# Kähler and para-Kähler structures associated with Finsler spaces of non-zero constant flag curvature

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## 1 Introduction

The first paper in the issue of the Houston Journal of Mathematics dedicated to the late Professor Shiing-Shen Chern, by R. L. Bryant, is entitled 'Some remarks on Finsler manifolds with constant flag curvature' [2]. The first of his four remarks is that there is a canonical Kähler structure on the space of geodesics, or path space, of a Finsler manifold with constant positive flag curvature. In this paper I wish to suggest a method of constructing a Kähler structure from a Finsler space with constant positive flag curvature which is different from Bryant's method, not least because it produces a structure on the slit tangent bundle rather than the path space; Bryant's structure arises by taking a quotient in an appropriate way. Moreover, this construction applies equally easily to a Finsler space with constant negative flag curvature, but gives in this case a para-Kähler rather than a Kähler structure. This para-Kähler structure has a number of interesting properties, which are explored later in the paper. In the course of the discussion surprisingly many (surprising to me at any rate) alternative equivalent conditions for a Finsler space to be of non-zero constant flag curvature are uncovered.

I can explain the idea behind the construction in the context simply of a spray space. For any vector field $X$ on the base manifold $M$ I denote by $X^{\mathrm{V}}$ and $X^{\mathrm{H}}$ its vertical and horizontal lifts to the slit tangent bundle $T^\circ M$ with respect to the horizontal distribution defined by the spray $\Gamma$. The usual way to define an almost complex structure $J$ in this situation (as in [7, 12] for example) is $J(X^{\mathrm{V}}) = X^{\mathrm{H}}$, $J(X^{\mathrm{H}}) = -X^{\mathrm{V}}$; but this has the drawback that $J$ does not have a homogeneity degree (for a vector field $X$ on the base manifold $M$, $X^{\mathrm{H}}$ is of degree 0 while $X^{\mathrm{V}}$ is of degree $-1$, so $J$ defined in this way changes homogeneity degree non-uniformly). To rectify this take some non-vanishing function $F$ of degree 1, and set

$$J(X^{\mathrm{V}}) = \frac{1}{F}X^{\mathrm{H}}, \qquad J(X^{\mathrm{H}}) = -FX^{\mathrm{V}};$$

$J$ so defined is homogeneous of degree 0, in the sense that $\mathcal{L}_\Delta J = 0$ where $\Delta$ is the Liouville field. (If this isn't self-evident it is easy to prove, as follows. For any vector field $\xi$ on $T^\circ M$, $\mathcal{L}_\Delta J(\xi) = [\Delta, J(\xi)] - J([\Delta, \xi])$; so if one can find a local basis of vector fields $\xi$ such that both $\xi$ and $J(\xi)$ are homogeneous of degree 0 then $\mathcal{L}_\Delta J = 0$. The vector fields $\{X^{\mathrm{H}}, FX^{\mathrm{V}}\}$ as $X$ ranges over a local basis on $M$ form such a basis.)

It is clear that $J^2 = -I$, so $J$ is an almost complex structure. I compute its Nijenhuis torsion $N$, which for an almost complex structure is given by

$$N(\xi, \eta) = -[\xi, \eta] - J([J(\xi), \eta]) - J([\xi, J(\eta)]) + [J(\xi), J(\eta)].$$

For any almost complex structure $J$ the Nijenhuis torsion satisfies $N(J(\xi), J(\eta)) = J(N(\xi, J(\eta))) = -N(\xi, \eta)$, so it is enough to compute say $N(X^{\mathrm{H}}, Y^{\mathrm{H}})$ where $X$ and $Y$ are vector fields on $M$; and in particular $N(H_i, H_j)$ where $H_i$ is the horizontal lift of the $i$th coordinate vector field on $M$. It will be convenient to write $V_i$ for the vertical lift of the $i$th coordinate vector field on $M$ (so that with respect to coordinates $(x^i, u^j)$ $V_i = \partial/\partial u^i$). We have

$$\begin{aligned}
N(H_i, H_j) &= -[H_i, H_j] + J([FV_i, H_j]) + J([H_i, FV_j]) + [FV_i, FV_j] \\
&= -[H_i, H_j] + F(J([V_i, H_j]) + J([H_i, V_j])) \\
&\quad + F^{-1}(H_j(F)H_i - H_i(F)H_j) - F(V_j(F)V_i - V_i(F)V_j) \\
&= R^k_{ij}V_k + F^{-1}(H_j(F)H_i - H_i(F)H_j) - F(V_j(F)V_i - V_i(F)V_j),
\end{aligned}$$

since $[V_i, H_j] = -\Gamma^k_{ij}V_k = [V_j, H_i]$. The necessary and sufficient conditions for the Nijenhuis torsion of $J$ to vanish, so that $J$ is a complex structure, are that

$$H_i(F) = 0 \quad \text{and} \quad R^k_{ij} = F(V_j(F)\delta^k_i - V_i(F)\delta^k_j).$$

(Without the extra factors $F$ the Nijenhuis torsion vanishes if and only if $[H_i, H_j] = 0$, that is, if and only if the spray is R-flat.) If $F$ is a Finsler function and the spray is its canonical geodesic spray then $H_i(F) = 0$. Conversely, a function $F$ which is homogeneous of degree 1 and satisfies $H_i(F) = 0$ is a Finsler function for the spray provided it is also strongly convex; but this part of the condition for the Nijenhuis torsion to vanish could be satisfied in more general circumstances.

Be that as it may, assume that $F$ is a Finsler function, with energy $E$; then $FV_i(F) = V_i(E) = g_{ij}u^j = u_i$, so the rest of the condition is $R^k_{ij} = u_j\delta^k_i - u_i\delta^k_j$, or equivalently $R^k_i = u_j u^j \delta^k_i - u_i u^k = F^2\delta^k_i - u_i u^k$, or

$$R_{ij} = g_{jk}R^k_i = g_{ik}R^k_j = (g_{kl}u^k u^l)g_{ij} - g_{ik}u^k g_{jl}u^l;$$

that is, the Finsler space is of flag curvature 1.

The Cartan 2-form of the energy, $\omega_E$, is given by $\omega_E = g_{ij}\phi^i \wedge dx^j$ where $\{dx^i, \phi^j\}$ is the 1-form basis dual to $\{H_i, V_j\}$. It is clear that $\omega_E(J(\cdot), J(\cdot)) = \omega_E(\cdot, \cdot)$; moreover $\omega_E$ is exact. Thus for a Finsler space of flag curvature 1 the complex structure $J$ and symplectic

2-form $\omega_E$ define a Kähler structure on $T^\circ M$. The associated Hermitian metric $G$, which is defined by $G(\xi,\eta) = \omega_E(J(\xi),\eta)$, is not quite the Sasaki metric generated by $g$; we have in fact

$$G(H_i, H_j) = F g_{ij}, \qquad G(V_i, V_j) = F^{-1} g_{ij}, \qquad G(H_i, V_j) = 0.$$

It is natural to ask whether a similar construction will work for Finsler spaces of constant negative flag curvature. I will show that similar steps lead to a para-Kähler rather than a Kähler structure.

## 2 The construction in general

Let's go back to the beginning and consider analoguous structures associated with Finsler spaces of constant flag curvature other than 1. For any constants $\varepsilon_1$, $\varepsilon_2$, define a type $(1,1)$ tensor field by

$$J(X^{\mathrm{V}}) = \varepsilon_1 \frac{1}{F} X^{\mathrm{H}}, \qquad J(X^{\mathrm{H}}) = \varepsilon_2 F X^{\mathrm{V}}.$$

Then $J$ is homogeneous of degree 0; $J^2 = \varepsilon_1\varepsilon_2 I$. The Nijenhuis torsion $N$ of $J$ is given by

$$N(\xi,\eta) = \varepsilon_1\varepsilon_2[\xi,\eta] - J([J(\xi),\eta]) - J([\xi,J(\eta)]) + [J(\xi),J(\eta)]$$

and satisfies $N(J(\xi),J(\eta)) = -J(N(\xi,J(\eta))) = \varepsilon_1\varepsilon_2 N(\xi,\eta)$. So provided $\varepsilon_1\varepsilon_2 \neq 0$, which I now assume to be the case, it is enough to compute $N(H_i,H_j)$ as before. We have

$$N(H_i,H_j) = -\varepsilon_1\varepsilon_2 R^k_{ij} V_k - \varepsilon_2^2 F(V_j(F)V_i - V_i(F)V_j).$$

So $J$ has vanishing torsion if and only if $-\varepsilon_1\varepsilon_2 R^k_{ij} = \varepsilon_2^2(u_j\delta^k_i - u_i\delta^k_j)$, that is, if and only if

$$R_{ij} = -\frac{\varepsilon_2}{\varepsilon_1}\left((g_{kl}u^k u^l)g_{ij} - g_{ik}u^k g_{jl}u^l\right);$$

so finally the torsion vanishes if and only if the Finsler space is of constant flag curvature $\kappa = -\varepsilon_2/\varepsilon_1$.

Now the effect of making a constant change of scale of $F$, say $F \mapsto \lambda F$, while keeping the same $J$, is to change $\varepsilon_1$ to $\lambda\varepsilon_1$ and $\varepsilon_2$ to $\varepsilon_2/\lambda$; thus $\kappa$ is transformed to $\lambda^2\kappa$. Thus once the cases $\kappa = \pm 1$ are analysed, all others with $\kappa \neq 0$ can then be accommodated by rescaling $F$. In particular, if we take

$$J(X^{\mathrm{V}}) = \frac{1}{F} X^{\mathrm{H}}, \qquad J(X^{\mathrm{H}}) = -\varepsilon F X^{\mathrm{V}}$$

with $\varepsilon = \pm 1$ then $J^2 = -\varepsilon I$ and $J$ has vanishing torsion if and only if the Finsler space has constant flag curvature $\varepsilon$. With $\varepsilon = 1$ we get a Finsler space of constant flag curvature 1, and associated with it a type $(1,1)$ tensor field $J$ on $T^\circ M$ such that $J^2 = -I$, whose Nijenhuis torsion vanishes, that is, a complex structure, as we saw before. When $\varepsilon = -1$, on the other hand, we get a Finsler space of constant flag curvature $-1$, and associated with it a type $(1,1)$ tensor field $J$ on $T^\circ M$ such that $J^2 = I$, whose Nijenhuis torsion vanishes. A type $(1,1)$ tensor field $J$ on $T^\circ M$ such that $J^2 = I$ is an almost product structure; when the torsion of $J$ vanishes we have a product structure.

Henceforth I restrict my attention to the cases $\varepsilon = \pm 1$.

The Cartan 2-form $\omega_E$ satisfies $\omega_E(J(\cdot), J(\cdot)) = \varepsilon\omega_E(\cdot,\cdot)$, or equivalently

$$\omega_E(J(\xi), \eta) + \omega_E(\xi, J(\eta)) = 0$$

(in either case). If we set $G(\xi, \eta) = \omega_E(J(\xi), \eta)$, then

$$G(\eta, \xi) = -\omega_E(\xi, J(\eta)) = -\varepsilon\omega_E(J(\xi), J^2(\eta)) = \omega_E(J(\xi), \eta) = G(\xi, \eta),$$

so $G$ is a symmetric covariant 2-tensor on $T^\circ M$. Moreover

$$G(J(\xi), J(\eta)) = \omega_E(J^2(\xi), J(\eta)) = \varepsilon G(\xi, \eta),$$

or equivalently

$$G(J(\xi), \eta) + G(\xi, J(\eta)) = 0$$

(in either case). In particular,

$$G(H_i, H_j) = F g_{ij}, \qquad G(V_i, V_j) = \varepsilon F^{-1} g_{ij}, \qquad G(H_i, V_j) = 0.$$

When $\varepsilon = 1$ we obtain a Kähler structure, as discussed in the Introduction. When $\varepsilon = -1$ on the other hand, $G$ is of split or neutral signature; that is to say, it defines an ultra-hyperbolic metric on $T^\circ M$. The structure we obtain in this case is called a para-Kähler structure [5], a hyperbolic Kähler structure [9] or a bi-Lagrangian structure [6].

In the case of a Kähler structure we have $\nabla J = 0$; I now show that this holds also in the para-Kähler case. The following calculation is based on that for the Kähler case given by Kobayashi and Nomizu [8], and works for both cases simultaneously. Using the definition of $G$ in terms of $\omega = \omega_E$ (it will be convenient to omit the subscript $E$ for now), and the fact that $\omega$ is closed, in the usual Koszul formula for $2G(\nabla_\xi \eta, \zeta)$ one finds that

$$\begin{aligned} 2G(\nabla_\xi \eta, \zeta) &= \xi\omega(J\eta, \zeta) + \omega(J[\xi, \eta], \zeta) - \omega(J[\xi, \zeta], \eta) \\ &\quad + (J\xi)\omega(\eta, \zeta) - \omega([J\xi, \eta], \zeta) + \omega([J\xi, \zeta], \eta). \end{aligned}$$

Thus

$$\begin{aligned} 2G((\nabla_\xi J)\eta, \zeta) &= 2G(\nabla_\xi (J\eta), \zeta) - 2G(J(\nabla_\xi \eta), \zeta) \\ &= 2G(\nabla_\xi (J\eta), \zeta) + 2G(\nabla_\xi \eta, J\zeta) \\ &= -\varepsilon\xi\omega(\eta, \zeta) + \omega(J[\xi, J\eta], \zeta) - \omega(J[\xi, \zeta], J\eta) \\ &\quad + (J\xi)\omega(J\eta, \zeta) - \omega([J\xi, J\eta], \zeta) + \omega([J\xi, \zeta], J\eta) \end{aligned}$$

$$
\begin{aligned}
&\quad + \xi\omega(J\eta, J\zeta) + \omega(J[\xi,\eta], J\zeta) - \omega(J[\xi, J\zeta], \eta) \\
&\quad + (J\xi)\omega(\eta, J\zeta) - \omega([J\xi,\eta], J\zeta) + \omega([J\xi, J\zeta], \eta) \\
&= -\varepsilon\xi\omega(\eta,\zeta) + \xi\omega(J\eta, J\zeta) + (J\xi)\omega(J\eta,\zeta) + (J\xi)\omega(\eta, J\zeta) \\
&\quad + \omega(J[\xi, J\eta],\zeta) - \omega([J\xi, J\eta],\zeta) + \omega(J[\xi,\eta], J\zeta) - \omega([J\xi,\eta], J\zeta) \\
&\quad - \omega(J[\xi,\zeta], J\eta) + \omega([J\xi,\zeta], J\eta) - \omega(J[\xi, J\zeta],\eta) + \omega([J\xi, J\zeta],\eta \\
&= \omega(J[\xi, J\eta],\zeta) - \omega([J\xi, J\eta],\zeta) - \omega([\xi,\eta],\zeta) + \omega(J[J\xi,\eta],\zeta) \\
&\quad + \omega([\xi,\zeta],\eta) - \omega(J[J\xi,\zeta],\eta) - \omega(J[\xi, J\zeta],\eta) + \omega([J\xi, J\zeta],\eta) \\
&= -\omega(N(\xi,\eta),\zeta) + \omega(N(\xi,\zeta),\eta).
\end{aligned}
$$

Thus if the Nijenhuis torsion $N$ of $J$ vanishes then $\nabla J = 0$. Conversely, assume that $\nabla J = 0$: then

$$
\begin{aligned}
\omega(N(\xi,\eta),\zeta) &= \omega(N(\xi,\zeta),\eta) = -\omega(N(\zeta,\xi),\eta) = -\omega(N(\zeta,\eta),\xi) \\
&= \omega(N(\eta,\zeta),\xi) = \omega(N(\eta,\xi),\zeta) = -\omega(N(\xi,\eta),\zeta),
\end{aligned}
$$

whence $N = 0$.

It follows from the relationship between $G$, $J$ and $\omega$ that the vanishing of $\nabla J$ is equivalent to the vanishing of $\nabla\omega$.

I now show that it is possible to express the condition for a Finsler space to have flag curvature $\pm 1$ in terms of the invariance of $J$ along the flow lines of the geodesic field $\Gamma$. To be exact, I shall show that $\mathcal{L}_{\hat\Gamma} J = 0$ where $\hat\Gamma = F^{-1}\Gamma$, a vector field which is homogeneous of degree 0.

Note first that

$$
\begin{aligned}
[\Gamma, H_i] &= [u^j H_j, H_i] = -u^j R^k_{ji} V_k + \Gamma^j_i H_j = R^j_i V_j + \Gamma^j_i H_j, \\
[\Gamma, V_i] &= [u^j H_j, V_i] = u^j \Gamma^k_{ij} V_k - H_i = \Gamma^j_i V_j - H_i.
\end{aligned}
$$

Also, for any tensor $T$ whose square is a constant multiple of the identity and any vector field $Z$, $(\mathcal{L}_Z T)\circ T = -T\circ(\mathcal{L}_Z T)$, so it is enough to calculate (say) $\mathcal{L}_{\hat\Gamma} J(H_i)$. But

$$
\begin{aligned}
\mathcal{L}_{\hat\Gamma} J(H_i) &= -\varepsilon[F^{-1}\Gamma, FV_i] - J([F^{-1}\Gamma, H_i]) \\
&= -\varepsilon[\Gamma, V_i] - \varepsilon F^{-1} V_i(F)\Gamma - F^{-1} J([\Gamma, H_i]) \\
&= -\varepsilon\Gamma^j_i V_j + \varepsilon H_i - \varepsilon F^{-1} V_i(F)\Gamma - F^{-1}(F^{-1} R^j_i H_j - \varepsilon F \Gamma^j_i V_j) \\
&= -F^{-2}(R^j_i - \varepsilon(F^2\delta^j_i - u_i u^j))H_j.
\end{aligned}
$$

Thus $R^j_i = \varepsilon(F^2\delta^j_i - u_i u^j)$, and the Finsler space is of flag curvature $\varepsilon$, if and only if $\mathcal{L}_{\hat\Gamma} J(H_i) = 0$, that is, if and only if $\mathcal{L}_{\hat\Gamma} J = 0$.

Recall that the metric $G$ is given by $G(\xi,\eta) = \omega_E(J(\xi),\eta)$. It follows that

$$
\mathcal{L}_{\hat\Gamma} G(\xi,\eta) = \mathcal{L}_{\hat\Gamma}\omega_E(J(\xi),\eta) + \omega_E(\mathcal{L}_{\hat\Gamma} J(\xi),\eta).
$$

But $\mathcal{L}_{\hat{\Gamma}}\omega_E = -d(F^{-1}dE) = -d(dF) = 0$, so $\mathcal{L}_{\hat{\Gamma}}G = 0$ if and only if $\mathcal{L}_{\hat{\Gamma}}J = 0$. But as we have just seen, the vanishing of $\mathcal{L}_{\hat{\Gamma}}J$ is a necessary and sufficient condition for the Finsler space to have constant flag curvature $\varepsilon$; so an alternative necessary and sufficient condition is that $\hat{\Gamma}$ is a Killing field of $G$. (This result has been obtained, for $\varepsilon = 1$, in somewhat different forms, by Bryant [2], and Bejancu and Farran [3].)

It is worth recording the various equivalent ways of stating that a Finsler function has flag curvature $\pm 1$ in the form of a theorem.

**Theorem** Consider a Finsler space with Finsler function $F$ and canonical geodesic vector field $\Gamma$. For $\varepsilon = \pm 1$, define a type $(1,1)$ tensor field $J$ on $T^\circ M$ by

$$J(X^{\mathrm{V}}) = \frac{1}{F}X^{\mathrm{H}}, \qquad J(X^{\mathrm{H}}) = -\varepsilon F X^{\mathrm{V}},$$

where $X$ is any vector field on $M$ and $X^{\mathrm{V}}$, $X^{\mathrm{H}}$ its vertical, horizontal lift (with respect to the horizontal distribution defined by $\Gamma$). Define the metric tensor $G$ on $T^\circ M$ by $G(\xi, \eta) = \omega_E(J(\xi), \eta)$ where $\omega_E$ is the Cartan 2-form of the energy of $F$, let $\nabla$ denote the covariant derivative operator of the Levi-Civita connection of $G$, and set $\hat{\Gamma} = F^{-1}\Gamma$.

The following conditions on the Finsler space are equivalent:

- the Nijenhuis torsion of $J$ vanishes;
- $\nabla J = 0$;
- $\nabla \omega_E = 0$;
- $\mathcal{L}_{\hat{\Gamma}}J = 0$;
- $\mathcal{L}_{\hat{\Gamma}}G = 0$;
- the space has constant flag curvature $\varepsilon$. ∎

# 3 Flag curvature 1: Bryant's construction

For a Finsler space of flag curvature 1 we have a Kähler structure: but it is defined on $T^\circ M$, whereas Bryant's [2] is on the path space. To obtain Bryant's Kähler structure one needs to pass to the path space. Bryant himself obtains the path space by first restricting to the indicatrix bundle and then quotienting by the geodesic spray; but equally one can quotient $T^\circ M$ directly by the 2-dimensional distribution $\mathcal{D}$ spanned by $\Gamma$ and $\Delta$ (which is integrable for any spray). (I should here enter the usual caveat that the quotient may not be well-defined as a differentiable manifold, but that this difficulty can be avoided either by restricting attention to those situations where it is, or by working locally.) In any case one has to consider the Lie derivative of $J$ by $\Gamma$, or some multiple of it: it is of course preferable to use $F^{-1}\Gamma = \hat{\Gamma}$, since as we know $\mathcal{L}_{\hat{\Gamma}}J = 0$.

Clearly $J(\mathcal{D}) \subset \mathcal{D}$; and we know already that $\mathcal{L}_\Delta J = 0$. It follows that for any vector field $\xi \in \mathcal{D}$ and for any vector field $\eta$ whatsoever, $\mathcal{L}_\xi J(\eta) \in \mathcal{D}$. Thus $J$ passes to the quotient to define a type $(1,1)$ tensor field $\tilde{J}$ there, which is easily seen to be a complex structure.

The symplectic 2-form won't pass to the quotient — for one thing, it is homogeneous of degree 1. Consider instead $\omega_F$, the exterior derivative of the Hilbert 1-form:

$$\omega_F = \frac{\partial^2 F}{\partial u^i \partial u^j}\phi^i \wedge dx^j = \frac{1}{F} h_{ij}\phi^i \wedge dx^j$$

where $h_{ij}$ is the angular metric,

$$h_{ij} = g_{ij} - \frac{1}{F^2} u_i u_j.$$

Now $\omega_F$ is singular, its characteristic distribution being $\mathcal{D}$, and it is of course closed; it follows that $\mathcal{L}_\xi \omega_F = 0$ for all $\xi \in \mathcal{D}$, so $\omega_F$ passes to the quotient, to define a non-singular 2-form $\tilde{\omega}_F$ on the path space. Moreover $\omega_F$ is clearly invariant by $J$, from which it follows that $\tilde{\omega}_F$ is invariant by $\tilde{J}$. Let $\pi$ be the projection onto the path space: then $\omega_F = \pi^*\tilde{\omega}_F$, so $\pi^* d\tilde{\omega}_F = d\omega_F = 0$, and since $\pi$ is surjective $\tilde{\omega}_F$ is closed (but not necessarily exact). So we have a Kähler structure on the path space.

Notice that

$$\omega_E = d\left(\frac{\partial E}{\partial u^i} dx^i\right) = d\left(F\frac{\partial F}{\partial u^i} dx^i\right) = F\omega_F + dF \wedge \left(\frac{\partial F}{\partial u^i} dx^i\right);$$

thus if $\mathcal{I}$ is the indicatrix sub-bundle of $T^\circ M$ (the level set of $F$ of value 1) and $\iota : \mathcal{I} \to T^\circ M$ is the imbedding then $\iota^*\omega_E = \iota^*\omega_F$: the 2-forms $\omega_E$ and $\omega_F$ agree when restricted to $\mathcal{I}$. So Bryant's construction is closely related to the one described here.

## 4 Flag curvature −1: properties of the para-Kähler structure

Associated with a Finsler space of constant flag curvature $-1$ there is a type $(1,1)$ tensor field $J$ on $T^\circ M$ such that $J^2 = I$, whose Nijenhuis torsion vanishes; that is to say, $T^\circ M$ is equipped with a product structure. The tensor $J$, considered as a linear map of tangent spaces, has eigenvalues $\pm 1$; in the case under consideration the eigenspace corresponding to the eigenvalue $+1$ is the $n$-dimensional space spanned by the vectors $H_i + FV_i$ at each point, while that corresponding to the eigenvalue $-1$ is spanned by the vectors $H_i - FV_i$. The corresponding distributions, say $\mathcal{D}^+$ and $\mathcal{D}^-$, are both integrable, since the torsion of $J$ vanishes. In fact it is easy to see by a direct calculation that

$$[H_i + FV_i, H_j + FV_j] = 0 = [H_i - FV_i, H_j - FV_j]$$

in a Finsler space of flag curvature $-1$. This gives another way of specifying that a Finsler space has flag curvature $-1$; to put it more invariantly, a Finsler space $(M, F)$ has flag curvature $-1$ if and only if the map $X \mapsto X^{\mathrm{H}} + FX^{\mathrm{V}}$ (from vector fields on $M$ to projectable vector fields on $T^{\circ}M$) is a Lie algebra homomorphism.

The fact that the eigenvector distributions $\mathcal{D}^{+}$ and $\mathcal{D}^{-}$ are both integrable is sufficient as well as necessary for $J$ to have vanishing Nijenhuis torsion $N$, as I now show. The torsion is given by

$$N(\xi, \eta) = [\xi, \eta] - J([J(\xi), \eta]) - J([\xi, J(\eta)]) + [J(\xi), J(\eta)].$$

Since $\mathcal{D}^{+}$ and $\mathcal{D}^{-}$ are complementary it is enough to consider $N(\xi, \eta)$ when $\xi$ and $\eta$ both come from the same distribution, and when one of $\xi$, $\eta$ comes from one, the other from the other. When $\xi \in \mathcal{D}^{+}$ and $\eta \in \mathcal{D}^{-}$

$$N(\xi, \eta) = [\xi, \eta] - J([\xi, \eta]) + J([\xi, \eta]) - [\xi, \eta] = 0,$$

which is to say that $N(\xi, \eta)$ vanishes without further conditions. When $\xi, \eta \in \mathcal{D}^{+}$

$$N(\xi, \eta) = [\xi, \eta] - J([\xi, \eta]) - J([\xi, \eta]) + [\xi, \eta],$$

so $N(\xi, \eta) = 0$ if and only if $J([\xi, \eta]) = [\xi, \eta]$, that is, if and only if $[\xi, \eta] \in \mathcal{D}^{+}$. Similarly, when $\xi, \eta \in \mathcal{D}^{-}$ one finds that $N(\xi, \eta) = 0$ if and only if $J([\xi, \eta]) = -[\xi, \eta]$, that is, if and only if $[\xi, \eta] \in \mathcal{D}^{-}$. Thus $N = 0$ if and only if $\mathcal{D}^{+}$ and $\mathcal{D}^{-}$ are both integrable.

So in addition to the various ways of specifying that a Finsler space has constant flag curvature given in the theorem in Section 2 we have the following theorem.

**Theorem** A Finsler space has flag curvature $-1$ if and only if any of the following equivalent properties holds:

- $[H_i + FV_i, H_j + FV_j] = 0$, or equally $[H_i - FV_i, H_j - FV_j] = 0$;
- either of the maps $X \mapsto X^{\mathrm{H}} \pm FX^{\mathrm{V}}$ is a Lie algebra homomorphism from vector fields on $M$ to projectable vector fields on $T^{\circ}M$;
- the eigendistributions $\mathcal{D}^{\pm}$ of $J$ (which are spanned by the $H_i \pm FV_i$) are both integrable. ∎

In the remainder of this section I shall deal with a number of properties of the para-Kähler structure.

## Projective Finsler metrics

The theorem has an interesting interpretation when $\Gamma$ is projectively flat, that is, projectively equivalent to the flat spray $u^i \partial/\partial x^i$; we are then dealing with a so-called projective

Finsler metric of flag curvature $-1$. Shen has discussed projective Finsler metrics of constant flag curvature in [11]. Suppose that

$$\Gamma = u^i \frac{\partial}{\partial x^i} - 2Pu^i \frac{\partial}{\partial u^i} = u^i \frac{\partial}{\partial x^i} - 2P\Delta,$$

in which case $P$ is the so-called projective factor of $F$. Shen shows (Lemma 3.2 for the case of flag curvature $-1$) that

$$\begin{aligned} \frac{\partial F}{\partial x^i} &= F\frac{\partial P}{\partial u^i} + P\frac{\partial F}{\partial u^i} \\ \frac{\partial P}{\partial x^i} &= P\frac{\partial P}{\partial u^i} + F\frac{\partial F}{\partial u^i}, \end{aligned}$$

are necessary and sufficient conditions for the flag curvature of the projective Finsler metric $F$ to be $-1$. Now in this case

$$H_i = \frac{\partial}{\partial x^i} - P\frac{\partial}{\partial u^i} - \frac{\partial P}{\partial u^i}\Delta,$$

whence

$$H_i(F) = \frac{\partial F}{\partial x^i} - P\frac{\partial F}{\partial u^i} - \frac{\partial P}{\partial u^i}F,$$

and the first of Shen's conditions merely expresses the fact that $\Gamma$ is the canonical geodesic field of $F$. Furthermore,

$$H_i + FV_i = \frac{\partial}{\partial x^i} + (F-P)\frac{\partial}{\partial u^i} - \frac{\partial P}{\partial u^i}\Delta,$$

whence (with $f = F - P$ for convenience)

$$\begin{aligned} [H_i + FV_i, H_j + FV_j] &= -\left(\frac{\partial f}{\partial x^j} + f\frac{\partial f}{\partial u^j}\right)\frac{\partial}{\partial u^i} + \left(\frac{\partial f}{\partial x^i} + f\frac{\partial f}{\partial u^i}\right)\frac{\partial}{\partial u^j} \\ &\quad - \left(\frac{\partial^2 P}{\partial x^i \partial u^j} - \frac{\partial^2 P}{\partial x^j \partial u^i}\right)\Delta. \end{aligned}$$

Thus provided the base dimension is at least 3, $[H_i + FV_i, H_j + FV_j] = 0$ if and only if

$$\frac{\partial f}{\partial x^i} + f\frac{\partial f}{\partial u^i} = 0 \quad \text{and} \quad \frac{\partial^2 P}{\partial x^i \partial u^j} - \frac{\partial^2 P}{\partial x^j \partial u^i} = 0.$$

The first of these is

$$\frac{\partial P}{\partial x^i} = \frac{\partial F}{\partial x^i} + (F-P)\left(\frac{\partial F}{\partial u^i} - \frac{\partial P}{\partial u^i}\right),$$

which gives Shen's second condition when account is taken of the first. When Shen's second condition holds, $\partial P/\partial x^i$ is of the form $\partial\phi/\partial u^i$, and so the second condition for the vanishing of $[H_i + FV_i, H_j + FV_j]$ is automatically satisfied. Thus Shen's second condition for the flag curvature of the projective Finsler metric $F$ to be $-1$ is equivalent to the vanishing of $[H_i + FV_i, H_j + FV_j]$.

## The eigendistributions and product structure

Since $\omega_E(J(\xi), J(\eta)) = -\omega_E(\xi, \eta)$ when $\varepsilon = -1$, we see that the eigendistributions $\mathcal{D}^\pm$ are both Lagrangian. Thus when the Finsler space has flag curvature $-1$ the symplectic form $\omega_E$ has a pair of transverse Lagrangian foliations, whose tangent vectors are the eigenvectors of $J$. In such a case the symplectic structure is said to be bi-Lagrangian [6].

I shall write $P_i$ for $H_i + FV_i$ and $Q_i$ for $H_i - FV_i$, for convenience, so that $\mathcal{D}^+ = \langle P_i \rangle$ and $\mathcal{D}^- = \langle Q_i \rangle$. The tensor $G$ given by $G(\xi, \eta) = \omega_E(J(\xi), \eta)$ defines an ultra-hyperbolic metric on $T^\circ M$ and we have

$$G(H_i, H_j) = F g_{ij}, \qquad G(V_i, V_j) = -F^{-1} g_{ij}, \qquad G(H_i, V_j) = 0,$$

whence

$$G(P_i, P_j) = 0 = G(Q_i, Q_j), \qquad G(P_i, Q_j) = 2F g_{ij}.$$

In particular, the distributions $\mathcal{D}^\pm$ are totally null, and therefore their integral submanifolds are totally null (that is to say, the symmetric 2-covariant tensor field induced by $G$ on any integral submanifold of either distribution is the zero field).

The product structure makes $T^\circ M$, at least locally, into a product of two $n$-dimensional manifolds $M^+ \times M^-$, where for each $z^- \in M^-$ the submanifold $(M^+, z^-)$ is a leaf of the distribution $\mathcal{D}^+$, and for each $z^+ \in M^+$ the submanifold $(z^+, M^-)$ is a leaf of the distribution $\mathcal{D}^-$.

Though $[P_i, P_j] = [Q_i, Q_j] = 0$, these vector fields are not coordinate fields because $[P_i, Q_j] \neq 0$: in fact

$$[P_i, Q_j] = -2(F\Gamma_{ij}^k + R_{ij}^k)V_k = -(\Gamma_{ij}^k + F^{-1}R_{ij}^k)(P_k - Q_k).$$

On the other hand, there are coordinates adapted to the product structure, say $(v^i, w^j)$, such that the $\partial/\partial v^i$ span $\mathcal{D}^+$ and the $\partial/\partial w^i$ span $\mathcal{D}^-$. I shall now show that $G$ has a simple expression in terms of such coordinates: in fact

$$G\left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial w^j}\right) = -2\frac{\partial^2 F}{\partial v^i \partial w^j}$$

and of course

$$G\left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j}\right) = G\left(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial w^j}\right) = 0.$$

Now

$$\frac{\partial}{\partial v^i} = A_i^k P_k, \qquad \frac{\partial}{\partial w^j} = B_j^l Q_l$$

for some non-singular-matrix-valued functions $A$ and $B$. Then

$$[A_i^k P_k, A_j^l B_l] = [B_i^k Q_k, B_j^l Q_l] = [A_i^k P_k, B_j^l Q_l] = 0,$$

whence $A_i^l P_l(A_j^k) = A_j^l P_l(A_i^k)$, $B_i^l Q_l(B_j^k) = B_j^l Q_l(B_i^k)$, and

$$A_i^l P_l(B_j^k) = B_i^l Q_l(A_j^k) = -A_i^m B_j^n(\Gamma_{mn}^{\ k} + F^{-1}R_{mn}^k).$$

Clearly

$$G\left(\frac{\partial}{\partial v^i},\frac{\partial}{\partial w^j}\right) = 2FA_i^kB_j^lg_{kl}.$$

On the other hand

$$\frac{\partial F}{\partial w^j} = B_j^lQ_l(F) = -B_j^lu_l,$$

whence

$$\frac{\partial^2 F}{\partial v^i\partial w^j} = -A_i^kP_k(B_j^lu_l) = A_i^kB_j^l((\Gamma_{kl}^m + F^{-1}R_{kl}^m)u_m - P_k(u_l)) = -FA_i^kB_j^lg_{kl}$$

since $R_{kl}^m u_m = (u_k\delta_l^m - u_l\delta_k^m)u_m = 0$ and $P_k(u_l) = \Gamma_{kl}^m u_m + Fg_{kl}$.

## Jacobi fields

The distributions $\mathcal{D}^{\pm}$ have an interesting relationship to the Jacobi fields of $\Gamma$. To all intents and purposes, a Jacobi field is a vector field along an integral curve of $\Gamma$ which is Lie transported by the flow of $\Gamma$. To be more precise, a Jacobi field is a vector field on $M$ along a geodesic of the Finsler space, that is, a base integral curve of $\Gamma$, which is the projection of a vector field along an integral curve of $\Gamma$ which is Lie transported. To obtain the Jacobi equation — the differential equation which determines Jacobi fields — one has therefore merely to consider the equation for Lie transport of a vector field $\xi$, namely $\mathcal{L}_\Gamma\xi = 0$, or $[\Gamma,\xi] = 0$. I shall deal with this equation by first expressing $\xi$ in terms of its horizontal and vertical components, so I write $\xi = \lambda^iH_i + \mu^iV_i$: then by the formulae for $[\Gamma, H_i]$ and $[\Gamma, V_i]$ given earlier

$$[\Gamma,\xi] = (\mathrm{D}\lambda^i - \mu^i)H_i + (\mathrm{D}\mu^i - F^2\lambda^i + (g_{jk}\lambda^ju^k)u^i)V_i.$$

Here D is the so-called dynamical covariant derivative of $\Gamma$:

$$\mathrm{D}\nu^i = \dot{\nu}^i + \Gamma_j^i\nu^j;$$

it is the Berwald covariant derivative in the direction of $\Gamma$, and has the following important properties (see for example [4, 10]): $\mathrm{D}u^i = 0$ and $\mathrm{D}g_{jk} = 0$. (This operator is closely related, but not identical, to the 'dynamical derivative' used by Foulon in [7].) The condition for $\xi$ to be Lie transported along an integral curve $\gamma(t)$ of $\Gamma$ therefore amounts to

$$\mu^i = \mathrm{D}\lambda^i, \quad \mathrm{D}\mu^i = \mathrm{D}^2\lambda^i = F^2\lambda^i - (g_{jk}\lambda^ju^k)u^i.$$

The latter equation is the Jacobi equation. Note that it is satisfied by $\lambda^i = u^i$ and $\lambda^i = tu^i$, as one might expect. Now

$$\frac{d^2}{dt^2}(g_{jk}\lambda^ju^k) = g_{jk}u^k\mathrm{D}^2\lambda^j = 0,$$

so if at $\gamma(0)$ (say) $g_{jk}\lambda^j u^k = 0$ and $d/dt(g_{jk}\lambda^j u^k) = 0$ then $g_{jk}\lambda^j u^k = 0$ all along the curve. I restrict my attention to those solutions of the Jacobi equation for which this condition holds: they evidently satisfy

$$\mathrm{D}^2\lambda^i = F^2\lambda^i.$$

Now consider the components of $\xi$ with respect to the vector field basis adapted to $\mathcal{D}^{\pm}$: say $\xi = \eta^i P_i + \zeta^i Q_i$. Then

$$\eta^i = \frac{1}{2F}(\mathrm{D}\lambda^i + F\lambda^i), \quad \zeta^i = -\frac{1}{2F}(\mathrm{D}\lambda^i - F\lambda^i).$$

Now $F$ is of course constant on integral curves of $\Gamma$, so we have

$$\begin{aligned} \mathrm{D}\eta^i &= \frac{1}{2F}(\mathrm{D}^2\lambda^i + F\mathrm{D}\lambda^i) = F\eta^i \\ \mathrm{D}\zeta^i &= -\frac{1}{2F}(\mathrm{D}^2\lambda^i - F\mathrm{D}\lambda^i) = -F\zeta. \end{aligned}$$

From the first of these, $\eta_0^i = \exp(-Ft)\eta^i$ is parallel along $\gamma$ in the sense that $\mathrm{D}\eta_0^i = 0$; or to put it the other way round $\eta^i = \exp(Ft)\eta_0^i$ where $\eta_0^i$ is parallel. Likewise $\zeta^i = \exp(-Ft)\zeta_0^i$ where $\zeta_0^i$ is parallel.

Evidently if $g_{ij}\eta^i u^j$ vanishes at any point of $\gamma$ it vanishes at every point of $\gamma$, and then $(\eta^i)$ is a Jacobi field, with corresponding Lie transported vector field $\eta^i P_i$; and similarly for $\zeta^i$. We therefore have the following result.

**Theorem** Let $\eta_0$ be a vector field along a geodesic which is parallel with respect to D and orthogonal to the geodesic with respect to $g$, and set $\eta = \exp(Ft)\eta_0$ ($\eta = \exp(-Ft)\eta_0$); then $\eta$ is a Jacobi field orthogonal to the geodesic, and the corresponding Lie transported vector field is $\eta^i P_i$ ($\eta^i Q_i$). Moreover, the Jacobi fields of these forms span the space of all Jacobi fields orthogonal to the geodesic. ∎

I imagine that this is the analogue in the negative flag curvature case of the 'finer structure' of a 'canonical circle of totally real $n$-planes passing through each point' of the path space in the positive flag curvature case discussed by Bryant [2].

We can take this line of argument somewhat further. Let me denote by $\mathcal{D}_0^{\pm}$ the sub-distributions of $\mathcal{D}^{\pm}$ consisting of those $\eta^i P_i$ ($\eta^i Q_i$) such that $g_{ij}\eta^i u^j = 0 = \eta^i u_i$. I shall show that the distributions $\mathcal{D}_0^{\pm}$ are both integrable also. The proof depends on the observation that $H_i(u_j) = \Gamma_{ij}^k u_k = H_j(u_i)$, which comes from the fact that $H_i(F) = 0$ on partial differentiation with respect to $u^j$, when one recalls that $[H_i, V_j] = \Gamma_{ij}^k V_k$. Now

$$[\eta^i P_i, \zeta^j P_j] = (\eta^j P_j(\zeta^i) - \zeta^j P_j(\eta^i))P_i;$$

I have to show that if $\eta^i u_i = \zeta^i u_i = 0$ then $(\eta^j P_j(\zeta^i) - \zeta^j P_j(\eta^i))u_i = 0$. But

$$\begin{aligned} (\eta^j P_j(\zeta^i) - \zeta^j P_j(\eta^i))u_i &= \eta^i\zeta^j(P_j(u_i) - P_i(u_j)) \\ &= \eta^i\zeta^j(H_j(u_i) - H_i(u_j) + F(V_j(u_i) - V_i(u_j))) = 0 \end{aligned}$$

since $V_i(u_j) = g_{ij}$. Thus $\mathcal{D}_0^+$ is integrable; a similar argument shows that $D_0^-$ is also integrable. We know from the previous theorem that $\mathcal{D}_0^+$ has a local basis consisting of vector fields Lie transported by $\Gamma$, so $\mathcal{D}_0^+$ is $\Gamma$-invariant, and $\mathcal{D}_0^-$ likewise. Finally, consider the restriction of $\mathcal{D}_0^\pm$ to the indicatrix bundle $\mathcal{I}$. We have

$$\eta^i P_i(F) = \eta^i(H_i(F) + FV_i(F)) = \eta^i u_i = 0,$$

so $\mathcal{D}_0^+$ is tangent to $\mathcal{I}$, and $\mathcal{D}_0^-$ likewise. So we have a $\Gamma$-invariant splitting $T\mathcal{I} = \langle\Gamma\rangle \oplus \mathcal{D}_0^+ \oplus \mathcal{D}_0^-$. Moreover, the restriction of the Hilbert 1-form

$$\theta_F = \frac{\partial F}{\partial u^i} dx^i$$

(or equally the Cartan 1-form $(\partial E/\partial u^i)dx^i$) to $\mathcal{I}$ is a contact form, with Reeb field $\Gamma$, and the $\mathcal{D}_0^\pm$ are integrable Legendrian distributions. We therefore have the following theorem.

**Theorem** The restriction to $\mathcal{I}$ of $\Gamma$ is a contact Anosov flow: the leaves of the integrable distribution $\mathcal{D}_0^+$ are the unstable manifolds, those of $\mathcal{D}_0^-$ the stable manifolds. ∎

This establishes the first step in the proof of the rigidity results of Akbar-Zadeh [1] for spaces of constant negative flag curvature by the methods of Foulon [7] (which of course apply in more general circumstances), and in particular identifies the stable and unstable manifolds as submanifolds of the leaves of the eigendistributions of the para-Kähler structure.

## The Levi-Civita connection of $G$

Finally, for the record, I determine the Levi-Civita connection of the ultra-hyperbolic metric $G$ of the para-Kähler structure. I work in terms of $P_i$ and $Q_i$ as defined above; recall that $JP_i = P_i$, $JQ_i = -Q_i$,

$$[P_i, P_j] = 0 = [Q_i, Q_j], \qquad [P_i, Q_j] = -2(F\Gamma_{ij}^k + R_{ij}^k)V_k = -(\Gamma_{ij}^k + F^{-1}R_{ij}^k)(P_k - Q_k).$$

Since for any $\xi$, $\eta$, $J(\nabla_\xi \eta) = \nabla_\xi(J\eta)$ we have $J(\nabla_\xi P_i) = \nabla_\xi P_i$ while $J(\nabla_\xi Q_i) = -\nabla_\xi Q_i$; thus $\nabla_\xi P_i$ is a linear combination of the $P_i$, $\nabla_\xi Q_i$ a linear combination of the $Q_i$. Thus using the symmetry of the Levi-Civita connection we can read off the expressions for $\nabla_{P_i} Q_j$ and $\nabla_{Q_i} P_j$ from the above formula for $[P_i, Q_j]$:

$$\nabla_{P_i} Q_j = (\Gamma_{ij}^k + F^{-1}R_{ij}^k)Q_k, \qquad \nabla_{Q_i} P_j = (\Gamma_{ij}^k - F^{-1}R_{ij}^k)P_k.$$

On the other hand

$$\begin{aligned} 2G(\nabla_{P_i} P_j, Q_k) &= P_i(G(P_j, Q_k)) + P_j(G(P_i, Q_k)) - G([P_i, Q_k], P_j) - G(P_i, [P_j, Q_k]) \\ &= 2P_i(Fg_{jk}) + 2P_j(Fg_{ik}) - 2Fg_{jl}(\Gamma_{ik}^l + R_{ik}^l) - 2Fg_{il}(\Gamma_{jk}^l + R_{jk}^l) \\ &= 2F(H_i(g_{jk}) - g_{jl}\Gamma_{ik}^l - g_{kl}\Gamma_{ij}^l) + 2F(H_j(g_{ik}) - g_{il}\Gamma_{jk}^l - g_{kl}\Gamma_{ij}^l) \\ &\quad + 4Fg_{kl}\Gamma_{ij}^l + 2F^2(g_{jk,i} + g_{ik,j}) + 2(u_i g_{jk} + u_j g_{ik} - g_{jl}R_{ik}^l - g_{il}R_{jk}^l) \\ &= 4Fg_{kl}\Gamma_{ij}^l + 2F(g_{jk|i} + g_{ik|j}) + 4F^2 C_{ijk} + 4g_{ij}u_k, \end{aligned}$$

where the solidus indicates a Berwald covariant derivative along a basic horizontal vector field, the comma a Berwald covariant derivative along a vertical coordinate vector field, and $C_{ijk}$ is the Cartan tensor. Now $g_{ij|k} = L_{ijk} = B^l_{ijk}u_l$ where $L$ is the Landsberg curvature and $B$ the Berwald curvature; $L$ is completely symmetric in its indices. (There are differences of numerical factors between these definitions and Shen's in [10].) Thus

$$\nabla_{P_i}P_j = (\Gamma^k_{ij} + S^k_{ij})P_k,$$

where $S$ is a tensor, symmetric in its lower indices, given explicitly by

$$S^k_{ij} = g^{kl}\left(L_{ijl} + FC_{ijl} + F^{-1}g_{ij}u_l\right) = L^k_{ij} + FC^k_{ij} + F^{-1}g_{ij}u^k.$$

Similarly

$$\begin{aligned}
2G(\nabla_{Q_i}Q_j, P_k) &= Q_i(G(Q_j, P_k)) + Q_j(G(Q_i, P_k)) - G([Q_i, P_k], Q_j) - G(Q_i, [Q_j, P_k]) \\
&= 2Q_i(Fg_{jk}) + 2Q_j(Fg_{ik}) - 2Fg_{jl}(\Gamma^l_{ki} + R^l_{ki}) - 2Fg_{il}(\Gamma^l_{kj} + R^l_{kj}) \\
&= 4Fg_{kl}\Gamma^l_{ij} + 2F(g_{jk|i} + g_{ik|j}) - 4F^2C_{ijk} \\
&\quad - 2(u_ig_{jk} + u_jg_{ik} - g_{jl}R^l_{ik} - g_{il}R^l_{jk}) \\
&= 4Fg_{kl}\Gamma^l_{ij} + 4FL_{ijk} - 4F^2C_{ijk} - 4g_{ij}u_k,
\end{aligned}$$

so that

$$\nabla_{Q_i}Q_j = (\Gamma^k_{ij} + T^k_{ij})Q_k$$

where the tensor $T$ is given by

$$T^k_{ij} = L^k_{ij} - FC^k_{ij} - F^{-1}g_{ij}u^k.$$

Thus the Levi-Civita connection of $G$ encodes the Cartan and Landsberg tensors of the Finsler space, as well as the Berwald connection. There is a certain superficial similarity between the expressions for the connection coefficients given above and those found by Bejancu and Farran in [3]; these authors deal however with spaces of flag curvature 1, and though they introduce an almost product structure it lacks the extra factors $F$ which are the key to the whole approach advocated here.

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