The Morse index theorem for general end conditions

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1 Introduction

Let (N, g) be a Riemannian manifold and P and Q submanifolds of N (of positive codimension). A curve $\gamma: [a, b] \to N$ is an extremal of the energy functional defined by the Riemannian metric g with the end conditions that $\gamma(a) \in P$, $\gamma(b) \in Q$, if and only if γ is a geodesic and the tangents to γ at its endpoints are normal to the submanifolds P and Q. We may assume that γ is parametrized by arc length.

Let γ be an extremal of the energy functional with these general end conditions. Let $\Sigma^P_{\gamma(a)}$ be the second fundamental form of P at $\gamma(a)$ with respect to the unit normal $-\dot\gamma(a)$, considered as a linear map on $T_{\gamma(a)}P$. A Jacobi field $J(t)$ along γ is called a P-Jacobi field if its initial values $J(a)$, $\nabla_{\dot\gamma(a)}J$ are adapted to P, in the sense that $J(a) \in T_{\gamma(a)}P$ and $\nabla_{\dot\gamma(a)}J - \Sigma^P_{\gamma(a)}J(a) \perp T_{\gamma(a)}P$; these are the conditions that $J(t)$ should be the variation field of a variation through geodesics which, like γ, are initially normal to P. A point $\gamma(t_P)$, $t_P \in (a, b]$, is called a focal point of P, or P-focal point, if there is a non-trivial P-Jacobi field $J(t)$ along γ such that $J(t_P) = 0$. The multiplicity of $\gamma(t_P)$ as a P-focal point is the dimension of the space of P-Jacobi fields vanishing at t_P.

The Morse index theorem in this situation should show how to calculate the Morse index of γ in terms of Jacobi fields along γ. It is clear that, unlike the case of fixed endpoints (where the Morse index is given by the number of points conjugate to one endpoint, counted according to their multiplicities), or the case of one variable and one fixed endpoint (where the Morse index is equal to the number of P-focal points, again counted according to their multiplicities), the Morse index for the general end condition problem cannot be obtained by a straightforward count of focal points. Kalish [3], [4] shows that the Morse index can be computed by adding to the count of P-focal points an additional term involving the second fundamental form of Q. Assuming that $\gamma(b)$ is not a P-focal point, one defines a bilinear form A on the subspace of the P-Jacobi fields $J(t)$ along γ such that $J(b) \in T_{\gamma(b)}Q$ by

$$A(J_1, J_2) = g(\nabla_{\dot{\gamma}} J_1, J_2)_{\gamma(b)} - \Sigma^Q (J_1, J_2)_{\gamma(b)}.$$

(Here Σ^Q is the second fundamental form of Q but regarded here, in contrast to the second paragraph, as a symmetric bilinear form. I shall not distinguish notationally between the two ways of representing a second fundamental form, trusting that which is in use will be clear from the context.) Then A is symmetric, by Lagrange's identity. As well as assuming that $\gamma(b)$ is not P-focal I shall assume that there are no non-trivial P-Q-Jacobi fields; that is, that if $J(t)$ is a P-Jacobi field and $J(b) \in T_{\gamma(b)}Q$, $\nabla_{\dot{\gamma}(b)} J - \Sigma^Q_{\gamma(b)} J(b) \perp T_{\gamma(b)}Q$ then $J \equiv 0$. This corresponds in the general end condition problem to the assumption that the endpoints are not conjugate in the fixed endpoint problem, and means that the Morse index form is non-singular. Note that the combined effect of the assumptions that $\gamma(b)$ is not P-focal and that there are no non-trivial P-Q-Jacobi fields is that A is non-singular: for from the first it follows that the evaluation map at b maps the space of P-Jacobi fields onto $T_{\gamma(b)}Q$, and so if J is P-Jacobi with $J(b) \in T_{\gamma(b)}Q$ and satisfies $A(J, \cdot) = 0$ then J is P-Q-Jacobi and therefore vanishes identically.

Under these assumptions the Morse index of γ for the general end condition problem is equal to the sum of the number of P-focal points along γ in (a, b), counted according to their multiplicities, and the index of the symmetric bilinear form A (see [3], [4]).

I shall show that a formally identical formula holds, not just in Riemannian geometry, but for any variational problem with general end conditions, provided that the Lagrangian is regular and its Hessian with respect to the velocity variables is positive definite, and also that appropriate generalizations of the assumptions made above hold. The result essentially follows from the formula for the second variation in Lagrangian geometry which I gave in [1], which is again formally identical to the Riemannian second variation formula. However, I shall give an independent proof of the Morse index theorem in this case, which uses results about curves of Lagrangian subspaces of a symplectic vector space to be found for example in Duistermaat's paper on the Morse index [2]. In the course of the argument I shall describe the appropriate generalizations of normality and of the second fundamental form to this very general situation.

2 Geometrical background

In this section I shall briefly review some results from the differential geometry of second-order differential equation fields and Euler-Lagrange fields which will be needed later. These will be treated throughout as vector fields on the bundle of 1-jets of sections of a manifold M fibred over the real line $\mathbb{R}$.

First, a notational matter: let $\pi: M \to B$ be a fibre bundle; then $V\pi$ will denote the vertical sub-bundle of TM, that is, the kernel of the bundle map $\pi_*: TM \to TB$. For $p \in M$, $V_p\pi$ will denote the vertical subspace of T_pM.

Let $\pi: M \to \mathbb{R}$ be a fibre bundle, $\pi_1^0: J^1\pi \to M$ its first-order jet bundle. A point of $J^1\pi$ can be regarded as a tangent vector to M of the form $\partial/\partial t + v^i\partial/\partial x^i$; the v^i can be taken as fibre coordinates on $J^1\pi$. The sequence

$$0 \longrightarrow V\pi_1^0 \longrightarrow TJ^1\pi \longrightarrow \pi_1^{0*}(TM) \longrightarrow 0$$

of vector bundles over $J^1\pi$, where $\pi_1^{0*}(TM)$ is the pull-back of the tangent bundle TM to $J^1\pi$ by π_1^0, is exact.

The jet bundle is also fibred over $\mathbb{R}$, with projection $\pi_1 = \pi \circ \pi_1^0$. Correspondingly, $V\pi_1^0$ is a sub-bundle of $V\pi_1$.

The fibre $(\pi_1^0)^{-1}(p)$ of $J^1\pi$ over $p \in M$ is an affine space modelled on $V_p\pi$; thus for any $v \in J^1\pi$, $V_v\pi_1^0$ can be canonically identified with $V_p\pi$, $p = \pi_1^0(v)$; and $V\pi_1^0$ can be canonically identified with the subbundle $\pi_1^{0*}(V\pi)$ of $\pi_1^{0*}(TM)$. Local sections of $\pi_1^{0*}(V\pi)$ over $J^1\pi$ will generally be denoted by X and Y. For such X, X^V will denote the corresponding π_1^0-vertical vector field on $J^1\pi$, its vertical lift.

A second-order differential equation field is a vector field on $J^1\pi$ whose integral curves are sections of π_1 which are the jets of their projections into M. The coordinate representation of a second-order differential equation field Γ takes the form

$$\Gamma = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i} + f^i\frac{\partial}{\partial v^i},$$

where $f^i = f^i(t, x^k, v^k)$. The integral curves of Γ are the 1-jets of the sections of π which are solutions of the equations $\ddot{x}^i = f^i(t, x^k, \dot{x}^k)$.

Any second-order differential equation field on $J^1\pi$ defines a horizontal distribution, that is to say, a splitting of the bundle exact sequence. Since this construction has been discussed extensively elsewhere (see for example [1] and references therein), I shall merely describe the horizontal vector fields here. The second-order differential equation field Γ itself is horizontal. Any X has a horizontal lift X^H to $J^1\pi$; in particular,

$$H_i = \left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial x^i} - \Gamma_i^j\frac{\partial}{\partial v^j} \quad \text{where } \Gamma_i^j = -\tfrac{1}{2}\frac{\partial f^j}{\partial v^i}.$$

The horizontal lift is of course $C^\infty(J^1\pi)$-linear.

It turns out that the horizontal component of $[\Gamma, X^V]$ is $-X^H$. If we set

$$[\Gamma, X^V] + X^H = \nabla X^V$$

then ∇ is an operator on local sections of $\pi_1^{0*}(V\pi)$ which satisfies

$$\nabla(fX) = f(\nabla X) + (\Gamma f)X, \quad f \in C^\infty(J^1\pi).$$

Thus ∇ behaves like a covariant derivative in the direction of Γ. It is called the dynamical covariant derivative associated with Γ.

It further turns out that the horizontal component of $[\Gamma, X^H]$ is ∇X^H. If we set

$$[\Gamma, X^H] - \nabla X^H = \Phi(X)^V$$

then Φ is an operator on local sections of $\pi_1^{0*}(V\pi)$ which satisfies

$$\Phi(fX) = f\Phi(X), \quad f \in C^\infty(J^1\pi),$$

and can therefore be considered as a type $(1,1)$ tensor field. It is called the Jacobi endomorphism of Γ. Suppose that ξ is a π_1-vertical vector field defined along an integral curve of Γ such that $\mathcal{L}_\Gamma \xi = 0$. Set $\xi = X^H + Y^V$. Then $Y = \nabla X$, and $\nabla^2 X + \Phi(X) = 0$, which is the generalization of Jacobi's equation to the case of a second-order differential equation field.

A Lagrangian L is a smooth function on $J^1\pi$. The 1-form Θ_L defined by

$$\Theta_L = L dt + \frac{\partial L}{\partial v^i}(dx^i - v^i dt) = L dt + dL \circ S, \quad S = \frac{\partial}{\partial v^i} \otimes (dx^i - v^i dt),$$

is called the Cartan 1-form of L. The Cartan 1-form is semi-basic, that is to say, it vanishes on π_1^0-vertical vector fields. The type $(1,1)$ tensor field S is the vertical endomorphism; we have $S(\Gamma) = 0$, $S(X^H) = X^V$, $S(X^V) = 0$.

A Lagrangian is said to be regular if its Cartan 2-form $d\Theta_L$ has maximal rank $2n$ where n is the fibre dimension of $\pi \colon M \to \mathbb{R}$. There is then a unique vector field Γ_L on $J^1\pi$, the Euler-Lagrange field of L, which is characteristic for $d\Theta_L$ and satisfies $\Gamma_L(t) = 1$. This vector field is a second-order differential equation field, so its integral curves are 1-jets of sections of π. The extremals of a regular Lagrangian L are the sections of π whose 1-jets are integral curves of Γ_L.

There is a close relation between the horizontal distribution corresponding to a Euler-Lagrange field Γ_L and the algebraic structure of $d\Theta_L$:

$$d\Theta_L(X^V, Y^V) = d\Theta_L(X^H, Y^H) = 0, \quad d\Theta_L(X^V, Y^H) = d\Theta_L(Y^V, X^H).$$

Thus if $g_L(X, Y) = d\Theta_L(X^V, Y^H)$ then g_L is a symmetric type $(0,2)$ tensor field on local sections of $\pi_1^{0*}(V\pi)$, which is non-singular if L is regular. In fact

$$g_{ij} = g_L\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 L}{\partial v^i \partial v^j}.$$

Since Γ_L is characteristic for $d\Theta_L$, $\mathcal{L}_{\Gamma_L} d\Theta_L = 0$; it follows from this, together with the formulae for the dynamical covariant derivative and the Jacobi endomorphism given above, that

$$\nabla g_L = 0 \quad \text{and} \quad g_L(\Phi(X), Y) = g_L(X, \Phi(Y)).$$

Furthermore, $d\Theta_L(S(\xi), \eta) + d\Theta_L(\xi, S(\eta)) = 0$ for any vector fields ξ and η.

The following coordinate formulae will be useful. Let

$$\Gamma_L = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial v^i}$$

be the Euler-Lagrange field of a Lagrangian $L(t, x^i, v^i)$. Then

$$f^i = g^{ij} \left(\frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial t \partial v^j} - \frac{\partial^2 L}{\partial x^k \partial v^j} v^k \right),$$

and therefore

$$\Gamma^i_j = -\tfrac{1}{2} \frac{\partial f^i}{\partial v^j} = \tfrac{1}{2} g^{ik} \left(A_{jk} + \Gamma(g_{jk}) \right), \quad \text{where } A_{ij} = -A_{ji} = \frac{\partial^2 L}{\partial x^i \partial v^j} - \frac{\partial^2 L}{\partial x^j \partial v^i}.$$

It follows that

$$H_i \left(\frac{\partial L}{\partial v^j} \right) = \tfrac{1}{2} \left(B_{ij} - \Gamma(g_{ij}) \right), \quad \text{where } B_{ij} = B_{ji} = \frac{\partial^2 L}{\partial x^i \partial v^j} + \frac{\partial^2 L}{\partial x^j \partial v^i};$$

note that this expression is symmetric in i and j.

3 General end condition variational problems

Let $a, b \in \mathbb{R}$, $a < b$. Let P be a submanifold of the fibre $\pi^{-1}(a)$ of M and Q a submanifold of $\pi^{-1}(b)$, both of dimension less than n. Suppose given a Lagrangian L on $J^1\pi$. For any local section σ of π defined on $[a, b] \subset \mathbb{R}$, let $\dot\sigma$ be its 1-jet, and set

$$\mathcal{I}(\sigma) = \int_a^b \dot\sigma^*(L dt) = \int_a^b \dot\sigma^* \Theta_L.$$

The variational problem for the Lagrangian L and end manifolds P and Q is the problem of finding those sections σ of π with $\sigma(a) \in P$ and $\sigma(b) \in Q$ for which $\mathcal{I}(\sigma)$ is stationary with respect to variations of σ through sections σ_s such that $\sigma_s(a) \in P$ and $\sigma_s(b) \in Q$, where $\sigma_0 = \sigma$. Let X be the variation field of such a variation: it is a π-vertical vector field along the image of σ with $X(a) \in T_{\sigma(a)}P$, $X(b) \in T_{\sigma(b)}P$. The first variation formula is

$$\frac{d\mathcal{I}}{ds}(0) = \int_a^b \dot\sigma^*(\tilde{X} \lrcorner\, d\Theta_L) + \Theta_L(\tilde{X}) \Big|_{\dot\sigma(a)}^{\dot\sigma(b)},$$

where $\tilde{X}$ is the variation field along the image of $\dot{\sigma}$ induced by X. In order for $\mathcal{I}(\sigma)$ to be stationary, we must have

$$\int_a^b \dot{\sigma}^*(\tilde{X} \lrcorner\, d\Theta_L) = 0, \quad \Theta_L(\tilde{X})_{\dot{\sigma}(b)} = 0, \quad \Theta_L(\tilde{X})_{\dot{\sigma}(a)} = 0$$

for all variation fields X such that $X(a) \in T_{\sigma(a)}P$, $X(b) \in T_{\sigma(b)}P$.

¿From the vanishing of the integral it follows that σ must be an extremal of L, that is, that $\dot{\sigma}$ is a segment of an integral curve of Γ_L.

To interpret the other conditions it is useful to introduce the idea of a transversal submanifold. I shall define the transversal submanifold of P, but the definition will clearly apply to any submanifold of a fibre of $\pi\colon M \to \mathbb{R}$. The transversal submanifold $\hat{P}$ of P is the submanifold of $\pi_1^{-1}(a)$ consisting of those $v \in J^1\pi$ such that $\pi_1(v) = a$, $p = \pi_1^0(v) \in P$ and $\Theta_L|_v(w) = 0$ for all $w \in V_v\pi_1$ such that $\pi^0_{1*}w \in T_pP$. (Since Θ_L is semi-basic, the last condition is satisfied if it holds for w in a subset of $V_v\pi_1$ which projects onto T_pP.) The conditions for $\mathcal{I}(\sigma)$ to be stationary are that σ is an extremal of L and that $\dot{\sigma}(a) \in \hat{P}$ and $\dot{\sigma}(b) \in \hat{Q}$.

Suppose that locally P is given in coordinates by $t = a$, $x^i = x^i(y^\alpha)$, $\alpha = 1, 2, \ldots \dim P$. Then $\hat{P}$ consists of points $(a, x^i(y^\alpha), v^i)$ such that

$$\frac{\partial x^i}{\partial y^\alpha}\frac{\partial L}{\partial v^i}(a, x^j(y^\beta), v^j) = 0.$$

In the Riemannian case, where L is the energy, $\hat{P}$ is the normal bundle of P, and so $\dot{\sigma}(a) \in \hat{P}$ generalizes normality in the Riemannian case.

If P is a point p, then $\hat{P} = (\pi_1^0)^{-1}(p)$.

In [1] I showed that the second variation formula can be written

$$\frac{d^2\mathcal{I}}{ds^2}(0) = \int_a^b \left\{ g_L(\nabla X, \nabla X)_{\dot{\gamma}(t)} - g_L(X, \Phi(X))_{\dot{\gamma}(t)} \right\} dt + X^H\left(\Theta_L(\tilde{X})\right)\Big|_{\dot{\gamma}(a)}^{\dot{\gamma}(b)},$$

where γ is an extremal of L with $\dot{\gamma}(a) \in \hat{P}$ and $\dot{\gamma}(b) \in \hat{Q}$, and X is a π-vertical vector field defined in a neighbourhood of the image of γ with $X_{\gamma(a)} \in T_{\gamma(a)}P$ and $X_{\gamma(b)} \in T_{\gamma(b)}Q$.

In fact the second variation formula depends only on the values of X along the image of γ. This is clear for the integral term. For the integrated terms I shall establish a slightly more general result.

Let X and Y be π-vertical vector fields on M, defined in a neighbourhood of a point p of P, and tangent to P at points of P. Then

$$X^H\Theta_L(Y^H) - Y^H\Theta_L(X^H) = d\Theta_L(X^H, Y^H) - \Theta_L([X^H, Y^H]).$$

Now Θ_L is semi-basic, and $[X^H, Y^H] - [X,Y]^H$ is vertical; moreover $d\Theta_L$ gives zero when both of its arguments are horizontal. Thus the right-hand side is just $-\Theta_L([X,Y]^H)$. Now let $v \in \hat{P}$: then $\Theta_L([X,Y]^H)_v = 0$ by definition of $\hat{P}$, since if X and Y are both tangent to P at points of it, so is $[X,Y]$. Moreover, for any function ϕ on M,

$$X^H\Theta_L((\phi Y)^H) = (X^H(\pi_1^{0*}\phi))\Theta_L(Y^H) + (\pi_1^{0*}\phi)X^H\Theta_L(Y^H),$$

so that for $v \in \hat{P}$, $p = \pi_1^0(v)$,

$$X^H\Theta_L((\phi Y)^H)_v = \phi(p)X^H\Theta_L(Y^H)_v.$$

It follows that $X^H\Theta_L(Y^H)_v$ depends only on the values of X and Y at p, and depends bilinearly and symmetrically on them. Thus if we set

$$\Sigma_v^P(X_p, Y_p) = -X^H\Theta_L(Y^H)_v, \quad v \in \hat{P}, \quad p = \pi_1^0(v)$$

where $X_p, Y_p \in T_pP$ and X, Y are any locally defined π-vertical vectorfields tangent to P such that $X(p) = X_p$, $Y(p) = Y_p$, then Σ_v^P is a well-defined symmetric bilinear form on T_pP, and Σ^P is well-defined over $\hat{P}$. We call Σ^P the second fundamental form of P. The sign is chosen to agree with Kalish's sign convention. (The first fundamental form of P at $v \in \hat{P}$ is the restriction of $g_L|_v$ to T_pP, $p = \pi_1^0(v)$. Its components are

$$g_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta},$$

where g_{ij} are the components of g_L.)

Now $\tilde{X} - X^H$ is π_1^0-vertical and Θ_L is semi-basic, so we may write the second variation formula as

$$\begin{aligned}
\frac{d^2\mathcal{I}}{ds^2}(0) &= \int_a^b \left\{ g_L(\nabla X, \nabla X)_{\dot\gamma(t)} - g_L(X, \Phi(X))_{\dot\gamma(t)} \right\} dt \\
&\quad -\Sigma^Q(X,X)_{\dot\gamma(b)} + \Sigma^P(X,X)_{\dot\gamma(a)} \\
&= -\int_a^b g_L(\nabla^2 X + \Phi(X), X)_{\dot\gamma(t)} dt \\
&\quad + \left(g_L(\nabla X, X)_{\dot\gamma(b)} - \Sigma^Q(X,X)_{\dot\gamma(b)}\right) - \left(g_L(\nabla X, X)_{\dot\gamma(a)} - \Sigma^P(X,X)_{\dot\gamma(a)}\right).
\end{aligned}$$

The expressions $g_L(\nabla X, X)_{\dot\gamma(b)}$ and $\Sigma^Q(X,X)_{\dot\gamma(b)}$ could be written more informatively as $g_L|_{\dot\gamma(b)}(\nabla X(b), X(b))$ and $\Sigma^Q_{\dot\gamma(b)}(X(b), X(b))$; the shorter forms will often be used below, as here.

In coordinates $\hat{P}$ is given by

$$\frac{\partial x^i}{\partial y^\alpha}\frac{\partial L}{\partial v^i}(a, x^j(y^\beta), v^j) = 0, \quad \alpha = 1, 2, \ldots \dim P.$$

Set

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i};$$

then we have

$$\begin{aligned}
\Sigma^P_{\alpha\beta} &= \Sigma^P\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right) = -\frac{\partial x^i}{\partial y^\alpha} H_i \left(\frac{\partial x^j}{\partial y^\beta}\frac{\partial L}{\partial v^j}\right) \\
&= -\frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}\frac{\partial L}{\partial v^i} - \tfrac{1}{2}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\left(B_{ij} - \Gamma(g_{ij})\right),
\end{aligned}$$

using a formula derived earlier.

In both the Riemannian and Finslerian cases this becomes

$$\Sigma^P_{\alpha\beta} = -g_{ij}v^j\left(\frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta}\Gamma^i_{kl}\right),$$

where v satisfies

$$g_{ij}(v)\frac{\partial x^i}{\partial y^\alpha}v^j = 0, \quad \alpha = 1, 2, \ldots \dim P.$$

In the Riemannian case v is normal to P at p; $\Sigma^P_{\alpha\beta}$ depends linearly on v^j, and is the component in the normal direction $-v$ of the covariant derivative of $\partial/\partial y^\beta$ along $\partial/\partial y^\alpha$, that is to say, it is the second fundamental form of P as usually defined. However, in Finsler geometry the g_{ij} depend on v in general, so the dependence on v^j is not linear; moreover, the second fundamental form is defined, by this construction, only over the transversal submanifold $\hat{P}$. The latter remarks are true a fortiori in the general case.

Assuming that L is regular, $\pi_1^{-1}(t)$ is a symplectic manifold for each $t \in \mathbb{R}$, with symplectic form obtained by restriction of the Cartan 2-form $d\Theta_L$. I now show that any transversal submanifold $\hat{P}$ is Lagrangian with respect to $d\Theta_L$. The tangent space $T_v\hat{P}$ to $\hat{P}$ at any point v clearly projects onto T_pP where $p = \pi_1^0(v)$. It follows from the coordinate specification of $\hat{P}$ that a vector

$$X^\alpha\left(\frac{\partial}{\partial y^\alpha}\right)^H + Y^i\frac{\partial}{\partial v^i} = X^\alpha\frac{\partial x^i}{\partial y^\alpha}H_i + Y^i\frac{\partial}{\partial v^i}$$

is tangent to $\hat{P}$ if and only

$$X^\alpha\frac{\partial x^i}{\partial y^\alpha}H_i\left(\frac{\partial x^j}{\partial y^\beta}\frac{\partial L}{\partial v^j}\right) + Y^i\frac{\partial x^j}{\partial y^\beta}\frac{\partial^2 L}{\partial v^i \partial v^j} = 0,$$

or in other words

$$Y^i g_{ij}\frac{\partial x^j}{\partial y_\beta} - X^\alpha\Sigma^P_{\alpha\beta} = 0.$$

It follows that when L is regular, $\dim\hat{P} = n$.

This condition just derived may be written equivalently as

$$g_L(Y,Z)_v - \Sigma^P(X,Z)_v = g_L(Y-\Sigma^P X, Z)_v = 0$$

for any $Z_p \in T_pP$, $p = \pi_1^0(v)$, $v \in \hat{P}$, where in the second expression Σ_v^P has been converted into a linear map on T_pP. That is to say, ${X_p}^H + {Y_p}^V \in T_v\hat{P}$ if and only if $X_p \in T_pP$

and $Y_p - \Sigma_v^P X_p \perp_v T_pP$, where $\perp_v$ means perpendicular with respect to $g_L|_v$.

Now take any two vectors $w, w' \in T_v\hat{P}$, where $w = {X_p}^H + {Y_p}^V$ with $X_p \in T_pP$ and $Y_p - \Sigma_v^P X_p \perp_v T_pP$, and similarly for w'. Then

$$d\Theta_L|_v(w,w') = g_L|_v(Y_p, X'_p) - g_L|_v(Y'_p, X_p) = \Sigma_v^P(X_p, X'_p) - \Sigma_v^P(X'_p, X_p) = 0.$$

Thus $T_v\hat{P}$ is isotropic. Its dimension is n, so it is Lagrangian.

4 The Morse index theorem

Let ξ be a π_1-vertical vector field defined along an integral curve of Γ such that

$$\mathcal{L}_{\Gamma_L}\xi = kS(\xi)$$

where k is constant and S is the vertical endomorphism. Set $\xi = X^H + Y^V$; then $Y = \nabla X$, and

$$\nabla^2 X + \Phi(X) = kX,$$

so that X is an eigenvector of the operator $\nabla^2 + \Phi$. Suppose that η is another vector field satisfying the same condition (with the same k). Then since $\mathcal{L}_{\Gamma_L} d\Theta_L = 0$,

$$\Gamma_L\left(d\Theta_L(\xi,\eta)\right) = d\Theta_L(\mathcal{L}_{\Gamma_L}\xi,\eta) + d\Theta_L(\xi,\mathcal{L}_{\Gamma_L}\eta) = k\left(d\Theta_L(S(\xi),\eta) + d\Theta_L(\xi,S(\eta))\right) = 0,$$

by a property of $d\Theta_L$ mentioned earlier. It follows that if $v \in \pi_1^{-1}(a)$, $a \in \mathbb{R}$, and one is given a Lagrangian subspace of $V_v\pi_1$ and propagates it by this rule along the integral curve of Γ_L through v then the propagated space remains Lagrangian.

Assume henceforth that g_L is positive definite. Consider an extremal $\gamma\colon [a,b] \to M$ of the variational problem with end manifolds P and Q. The Morse index of γ is the number of positive eigenvalues, counted according to their multiplicities, of the operator $\nabla^2 + \Phi$ on the space of vector fields X along γ satisfying the end conditions $X(a) \in T_{\gamma(a)}P$, $\nabla X(a) - \Sigma^P_{\dot\gamma(a)} X(a) \perp_{\dot\gamma(a)} T_{\gamma(a)}P$ and $X(b) \in T_{\gamma(b)}Q$, $\nabla X(b) - \Sigma^Q_{\dot\gamma(b)} X(b) \perp_{\dot\gamma(b)} T_{\gamma(b)}Q$. These end conditions may be written equivalently as $X(a)^H + \nabla X(a)^V \in T_{\dot\gamma(a)}\hat{P}$ and $X(b)^H + \nabla X(b)^V \in T_{\dot\gamma(b)}\hat{Q}$. In other words, the Morse index is the number of positive values of k, counted according to their multiplicities, for which there are non-trivial solutions ξ of the equation $\mathcal{L}_{\Gamma_L}\xi = kS(\xi)$ along the integral curve $\dot\gamma$ of Γ_L, such that $\xi(a) \in T_{\dot\gamma(a)}\hat{P}$, $\xi(b) \in T_{\dot\gamma(b)}\hat{Q}$. Now $T_{\dot\gamma(a)}\hat{P}$ is Lagrangian; so for each k the subspace $\Lambda^P(k)$ of the symplectic vector space $V_{\dot\gamma(b)}\pi_1$ defined by

$$\Lambda^P(k) = \left\{\xi(b) \,\middle|\, \mathcal{L}_{\Gamma_L}\xi = kS(\xi), \, \xi(a) \in T_{\dot\gamma(a)}\hat{P}\right\}$$

is Lagrangian. It follows that the Morse index I_M of γ for the general end condition problem is the intersection number of the curve $k \mapsto \Lambda^P(k)$, $k > 0$, of Lagrangian subspaces of $V_{\dot\gamma(b)}\pi_1$ with the fixed Lagrangian subspace $\Lambda_1 = T_{\dot\gamma(b)}\hat{Q}$. To be a little more precise: there are only finitely many positive values of k for which $\Lambda^P(k)$ has non-zero intersection with Λ_1; the Morse index is the sum of the dimensions of these intersections. Let $K > 0$ be sufficiently large that all positive values of k for which $\Lambda^P(k)$ has non-zero intersection with Λ_1 lie in $(0, K)$: then the Morse index of γ for the general end condition problem is $[\Lambda^P\colon\Lambda_1]$, the intersection number of the curve of Lagrangian subspaces $k \mapsto \Lambda^P(k)$, $k \in [0, K]$, with Λ_1.

On the other hand, the number of P-focal points along γ in (a, b) (in the obvious sense) is $[\Lambda^P\colon\Lambda_2]$, the intersection number of $\Lambda^P(k)$, $k \in [0, K]$, with Λ_2, the π_1^0-vertical subspace $V_{\dot\gamma(b)}\pi_1^0$ of $V_{\dot\gamma(b)}\pi_1$, which of course is also Lagrangian. Here K is required to be sufficiently large that each of the finitely many positive values of k for which $\Lambda^P(k)$ has non-zero intersection with Λ_2 also lies in $(0, K)$. Denote the P-focal point index by I_F; then

$$I_M = I_F + [\Lambda^P\colon\Lambda_1] - [\Lambda^P\colon\Lambda_2].$$

To determine the correction term we need a formula for the difference in the

intersection numbers. Such a formula has been given by Hörmander. According to his formula the difference of the intersection numbers can be expressed in terms of the signatures of two symmetric bilinear forms, one associated with the triplet of Lagrangian subspaces Λ_1, Λ_2, $\Lambda^P(0)$, the other with Λ_1, Λ_2, $\Lambda^P(K)$. The following description of Hörmander's formula is based on that given by Duistermaat [2].

Let U be a symplectic vector space with symplectic form Ω, and let Λ_1, Λ_2 and Λ be three Lagrangian subspaces of U. Suppose first that $\Lambda_1 \cap \Lambda_2 = \{0\}$, so that Λ_1 and Λ_2 are complementary. Suppose further that $\Lambda \cap \Lambda_2 = \{0\}$. Then as a subspace of U, Λ may be represented as the graph of a linear map $C\colon \Lambda_1 \to \Lambda_2$; that is,

$$\Lambda = \{u + Cu \mid u \in \Lambda_1\}.$$

Moreover, Λ is Lagrangian if and only if the bilinear form B on Λ_1 defined by

$$B(u, u') = \Omega(Cu, u'), \quad u, u' \in \Lambda_1,$$

is symmetric. Now suppose that $\Lambda_1 \cap \Lambda_2 = W \neq \{0\}$ (but that $\Lambda \cap \Lambda_2 = \{0\}$ still holds). Then W is an isotropic subspace of U. Let W° be its symplectic orthogonal;

then $W \subset W^\circ$, and Ω defines a symplectic structure on $V = W^\circ/W$. Let $\rho\colon W^\circ \to V$ be the projection. Then for any Lagrangian subspace $\Lambda \subset U$, $\rho(\Lambda \cap W^\circ)$ is a Lagrangian subspace of V. We have $\Lambda_1 \subset W^\circ$, and likewise for Λ_2; $\rho(\Lambda_1) \cap \rho(\Lambda_2) = \{0\}$ and $\rho(\Lambda \cap W^\circ) \cap \rho(\Lambda_2) = \{0\}$, so we can adapt the definition of B in the case $\Lambda_1 \cap \Lambda_2 = \{0\}$ to obtain a symmetric bilinear form, also denoted by B, on $\rho(\Lambda_1)$.

Now let $\Lambda(k)$, $k \in [0, K]$, be a curve of Lagrangian subspaces of U, such that $\Lambda(0) \cap \Lambda_2 = \{0\}$ and $\Lambda(K) \cap \Lambda_2 = \{0\}$. Let B_0 be the symmetric bilinear form on $\rho(\Lambda_1)$ defined by the construction described above for the triplet of Lagrangian subspaces Λ_1, Λ_2 and $\Lambda(0)$, and B_K the form defined by Λ_1, Λ_2 and $\Lambda(K)$. Then Hörmander's formula is

$$[\Lambda\colon \Lambda_1] - [\Lambda\colon \Lambda_2] = \tfrac{1}{2}(\operatorname{sgn} B_K - \operatorname{sgn} B_0).$$

I now compute B_k, the symmetric bilinear form defined by the triplet of Lagrangian subspaces $\Lambda_1 = T_{\dot\gamma(b)}\hat{Q}$, $\Lambda_2 = V_{\dot\gamma(b)}\pi_1^0$ and $\Lambda^P(k)$ of $V_{\dot\gamma(b)}\pi_1$, assuming that k is such that $\Lambda^P(k) \cap \Lambda_2 = \{0\}$ (which will be so in the cases of interest $k = 0$ and $k = K$). The first step is to identify $T_{\dot\gamma(b)}\hat{Q} \cap V_{\dot\gamma(b)}\pi_1^0$. Now

$$T_{\dot\gamma(b)}\hat{Q} = \left\{ X_{\gamma(b)}{}^H + Y_{\gamma(b)}{}^V \;\middle|\; X_{\gamma(b)} \in T_{\gamma(b)}Q,\, Y_{\gamma(b)} - \Sigma^Q_{\dot\gamma(b)} X_{\gamma(b)} \perp_{\dot\gamma(b)} T_{\gamma(b)}Q \right\},$$

and therefore

$$T_{\dot\gamma(b)}\hat{Q} \cap V_{\dot\gamma(b)}\pi_1^0 = \left(T_{\gamma(b)}Q^{\perp_{\dot\gamma(b)}}\right)^V.$$

Notice that the subspace

$$\left\{ X_{\gamma(b)}{}^H + \Sigma^Q_{\dot\gamma(b)} X_{\gamma(b)}{}^V \;\middle|\; X_{\gamma(b)} \in T_{\gamma(b)}Q \right\}$$

is a complement to $T_{\dot\gamma(b)}\hat{Q} \cap V_{\dot\gamma(b)}\pi_1^0$ in $T_{\dot\gamma(b)}\hat{Q}$.

So we take $W = \left(T_{\gamma(b)}Q^{\perp_{\dot\gamma(b)}}\right)^V$, and $W^\circ = (T_{\gamma(b)}Q)^H \oplus V_{\dot\gamma(b)}\pi_1^0$. It follows that W°/W may be identified with $(T_{\gamma(b)}Q)^H \oplus (T_{\gamma(b)}Q)^V \subset V_{\dot\gamma(b)}\pi_1$, and $\rho\colon W^\circ \to W^\circ/W$ with perpendicular projection in the vertical part. We have

$$\rho(\Lambda_1) = \rho(T_{\dot\gamma(b)}\hat{Q}) = \left\{ X_{\gamma(b)}{}^H + \Sigma^Q_{\dot\gamma(b)} X_{\gamma(b)}{}^V \;\middle|\; X_{\gamma(b)} \in T_{\gamma(b)}Q \right\},$$

while

$$\rho(\Lambda_2) = \rho(V_{\dot\gamma(b)}\pi_1^0) = (T_{\gamma(b)}Q)^V.$$

Furthermore,

$$\begin{aligned}\Lambda^P(k) \;&=\; \Big\{ X(b)^H + \nabla X(b)^V \;\Big|\; \nabla^2 X + \Phi(X) = kX,\, X(a) \in T_{\gamma(a)}P, \\ &\qquad \nabla X(a) - \Sigma^P_{\dot\gamma(a)} X(a) \perp_{\dot\gamma(a)} T_{\gamma(a)}P \Big\},\end{aligned}$$

and therefore

$$\begin{aligned}\Lambda^P(k)\cap W^\circ &= \Big\{X(b)^H+\nabla X(b)^V \,\Big|\, \nabla^2X+\Phi(X)=kX,\ X(a)\in T_{\gamma(a)}P,\\ &\qquad \nabla X(a)-\Sigma^P_{\dot\gamma(a)}X(a)\perp_{\dot\gamma(a)} T_{\gamma(a)}P,\ X(b)\in T_{\gamma(b)}Q\Big\}.\end{aligned}$$

For $X(b)^H+\nabla X(b)^V\in\Lambda^P(k)\cap W^\circ$,

$$\rho(X(b)^H+\nabla X(b)^V)=X(b)^H+\Pi_{\dot\gamma(b)}\nabla X(b)^V$$

where $\Pi_{\dot\gamma(b)}\colon V_{\gamma(b)}\pi\to T_{\gamma(b)}Q$ is perpendicular projection with respect to $g_L|_{\dot\gamma(b)}$. We may write

$$\rho(X(b)^H+\nabla X(b)^V)=\left(X(b)^H+\Sigma^Q_{\dot\gamma(b)}X(b)^V\right)+\left(\Pi_{\dot\gamma(b)}\nabla X(b)-\Sigma^Q_{\dot\gamma(b)}X(b)\right)^V,$$

where $X(b)^H+\Sigma^Q_{\dot\gamma(b)}X(b)^V\in\rho(\Lambda_1)$ and $(\Pi_{\dot\gamma(b)}\nabla X(b)-\Sigma^Q_{\dot\gamma(b)}X(b))^V\in\rho(\Lambda_2)$. It follows from the assumption that $\Lambda^P(k)\cap\Lambda_2=\{0\}$ that given $w\in T_{\gamma(b)}Q$ there is a unique solution of $\nabla^2X+\Phi(X)=k$ satisfying $X(a)\in T_{\gamma(a)}P$, $\nabla X(a)-\Sigma^P_{\dot\gamma(a)}X(a)\perp_{\dot\gamma(a)}T_{\gamma(a)}P$, such that $X(b)=w$. Thus if we set

$$C(X(b)^H+\Sigma^Q_{\dot\gamma(b)}X(b)^V)=(\Pi_{\dot\gamma(b)}\nabla X(b)-\Sigma^Q_{\dot\gamma(b)}X(b))^V$$

then C is a well-defined linear map from $\rho(\Lambda_1)$ to $\rho(\Lambda_2)$. Thus the bilinear form B_k may be expressed as follows:

$$\begin{aligned}B_k(X,Y) &= d\Theta_L|_{\dot\gamma(b)}\left((\Pi_{\dot\gamma(b)}\nabla X(b)-\Sigma^Q_{\dot\gamma(b)}X(b))^V, Y(b)^H+\Sigma^Q_{\dot\gamma(b)}Y(b)^V\right)\\ &= \left(g_L(\nabla X,Y)-\Sigma^Q(X,Y)\right)_{\dot\gamma(b)}\end{aligned}$$

where X and Y are vector fields along γ satisfying $\nabla^2X+\Phi(X)=kX$, $X(a)\in T_{\gamma(a)}P$, $\nabla X(a)-\Sigma^P_{\dot\gamma(a)}X(a)\perp_{\dot\gamma(a)}T_{\gamma(a)}P$, $X(b)\in T_{\gamma(b)}Q$, and similarly for Y.

For $k=0$ the vector fields X and Y are P-Jacobi fields in the obvious sense, and so B_0 corresponds to Kalish's A. Assume that $\Lambda^P(0)\cap\Lambda_1=\{0\}$, so that there are no non-trivial P-Q-Jacobi fields: then B_0 is non-singular, and $\operatorname{sgn}B_0=\dim Q-2\operatorname{ind}B_0$.

I shall show that for K sufficiently large B_K will be positive definite, so that $\operatorname{sgn}B_K=\dim Q$. The idea of the proof is to show that $g_L(\nabla X,X)_{\dot\gamma(b)}/g_L(X,X)_{\dot\gamma(b)}$ can be made large enough to swamp $\Sigma^Q(X,X)_{\dot\gamma(b)}/g_L(X,X)_{\dot\gamma(b)}$ by choosing K sufficiently large; note that the latter has a fixed finite maximum value that is independent of K, as $X(b)$ ranges over $T_{\gamma(b)}Q$.

I begin by taking K sufficiently large that if $X(t)$ satisfies $\nabla^2X+\Phi(X)=k$ for $k\geq K$, $X(a)\in T_{\gamma(a)}P$, $\nabla X(a)-\Sigma^P_{\dot\gamma(a)}X(a)\perp_{\dot\gamma(a)}T_{\gamma(a)}P$, then $X(t)\neq 0$ for $a<t\leq b$. Then the function

$$x(t)=\frac{g_L(\nabla X,X)_{\dot\gamma(t)}}{g_L(X,X)_{\dot\gamma(t)}}$$

is well-defined for $a < t \leq b$. It satisfies

$$\frac{dx}{dt} = \frac{g_L(\nabla^2 X, X) + g_L(\nabla X, \nabla X)}{g_L(X,X)} - 2\frac{g_L(\nabla X, X)^2}{g_L(X,X)^2},$$

using the fact that $\nabla g_L = 0$, and so

$$\frac{dx}{dt} + 2x^2 = k - \frac{g_L(\Phi(X), X)}{g_L(X,X)} + \frac{g_L(\nabla X, \nabla X)}{g_L(X,X)}.$$

I shall compare this with the Riccati equation

$$\frac{dy}{dt} + 2y^2 = \tfrac{1}{2}\mu^2,$$

where μ is a positive constant. The value at b of the solution of this equation with initial value $y(a)$ is

$$y(b) = \mu\left(\frac{(\mu + y(a))e^{\mu(b-a)} - (\mu - y(a))e^{-\mu(b-a)}}{(\mu + y(a))e^{\mu(b-a)} + (\mu - y(a))e^{-\mu(b-a)}}\right);$$

then given any S we can find $\mu > 0$ such that $y(b) > S$. The same is true for the solution which corresponds to taking $y(a) = \infty$, for which

$$y(b) = \mu\left(\frac{e^{\mu(b-a)} + e^{-\mu(b-a)}}{e^{\mu(b-a)} - e^{-\mu(b-a)}}\right).$$

Now set $z(t) = x(t) - y(t)$, where $y(a) = x(a)$ so that $z(a) = 0$; it can be shown that if $X(a) = 0$ and $y(t)$ is the solution corresponding to $y(a) = \infty$ then $z(t) \to 0$ as $t \to a$. Then

$$\frac{dx}{dt} + 2x^2 = \frac{dy}{dt} + 2y^2 + \frac{dz}{dt} + 2z^2 + 4yz,$$

so that

$$\frac{dz}{dt} + 2z^2 + 4yz = k - \frac{g_L(\Phi(X), X)}{g_L(X,X)} - \tfrac{1}{2}\mu^2 + \frac{g_L(\nabla X, \nabla X)}{g_L(X,X)},$$

where μ is chosen so that $y(b) > S$. Let χ be the maximum value of $g_L(\Phi(\bar{X}), \bar{X})_{\dot{\gamma}(t)}$ over vectors $\bar{X}_{\gamma(t)} \in V_{\gamma(t)}\pi$, $a \leq t \leq b$ which are unit with respect to $g_L|_{\dot{\gamma}(t)}$; then for any $k > \chi + \frac{1}{2}\mu^2$ the right-hand side of this equation is positive. Thus whenever $z = 0$, $dz/dt > 0$. Since $z(a) = 0$, $dz/dt(a) > 0$, so $z(a+) > 0$. Suppose that $z(b)$ were negative: then there would be some t, $a < t < b$, such that $z(t) = 0$, $z(t-) > 0$, $z(t+) < 0$, which is not possible if $dz/dt > 0$ whenever $z = 0$. So $z(b) \geq 0$, and $x(b) > S$. Thus for any non-zero initial vector $w \in T_{\dot{\gamma}(a)}\hat{P}$, there is $K(w)$ such that if $k \geq K(w)$ and $X(t)$ is the solution of $\nabla^2 X + \Phi(X) = kX$ with $X(a)^H + \nabla X(a)^V = w$, $x(b) > S$. Now for $\lambda \in \mathbb{R} \neq 0$, $K(\lambda w) = K(w)$, so we may assume that w lies on some unit sphere in $T_{\dot{\gamma}(b)}\hat{P}$. Then by compactness and continuous dependence of solutions on initial conditions we can find K such that the inequality $x(b) > S$ holds for all $X(t)$ such that $\nabla^2 X + \Phi(X) = KX$, $X(a) \in T_{\gamma(a)}P$, $\nabla X(a) - \Sigma^P_{\dot{\gamma}(a)}X(a) \perp_{\dot{\gamma}(a)} T_{\gamma(a)}P$.

Now let S be the maximum value of $\Sigma^Q_{\dot\gamma(b)}(\bar X_b, \bar X_b)$ over vectors $\bar X_b \in T_{\gamma(b)}Q$ which are unit with respect to $g_L|_{\dot\gamma(b)}$. Then for all non-zero X such that $\nabla^2 X + \Phi(X) = KX$, $X(a) \in T_{\gamma(a)}P$, $\nabla X(a) - \Sigma^P_{\dot\gamma(a)}X(a) \perp_{\dot\gamma(a)} T_{\gamma(a)}P$, and $X(b) \in T_{\gamma(b)}$,

$$\begin{aligned}
&g_L|_{\dot\gamma(b)}(\nabla X(b), X(b)) - \Sigma^Q_{\dot\gamma(b)}(X(b), X(b)) \\
&\quad = g_L|_{\dot\gamma(b)}(X(b), X(b))(x(b) - \Sigma^Q_{\dot\gamma(b)}(\bar X(b), \bar X(b))) \\
&\quad \geq g_L|_{\dot\gamma(b)}(X(b), X(b))(x(b) - S) > 0,
\end{aligned}$$

where $\bar X(b)$ is the unit vector in the direction of $X(b)$; so B_K is positive definite.

So finally

$$\begin{aligned}
I_M &= I_F + [\Lambda^P : \Lambda_1] - [\Lambda^P : \Lambda_2] \\
&= I_F + \tfrac{1}{2}(\operatorname{sgn} B_K - \operatorname{sgn} B_0) \\
&= I_F + \tfrac{1}{2}(\dim Q - (\dim Q - 2 \operatorname{ind} B_0)) \\
&= I_F + \operatorname{ind} B_0,
\end{aligned}$$

which is the generalization of Kalish's formula.

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