Gaussian curvature
as a null homogeneous second-order Lagrangian

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August 5, 2002

1 Introduction

This paper is a contribution to the study of null Lagrangians in the context of homogeneous problems in the calculus of variations, that is, problems in which the fundamental, or variational, integral is independent of the choice of parameters. The variational problems of geometry are usually of this type, foremost among them the problem of the variation of the length integral in Riemannian geometry, which leads to the geodesic equations in parameter-independent form, that is, without the assumption that the parameter is arc length (or even that the geodesics are affinely parametrized). Finsler geometry is a natural generalization, and the positive-homogeneity requirement for a Finsler function is the basic example of the kind of condition that parameter independence imposes on a Lagrangian. It is described for example in [5] how the idea of a homogeneous, or parameter-independent, variational problem can be extended to multiple integral problems, of first order in derivatives. Taking another point of view, one could say that a manifold equipped with a homogeneous first-order Lagrangian of the type that occurs in this theory is an areal space as described for example in [6]. For some recent work on homogeneous Lagrangians, of which the present paper is an outgrowth, see [1, 2].

A null Lagrangian is one whose Euler-Lagrange equations vanish identically. Null Lagrangians have an important role to play in the study of symmetries of Lagrangian systems, in Carathéodory's approach to the theory of fields of extremals, and in the construction of integral invariants: see for example [4] for a general discussion, and [3] for a recent account of the homogeneous first-order case.

I shall devote the first, and larger, part of this paper to the development of a theory of homogeneous multiple-integral variational problems for second-order Lagrangians, that is, those depending on derivatives of the dependent variables up to and including the second. The endpoint of the discussion will be the derivation of a simple sufficient condition for a homogeneous second-order Lagrangian to be null.

I shall illustrate this theory with a familiar example from the geometry of surfaces, here regarded as defined by local imbeddings of $\mathbf{R}^2$ in $\mathbf{R}^3$. From the point of view of the calculus of variations, the most obvious conclusion to be drawn from the Gauss-Bonnet formula, which expresses the integral $\int \kappa dA$ of the Gaussian curvature κ with respect to the area form dA over a compact region of a surface purely in terms of quantities defined on the boundary of the region, is that the integrand $\kappa\sqrt{g}$, where g is the determinant of the metric, is a null Lagrangian. It is of second order in the imbedding functions, and is pretty evidently homogeneous. It is of interest to see how this observation fits into the general theory of null homogeneous second-order Lagrangians developed in the first part of the paper. I shall discuss this question in the second part of the paper.

I shall use the Einstein summation convention for repeated indices, whichever alphabet the indices are taken from.

2 Homogeneous second-order Lagrangians

Suppose given a manifold E of dimension n. The bundle of 2-jets at 0 of smooth maps $\mathbf{R}^m \to E$, for some integer $m < n$, is denoted by $T^2_{(m)}E$. The 2-jet of an immersion will be called a second-order m-frame, and the bundle of second-order m-frames over E will be denoted by $\mathcal{F}^2_{(m)}E$; it is an open submanifold of $T^2_{(m)}E$, and is its restriction to $\mathcal{F}^1_{(m)}E \subset T^1_{(m)}E$, where $T^1_{(m)}E$ and $\mathcal{F}^1_{(m)}E$ are the corresponding first-order objects. I denote by τ^1_2 the projection $\mathcal{F}^2_{(m)}E \to \mathcal{F}^1_{(m)}E$.

I write the coordinates on both $T^1_{(m)}E$ and $\mathcal{F}^1_{(m)}E$ as (u^A, u^A_i), where u^A, $A = 1, 2, \ldots, n$ are coordinates on E and, for any σ defining the 1-jet,

$$u^A_i = \frac{\partial \sigma^A}{\partial x^i}(0),$$

where the x^i are natural coordinates on $\mathbf{R}^m$. Then $\mathcal{F}^1_{(m)}E$ is defined by the condition that the matrix (u^A_i) has rank m. The extra fibre coordinates in $\mathcal{F}^2_{(m)}E$ will be denoted by u^A_{ij}, with the understanding that $u^A_{ji} = u^A_{ij}$ when $i \neq j$.

There is no loss of generality in restricting attention to jets at 0, for if $\sigma: \mathbf{R}^m \to E$ is a smooth map, the 2-jet of σ at $x \in \mathbf{R}^m$ is the 2-jet of $\sigma \circ t_x$ at 0, where $t_x: \mathbf{R}^m \to \mathbf{R}^m$ is the translation $t_x y = x + y$. Any smooth σ defines a map $\hat{\sigma}: \mathbf{R}^m \to T^2_{(m)}E$ which sends $x \in \mathbf{R}^m$ to the 2-jet of σ at x, or equivalently the 2-jet of $\sigma \circ t_x$ at 0. Similar considerations apply to the 1-jet bundles.

Each point $\xi \in \mathcal{F}^2_{(m)}E$ determines an ordered set of m linearly independent elements of $T_{\tau^1_2(\xi)}\mathcal{F}^1_{(m)}E$, say $\{\xi_1, \xi_2, \ldots, \xi_m\}$, as follows. Let σ be a map whose 2-jet at 0 is ξ. We extend σ to a map $\hat{\sigma}\colon \mathbf{R}^m \to \mathcal{F}^1_{(m)}E$, as described above, and set

$$\xi_i = \hat{\sigma}_{*0}\left(\frac{\partial}{\partial x^i}\right).$$

This clearly depends only on the 2-jet of σ at 0, and in fact in coordinates

$$\xi_i = u^A_i \frac{\partial}{\partial u^A} + u^A_{ij}\frac{\partial}{\partial u^A_j}$$

(where (u^A, u^A_i, u^A_{jk}) are the coordinates of ξ). The expression on the right defines, as the coordinates vary, a vector field along the map τ^1_2, which will be denoted by Δ_i.

The terminology (calling a 2-jet a frame) comes from this representation, and uses the common meaning of the term 'frame', namely a linearly independent set of vectors. Thus points of $\mathcal{F}^1_{(m)}E$ can be considered as m-frames (of tangent vectors) at points of E; points of $\mathcal{F}^2_{(m)}E$ as m-frames at points of $\mathcal{F}^1_{(m)}E$ (though it should be remembered that such frames are not arbitrary but must satisfy the symmetry conditions $\langle \xi_i, du^A_j\rangle = \langle \xi_j, du^A_i\rangle$).

Let $\psi\colon \mathbf{R}^m \to \mathbf{R}^m$ be an orientation-preserving diffeomorphism of a neighbourhood of 0 that leaves 0 fixed, and σ a smooth map $\mathbf{R}^m \to E$. Then if (u^A_i, u^A_{jk}) are the jet coordinates of the 2-jet of σ at 0, those of $\sigma\circ\psi$ are (v^A_i, v^A_{jk}) where

$$v^A_i = u^A_l \frac{\partial \psi^l}{\partial x^i}(0), \quad v^A_{jk} = u^A_{mn}\frac{\partial\psi^m}{\partial x^j}(0)\frac{\partial \psi^n}{\partial x^k}(0) + u^A_p \frac{\partial^2\psi^p}{\partial x^j \partial x^k}(0).$$

This generates a right action of a certain group G_2 on $\mathcal{F}^2_{(m)}E$. The group G_2 consists of pairs (a, b) where a is a constant $m\times m$ matrix of positive determinant and b a constant type $(1,2)$ tensor symmetric in its lower indices. The group multiplication law is $(a,b)\cdot(\hat{a},\hat{b}) = (a\cdot\hat{a}, c)$ where $a\cdot\hat{a}$ is the matrix product of a and $\hat{a}$ and c is the tensor given by

$$c^i_{jk} = b^i_{lm}\hat{a}^l_j\hat{a}^m_k + a^i_n\hat{b}^n_{jk}.$$

Thus G_2 is the semi-direct product of $GL(m)^+$, the group of $m\times m$ matrices of positive determinant, with the space of symmetric type $(1,2)$ tensors, determined by the action of the former on the latter specified above. The group G_2 acts on $\mathcal{F}^2_{(m)}E$ by

$$(u^A, u^A_i, u^A_{jk}) \mapsto (u^A, u^A_l a^l_i, u^A_{mn}a^m_j a^n_k + u^A_p b^p_{jk}).$$

I denote this action by $R_{(a,b)}$. It makes $\mathcal{F}^2_{(m)}E$ into a principal fibre bundle. I call the base (that is, the quotient of $\mathcal{F}^2_{(m)}E$ under the action) the second-order sphere bundle over E, and denote it by $\mathcal{S}^2_{(m)}E$. A point of $\mathcal{S}^2_{(m)}E$ can be regarded as an oriented m-dimensional second-order contact element at a point of E.

The action of G_2 on $\mathcal{F}^2_{(m)}E$ is fibred over an action of $G_1 = GL(m)^+$ on $\mathcal{F}^1_{(m)}E$:

$$\tau^1_2 \circ R_{(a,b)} = R_a \circ \tau^1_2,$$

where R_a denotes the right action of $a \in GL(m)^+$ on $\mathcal{F}^1_{(m)}E$. This action makes $\mathcal{F}^1_{(m)}E$ into a principle fibre bundle over the first-order sphere bundle $\mathcal{S}^1_{(m)}E$. The diagram

$$\begin{array}{ccc} \mathcal{F}^2_{(m)}E & \longrightarrow & \mathcal{F}^1_{(m)}E \\ \downarrow & & \downarrow \\ \mathcal{S}^2_{(m)}E & \longrightarrow & \mathcal{S}^1_{(m)}E \end{array}$$

is commutative. I shall denote by $\rho_1 \colon \mathcal{F}^1_{(m)}E \to \mathcal{S}^1_{(m)}E$ the projection.

A function L on $\mathcal{F}^2_{(m)}E$ is said to be homogeneous if it satisfies

$$L \circ R_{(a,b)} = (\det a) L$$

for all $(a, b) \in G_2$. The coordinate expression of this condition is

$$L(u^A, u^A_l a^l_i, u^A_{mn} a^m_j a^n_k + u^A_p b^p_{jk}) = \det(a^n_m) L(u^A, u^A_i, u^A_{jk}).$$

The homogeneity of L is the necessary and sufficient condition that the variational integral it defines does not depend on the parametrization, provided the orientation is unchanged.

A homogeneous Lagrangian does not of course pass to the quotient under the G_2 action to define a function on $\mathcal{S}^2_{(m)}E$. It can, however, be described as an object on $\mathcal{S}^2_{(m)}E$, as follows: $(a, b) \mapsto \det a$ is a representation of G_2, and a homogeneous Lagrangian is a cross-section of the line bundle over $\mathcal{S}^2_{(m)}E$ associated with the principle bundle $\mathcal{F}^2_{(m)}E$ by this representation.

The space of fundamental vector fields on $\mathcal{F}^2_{(m)}E$ corresponding to the G_2 action is spanned by

$$\Delta^j_i = u^A_i \frac{\partial}{\partial u^A_j} + u^A_{ik} \frac{\partial}{\partial u^A_{jk}}, \quad \Delta^{jk}_i = u^A_i \frac{\partial}{\partial u^A_{jk}}.$$

The first of these vector fields is τ^1_2-related to a vector field on $\mathcal{F}^1_{(m)}E$, denoted by the same symbol, which is a fundamental vector field corresponding to the G_1 action. The second projects to zero.

The homogeneity condition can be expressed in differential form using these fundamental vector fields: it is equivalent to the conditions

$$\Delta^j_i L = \delta^j_i L, \quad \Delta^{jk}_i L = 0$$

(where δ^j_i is the Kronecker delta).

It turns out that any m-form λ on $\mathcal{S}^1_{(m)}E$ can be used to construct a homogeneous Lagrangian L on $\mathcal{F}^2_{(m)}E$: I shall say that λ is a Lagrangian form corresponding to the Lagrangian function L. The construction of a Lagrangian function from a Lagrangian form proceeds as follows. Let $\xi \in \mathcal{F}^2_{(m)}E$ and let $\{\xi_i\}$ be the corresponding m-frame at $\tau^1_2(\xi) \in \mathcal{F}^1_{(m)}E$. I shall write $\rho^1_2 = \rho_1 \circ \tau^1_2$: note that

$$\rho^1_2 \circ R_{(a,b)} = \rho_1 \circ R_a \circ \tau^1_2 = \rho_1 \circ \tau^1_2 = \rho^1_2.$$

Set

$$L(\xi) = \lambda_{\rho^1_2(\xi)}(\rho_{1*}\xi_1, \rho_{1*}\xi_2, \ldots, \rho_{1*}\xi_m).$$

Then L is a function on $\mathcal{F}^2_{(m)}E$, and we can write

$$L = (\rho^*_1\lambda)(\Delta_1, \Delta_2, \ldots, \Delta_m).$$

I shall now show that L is homogeneous, that is to say, that $L \circ R_{(a,b)} = (\det a)L$ for every $(a,b) \in G_2$. The frame corresponding to the point $R_{(a,b)}(\xi)$ is $\{R_{(a,b)*}\xi_i\}$. From the representation

$$\xi_i = u^A_i \frac{\partial}{\partial u^A} + u^A_{ij}\frac{\partial}{\partial u^A_j}$$

of ξ_i and the formula

$$(u^A, u^A_i, u^A_{jk}) \mapsto (u^A, u^A_l a^l_i, u^A_{mn} a^m_j a^n_k + u^A_p b^p_{jk}).$$

for the action of (a,b) it can be seen that

$$R_{(a,b)*}\xi_i = a^j_i \xi_j + b^k_{ij}\Delta^j_k.$$

Thus $\rho_{1*}R_{(a,b)*}\xi_i = a^j_i\xi_j$, from which it follows that

$$\begin{aligned} L(R_{(a,b)}(\xi)) &= \lambda_{\rho^1_2(R_{(a,b)}\xi)}(a^{i_1}_1\xi_{i_1}, a^{i_2}_2\xi_{i_2}, \ldots, a^{i_m}_m\xi_{i_m}) \\ &= (\det a)\lambda_{\rho^1_2(\xi)}(\xi_1, \xi_2, \ldots, \xi_m) \\ &= (\det a)L(\xi) \end{aligned}$$

as claimed.

The variational problem associated with a Lagrangian function L is the problem of finding extremals $\sigma: C \to E$ of the integral $\int_C(\hat{\sigma}^*L)d^m x$, where C is a compact connected m-dimensional submanifold of $\mathbf{R}^m$ with boundary ∂C and $d^m x$ is the canonical volume form, subject to variations that leave $\sigma(\partial C)$ fixed. It is easy to see that when L is the Lagrangian function of a Lagrangian form λ,

$$(\hat{\sigma}^*L)d^m x = (\rho^1_2 \circ \hat{\sigma})^*\lambda.$$

If λ happens to be an exact form, then so is $(\hat{\sigma}^*L)d^m x$ for any σ; the value of the integral $\int_C(\hat{\sigma}^*L)d^m x$ then depends only on the value of the integrand on the boundary, and so the variational problem for variations that leave $\sigma(\partial C)$ fixed is trivial. Thus the Lagrangian function corresponding to an exact Lagrangian form is null. In fact it is not necessary to check that λ is exact: it is enough to know, for example, that $\rho^*_1\lambda$ is exact to draw the same conclusion.

Gaussian curvature

The integrand of the integral $\int \kappa dA$ that appears in the Gauss-Bonnet formula defines a homogeneous Lagrangian of order 2: that is, there is a homogeneous Lagrangian L on $\mathcal{F}^2_{(m)}\mathbf{R}^3$ such that

$$\hat{\sigma}^* L = \kappa\sqrt{\det(g_{ij})}$$

where κ is the Gaussian curvature of the surface given by the embedding $\sigma\colon \mathbf{R}^2 \to \mathbf{R}^3$, and g_{ij} its metric. According to the Gauss-Bonnet formula this Lagrangian must be null. I shall show that this result can be derived by using the construction of a null homogeneous second-order Lagrangian function from an exact Lagrangian form as described above.

Consider the 2-form μ on $\mathcal{F}^1_{(2)}\mathbf{R}^3$ defined by

$$\mu = \frac{(\mathbf{n}\cdot d\mathbf{u}_1)\wedge(\mathbf{n}\cdot d\mathbf{u}_2)}{\sqrt{g}},$$

where $\mathbf{u}_i$, $i = 1, 2$, are the vector-valued functions on $\mathcal{F}^1_{(2)}\mathbf{R}^3$ whose components are u^A_i and $\mathbf{n}$ is the vector-valued function on $\mathcal{F}^1_{(2)}\mathbf{R}^3$ given by

$$\mathbf{n} = \frac{\mathbf{u}_1\times\mathbf{u}_2}{\sqrt{g}}$$

with $g_{ij} = \mathbf{u}_i\cdot\mathbf{u}_j$ and $g = \det(g_{ij})$. Then for an embedding σ defining a surface, $\hat{\sigma}^*\mathbf{n}$ is a unit normal field on the surface, and $\hat{\sigma}^*\mu = \kappa dA$ (using the definition of κ in terms of the determinant of the second fundamental form). Now μ is a 2-form on $\mathcal{F}^1_{(2)}\mathbf{R}^3$, whereas we seek a 2-form on $\mathcal{S}^1_{(2)}\mathbf{R}^3$ to serve as the Lagrangian form. In order for μ to pass to the quotient, that is to be the pull-back of a 2-form on $\mathcal{S}^1_{(2)}\mathbf{R}^3$, it must be invariant under the action of $GL(2)^+$, and must give zero when any of its vector arguments is vertical with respect to the projection $\rho_1\colon \mathcal{F}^1_{(2)}\mathbf{R}^3 \to \mathcal{S}^1_{(2)}\mathbf{R}^3$. It is clear that μ is invariant: when one acts with an element a of $GL(2)^+$ both the numerator and the denominator get multiplied by $\det a$. Furthermore, for the fundamental vector fields Δ^j_i generated by the action,

$$\langle\Delta^j_i, \mathbf{n}\cdot d\mathbf{u}_k\rangle = \delta^j_k \mathbf{n}\cdot\mathbf{u}_i = 0.$$

There is therefore a well-defined 2-form λ on $\mathcal{S}^1_{(2)}\mathbf{R}^3$ such that $\rho_1^*\lambda = \mu$, and λ is the required Lagrangian form.

One can show that μ is closed by a direct calculation, with a certain amount of effort. However, it is also possible to find a 1-form of which it is the exterior derivative, by using some equations related to the second structure equation. In fact one can introduce these 'structure equations' on $\mathcal{F}^1_{(2)}\mathbf{R}^3$, or indeed on any manifold on which one can find a set of three orthonormal $\mathbf{R}^3$-valued functions.

Suppose that $\mathbf{v}_a$, $a = 1, 2, 3$, are $\mathbf{R}^3$-valued functions on any manifold M, which are orthonormal, that is, they satisfy $\mathbf{v}_a \cdot \mathbf{v}_b = \delta_{ab}$ where δ is the matrix of the Euclidean metric on $\mathbf{R}^3$. Then for any vector field X on M we have

$$X(\mathbf{v}_a) \cdot \mathbf{v}_b + \mathbf{v}_a \cdot X(\mathbf{v}_b) = 0.$$

But for any vector-valued function $\mathbf{v}$, $X(\mathbf{v})$ is a vector-valued function, and may be expressed in terms of the basis of vector-valued functions $\mathbf{v}_a$; thus there are 1-forms ω_a^b on M such that

$$X(\mathbf{v}_a) = \langle X, \omega_a^b \rangle \mathbf{v}_b$$

(summation over b assumed): in fact we can write

$$\delta_{ac}\omega_b^c = d\mathbf{v}_a \cdot \mathbf{v}_b.$$

Then from the differentiation formula above,

$$\delta_{ac}\omega_b^c + \delta_{bc}\omega_a^c = 0.$$

Furthermore, for any pair of vector fields X, Y on M,

$$X(Y(\mathbf{v}_a)) = (X\langle Y, \omega_a^b \rangle + \langle Y, \omega_a^c \rangle \langle X, \omega_c^b \rangle)\mathbf{v}_b,$$

from which follow the 'structure equations'

$$d\omega_a^b + \omega_c^b \wedge \omega_a^c = 0.$$

Now let $\mathbf{v}_a$ be orthonormal vector-valued functions on $\mathcal{F}^1_{(2)}\mathbf{R}^3$ such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$ and $\mathbf{v}_3 = \mathbf{n}$; for example, we could obtain the $\mathbf{v}_i$ from the $\mathbf{u}_i$ by the Gram-Schmidt process. Then I claim that

$$\mu = d\omega_2^1 = -d\omega_1^2.$$

We have

$$d\omega_2^1 = \omega_3^1 \wedge \omega_3^2,$$

from the structure equations. But

$$\omega_3^i = \mathbf{n} \cdot d\mathbf{v}_i.$$

We can write $\mathbf{v}_i = V_i^j \mathbf{u}_j$ for some functions V_i^j, where by orthonormality

$$V_i^k V_j^l \mathbf{u}_k \cdot \mathbf{u}_l = g_{kl} V_i^k V_j^l = \delta_{ij}.$$

Thus

$$\mathbf{n} \cdot d\mathbf{v}_i = \mathbf{n} \cdot (V_i^j d\mathbf{u}_j + \mathbf{u}_j dV_i^j) = V_i^j(\mathbf{n} \cdot d\mathbf{u}_j),$$

whence

$$d\omega_2^1 = \omega_3^1 \wedge \omega_3^2 = (\det V)(\mathbf{n} \cdot d\mathbf{u}_1) \wedge (\mathbf{n} \cdot d\mathbf{u}_2).$$

But from the orthogonality conditions $g(\det V)^2 = 1$, so that

$$d\omega_2^1 = \frac{(\mathbf{n}\cdot d\mathbf{u}_1)\wedge(\mathbf{n}\cdot d\mathbf{u}_2)}{\sqrt{g}} = \mu$$

as claimed.

On a surface ω_1^2 becomes the connection form (that is, $\hat{\sigma}^*\omega_1^2$ is the connection form of the surface defined by σ, with respect to the orthonormal frame induced from $\mathbf{v}_1$ and $\mathbf{v}_2$). Thus the equation $d\omega_1^2 = -\omega_3^1 \wedge \omega_3^2$ is a sort of master version of Gauss's equation, and $d\omega_1^2 = -\mu$ of the second structure equation, of surface theory; I emphasise that these equations relate geometrical objects defined on $\mathcal{F}^1_{(2)}\mathbf{R}^3$, not on any particular surface. These objects do depend on the choice of $\mathbf{v}_1$ and $\mathbf{v}_2$; however, it is easy to see that replacing them by another orthonormal pair obtained by applying a variable rotation has the effect of adding $d\theta$ to ω_2^1, where the function θ is the angle through which they are rotated to obtain the new pair; $\omega_3^1 \wedge \omega_3^2$ is unchanged.

The observation that the Gauss-Bonnet integrand defines a null Lagrangian is of course not new. It is to be found, amongst other places no doubt, in [4], together with a proof of the result in something of the spirit of the foregoing. However, the authors say of their proof that 'any computation will be somewhat artificial if it is not guided by geometric intuition', and that lacking such intuition their proof 'is merely clever guesswork'. What their approach lacks is firstly an appreciation of the importance of the fact that the integrand is homogeneous, and secondly the realization that surface geometry itself provides the clue to the understanding of the geometrical objects involved.

References

[1] Crampin, M. and D. J. Saunders The Hilbert-Carathéodory form for parametric multiple integral problems in the calculus of variations Accepted for publication in *Acta Appl. Math.*

[2] Crampin, M. and D. J. Saunders The Hilbert-Carathéodory and Poincaré-Cartan forms for higher-order multiple-integral variational problems Preprint: The Open University (2002).

[3] Crampin, M. and D. J. Saunders On null Lagrangians Preprint: The Open University (2002).

[4] Giaquinta, M. and S. Hildebrandt *Calculus of Variations I* (Springer, 1996). Chapter 1 Sections 4 and 5

[5] Rund, H. *The Hamilton-Jacobi Theory in the Calculus of Variations* (Van Nostrand, 1966). Chapter 4

[6] Tamássy, L. Area and metrical connections in Finsler space In: *Finslerian Geometries*, ed P. L. Antonelli (Kluwer, 2000), 263–280.