Concircular vector fields and special conformal Killing tensors

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Abstract

I explain why there are lacunae in the ranges of possible dimensions of the spaces of concircular vector fields and of special conformal Killing tensors on a Riemannian manifold.

1 Introduction

The aim of this mainly expository paper is to describe the results of the Russian mathematician I. G. Shandra [15] on the lacunae in the range of possible dimensions of the solution space of the tensor differential equation

$$L_{ij|k} = \tfrac{1}{2}(g_{jk}\lambda_i + g_{ik}\lambda_j);$$

here L_{ij} is a symmetric tensor of type $(0, 2)$ on a Riemannian manifold whose metric tensor is g_{ij}, and the problem is to find tensors L_{ij} satisfying this equation for some covariant vector λ_i. Solutions L_{ij} of this equation are conformal Killing tensors, albeit of a special type, and are therefore called special conformal Killing tensors. Special conformal Killing tensors play an important role in at least three areas of research:

- the projective equivalence of Riemannian manifolds: two Riemannian manifolds are said to be projectively equivalent if they have the same geodesics up to reparametrization; this situation occurs if and only if a certain tensor formed out of the two metric tensors is a special conformal Killing tensor [2,4,13,15];
- separation of variables in the Hamilton-Jacobi equation: when a Riemannian manifold admits a special conformal Killing tensor whose eigenfunctions are

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simple and functionally independent, those eigenfunctions are orthogonal separation coordinates for the Hamilton-Jacobi equation for the geodesics of the manifold [1,3,6,10];
- completely integrable dynamical systems: special conformal Killing tensors are involved in the definition of certain nonconservative Lagrangian systems which provide interesting examples of completely integrable systems [5,11,14].

My method is to illustrate and motivate the relevant results about special conformal Killing tensors by first considering a similar but simpler problem, that of finding so-called concircular vector fields on an affinely-connected manifold. When the manifold is a Riemannian space with the Levi-Civita connection the theory of concircular vector fields provides a model for the theory of special conformal Killing tensors; but in fact the relationship is somewhat closer than this suggests, because concircular vector fields can be used to construct special conformal Killing tensors.

I give the definition and discuss the general theory of concircular vector fields in Section 2, and specialize to the Riemannian case in Section 3. Section 4 is devoted to the analogous results for special conformal Killing tensors. Some use is made of the method of structural equations for obtaining properties of solutions of systems of partial differential equations; I give a brief discussion of this subject in an appendix.

I shall explain the origin of the name ‘concircular vector field’ in due course; for the present I shall just remark that, though it is treated as standard terminology in [15], it is misleading: there will be very little to do with circles in this paper.

2 Concircular vector fields: the general case

Recall that for any vector field X on a manifold M equipped with a symmetric affine connection with covariant derivative operator ∇, ∇X is a type $(1,1)$ tensor field.

Definition 1 *A vector field X is* concircular *if ∇X is a scalar multiple of the identity, say $\nabla X = \rho I$.*

I shall denote by $\mathcal{C}(M,\nabla)$ the set of concircular vector fields on a manifold M with connection ∇; it is evidently a vector space over the reals $\mathbf{R}$, and it contains the space of covariant-constant, or parallel, vector fields, which will be denoted by $\mathcal{C}_0(M,\nabla)$.

A simple example of a concircular vector field is the vector field

$$X = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + \cdots + x^n\frac{\partial}{\partial x^n}$$

on $\mathbf{R}^n$ with the standard flat connection: we have

$$\nabla_{\partial/\partial x^i} X = \frac{\partial}{\partial x^i}, \quad \text{or} \quad \nabla_Z X = Z, \quad \text{or} \quad \nabla X = I, \quad \text{or} \quad X^i_{|j} = \delta^i_j.$$

In fact in $\mathbf{R}^n$

$$\frac{\partial X^i}{\partial x^j} = \rho \delta^i_j \quad \Longrightarrow \quad \frac{\partial^2 X^i}{\partial x^j \partial x^k} = \frac{\partial \rho}{\partial x^k}\delta^i_j = \frac{\partial \rho}{\partial x^j}\delta^i_k$$
$$\Longrightarrow \quad \frac{\partial \rho}{\partial x^k} = 0 \quad \Longrightarrow \quad X^i = \rho x^i + c^i.$$

So the concircular vector fields on $\mathbf{R}^n$ with respect to the flat connection form a vector space of dimension $n+1$. We can choose as a basis $\{X_0, X_i\}$ where

$$X_0 = x^i \frac{\partial}{\partial x^i}, \quad \nabla X_0 = I \quad ; \quad X_i = \frac{\partial}{\partial x^i}, \quad \nabla X_i = 0.$$

A concircular vector field X that satisfies $\nabla X = I$, like X_0 above, is said to be *converging*. Evidently a concircular vector field that satisfies $\nabla X = kI$ where k is a non-zero constant can be renormalized to a converging field.

The concircular vector fields in the case just discussed are geodesic. This is generally true: if $\nabla X = \rho I$ then $\nabla_X X = \rho X$, so the integral curves of X are geodesic (but not affinely parametrized in general).

I shall shortly show that in general the vector space $\mathcal{C}(M, \nabla)$ is of finite dimension, just as is the case in $\mathbf{R}^n$. To do so, I shall use the method of structural equations. I give a brief description of this method in an appendix; for more detailed accounts the reader may consult [8,16,17]. The method in this case involves the use of the Ricci identities and the Ricci tensor, so I had better make it clear which sign conventions I use for curvature. Since I shall employ tensorial methods for calculations I adopt the sign conventions of Eisenhart [7], which are given in tensorial form as follows:

$$X^i_{|jk} - X^i_{|kj} = -R^i_{ljk} X^l; \quad R_{ij} = R^k_{ijk}.$$

Theorem 2 *The equations*

$$X^i_{|j} = \rho \delta^i_j$$
$$\rho_{|k} = \frac{1}{n-1} R_{lk} X^l,$$

for the variables (X^i, ρ), are structural equations for concircular vector fields. The solution space of these structural equations is a vector space of maximum dimension $n+1$; the maximum dimension is achieved when the Ricci tensor of the connection is symmetric and the connection is projectively flat.

Proof. Consider the defining equation $X^i_{|j} = \rho \delta^i_j$; differentiate covariantly to obtain $X^i_{|jk} = \rho_{|k} \delta^i_j$, from which it follows that $R^i_{ljk} X^l = -\rho_{|k}\delta^i_j + \rho_{|j}\delta^i_k$, by the

Ricci identity. Taking the trace on i and j gives $-(n-1)\rho_{|k} = -R_{lk}X^l$. The equations

$$X^i_{|j} = \rho\delta^i_j, \quad \rho_{|k} = \frac{1}{n-1}R_{lk}X^l$$

evidently have the required form to be structural equations, and X is a concircular vector field with corresponding scalar ρ if and only if X^i and ρ satisfy them. Since there are $n+1$ variables the solution space of the structural equations is of maximum dimension $n+1$. The integrability conditions are

$$R^i_{ljk}X^l = -\rho_{|k}\delta^i_j + \rho_{|j}\delta^i_k = \frac{1}{n-1}\left(R_{lj}\delta^i_k - R_{lk}\delta^i_j\right)X^l$$

and

$$0 = (n-1)(\rho_{|jk} - \rho_{|kj}) = (R_{lj|k} - R_{lk|j})X^l + (R_{kj} - R_{jk})\rho.$$

So the maximum dimension is achieved when

$$R^i_{jkl} - \frac{1}{n-1}\left(R_{jk}\delta^i_l - R_{jl}\delta^i_k\right) = 0, \quad R_{ij|k} - R_{ik|j} = 0, \quad R_{ij} - R_{ji} = 0.$$

In a Ricci-symmetric space

$$R^i_{jkl} - \frac{1}{n-1}\left(R_{jk}\delta^i_l - R_{jl}\delta^i_k\right) = P^i_{jkl},$$

the projective curvature tensor. If $n > 2$, $P^i_{jkl} = 0 \Longrightarrow R_{ij|k} - R_{ik|j} = 0$; if $n = 2$, $P^i_{jkl} = 0$ identically and $R_{ij|k} - R_{ik|j} = 0$ is the condition for projective flatness. □

I shall call a pair (X, ρ) consisting of a vector X and a scalar ρ which together satisfy the structural equations a *concircular pair*.

The appearance of the projective curvature tensor raises the question of what relationship there may be between concircular vectors and projective differential geometry; it may be answered as follows. A projective transformation of a connection takes the form

$$\Gamma^i_{jk} \mapsto \Gamma^i_{jk} + \psi_j\delta^i_k + \psi_k\delta^i_j.$$

If ψ_k is a gradient, in which case we are dealing with a so-called restricted projective transformation, then

$$\hat{\nabla}(e^{-\psi}X) = e^{-\psi}(\nabla X + X(\psi)I).$$

Moreover, restricted projective transformations preserve Ricci symmetry. So

$$(X, \rho) \mapsto e^{-\psi}(X, \rho + X(\psi))$$

is a 1-1 correspondence of concircular pairs on Ricci-symmetric spaces, under restricted projective transformations.

3 Concircular vector fields: the Riemannian case

When dealing with a Riemannian space, with metric g, I shall write $\mathcal{C}(g)$ instead of $\mathcal{C}(M, \nabla)$ and $\mathcal{C}_0(g)$ instead of $\mathcal{C}_0(M, \nabla)$. The connection is of course taken to be the Levi-Civita connection.

Recall that every Riemannian space is Ricci-symmetric, and that a Riemannian space with $n > 2$ is projectively flat if and only if it is of constant curvature. So $\mathcal{C}(g) \leq n + 1$, and for $n > 2$, $\mathcal{C}(g) = n + 1$ if and only of the space is of constant curvature.

Using just this last result and the remarks about projective transformations above, one can give a very neat and short proof of Beltrami's Theorem. (A conventional tensorial proof is to be found in [7]. An unconventional proof was given by Matveev in [12]. The proof below is, like Matveev's, unconventional; it is however somewhat shorter.)

Theorem 3 (Beltrami) *A Riemannian metric on a manifold of dimension greater than 2 which is projectively equivalent to one of constant curvature is itself of constant curvature.*

Proof. If two Riemannian metrics g and $\hat{g}$ are projectively equivalent, the projective transformation relating their Levi-Civita connections is of the restricted type. The corresponding map $X \mapsto e^{-\psi}X$ then gives a linear map $\mathcal{C}(g) \to \mathcal{C}(\hat{g})$, which is evidently an isomorphism. Let us suppose that g is of constant curvature. Then $\dim \mathcal{C}(\hat{g}) = \dim \mathcal{C}(g) = n+1$, and so $\hat{g}$ is of constant curvature also. $\square$

The defining equation for a concircular vector field can be written in covariant form

$$X_{i|j} = \rho g_{ij}.$$

Thus $X_{j|i} = X_{i|j}$, and so X is a gradient. It is also a conformal Killing vector.

I can now explain the terminology. A transformation of a Riemannian manifold (M, g) preserves geodesic circles if and only if it is conformal, and the gradient of the conformal factor is concircular, that is, satisfies the equation above (see for example [9]). (It might be argued that the term 'concircular vector field' describes better the infinitesimal generator of transformations preserving geodesic circles, and this indeed is how the term is used in [9].)

In the Riemannian case one can say a good deal more about concircular vector fields, as one might suspect.

Lemma 4 *In a Riemannian space, for any concircular pair (X, ρ)*

$$\rho_{|i} = \kappa X_i$$

for some scalar κ. Moreover, $\kappa_{|i} \propto X_i$ also.

Proof. Recall that $R^i_{ljk}X^l = -\rho_{|k}\delta^i_j + \rho_{|j}\delta^i_k$, or

$$R_{lijk}X^l = \rho_{|k}g_{ij} - \rho_{|j}g_{ik}.$$

But R_{lijk} is skew in i and l, so

$$0 = X^i(\rho_{|k}g_{ij} - \rho_{|j}g_{ik}) = \rho_{|k}X_j - \rho_{|j}X_k,$$

from which the first result follows. For the second, note that

$$0 = \rho_{|ij} - \rho_{|ji} = \kappa_{|j}X_i + \kappa\rho g_{ij} - \kappa_{|i}X_j - \kappa\rho g_{ji} = \kappa_{|j}X_i - \kappa_{|i}X_j,$$

whence $\kappa_{|i} \propto X_i$ also. □

When the space is of constant curvature, κ is the curvature.

Corollary 5 *In dimension 2, if $\kappa = 0$ (or in other words ρ is constant) the space must be flat.*

Proof. If $\kappa = 0$ then $R_{lijk}X^l = 0$. In dimension 2, if this holds for non-zero X the space is flat: $R_{1212}X^1 = R_{2112}X^2 = 0$, whence $R_{1212} = 0$. □

The lemma has important and interesting consequences when $\dim \mathcal{C}(g) \geq 2$.

Theorem 6 *If $\dim \mathcal{C}(g) \geq 2$, with (X, ρ) and (Y, σ) independent concircular pairs, then $\rho_{|i} = \kappa X_i$, $\sigma_{|i} = \kappa Y_i$ (with the same κ) and κ is constant.*

Proof. We have $R_{lijk}X^lY^i = \kappa_X(X_kY_j - X_jY_k) = -\kappa_Y(Y_kX_j - Y_jX_k)$, so $\kappa_X = \kappa_Y$. Then $\kappa_{|i} \propto X_i$ and $\kappa_{|i} \propto Y_i$; but since X and Y are independent, $\kappa_{|i} = 0$. □

So if $\dim \mathcal{C}(g) \geq 2$ there is a constant κ such that for every concircular pair (X, ρ), $\rho_{|i} = \kappa X_i$. Various cases arise:

(1) $\kappa = 0$, so all ρ are constant: then
  (a) all $\rho = 0$, so $\mathcal{C}(g) = \mathcal{C}_0(g)$; or
  (b) some $\rho \neq 0$, whence we can choose a basis with one element converging, the rest parallel: $\dim \mathcal{C}(g) = \dim \mathcal{C}_0(g) + 1$;

(2) $\kappa \neq 0$.

In the latter case, consider the (possibly pseudo-Riemannian) metric

$$e^{2\kappa x^0}\left((dx^0)^2 - \frac{1}{\kappa}g_{ij}dx^idx^j\right), \quad 1 \leq i, j \leq n;$$

here κ is a non-zero constant, and the g_{ij} are independent of x^0. The Christoffel symbols of such a metric take the form

$$\Gamma^0_{00} = \kappa, \quad \Gamma^0_{ij} = g_{ij}, \quad \Gamma^i_{j0} = \kappa\delta^i_j,$$

while Γ^i_{jk} are the Christoffel symbols of g_{ij}. The components of the covariant differential of a covector field Z_a, $a = 0, 1, \cdots, n$ are given by

$$Z_{0\|0} = \frac{\partial Z_0}{\partial x^0} - \kappa Z_0 \qquad Z_{i\|0} = \frac{\partial Z_i}{\partial x^0} - \kappa Z_i$$
$$Z_{0\|i} = \frac{\partial Z_0}{\partial x^i} - \kappa Z_i \qquad Z_{i\|j} = Z_{i|j} - g_{ij} Z_0.$$

Thus

- If $Z_a = (e^{2\kappa x^0}/\kappa, 0)$ then $Z_{a\|b} = g_{ab}$ or $\nabla Z = I$.
- If $Z_a = e^{\kappa x^0}(\rho, X_i)$ for some concircular pair (X, ρ) on the original Riemannian space, then $Z_{a\|b} = 0$; and conversely.

So this incorporates case (2) in case 1(b) (ignoring the question of signature).

I call the $(n+1)$-dimensional manifold with this metric the *augmented space.*

Using these results, I now show that there are lacunae in the range of possible values of $\dim \mathcal{C}(g)$.

Theorem 7 *For* $n \geq 3$, *either* $\dim \mathcal{C}(g) \leq n-2$ *or* $\dim \mathcal{C}(g) = n+1$.

Proof. Assume *per contra* that $n + 1 > \dim \mathcal{C}(g) > n - 2$.

Any Riemannian manifold (M, g) has a partial de Rham decomposition: it can be written as an orthogonal product $M = M_0 \times M_1$, $g = g_0 + g_1$, where (M_0, g_0) is flat and for each $x_1 \in M_1$, $M_0 \times \{x_1\}$ is a leaf of the integrable distribution spanned by the covariant-constant vector fields on M. Then $\dim M_0 = \dim \mathcal{C}_0(g)$. Moreover $\dim M_1 \geq 2$, or (M, g) would be flat.

If $\dim \mathcal{C}(g) > n-2$ then $\dim \mathcal{C}(g) \geq 2$. I consider the various cases.

$\kappa = 0$, $\mathcal{C}(g) = \mathcal{C}_0(g)$. We should have $\dim M_1 \geq 2$: but by assumption

$$\dim M_1 = n - \dim M_0 = n - \dim \mathcal{C}_0(g) = n - \dim \mathcal{C}(g) < 2.$$

$\kappa = 0$, $\dim \mathcal{C}(g) = \dim \mathcal{C}_0(g) + 1$, $\nabla X = I$. With respect to the decomposition we have $\nabla = \nabla^0 + \nabla^1$, that is

$$\nabla_{Z_0 + Z_1}(X_0 + X_1) = \nabla^0_{Z_0} X_0 + \nabla^1_{Z_1} X_1;$$

but $\nabla_{Z_0+Z_1}(X_0 + X_1) = Z_0 + Z_1$, so $\nabla^1 X_1 = I$. Thus (M_1, g_1) admits a converging field, which is a concircular field with constant ρ; so $\dim M_1 \geq 3$, or (M, g) would be flat. But by assumption

$$\dim M_1 = n - \dim M_0 = n - \dim \mathcal{C}_0(g) = n - (\dim \mathcal{C}(g) - 1) < 3.$$

$\kappa < 0$. The augmented space is Riemannian. Apply the previous case but taking M as the augmented space. Then $\dim M_0 = \dim \mathcal{C}(g)$, $\dim M = n+1$, and there is a converging vector field so $\dim M_1 \geq 3$, or (M, g) would be flat. But by assumption

$$\dim M_1 = n + 1 - \dim M_0 = n + 1 - \dim \mathcal{C}(g) < 3.$$

$\kappa > 0$. A similar argument applies, but one has to be careful about the signature.

In each case we obtain a contradiction. □

I give below an alternative proof, which has its points of interest, but doesn't fit so well with what comes later.

Proof. Let $\{(X_\alpha, \rho_\alpha)\}$, $\alpha = 1, 2, \ldots, m \leq n$ be a set of linearly independent solutions of the structural equations. I show that the vector fields X_α are linearly independent over $C^\infty(M)$ almost everywhere, in the topological sense: that is to say, on a set whose complement contains no interior points. Consider a point $x \in M$ where the $X_\alpha(x)$ are linearly dependent, say $f^\alpha X_\alpha(x) = 0$ for some numbers f^α, not all zero. Suppose that there is an open neighbourhood of x on which they remain linearly dependent: then there are functions F^α such that $F^\alpha X_\alpha = 0$, $F^\alpha(x) = f^\alpha$. There is some $v \in T_xM$ linearly independent of the $X_\alpha(x)$, and

$$0 = \nabla_v(F^\alpha X_\alpha) = v(F^\alpha)X_\alpha(x) + (f^\alpha \rho_\alpha(x))v,$$

so $f^\alpha \rho_\alpha(x) = 0$. But then $f^\alpha(X_\alpha(x), \rho_\alpha(x)) = (0, 0)$, contradicting the assumption that $\{(X_\alpha, \rho_\alpha)\}$ is a linearly independent set.

Now suppose that $\dim \mathcal{C}(g) \geq n - 1$: then there are $n - 1$ concircular vector fields X_α, linearly independent almost everywhere. Since $n \geq 3$, $\dim \mathcal{C}(g) \geq 2$, so for any concircular vector field X

$$\begin{aligned} 0 &= R_{lijk}X^l - (\rho_{|k}g_{ij} - \rho_{|j}g_{ik}) = R_{lijk}X^l - \kappa(X_k g_{ij} - X_j g_{ik}) \\ &= (R_{lijk} - \kappa(g_{ij}g_{kl} - g_{ik}g_{jl}))X^l \\ &= K_{lijk}X^l \end{aligned}$$

say: κ is the same for all X, and constant. Note that $K_{jikl} = -K_{ijkl}$. Now $K_{ijkl}X^i_\alpha = 0$, $\alpha = 1, 2, \ldots, n - 1$. Let Y be linearly independent of the X_α. Then $K_{ijkl}Y^i = 0$ also: for $K_{ijkl}Y^i X^j_\alpha = -K_{ijkl}X^i_\alpha Y^j = 0$, and $K_{ijkl}Y^i Y^j = 0$. Thus $K_{ijkl} = 0$ almost everywhere, and so everywhere; the space is of constant curvature κ, and $\dim \mathcal{C}(g) = n + 1$. □

4 Special conformal Killing tensors

Definition 8 *A* special conformal Killing tensor *on a Riemannian manifold (M, g) is a tensor L of valence 2 such that*

$$L_{ij|k} = \tfrac{1}{2}(g_{jk}\lambda_i + g_{ik}\lambda_j)$$

for some covector λ.

The importance of special conformal Killing tensors has been explained in the introduction.

The theory of concircular vector fields in the Riemannian case is related to that of special conformal Killing tensors in two ways: on the one hand, as a simplified or model version; on the other, because concircular vector fields can be used to define special conformal Killing tensors.

I take up the latter point first. Let X_α, $\alpha = 1, 2, \ldots, m$, be elements of $\mathcal{C}(g)$ (which we may as well take to be linearly independent), and let $(c_{\alpha\beta})$ be a constant symmetric matrix. Set $L_{ij} = \frac{1}{2}\sum_{\alpha,\beta} c_{\alpha\beta} X_i^\alpha X_j^\beta$. Then

$$\begin{aligned} L_{ij|k} &= \tfrac{1}{2}\sum_{\alpha,\beta} c_{\alpha\beta}(X^\alpha_{i|k}X^\beta_j + X^\alpha_i X^\beta_{j|k}) \\ &= \tfrac{1}{2}\sum_{\alpha,\beta} c_{\alpha\beta}(\rho^\alpha g_{ik} X^\beta_j + \rho^\beta X^\alpha_i g_{jk}) \\ &= \tfrac{1}{2}(g_{jk}\lambda_i + g_{ik}\lambda_j) \end{aligned}$$

where

$$\lambda_i = \sum_{\alpha,\beta} c_{\alpha\beta}\rho^\alpha X^\beta_i.$$

So L is a special conformal Killing tensor. If $1 \le \alpha, \beta \le m \le n-2$ then L is of rank m. The dimension of the space of special conformal Killing tensors of this form is $\frac{1}{2}m(m+1)$ if $\dim \mathcal{C} = m$.

I now consider concircular vector fields as a model theory. I have discussed the following features of concircular vector fields:

(1) structural equations may be formulated, and results on dimension drawn;
(2) simplifications occur when $\dim \mathcal{C}(g) \ge 2$;
(3) there is a correspondence with covariant-constant vector fields of the augmented space;
(4) the range of possible values of $\dim \mathcal{C}(g)$ has lacunae.

I shall show that the theory of special conformal Killing tensors has similar features.

4.1 Structural equations

The following equations, for the unknowns L_{ij}, λ_i and μ, are structural equations for special conformal Killing tensors ([13,16]):

$$L_{ij|k} = \tfrac{1}{2}(g_{jk}\lambda_i + g_{ik}\lambda_j)$$

$$\lambda_{i|j} = \frac{1}{n}\left(2R_j^k L_{ik} - 2g^{kl}R^m{}_{ijk}L_{lm} + g_{ij}\mu\right)$$
$$\mu_{|i} = \frac{2}{n-1}\left(g^{jl}(2R^k_{i|l} - R^k_{l|i})L_{jk} + (n+1)R_i^j\lambda_j\right).$$

The space of special conformal Killing tensors is therefore a vector space $\mathcal{B}(g)$ of maximum dimension $\frac{1}{2}n(n+1)+n+1 = \frac{1}{2}(n+1)(n+2)$. The maximum is achieved if and only if (M, g) is a space of constant curvature. This follows from the integrability conditions of the structural equations, as shown in [16]. In that case $\dim\mathcal{C}(g) = n+1$, so every special conformal Killing tensor can be written $\frac{1}{2}\sum_{\alpha,\beta} c_{\alpha\beta}X_i^\alpha X_j^\beta$, $X^\alpha \in \mathcal{C}(g)$. Otherwise,

$$\dim\mathcal{B}(g) \geq \tfrac{1}{2}m(m+1)+1, \quad m = \dim\mathcal{C}(g),$$

taking account of the fact that any constant multiple of the metric tensor is a special conformal Killing tensor, or in other words $(g_{ij}, 0, 0)$ is a solution of the structural equations.

4.2 *Simplification when* $\dim\mathcal{B}(g) \geq 3$

I have shown that when $\dim\mathcal{C}(g) \geq 2$ the second structural equation for a concircular field simplifies to $\rho_{|k} = \kappa X_k$ with κ a constant. A similar result holds for special conformal Killing tensors. The condition on the dimension is now $\dim\mathcal{B}(g) \geq 3$: this change is just a consequence of the fact noted above that g is a special conformal Killing tensor.

Theorem 9 (A. S. Solodovnikov) *If* $\dim\mathcal{B}(g) \geq 3$ *(i.e. if there are two special conformal Killing tensors linearly independent of g and each other) then* $\lambda_{i|j} = 2\kappa L_{ij} + \nu g_{ij}$ *for some constant* κ *and scalar* ν*; and* $\nu_{|i} = 2\kappa\lambda_i$.

I do not give a proof here, but merely some examples.

Firstly, it is easy to see that for a space of constant curvature κ we have

$$\lambda_{i|j} = 2\kappa L_{ij} + \nu g_{ij}, \quad \nu = \frac{1}{n}\left(\mu - 2\kappa\lambda\right),$$

whence $\nu_{|i} = 2\kappa\lambda_i$.

Consider next a special conformal Killing tensor derived from concircular vector fields, when $\dim\mathcal{C}(g) \geq 2$. We have $\lambda_i = \sum_{\alpha,\beta} c_{\alpha\beta}\rho^\alpha X_i^\beta$, so

$$\lambda_{i|j} = \sum_{\alpha,\beta} c_{\alpha\beta}(\kappa X_j^\alpha X_i^\beta + \rho^\alpha\rho^\beta g_{ij}) = 2\kappa L_{ij} + \nu g_{ij},$$

with $\nu = \sum_{\alpha,\beta} c_{\alpha\beta}\rho^\alpha\rho^\beta$ (and κ constant). Then

$$\nu_{|i} = 2\kappa\sum_{\alpha,\beta} c_{\alpha\beta}\rho^\alpha X_i^\beta = 2\kappa\lambda_i.$$

4.3 Augmentation

For $\kappa \neq 0$, the symmetric covariant-constant 2-tensors of the augmented metric

$$e^{2\kappa x^0}\left((dx^0)^2 - \frac{1}{\kappa} g_{ij} dx^i dx^j\right)$$

are in 1-1 correspondence with special conformal Killing tensors of g_{ij} satisfying Solodovnikov's theorem, via $(L_{ij}, \lambda_i, \nu) \leftrightarrow \tilde{L}_{ab}$ where

$$\tilde{L}_{ab} = e^{2\kappa x^0} \begin{pmatrix} \nu & \lambda_i \\ \lambda_j & L_{ij} \end{pmatrix}.$$

4.4 Lacunae

I shall write $\mathrm{int}(r)$ for the integer part of r.

Theorem 10 (I. G. Shandra [15]) *Set $m = \dim \mathcal{C}(g)$. If $m \neq n+1$*

$$\tfrac{1}{2}m(m+1) + 1 \leq \dim \mathcal{B}(g) \leq \tfrac{1}{2}m(m+1) + \mathrm{int}(\tfrac{1}{3}(n+1-m)).$$

I give below a partial proof, following [15].

Proof. Let $\mathcal{B}_0(g)$ be the space of covariant-constant symmetric 2-tensors. The idea is to reduce the problem to finding $\dim \mathcal{B}_0(g)$ on a suitable space (M, g) which admits a converging field.

Consider such a space (M, g). Its de Rham decomposition is

$$M = M_0 \times M_1 \times \cdots \times M_p, \quad g = g_0 + g_1 + \cdots + g_p, \quad \text{where}$$

- (M_0, g_0) is flat, $\dim M_0 = \dim \mathcal{C}_0(g) = k$ say;
- each (M_α, g_α) with $\alpha = 1, 2, \ldots, p$ is irreducible; since by assumption there is a converging field, each component (M_α, g_α) admits a converging field, so that $\dim M_\alpha \geq 3$.

Any covariant-constant symmetric 2-tensor decomposes as

$$B_0 + \sum_{\alpha=1}^{p} c_\alpha g_\alpha,$$

where B_0 is a constant symmetric tensor on the flat space M_0, and the c_α are constants. Since each factor M_α has dimension at least 3, there are at most $\mathrm{int}(\frac{1}{3}(n-k))$ of them. We therefore have the following lemma.

Lemma 11 *If the space is not flat (so that $p \geq 1$)*

$$\tfrac{1}{2}k(k+1) + 1 \leq \dim \mathcal{B}_0(g) \leq \tfrac{1}{2}k(k+1) + \mathrm{int}(\tfrac{1}{3}(n-k)).$$

I show how this leads to the required result in a couple of cases.

In the case $\kappa < 0$ I apply the lemma to the augmented space; then $k = \dim \mathcal{C}(g) = m$, $\mathcal{B}_0(g)$ becomes $\mathcal{B}(g)$ and n becomes $n+1$ and we obtain

$$\tfrac{1}{2}m(m+1) + 1 \le \dim \mathcal{B}(g) \le \tfrac{1}{2}m(m+1) + \operatorname{int}(\tfrac{1}{3}(n+1-m)).$$

If $\kappa = 0$ then $\lambda_{i|j} = \nu g_{ij}$ with $\nu_{|i} = 0$; assume that some $\nu \neq 0$. Then (by scaling) the corresponding λ_i may be taken to be a converging field. Let $\{X^\alpha\}$, $\alpha = 1, 2, \ldots, m$ be a basis for $\mathcal{C}(g)$, with $X^1_{i|j} = g_{ij}$, $X^\alpha_{i|j} = 0$ for $\alpha \ge 2$. For any $L_{ij} \in \mathcal{B}(g)$, $\lambda_i \in \mathcal{C}(g)$, so

$$\lambda_i = \sum_{\alpha=1}^{m} c_\alpha X^\alpha_i \quad \text{for some constants } c_\alpha.$$

Set

$$M_{ij} = 2L_{ij} - c_1 X^1_i X^1_j - \sum_{\alpha=2}^{m} c_\alpha (X^1_i X^\alpha_j + X^\alpha_i X^1_j).$$

Then

$$\begin{aligned} M_{ij|k} &= 2L_{ij|k} - c_1 X^1_{i|k} X^1_j - c_1 X^1_i X^1_{j|k} - \sum_{\alpha=2}^{m} c_\alpha (X^1_{i|k} X^\alpha_j + X^\alpha_i X^1_{j|k}) \\ &= \lambda_i g_{jk} + \lambda_j g_{ik} - c_1 X^1_j g_{ik} - c_1 X^1_i g_{jk} - \sum_{\alpha=2}^{m} c_\alpha (X^\alpha_j g_{ik} + X^\alpha_i g_{jk}) \\ &= 0. \end{aligned}$$

We have a linear map $\mathcal{B}(g) \to \mathcal{B}_0(g)$ whose kernel consists of all L_{ij} of the form

$$c_1 X^1_i X^1_j - \sum_{\alpha=2}^{m} c_\alpha (X^1_i X^\alpha_j + X^\alpha_i X^1_j);$$

so $\dim \mathcal{B}(g) = \dim \mathcal{B}_0(g) + \dim \mathcal{C}(g) = \dim \mathcal{B}_0(g) + m$. Moreover, we have a converging field; and so $\dim \mathcal{C}(g) = \dim \mathcal{C}_0(g) + 1$. So by the lemma

$$\tfrac{1}{2}(m-1)m + 1 \le \dim \mathcal{B}_0(g) \le \tfrac{1}{2}(m-1)m + \operatorname{int}(\tfrac{1}{3}(n-(m-1)));$$

add m throughout to get

$$\tfrac{1}{2}m(m+1) + 1 \le \dim \mathcal{B}(g) \le \tfrac{1}{2}m(m+1) + \operatorname{int}(\tfrac{1}{3}(n+1-m)). \qquad \square$$

This result severely limits the possibilities for special conformal Killing tensors, particularly in low dimensions. For example, for $n = 3$, m can take only the values 0, 1 and 4, the latter corresponding to a space of constant curvature. We have $\operatorname{int}(\frac{1}{3}(n+1-m)) = 1$ for both $m = 0$ and $m = 1$. So for $m = 0$, $\dim \mathcal{B}(g) = 1$, and the only special conformal Killing tensor is the trivial one, namely the metric (up to a constant multiple). For $m = 1$, $\dim \mathcal{B}(g) = 2$, and the only special conformal Killing tensor, apart from the trivial one, is the one formed from the concircular vector field. For a space of constant curvature, on the other hand, $\dim \mathcal{B}(g) = 10$. In particular, when $n = 3$ there are no non-trivial non-singular special conformal Killing tensors except in a space of constant curvature.

Appendix: structural equations

Suppose that tensorial quantities F^A satisfy a system of equations

$$F^A_{|i} = \Gamma^A_{Bi} F^B$$

where the coefficients Γ^A_{Bi} are tensorial quantities independent of the F^A; structural equations take this form. Think of the F^A as components of a section of a vector bundle $E \to M$, the Whitney sum of tensor bundles over M. The subscript $|i$ indicates covariant differentiation with respect to a given connection, acting on E. We can define a new covariant derivative on E by

$$F^A_{\|i} = F^A_{|i} - \Gamma^A_{Bi} F^B.$$

The structural equations are the equations for parallel sections under the new connection. We can draw the following conclusions about the solutions of the structural equations.

- If, for any $x \in M$, there is a solution with prescribed values $F^A(x)$ it is unique.
- The space of solutions has maximum dimension equal to the fibre dimension of E.
- The maximum dimension of the solution space is attained if and only if the curvature of the new connection vanishes.
- The curvature of the new connection vanishes if and only if the 'integrability conditions' of the structural equations are always satisfied.

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