On Landsberg spaces and the Landsberg-Berwald problem

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1 Introduction

A Berwald space is a Finsler space whose canonical Berwald connection is affine; a Landsberg space is one whose Berwald connection, while not affine, satisfies a property reminiscent of that which typifies the Levi-Civita connection in Riemannian geometry. Every Berwald space is a Landsberg space. Whether there are Landsberg spaces which are not of Berwald type is a long-standing question in Finsler geometry, which is still far from being resolved in all generality; for the sake of brevity I have called it in the title the Landsberg-Berwald problem. This paper contains several not very closely related observations and results about Landsberg spaces, which are connected however by the fact that they were inspired by consideration of that problem; they should nevertheless be of value in their own right, at least until the problem is finally settled.

Interest in the Landsberg-Berwald problem has been rekindled recently by the discovery by Asanov [1] of examples of non-Berwaldian Landsberg spaces, of dimension at least 3. The reason why this development has rekindled interest rather than resolving the whole issue is that in Asanov's examples the Finsler functions are not defined for all values of the fibre coordinates y^i: in the jargon, they are y-local (as opposed to y-global).

Whether or not there are y-global non-Berwaldian Landsberg spaces remains an open question, therefore. Partly inspired by Asanov's results, Shen has studied the class of (α, β) metrics of Landsberg type, of which Asanov's examples are particular cases; he finds [2, 12] that though there are y-local non-Berwaldian Landsberg spaces with (α, β) metrics, there are no y-global ones. Bao [3] has initiated a different line of attack on the problem, involving a method of constructing non-Berwaldian Landsberg spaces by successive approximation; but this method has so far failed to produce even an approximate metric which is y-global.

The elusiveness of y-global non-Berwaldian Landsberg spaces leads Bao to describe them as the unicorns of Finsler geometry. Indeed, the evidence quoted above suggests that, just like unicorns, in reality there are none, at least of dimension greater than or equal to 3. This paper contains, among other things, some further additions to that evidence. (As Asanov's results suggest, dimension 2 is special, and I will not consider it here. To avoid excessive repetition I state now that all Finsler spaces considered below are over manifolds of dimension greater than or equal to 3.)

In fact the role of y-globality in the Landsberg-Berwald problem is more complicated than the discussion above may suggest. It is true that the general form of the non-existence conjecture must be 'there are no y-global non-Berwaldian Landsberg spaces', as Asanov's examples show. On the other hand, the best known partial result in this direction, namely that every Landsberg space with vanishing Douglas tensor is a Berwald space, is y-local: I give a manifestly y-local proof in an appendix to this paper to confirm that this is the case.

If one's aim is to prove that the canonical connection of a Finsler space of a certain type is affine, to make an auxiliary assumption that the space is projectively affine seems very natural. This is of course equivalent to assuming that its Douglas tensor vanishes, the situation just discussed. In the same context, an alternative natural auxiliary assumption to make is that the space's Riemann curvature, like that of any space with an affine connection, depends only on position. Such a space is said to be R-quadratic. More particularly, a space whose Riemann curvature vanishes is called R-flat. Another partial result, due to Shen ([11] Theorem 10.3.7), states that a forward-complete R-flat Landsberg space in which certain boundedness conditions are satisfied is a Berwald space. In Shen's theorem as it is stated, the R-flat Landsberg space is not required to be y-global; however, the most obvious way of satisfying the conditions is to take the space to be y-global and the underlying manifold to be compact, and the theorem is then a y-global one. My first result in this paper is the extension of Shen's theorem to R-quadratic spaces, using his methods. This will be found in Section 3.

Each indicatrix of a y-global Finsler space is a compact Riemannian manifold; one characteristic property of a Landsberg space is that its holonomy groups are composed of isometries of the indicatrices [9]. This brings into play the geometrical features of the indicatrices, and in particular their isometries, which one can study in infinitesimal form via Killing's equation. Landsbergian geometry differs from Riemannian geometry in that the isometry algebra of the indicatrix is not predetermined; indeed, there is no reason to assume that there are any (non-zero) infinitesimal isometries at all. But if this is the case the Landsberg space, if y-global and over a compact base, must be Berwaldian. I discuss these matters, beginning with an account of the holonomy group of a Landsberg space and its Lie algebra, in Section 4.

It is well known that for any Berwald space one can find a Riemannian metric whose Levi-Civita connection coincides with the Berwaldian affine connection. One way of constructing such a metric, devised by Vincze [16], involves averaging the fundamental tensor over indicatrices; to do this, of course, one requires the space to be y-global. This construction can in fact be carried out in any y-global Landsberg space; the Levi-Civita connection of the resulting metric is then an obvious candidate for the Berwaldian affine connection, if one seeks to prove that the Landsberg space is actually a Berwald space. As a step in this direction I show in Section 5 that the Berwald connection coefficients of a Landsberg space, written in a suitable form, can be averaged over indicatrices, and the result is the Levi-Civita connection of the averaged metric.

When I first wrote this paper I was not aware that Szabó had recently claimed, mistakenly as it turned out, to have proved that y-global Landsberg spaces are always Berwaldian (Szabó's original paper is at [13], Matveev's refutation at [10], and Szabó's acceptance of his error at [14]). Since the claim did turn out to be wrong, and the Landsberg-Berwald problem remains open as Matveev, and now Szabó, state, I have not found it necessary to make any changes to my paper apart from the insertion of this paragraph. I do want to point out, however, that Szabó's original argument was based on the construction described in Section 5; his error was to claim that the Berwald connection coincides with the Levi-Civita connection of the averaged metric, which need not be the case; and that my result should perhaps be seen as the strongest correct general statement that can be made about the relation between the two connections.

The paper begins with a brief account of the definitions and main local properties of Landsberg spaces, in tensorial form, which serves the dual purpose of establishing notations and of making the paper reasonably self-contained by providing a compendium of known results.

2 Definitions and known results

Let M be a differential manifold, with $\dim M \geq 3$. I denote by $\pi : TM \to M$ the tangent bundle of M, and by $\pi^o : T^\circ M \to M$ the tangent bundle with the zero section deleted. Coordinates on $T^\circ M$ will generally be written (x^i, y^i).

Most of the geometric objects of interest in this paper are tensors along π^o. A contravariant vector field along π^o, for example, is a section of the pull-back bundle $\pi^{o*}TM \to T^\circ M$. More generally, a tensor field along π^o is a section of the pull-back of a tensor bundle over M. To put it more crudely, a tensor along π^o has components which transform as those of a tensor on M but which are functions of the y^i as well as the x^i. Such objects will just be called tensors for brevity, even though this is strictly incorrect.

A spray Γ on $T^\circ M$ is a second-order differential equation field

$$y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i}$$

such that the coefficients Γ^i are positively homogeneous of degree 2 in the y^i; if they are quadratic then the spray is affine. (Homogeneity, below, will always be positive homogeneity unless it is explicitly mentioned otherwise.)

The Berwald curvature is the tensor whose components B^i_{jkl} are given by

$$B^i_{jkl} = \frac{\partial^3 \Gamma^i}{\partial y^j \partial y^k \partial y^l}.$$

The Berwald curvature is symmetric in its lower indices, and by homogeneity $B^i_{jkl} y^j = 0$. The vanishing of B is the necessary and sufficient condition for the spray to be affine.

The horizontal distribution associated with a spray is spanned by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial y^j}, \qquad \Gamma^j_i = \frac{\partial \Gamma^j}{\partial y^i};$$

Γ^j_i is of homogeneity degree 1. The horizontal lift of $v \in T_x M$ to $(x, y) \in T^\circ M$ is $v^{\mathrm{H}} = v^i H_i(x, y)$ (where $v = v^i \partial/\partial x^i$); and similarly for other appropriate objects on M. The bracket of a pair of horizontal vector fields from the basis is given by

$$[H_i, H_j] = -R^l_{kij} y^k \frac{\partial}{\partial y^l} = -R^l_{ij} \frac{\partial}{\partial y^l}$$

where the R^l_{kij} are the components of the Riemann curvature of the spray (that is, the Riemann curvature of the associated Berwald connection):

$$R^l_{kij} = H_i\left(\Gamma^l_{jk}\right) - H_j\left(\Gamma^l_{ik}\right) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}, \qquad \Gamma^i_{jk} = \Gamma^i_{kj} = \frac{\partial^2 \Gamma^i}{\partial y^j \partial y^k}.$$

More generally, for vector fields X, Y on M,

$$[X^{\mathrm{H}}, Y^{\mathrm{H}}] = [X, Y]^{\mathrm{H}} - R^i(X, Y)\frac{\partial}{\partial y^i}, \quad R^i(X, Y) = R^i_{jk} X^j Y^k.$$

The Riemann curvature satisfies the cyclic identity

$$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0.$$

If $R^l_{kij} = 0$ the space is said to be R-flat; if $\partial R^l_{kij}/\partial y^m = 0$ it is said to be R-quadratic (because then R^l_{ij} depends quadratically on the y^i).

The covariant derivatives of a contravariant vector V with respect to the Berwald connection associated with a spray may be written as follows:

$$V^i{}_{;j} = H_j(V^i) + \Gamma^i_{jk} V^k, \qquad V^i{}_{,j} = \frac{\partial V^i}{\partial y^j}.$$

The semicolon will be used to indicate covariant differentiation in a horizontal direction, the comma covariant differentiation in a vertical direction. Covariant differentiation of other tensors follows the usual pattern. It is a consequence of the homogeneity of the connection coefficients that $y^i{}_{;j} = 0$. The spray Γ is horizontal, and $T_{\dots;k} y^k$ are the components of the covariant derivative of the tensor T along Γ; it will be convenient to denote this by $\nabla T_{...}$, where ∇ is the so-called dynamical covariant derivative associated with Γ. Just as for an ordinary covariant derivative one can think of ∇ as an operator on any vector field defined along any integral curve of Γ, or indeed any base integral curve (projection of an integral curve into M); we call such curves in M geodesics of Γ. A vector field V along a geodesic of Γ which satisfies $\nabla V = 0$ is said to be parallel. As a consequence of the fact that $y^i{}_{;j} = 0$ we see that the tangent vector field to a geodesic is parallel.

We have the following Ricci identities for repeated covariant differentiation:

$$\begin{aligned} V^i{}_{,jk} - V^i{}_{,kj} &= 0 \\ V^i{}_{;j,k} - V^i{}_{,k;j} &= B^i_{jkl} V^l \\ V^i{}_{;jk} - V^i{}_{;kj} &= -R^i_{ljk} V^l + R^l_{jk} V^i{}_{,l}, \end{aligned}$$

the final term coming from the bracket of horizontal fields. The Bianchi identities are

$$\begin{aligned} B^m_{ijk,l} - B^m_{ijl.k} &= 0 \\ B^m_{ijk;l} - B^m_{ijl;k} &= -R^m_{jkl,i} \\ R^m_{ijk;l} + R^m_{ikl;j} + R^m_{ilj;k} &= 0. \end{aligned}$$

The middle one of these implies that $R^m_{jkl,i} = R^m_{ikl,j}$; but this follows from the relation

$$R^m_{ijk} = \frac{\partial}{\partial y^i}\left(R^m_{jk}\right).$$

Let F be a Finsler function on $T^\circ M$, $E = \frac{1}{2}F^2$ its energy and g its fundamental tensor, with components g_{ij}. It will be convenient to write y_i for $g_{ij}y^j$; note that $y_i y^i = F^2$ and $\partial y_i / \partial y^j = g_{ij}$.

The canonical geodesic spray of F is the unique spray such that

$$\Gamma\left(\frac{\partial E}{\partial y^i}\right) - \frac{\partial E}{\partial x^i} = 0.$$

A Berwald space is a Finsler space whose geodesic spray is affine, or equivalently whose Berwald connection is affine; in other words, one whose Berwald curvature vanishes.

It is easy to see that in any Finsler space the horizontal vector fields associated with Γ satisfy $H_i(E) = 0$, and thence that

$$g_{ij;k} = y_l B^l_{ijk}.$$

Notice that $\nabla g_{ij} = 0$. It follows, in the usual way, that along geodesics parallelism preserves scalar products determined by g.

The Cartan tensor $C_{ijk} = g_{ij,k}$ satisfies

$$C_{ijk;l} = g_{ij;l,k} + B^m_{jkl} g_{im} + B^m_{ikl} g_{jm}$$

by the second Ricci identity, so that

$$\nabla C_{ijk} = C_{ijk;l}y^l = (y_m B^m_{ijl})_{,k} y^l = y_m B^m_{ijl,k} y^l = y_m B^m_{ijk,l} y^l = -y_m B^m_{ijk},$$

since $B^m_{ijl}y^l = 0$ and B^m_{ijk} is homogeneous of degree -1. We have the following (well-known) result.

Proposition 1. *The following conditions on a Finsler space are equivalent:*

1. *the fundamental tensor is covariant constant along horizontal curves;*
2. *the Berwald tensor is such that at any $(x, y) \in T^\circ M$, for any vertical vectors u, v, w at $y \in T_x M$, $B_y(u, v, w)$ is orthogonal to the radial vector y (with respect to the fundamental tensor);*
3. *the Cartan tensor is covariant constant along geodesics.*

A Finsler space in which one, and hence all, of these properties holds is a Landsberg space. Every Berwald space is a Landsberg space. The question of interest is whether the converse holds.

In a Landsberg space, parallelism preserves scalar products determined by g along any horizontal curve.

Corollary 1. *In a Landsberg space the Berwald connection coincides with the Chern-Rund connection.*

Proof. The difference tensor between the two connections is essentially

$$C^i_{jk;l}y^l = (g^{im}C_{mjk})_{;l}y^l = g^{im}C_{mjk;l}y^l$$

(see [5, 7]). □

I set $B_{ijkl} = g_{im}B^m_{jkl}$ and $R_{ijkl} = g_{im}R^m_{jkl}$.

Corollary 2. *In a Landsberg space*

1. $B_{ijkl} + B_{jikl} = C_{ijk;l}$;
2. $R_{ijkl} + R_{jikl} = -R^m_{kl}C_{ijm}$.

Proof. Apply the second Ricci identity to $g_{ij;k,l} - g_{ij,l;k}$, and the third Ricci identity to $g_{ij;kl} - g_{ij;lk}$, respectively. □

The following two results are due to Vattamány [15].

Proposition 2. *A Finsler space is a Landsberg space if and only if B_{ijkl} is completely symmetric in its indices.*

Proof. Clearly, if $g_{im}B^m_{jkl} = g_{jm}B^m_{ikl}$, then $y^i g_{im}B^m_{jkl} = y^i g_{jm}B^m_{ikl} = 0$ and the space is Landsberg. Conversely, suppose that the space is Landsberg. Then

$$\begin{aligned} 0 = (y^m B_{mjkl})_{,i} &= B_{ijkl} + y^m (g_{mn}B^n_{jkl})_{,i} \\ &= B_{ijkl} + y^m (C_{mni}B^n_{jkl} + g_{mn}B^n_{jkl,i}) \\ &= B_{ijkl} + y_n B^n_{jkl,i}. \end{aligned}$$

But $B^n_{jkl,i}$ is completely symmetric in its lower indices, so B_{ijkl} is completely symmetric in its indices. □

Corollary 3. *A Finsler space is a Landsberg space if and only if $C_{ijk;l}$ is completely symmetric in its indices.*

Proof. If $C_{ijk;l} = C_{ijl;k}$ then $\nabla C_{ijk} = C_{ijl;k}y^l = (C_{ijl}y^l)_{;k} = 0$ and the space is Landsberg. If the space is Landsberg, so that B_{ijkl} is completely symmetric, it follows from Corollary 2 part 1 that $C_{ijk;l} = 2B_{ijkl}$ and so $C_{ijk;l}$ is completely symmetric. □

Corollary 4. *Let B_{ijkl} be the Berwald tensor of a Landsberg space. Suppose that for $x \in M$, for every $y \in T^\circ_x M$ such that $F(x,y) = 1$ (that is, every point of the indicatrix) and for every $v \in T_x M$ such that $g_{(x,y)}(v,v) = 1$ (that is every unit vector) we have $B_{ijkl}(x,y)v^i v^j v^k v^l = 0$. Then $B_{ijkl} = 0$ all over $T_x M$. If this property holds at every x then the Landsberg space is a Berwald space.*

Proof. This is a simple consequence of the symmetry and homogeneity of B_{ijkl}. □

In fact since $B_{ijkl}y^l = 0$ it is enough to have v satisfy $g_{(x,y)}(y,v) = 0$, that is, to be tangent to the indicatrix.

3 R-quadratic Landsberg spaces

Shen has shown that a forward-complete R-flat Landsberg space in which the Cartan tensor satisfies a certain boundedness condition must be a Berwald (in fact a Minkowski) space (Theorem 10.3.7 of [11]). By generalizing his method I extend his result to R-quadratic Landsberg spaces, which I assume to be y-global.

Consider the submanifold N of the pull-back of TM to SM, the sphere bundle or indicatrix bundle over M, which consists of those $v \in T_x M$, $(x,y) \in SM$, such that $g_{(x,y)}(v,v) = 1$. Any symmetric covariant tensor A defines a function α on N by $\alpha(x,y,v) = A_{ij\dots}(x,y)v^i v^j \cdots$. The tensor A is said to be bounded if the function α

is bounded. In particular, if M is compact so will N be, and then any symmetric tensor is automatically bounded. Furthermore, a Finsler space over a compact base is necessarily forward complete, indeed complete in both directions.

Theorem 1. *A forward-complete, R-quadratic Landsberg space for which the symmetric tensor $C_{ijk,l}$ is bounded is a Berwald space. In particular, an R-quadratic Landsberg space over a compact base is a Berwald space.*

Proof. In an R-quadratic space we have $B^m_{ijk;l} - B^m_{ijl;k} = 0$, by the second Bianchi identity, since $R^m_{jkl,i} = 0$. Thus in an R-quadratic Landsberg space $\nabla B_{ijkl} = 0$. Furthermore, by the second Ricci identity

$$C_{ijk,l;m} - C_{ijk;m,l} = B^n_{ilm} C_{njk} + B^n_{jlm} C_{ink} + B^n_{klm} C_{ijn},$$

and therefore in any Landsberg space

$$\nabla C_{ijk,l} = -C_{ijk;l} = -2B_{ijkl}.$$

So in an R-quadratic Landsberg space $\nabla^2 C_{ijk,l} = 0$.

Now consider any geodesic $\gamma(t)$ of Γ, parametrized by arc length (so that $g(\dot\gamma, \dot\gamma) = 1$), and let $v(t)$ be any unit parallel vector field along γ. Define a function $C(t)$ by

$$C(t) = C_{ijk,l}(\gamma(t), \dot\gamma(t)) v^i(t) v^j(t) v^k(t) v^l(t);$$

we can think of $C(t)$ as being defined along a curve in N. Then

$$\frac{d^2 C}{dt^2} = 0,$$

so $C(t) = \dot C(0)t + C(0)$. Now t can take any positive value, but by assumption $C(t)$ is bounded: thus $\dot C(0)$ must vanish. But $\gamma(0) = x$, $\dot\gamma(0) = y$ and $v(0) = v$ may be chosen arbitrarily (subject to the conditions $F(x, y) = 1$ and $g_{(x,y)}(v, v) = 1$), so $\dot C(0) = -2B_{ijkl}(x, y) v^i v^j v^k v^l = 0$ for every x, y and v. By Corollary 4 the space is a Berwald space. □

Now according to Shen's Theorem 10.3.2 a forward-complete, R-quadratic Finsler space for which the symmetric tensor C_{ijk} is bounded is a Landsberg space. This is proved by an argument very similar to the one above (indeed, an argument on which the one above is partly based). In any R-quadratic space $\nabla B^l_{ijk} = 0$, and since $\nabla y_l = \nabla(g_{lm} y^m) = 0$ we have $\nabla(y_l B^l_{ijk}) = 0$. But $\nabla C_{ijk} = -y_l B^l_{ijk}$, so $\nabla^2 C_{ijk} = 0$. The argument now proceeds as before but with C_{ijk} in place of $C_{ijk,l}$. The conclusion is that $y_l B^l_{ijk} = 0$, so the space is Landsberg. We therefore have the following result.

Corollary 5. *A forward-complete, R-quadratic Finsler space for which the symmetric tensors C_{ijk} and $C_{ijk,l}$ are both bounded is a Berwald space. In particular, an R-quadratic Finsler space over a compact base is a Berwald space.*

4 Holonomy

Holonomy in a Finsler space is defined as follows (for more details see [4, 8, 9]).

Take a pair of points $x, x' \in M$, and any piecewise smooth curve c in M with $c(0) = x$, $c(1) = x'$. For any $y \in T_x^\circ M$ let $t \mapsto c^{\mathrm{H}}(t, y)$ be the horizontal lift of c through y. We may define a smooth map, indeed diffeomorphism, $\rho : T_x^\circ M \to T_{x'}^\circ M$ by $\rho(y) = c^{\mathrm{H}}(1, y)$.

I denote by S_x the indicatrix at x, $S_x = \{y \in T_x^\circ M : F(x, y) = 1\}$. In any Finsler space, for any $v \in T_x M$, $v^{\mathrm{H}}(F) = 0$. It follows that ρ maps S_x onto $S_{x'}$.

Next, consider the differential of ρ. Since $T_x^\circ M$ is a vector space (though deprived of its origin), its tangent space at any point may be canonically identified with itself (with origin restored); so for any $y \in T_x^\circ M$, ρ_{*y} may be regarded as a (linear) map $T_x M \to T_{x'} M$. This map may be described as follows. For $v \in T_x M$, $y \in T_x^\circ M$, let $v(t)$ be the parallel field along $c^{\mathrm{H}}(t, y)$ with $v(0) = v$. Then $\rho_{*y}(v) = v(1)$.

In any Finsler space the fundamental tensor $g = (g_{ij})$ defines on each fibre $T_x^\circ M$ a Riemannian metric, namely $g_x = g_{ij}(x, \cdot) dy^i \odot dy^j$. In a Landsberg space, parallel transport along horizontal curves is isometrical. Thus ρ is an isometry of $T_x^\circ M$ onto $T_{x'}^\circ M$, where these spaces are equipped with the Riemannian metrics g_x and $g_{x'}$. Moreover, since ρ maps S_x onto $S_{x'}$, it is an isometry of these manifolds equipped with the metrics induced on them by g_x and $g_{x'}$.

Now specialize to $x' = x$, so that the curves in M under consideration are closed. Combining such curves in the usual way gives rise to a multiplication of corresponding transformations ρ, with respect to which they form a group, the holonomy group at x. For a Landsberg space this is a closed subgroup of the group of isometries of the compact Riemannian manifold S_x, and is therefore a Lie group.

At each $x \in M$, the Lie algebra of the holonomy group at x may be realised as an $\mathbb{R}$-linear space of vector fields on $T_x^\circ M$, tangent to S_x (indeed, to each level set of F), which are infinitesimal isometries of g_x, that is, which satisfy Killing's equation. Moreover, for any pair of points $x, x' \in M$ and any curve joining them the corresponding map ρ defines an isomorphism of both isometry groups and holonomy subgroups, and its differential an isomorphism of isometry algebras and holonomy subalgebras.

I should like to propose a possible candidate for a (fairly) explicit construction of the holonomy algebras, as follows.

I first define a collection $\mathcal{I}$ of vertical vector fields on $T^\circ M$ by the following requirements:

1. for any pair of vector fields X, Y on M, $\mathcal{I}$ contains $[X, Y]^{\mathrm{H}} - [X^{\mathrm{H}}, Y^{\mathrm{H}}]$ (which is vertical);

2. for every ξ that $\mathcal{I}$ contains, it also contains $[X^{\mathrm{H}}, \xi]$ (which is vertical);

3. $\mathcal{I}$ is closed under addition and scalar multiplication by smooth functions on M;

4. $\mathcal{I}$ is closed under bracket;

5. $\mathcal{I}$ is the smallest set of vertical vector fields with these properties.

It is not to be assumed that $\mathcal{I}$ is a distribution, or in other words that it is closed under multiplication by functions on $T^\circ M$ (though of course there is at least one distribution of vertical vector fields which satisfies requirements 1 to 4, namely the vertical distribution itself). But by requirement 3, $\mathcal{I}$ is a $C^\infty(M)$-module. Note that for $f \in C^\infty(M)$,

$$\begin{aligned}[X, fY]^{\mathrm{H}} - [X^{\mathrm{H}}, fY^{\mathrm{H}}] &= f[X,Y]^{\mathrm{H}} + X(f)Y^{\mathrm{H}} - f[X^{\mathrm{H}}, Y^{\mathrm{H}}] - X^{\mathrm{H}}(f)Y^{\mathrm{H}} \\ &= f\left([X,Y]^{\mathrm{H}} - [X^{\mathrm{H}}, Y^{\mathrm{H}}]\right)\end{aligned}$$

since of course $X^{\mathrm{H}}(f) = X(f)$. Secondly

$$[X^{\mathrm{H}}, f\xi] = X(f)\xi + f[X^{\mathrm{H}}, \xi].$$

So requirement 3 is consistent with requirements 1 and 2, if not actually forced to hold by them. Moreover, if $\xi, \eta \in \mathcal{I}$ and $f \in C^\infty(M)$, $[\xi, f\eta] = f[\xi, \eta]$.

The idea is that for $x \in M$, $\mathcal{I}_x$, the restriction of $\mathcal{I}$ to $T_x^\circ M$, should be the holonomy algebra at x, realised as a Lie algebra of vector fields on $T_x^\circ M$. It is implicit in this statement, of course, that each vector field in $\mathcal{I}_x$ can be extended to a vertical vector field on $T^\circ M$ which is an element of $\mathcal{I}$. However, for the purpose of specifying the holonomy algebra at x one should really be interested only in vector fields defined locally near x: for example, it should be enough to take X and Y in requirement 1 to be defined in a neighbourhood of x. We can however extend such vector fields to the whole of M by using bump functions defined around x. This justifies the use below of locally-defined vector fields, such as coordinate fields, in discussing the properties of $\mathcal{I}_x$. Indeed, the requirement that $\mathcal{I}$ should be a $C^\infty(m)$-module, combined with the availability of bump functions on M, ensures that $\mathcal{I}_x$ is a local construct. This point may be expressed more precisely as follows. Let $\hat{\mathcal{I}}_x$ be the set of germs of elements of $\mathcal{I}$ at x. (By a germ of an element of $\mathcal{I}$ at x I mean the following. An element of $\mathcal{I}$ is a vertical vector field on $T^\circ M$. Define an equivalence relation on elements of $\mathcal{I}$ by setting two elements equivalent if as vector fields they agree over some neighbourhood of x; a germ is an equivalence class.) Then $\hat{\mathcal{I}}_x$ is a vector space (over $\mathbb{R}$), and each element of $\hat{\mathcal{I}}_x$ determines a vector field on $T_x^\circ M$. Now let $\hat{\mathcal{I}}_x^0$ be the subspace of $\hat{\mathcal{I}}_x$ consisting of those elements for which the corresponding vector field on $T_x^\circ M$ vanishes. Then $\mathcal{I}_x = \hat{\mathcal{I}}_x / \hat{\mathcal{I}}_x^0$.

The necessity of including requirement 1 in the specification of $\mathcal{I}$ may be seen as follows. Suppose that the vector fields X and Y commute (they may be coordinate fields, for example). Consider a piecewise smooth closed curve in M which is a 'square' whose sides are integral curves alternately of X and Y, of parametric length t; take x to be one vertex. The horizontal lift of this curve to $T^\circ M$, with the initial end at a fixed point (x, y) say, will not in general be closed. As t varies the other end sweeps out a curve $(x, y(t))$ in $T_x^\circ M$, which is the image of (x, y) under the action of a curve in the holonomy group at x. The tangent vector to this curve at $t = 0$ vanishes, but its second-order tangent is just $[X^{\mathrm{H}}, Y^{\mathrm{H}}](x, y)$.

Requirement 2 is, roughly speaking, the infinitesimal version of the fact that the map ρ corresponding to the horizontal lift of some curve in M is an isomorphism of holonomy groups. I shall discuss this point in more detail below.

Since X^{H} is homogeneous of degree 0, and this property is preserved under addition, multiplication by functions on M and taking brackets, all members of $\mathcal{I}$ are homogeneous of degree 0.

Since $X^{\mathrm{H}}(F) = 0$, and this property too is preserved under addition, multiplication by functions on M and taking brackets, all members of $\mathcal{I}$ annihilate F, and so on restriction to $T_x^\circ M$ are tangent to level sets of F.

I shall present three further pieces of evidence in support of the claim that for a Landsberg space $\mathcal{I}_x$ is the holonomy algebra at x. The first is that if the Landsberg space reduces to a Berwald space then $\mathcal{I}_x$ is indeed the holonomy algebra of the corresponding affine connection. Notice that

$$[H_i, \xi] = (H_i(\xi^j) + \Gamma^j_{ik}\xi^k)\frac{\partial}{\partial y^j} = \xi^j_{;i}\frac{\partial}{\partial y^j}.$$

The construction of $\mathcal{I}$ starts with

$$R^m_{lij}y^l\frac{\partial}{\partial y^m}.$$

It follows that when the connection is affine, by repeated bracketing with horizontal fields and judicious use of requirement 3, starting with the vector field above we obtain all vector fields of the form

$$R^m_{lij;k_1k_2\ldots k_r}y^l\frac{\partial}{\partial y^m}.$$

These vector fields (and indeed all of those in $\mathcal{I}$) are linear in y^k, and so for each x, $\mathcal{I}_x$ is the Lie algebra of linear vector fields generated by the curvature and its covariant differentials; but in the affine case this is just the holonomy algebra at x.

This is one piece of evidence that $\mathcal{I}_x$ should be regarded as the holonomy algebra at x. For a second I show that in general in a Landsberg space the elements of $\mathcal{I}_x$, though they can no longer be assumed to be linear vector fields, are infinitesimal isometries of g_x, or Killing (vector) fields.

Theorem 2. *In a Landsberg space, $\mathcal{I}_x$ is a subalgebra of the Lie algebra of Killing fields of the metric g_x on $T_x^\circ M$.*

Proof. The condition for $\xi = \xi^k\partial/\partial y^k$ to be a Killing field is

$$\xi^k\frac{\partial g_{ij}}{\partial y^k} + g_{ik}\frac{\partial \xi^k}{\partial y^j} + g_{jk}\frac{\partial \xi^k}{\partial y^i} = \xi^k C_{ijk} + g_{ik}\xi^k_{,j} + g_{jk}\xi^k_{,i} = 0.$$

In the proof below ξ is to be taken to be defined over a neighbourhood U of x in M, and to satisfy this equation (Killing's equation) for all $y \neq 0$ and all $x' \in U$. Then the restriction of ξ to $T_x^\circ M$ will certainly be a Killing field of g_x.

Now sums, constant multiples and brackets of Killing fields are Killing fields; moreover, if ξ satisfies Killing's equation over a neighbourhood U of x in M and f is a function defined on U then $f\xi$ also satisfies the equation on U. So it is enough to show that $\xi^k = R^k_{lm}$ satisfies Killing's equation and that if ξ satisfies it so does $[H_i, \xi]$.

Firstly, with $\xi^k = R^k_{lm}$ the left-hand side of Killing's equation is

$$R^k_{lm}C_{ijk} + g_{ik}R^k_{jlm} + g_{jk}R^k_{ilm} = R_{ijlm} + R_{jilm} + R^k_{lm}C_{ijk},$$

which is zero in a Landsberg space by Corollary 2 part 2. Now suppose that ξ satisfies Killing's equation and consider $[H_i, \xi] = \xi^j_{;i}\partial/\partial y^j$. Killing's equation is supposed to hold over some neighbourhood of x; we may therefore take the horizontal covariant derivative, to get

$$\xi^k_{;l}C_{ijk} + \xi^k C_{ijk;l} + g_{ik}\xi^k_{,j;l} + g_{jk}\xi^k_{,i;l} = 0$$

(using the fact that $g_{ij;l} = 0$). When the second Ricci identity is applied to the last two terms we obtain

$$\xi^k_{;l}C_{ijk} + \xi^k C_{ijk;l} + g_{ik}(\xi^k_{;l,j} - B^k_{jlm}\xi^m) + g_{jk}(\xi^k_{;l,i} - B^k_{ilm}\xi^m) = 0.$$

The terms involving undifferentiated ξs cancel by Corollary 2 part 1. The remainder says that for each l, $\xi^k_{;l}\partial/\partial y^k$ is a Killing field. □

So for each $x \in M$, $\mathcal{I}_x$ is a subalgebra of the isometry algebra of g_x, and in particular is finite dimensional and a Lie algebra. Moreover, since each element of $\mathcal{I}_x$ is tangent to the indicatrix it is a Killing vector of the induced metric on the indicatrix. The maximum dimension of $\mathcal{I}_x$ is therefore $\frac{1}{2}n(n-1)$, and if this is achieved the indicatrix is a space of positive constant curvature.

The fact that $\mathcal{I}_x$ is finite dimensional for each x suggests that $\mathcal{I}$ should be finite dimensional as a $C^\infty(M)$-module. Let us suppose that this is indeed the case, so that for some integer p there are elements Ξ_a, $a = 1, 2, \ldots, p$ of $\mathcal{I}$ (perhaps defined only locally over M) such that each element ξ of $\mathcal{I}$ may be expressed as $\xi = \sum_{a=1}^p \xi^a \Xi_a$ with $\xi^a \in C^\infty(M)$. Suppose further that for each x, $\{\Xi_a(x)\}$ is a basis for $\mathcal{I}_x$. Using these assumptions I shall now show that for any points $x, x' \in M$ the spaces $\mathcal{I}_x$ and $\mathcal{I}_{x'}$ are isomorphic via the horizontal lift of any piecewise smooth curve joining them. It will be evident from the argument below that this is primarily a consequence of requirement 2.

As before, take any piecewise smooth curve c in M with $c(0) = x$, $c(1) = x'$, and let $\rho : T^\circ_x M \to T^\circ_{x'}M$ be the corresponding diffeomorphism. Then $\rho_*(\mathcal{I}_x)$ is certainly a Lie algebra of Killing fields on $T^\circ_{x'}M$; I have to show that it is the restriction of $\mathcal{I}$ to $T^\circ_{x'}M$. I shall first assume that c is smooth and can be embedded in the flow of some local vector field X on M.

We know that $[X^{\mathrm{H}}, \Xi_a]$ belongs to $\mathcal{I}$; it may therefore be expressed as a linear combination of the Ξ_a, say $[X^{\mathrm{H}}, \Xi_a] = K^b_a \Xi_b$ for some functions K^b_a on M. I propose to modify the Ξ_a so as to obtain a new basis $\{\Xi'_a\}$ for $\mathcal{I}$ such that $[X^{\mathrm{H}}, \Xi'_a] = 0$. Consider the equations

$$X(\Lambda^b_a) + \Lambda^c_a K^b_c = 0$$

for functions Λ_a^b on M. These are effectively first-order linear ordinary differential equations for the unknowns Λ_a^b along the integral curves of X, and have a unique solution for initial conditions specified on a codimension 1 submanifold of M transverse to X. Moreover, if we take the initial conditions to be $\Lambda_a^b(0) = \delta_a^b$ then the solution, considered as a matrix, will be nonsingular at all points sufficiently close to the initial submanifold. We can take the initial submanifold to pass through x, and assume that x' lies within the neighbourhood on which (Λ_a^b) is nonsingular. Then if $\Xi'_a = \Lambda_a^b \Xi_b$, $\{\Xi'_a\}$ is a new (local) basis for $\mathcal{I}$ such that $[X^{\mathrm{H}}, \Xi'_a] = 0$. But this is just the condition for Ξ'_a to be parallel along the integral curves of X^{H}. Thus any vector field which has constant coefficients with respect to $\{\Xi'_a\}$ is an element of $\mathcal{I}$ and is parallel along integral curves of X^{H}. Given any element ξ_x of $\mathcal{I}_x$, we can take the element ξ of $\mathcal{I}$ which has the same coefficients with respect to $\{\Xi'_a\}$, and let $\xi_{x'}$ be its value at x'. But ρ_* is defined by parallel transport along horizontal lifts of c, which by assumption is an integral curve X; so $\xi_{x'} = \rho_* \xi_x$. Thus $\rho_*(\mathcal{I}_x) \subseteq \mathcal{I}_{x'}$. By reversing the direction of c we obtain $\rho_*^{-1}(\mathcal{I}_{x'}) \subseteq \mathcal{I}_x$, and so $\rho_*(\mathcal{I}_x) = \mathcal{I}_{x'}$. For any points $x, x' \in M$ and any piecewise smooth curve joining them, by breaking the curve up into sufficiently short smooth portions and applying the argument above to each portion in turn we obtain the same result in the general case. Thus a third necessary condition for the $\mathcal{I}_x$ to be holonomy algebras is satisfied.

In a Riemannian space the indicatrix is to all intents and purposes a Euclidean sphere, and so its isometry algebra has the maximum dimension quoted above. However, in a Finsler space we have no a priori information about how symmetric the indicatrix is. This strikes me as an interesting and so far unconsidered aspect of the structure of a Landsberg space. To repeat: in a Riemannian space the isometry algebra of the indicatrix is always $\mathrm{so}(n-1)$, the holonomy algebra may in principle be any subalgebra of it; whereas in a Landsberg space the isometry algebra of the indicatrix is not determined in advance, though again the holonomy algebra may in principle be any subalgebra of it.

In fact there is no obvious reason why the indicatrix of a Landsberg space should have any non-zero Killing vectors. But if the indicatrix at one, and hence every, point x should have no non-zero Killing vectors then $\mathcal{I}_x$ consists of just the zero vector, and so $R_{ij}^k = 0$. The space is therefore R-flat. We therefore have the following corollary of Theorems 1 and 2.

Corollary 6. *If in a Landsberg space which is forward complete and satisfies the boundedness condition, the indicatrix at some point $x \in M$ should admit no non-zero Killing vectors then the space is a Berwald space.*

Indeed, the space is even a Minkowski space, according to Shen.

So a Landsberg space whose indicatrices have no symmetry is a Berwald space. At the other end of the spectrum, it would be of interest to know more about those Landsberg spaces whose indicatrices have the maximum symmetry, that is, are spaces of constant curvature. (It is important to be clear that I am talking about the curvature of the indicatrix calculated with respect to the metric induced on it by the fundamental tensor g_x regarded as a metric on $T_x^\circ M$. There is no suggestion that the Finsler space is a space of scalar curvature in the usual sense.) Here we make contact again with Asanov's examples.

The Finsler function for any of Asanov's non-Berwaldian Landsberg spaces depends on a constant say ϑ, which we can take to satisfy $-\pi/2 < \vartheta < \pi/2$. The Finsler function is given by

$$F(x,y) = \exp(\tan\vartheta\,\Psi)\sqrt{\beta^2 + 2\sin\vartheta\,\beta q + q^2}$$

where

$$\alpha = \sqrt{a_{ij}(x)y^i y^j}, \quad \beta = b_i(x)y^i, \quad q = \sqrt{\alpha^2 - \beta^2},$$

the a_{ij} are the components of a Riemannian metric, and Ψ is a certain function whose exact form need not concern us. For $n \geq 3$ this is the Finsler function of a Landsberg space, which for $\vartheta \neq 0$ is not a Berwald space, but is only y-local. When $\vartheta = 0$, on the other hand, the Finsler function reduces to that of the Riemannian metric (a_{ij}), and of course the space is y-global. It turns out that such a Finsler space has the property that its indicatrices are spaces of constant curvature $C = \cos^2\vartheta$ (see [1, 4]). Notice firstly that all indicatrices have the same constant curvature: this is to be expected, since in a Landsberg space the indicatrices are isometric. Notice secondly that $C \leq 1$, with equality only in the Riemannian case $\vartheta = 0$. Thus within the class of Asanov's Landsberg spaces the only y-global one has $C = 1$, and it is Riemannian; those with $C < 1$ are non-Berwaldian but y-local.

There is one result on Finsler spaces with indicatrices of constant curvature which it is worth mentioning in this context: there can be no y-global absolutely homogeneous Finsler space whose indicatrices have constant curvature less than 1, for the following reason. For each x, $T_x^\circ M$ is a Minkowski space whose Minkowski norm is absolutely homogeneous. In such a Minkowski space

$$\mathrm{vol}_{\dot g}(S) \leq \mathrm{vol}(\mathbb{S}),$$

where S is the (unit) indicatrix, $\mathrm{vol}_{\dot g}(S)$ its volume measured by the induced metric $\dot g$, $\mathbb{S}$ is the standard unit sphere in Euclidean n-space and $\mathrm{vol}(\mathbb{S})$ its Euclidean volume; and equality holds if and only if the norm is that of an inner product (this is Proposition 14.9.1 of Bao, Chern and Shen [5]). If S has constant curvature $1/r$ then it is isometric to the standard sphere $\mathbb{S}_r$ of radius r, and then $\mathrm{vol}_{\dot g}(S) = \mathrm{vol}(\mathbb{S}_r)$, and of course $\mathrm{vol}(\mathbb{S}_r) > \mathrm{vol}(\mathbb{S})$ if $r > 1$. So if a y-global absolutely homogeneous Finsler space has indicatrices of constant curvature then their curvatures must be greater than or equal to 1, and equality holds everywhere if and only if the space is Riemannian.

The remarks in the preceeding paragraph have nothing directly to do with Landsberg spaces; nor do they have any direct relevance to Asanov's examples (which are not absolutely homogeneous except, again, for the Riemannian case $\vartheta = 0$, $C = 1$). Nevertheless they are quite striking.

5 The averaged metric in a Landsberg space

Vincze showed in [16] that in a y-global Berwald space one can find a Riemannian metric whose Levi-Civita connection is the Berwald connection by averaging the fundamental tensor over the indicatrix. In fact one can carry out the averaging construction in any y-global Finsler space. I shall show that, roughly speaking, in a Landsberg space the Levi-Civita connection of the averaged metric is obtained by averaging the Berwald connection over the indicatrix.

For $x \in M$ let $S_x \subset T^\circ_x M$ be the indicatrix at x and ω_x the volume form on S_x induced by g_x (that is, induced by the volume form on $T^\circ_x M$ determined by g_x). For any function f on S_x I denote by $\overline{f}$ its average over S_x,

$$\overline{f} = \frac{\int_{S_x} f(y)\, \omega_x}{\int_{S_x} \omega_x},$$

and for any function f on $T^\circ M$ I denote by $\overline{f}$ the function on M obtained by averaging over indicatrices. I shall in fact apply the averaging process componentwise to geometric objects on $T^\circ M$ whose transformation laws under coordinate transformations on M depend only on the coordinates on M. Then the averaging process defines a geometric object on M with the same transformation law. For example, the g_{ij} transform like the components of a type $(0,2)$ tensor on M, and so their averages are the components of a type $(0,2)$ tensor on M.

Set $\overline{g}_{ij} = \overline{g_{ij}}$. Then $\overline{g}_{ij}$ are the components of a symmetric type $(0,2)$ tensor field $\overline{g}$ on M. Moreover, for any $v \in T_x M$

$$\overline{g}_{ij}(x) v^i v^j = \frac{\int_{S_x} g_{ij}(x,y) v^i v^j\, \omega_x}{\int_{S_x} \omega_x} \geq 0,$$

with equality if and only if $v = 0$, so $\overline{g}$ is a Riemannian metric on M.

Proposition 3. *In a Landsberg space, for any function f on $T^\circ M$ and any $v \in T_x M$,*

$$v(\overline{f}) = \overline{v^{\mathrm{H}}(f)}.$$

Proof. As I have pointed out before, in any Finsler space, for any $v \in T_x M$, $v^{\mathrm{H}}(F) = 0$, and in a Landsberg space, parallel transport along horizontal curves is isometrical. Take a curve c in M with $c(0) = x$, $\dot{c}(0) = v$, and let $t \mapsto c^{\mathrm{H}}(t,y)$ be the horizontal lift of c through $y \in T^\circ_x M$. For t in the domain of c define a map $\rho(t) : T^\circ_x M \to T^o_{c(t)} M$ by $\rho(t)(y) = c^{\mathrm{H}}(t,y)$: then $\rho(t)$ maps S_x isometrically onto $S_{c(t)}$. Thus in particular $\rho(t)$ is volume preserving: $\rho(t)^* \omega_{c(t)} = \omega_x$. It follows that

$$\int_{S_{c(t)}} f\, \omega_{c(t)} = \int_{\rho(t)(S_x)} f\, \omega_{c(t)} = \int_{S_x} \rho(t)^*(f\, \omega_{c(t)}) = \int_{S_x} \rho(t)^*(f)\, \omega_x.$$

Clearly $\int_{S_{c(t)}} \omega_{c(t)} = \int_{S_x} \omega_x$ (this is in fact a well-known result of Bao and Shen [6]). Thus

$$\overline{f}(c(t)) = \overline{\rho(t)^*(f)}.$$

Now $\rho(t)^*(f)(x,y) = f(c^{\mathrm{H}}(t,y))$, so on differentiating with respect to t at $t = 0$ we obtain the stated result. □

For the Berwald connection coefficients Γ^i_{jk}, set $\Gamma_{ijk} = g_{il}\Gamma^l_{jk}$ (the order of indices is important: in particular Γ_{ijk} is symmetric in the *second* pair of indices). Let $\overline{\Gamma}^i_{jk}$ be the coefficients of the Levi-Civita connection of $\overline{g}$, and set $\overline{\Gamma}_{ijk} = \overline{g}_{il}\overline{\Gamma}^l_{jk}$. Now in a Landsberg space

$$H_k(g_{ij}) = \Gamma_{ijk} + \Gamma_{jik},$$

from which it follows by the usual method that

$$\Gamma_{ijk} = \tfrac{1}{2}\left(H_j(g_{ik}) + H_k(g_{ij}) - H_i(g_{jk})\right);$$

and of course

$$\overline{\Gamma}_{ijk} = \tfrac{1}{2}\left(\frac{\partial \overline{g}_{ik}}{\partial x^j} + \frac{\partial \overline{g}_{ij}}{\partial x^k} - \frac{\partial \overline{g}_{ij}}{\partial x^k}\right).$$

Since H_i is the horizontal lift of $\partial/\partial x^i$, from the lemma we have the following result.

Theorem 3. *In a Landsberg space*

$$\overline{\Gamma}_{ijk} = \overline{\Gamma_{ijk}}.$$

Appendix: Landsberg spaces with vanishing Douglas tensor

The proof below is of interest for the following reasons. In the first place, it is manifestly y-local in character. Secondly, unlike two recently published proofs [11, 15] it makes no appeal to Deicke's Theorem, and in that sense it is elementary. Thirdly, it uses what is to my mind the characteristic property of the Douglas tensor, namely that its vanishing is the necessary and sufficient condition for the spray from which it is derived to be (locally) projectively equivalent to an affine spray — and indeed this seems a natural way to start the proof of an assertion which amounts to the claim that a certain spray is affine.

Theorem 4. *A Landsberg space over a manifold M with $\dim M \geq 3$, whose Douglas tensor vanishes, is a Berwald space.*

Proof. A Finsler space whose Douglas tensor vanishes is projectively affine, that is, the coefficients of its geodesic spray are of the form $G^i = G^i_0 + \lambda y^i$ where the G^i_0 are those of an affine spray, and λ is a function of homogeneity degree 1. Now G_0^i does not contribute to the Berwald tensor, which takes the form

$$B^i_{jkl} = \lambda_{,jkl}y^i + \lambda_{,kl}\delta^i_j + \lambda_{,jl}\delta^i_k + \lambda_{,jk}\delta^i_l;$$

of course for any pair of indices, $\lambda_{,\dots k\dots l\dots} = \lambda_{,\dots l\dots k\dots}$. The condition $B_{jikl} = B_{ijkl}$ (Proposition 2) gives

$$y_i\lambda_{,jkl} + g_{ik}\lambda_{,jl} + g_{il}\lambda_{,jk} = y_j\lambda_{,ikl} + g_{jk}\lambda_{,il} + g_{jl}\lambda_{,ik}.$$

On contracting this with y^i and using homogeneity one obtains the following expression for $\lambda_{,jkl}$:

$$F^2\lambda_{,jkl} = -\lambda_{,kl}y_j - \lambda_{,jl}y_k - \lambda_{,jk}y_l.$$

With the aid of this expression the condition above can be reduced to

$$\lambda_{,kl}h_{jm} + \lambda_{,jl}h_{km} = \lambda_{,km}h_{jl} + \lambda_{,jm}h_{kl}$$

where h is the angular metric,

$$h_{ij} = g_{ij} - \frac{1}{F^2}y_iy_j.$$

Now $g^{jk}h_{ik} = \delta^j_i - F^{-2}y_iy^j$, and $g^{ij}h_{ij} = (n-1)$ where $n = \dim M$. Contract the equation for the second derivatives of λ with g^{jm} and use homogeneity again to get

$$(n-1)\lambda_{,kl} = Lh_{kl}, \qquad L = g^{ij}\lambda_{,ij};$$

L is of homogeneity degree -1. Now it follows from the definition of h_{ij}, and the fact that $g_{kl,m} = g_{km,l}$, that

$$h_{kl,m} - h_{km,l} = \frac{1}{F^2}(g_{km}y_l - g_{kl}y_m).$$

Differentiate the equation for $\lambda_{,kl}$ again and use this fact, and symmetry, to obtain

$$L_{,m}h_{kl} - L_{,l}h_{km} + \frac{L}{F^2}(g_{kl}y_m - g_{km}y_l) = 0.$$

Contract with g^{kl} to get

$$(n-2)\left(L_{,m} + \frac{L}{F^2}y_m\right) = 0.$$

Then for $n > 2$ we have $(FL)_{,m} = 0$, or in other words $L = K/F$ for some function K on M. Thus

$$(n-1)\lambda_{,kl} = \frac{K}{F}h_{kl}.$$

Then by differentiating

$$\begin{aligned}(n-1)\lambda_{,jkl} &= -\frac{K}{F^3}h_{kl}y_j + \frac{K}{F}h_{kl,j}\\ &= \frac{K}{F}g_{kl,j} - \frac{K}{F^3}(g_{kl}y_j + g_{lj}y_k + g_{jk}y_l) + \frac{3K}{F^5}y_jy_ky_l,\end{aligned}$$

while from the formula $F^2\lambda_{,jkl}+\lambda_{,kl}y_j+\lambda_{,lj}y_k+\lambda_{,jk}y_l=0$ we have

$$\begin{aligned}(n-1)\lambda_{,jkl} &= -\frac{K}{F^3}(h_{kl}y_j+h_{lj}y_k+h_{jk}y_l)\\ &= -\frac{K}{F^3}(g_{kl}y_j+g_{lj}y_k+g_{jk}y_l)+\frac{3K}{F^5}y_jy_ky_l.\end{aligned}$$

It follows that $Kg_{kl,j}=0$, so either $g_{kl,j}=0$ and the space is Riemannian, or $K=0$, whence $\lambda_{,kl}=0$, λ is linear in the y^i, the geodesic spray is affine, and the space is Berwaldian. □

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