now a symmetric matrix cannot change signature without becoming singular; thus B has the same signature everywhere on its coordinate patch, and B and \hat{B} have the same signature on the intersection of coordinate patches; and similarly for all coordinate patches. So the coefficient matrix has the same signature everywhere. \square

Theorem 1. *Suppose that ω is a symplectic 2-form on Γ which vanishes on all submanifolds corresponding to lines through a point of \mathbf{R}^n . Then*

1. $\pi^*\omega = \Omega$ satisfies the hypotheses of Proposition 3 and determines a projective pseudo-Finsler function F on $T^\circ\mathbf{R}^n$ which has the same signature as ω , and $\pi^*\omega$ is the Hilbert 2-form of F ;
2. if $\rho^*\omega = -\omega$ and ω is positive definite then there is a unique projective absolutely homogeneous Finsler function F on $T^\circ\mathbf{R}^n$ such that $\pi^*\omega$ is the Hilbert 2-form of F .

Proof. Consider the pull-back of ω from U_n^+ . To find an expression for it we just have to substitute for ξ^a and η^a in terms of x^i and y^i . Actually it is simpler to substitute just for ξ^a in the first instance. We have $\xi^a = (x^ay^n - x^ny^a)/y^n = x^a - x^n\eta^a$, whence

$$d\xi^a = dx^a - \eta^a dx^n - x^n d\eta^a,$$

so that

$$\pi^*\omega = B_{ab}d\eta^a \wedge (dx^b - \eta^b dx^n - x^n d\eta^b) = B_{ab}d\eta^a \wedge (dx^b - \eta^b dx^n)$$

by symmetry of B_{ab} . Thus

$$\pi^*\omega = (y^n)^{-3}B_{ab}(y^n dy^a - y^a dy^n) \wedge (y^n dx^b - y^b dx^n),$$

so that $\pi^*\omega$ takes the desired form: $\pi^*\omega = \Omega_{ij}dy^i \wedge dx^j$ where

$$\Omega_{ab} = (y^n)^{-1}B_{ab}, \quad \Omega_{an} = -(y^n)^{-2}y^b B_{ab} = \Omega_{na}, \quad \Omega_{nn} = (y^n)^{-3}y^a y^b B_{ab}.$$

The coefficients Ω_{ij} are symmetric in their indices. Moreover

$$\Omega_{aj}u^j = \Omega_{ab}u^b + \Omega_{an}u^n = (y^n)^{-2}B_{ab}(y^n u^b - y^b u^n),$$

which vanishes if and only if u is a scalar multiple of y . It is easy to see that a similar result holds for $\Omega_{nj}u^j = 0$. These results have been established only for one coordinate patch; but of course Ω is globally well-defined (as $\pi^*\omega$), and the calculations above represent fairly what happens on each coordinate patch. Finally, $d\pi^*\omega = \pi^*d\omega = 0$. So $\pi^*\omega$ satisfies the conditions of Proposition 3. The remaining results follow from that proposition and its second corollary. \square

It would be nice to have an intrinsic definition of what it would mean for a symplectic form ω on Γ satisfying the Lagrangian submanifold condition to be positive definite. According to Álvarez Paiva this can be done in terms of 2-planes in \mathbf{R}^n , as follows. Let Π be a 2-plane in \mathbf{R}^n . The set of all oriented lines in Π defines a 2-dimensional submanifold P of Γ . One then considers, for any point l of P (i.e. line l in Π), the restriction of ω to $T_l P$. I now show what happens in my formalism.

Take a 2-plane Π in \mathbf{R}^n . This determines a submanifold $\hat{\Pi}$ of $T^\circ\mathbf{R}^n$ as follows: $(x, y) \in \hat{\Pi}$ if $x \in \Pi$, $y \in T_x\Pi$. Then $\hat{\Pi}$ is 4-dimensional, but both S and C are tangent to it, and its projection into Γ (which is P) is 2-dimensional. Let $(x_0, y_0) \in \hat{\Pi}$. Then Π contains the line $s \mapsto x_0 + sy_0$. Let $u \in T_{x_0}\Pi$ with u linearly independent of y_0 ; then Π is the image of the map $\mathbf{R}^2 \rightarrow \mathbf{R}^n$ by $(s, t) \mapsto x_0 + sy_0 + tu$, and $\hat{\Pi}$ is the image of the map $\mathbf{R}^4 \rightarrow T^\circ\mathbf{R}^n$ by

$$(s, t, k, l) \mapsto (x_0 + sy_0 + tu, ky_0 + lu).$$

The tangent space to $\hat{\Pi}$ at (x_0, y_0) is spanned by

$$y_0^i \frac{\partial}{\partial x^i} = S_{(x_0, y_0)}, \quad y_0^i \frac{\partial}{\partial y^i} = C_{(x_0, y_0)}, \quad u^i \frac{\partial}{\partial x^i}, \quad u^i \frac{\partial}{\partial y^i}.$$

I assume that $y_0^n \neq 0$. Then without loss of generality I can take $x_0^n = 0$, $u^n = 0$. I next determine $T_{\pi(x_0, y_0)}P$. Using coordinates (ξ^a, η^a) corresponding to U_n^+ we have

$$\pi_* \left(\frac{\partial}{\partial x^a} \right) = \frac{\partial}{\partial \xi^a}, \quad \pi_* \left(\frac{\partial}{\partial y^a} \right) = \frac{1}{y^n} \left(\frac{\partial}{\partial \eta^a} - x^n \frac{\partial}{\partial \xi^a} \right).$$

Thus with $x_0^n = 0$ and $u^n = 0$

$$\pi_* \left(u^i \frac{\partial}{\partial x^i} \right) = u^a \frac{\partial}{\partial \xi^a}, \quad \pi_* \left(u^i \frac{\partial}{\partial y^i} \right) = \frac{1}{y^n} u^a \frac{\partial}{\partial \eta^a}$$

So $T_{\pi(x_0, y_0)}P$ is spanned by

$$u^a \frac{\partial}{\partial \xi^a} = \mu, \quad u^a \frac{\partial}{\partial \eta^a} = \nu$$

say. Thus at $\pi(x_0, y_0)$

$$\omega(\mu, \nu) = -B_{ab}u^a u^b.$$

The value of ω on any pair of independent vectors in $T_{\pi(x_0, y_0)}P$ is a nonzero multiple of $B_{ab}u^a u^b$. (There is no essential difference in U_n^- , though some signs are changed). Thus if ω never vanishes when restricted to any such 2-dimensional submanifold P then (B_{ab}) is definite (positive or negative). We cannot determine which on the basis of these data (since one can clearly change the sign of ω without disturbing anything else). However, whichever it is, it is the same everywhere. We have thus established the following theorem of Álvarez Paiva (Theorem 3.1 of [1], with some necessary modifications of the statement).

Theorem 2. *Let ω be a symplectic form on the space of oriented lines of \mathbf{R}^n which has the property that the lines through any given point form a Lagrangian submanifold, and which satisfies $\rho^*\omega = -\omega$. If the pull-back of ω to the space of oriented lines lying on an arbitrary plane never vanishes, then either ω or $-\omega$ is the symplectic form induced by some projective absolutely homogeneous Finsler function on its space of geodesics.*

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References

- [1] J. C. Álvarez Paiva, *Symplectic geometry and Hilbert's fourth problem*, J. Diff. Geom. **69** (2005) 353–378.
- [2] D. Bao, S.-S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer (2000).
- [3] M. Crampin and D.J. Saunders, *Path geometries and almost Grassmann structures*, Adv. Stud. Pure Math. **48** (2007) 225–261.
- [4] M. Dunajski, *Oriented straight lines and twistor correspondence*, Geom. Ded. **112** (2005) 243–251.
- [5] R. L. Lovas, *A note on Finsler-Minkowski norms*, Houston J. Math. **33** (2007) 701–707.
- [6] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer (2001).
- [7] Z. Shen, *Projectively flat Finsler metrics of constant flag curvature*, Trans. Amer. Math. Soc. **355** (2003) 1713–1728.
- [8] J. Szilasi, *Calculus along the tangent bundle projection and projective metrizable*, invited talk at the 10th International Conference on Differential Geometry and its Applications, Olomouc, Czech Republic, 2007; to appear in the Proceedings.