On projective connections: the general case

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Abstract

We derive a Cartan normal projective connection for a system of second-order ordinary differential equations (extending the results of Cartan from a single equation to many)); we generalize the concept of a normal Thomas-Whitehead connection from affine to general sprays; and we show how to obtain the former from the latter by a global construction. This completes a study of projective connections begun in [4].

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1 Introduction

This paper is the second of a pair devoted to the study of the relationship between the two classical approaches to projective differential geometry. One of these is associated almost entirely with the name of Cartan, and is described in 'Sur les variétés à connexion projective' [1]. Many authors have contributed to the other, but for our present purposes the most appropriate choice of representative paper is Douglas's 'The general geometry of paths' [5].

In our first paper [4] we discussed the affine case, the geometry of what Douglas calls restricted path spaces; that is, the projective differential geometry of affine connections and their geodesics, which is the subject of the first and longer part of Cartan's paper. We based our discussion of the path space approach on Roberts's exposition [8] of the work of Thomas and Whitehead, which is crystallized in the concept of a Thomas-Whitehead connection (TW-connection). We showed how to define intrinsically for any manifold M a principal fibre bundle $\mathcal{C}M \to M$ with group the projective group $\mathrm{PGL}(m+1)$, $m = \dim M$, which is the carrier space of the global connection form of any Cartan projective connection on M; we called this bundle the Cartan bundle. We showed further how, given a projective equivalence class of affine connections, to construct from the Ehresmann connection form of the corresponding TW-connection a global Cartan connection form on the Cartan bundle.

In the present paper we will extend these results to the case of a general path space, that is, to the projective differential geometry of sprays in general. In doing so we have had to face two problems. In the first place, though Cartan dealt in [1] with the affine case in arbitrary dimension, his account there of the general case is restricted to dimension 2. Secondly, the theory of the TW-connection applies only to the affine case. We have therefore had to develop a theory of Cartan projective connections in the general case for dimension greater than 2, and to generalize the theory of TW-connections from scratch. On the other hand, the bundle constructions we used in the first paper turn out, perhaps somewhat surprisingly, to serve their turn here as well, mutatis mutandis. A preliminary account of some of our results has been given in [9].

What we have described above as Cartan's theory of the projective connection in the general case in dimension 2 actually appears in [1] as being concerned with a single second-order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

One interpretation of the theory is that it is a method of obtaining invariants of such equations under coordinate transformations. From this point of view it is of course important to be clear what class of transformations is considered: it is the class of so-called point transformations, that is, transformations of the form $\hat{x} = \hat{x}(x, y)$, $\hat{y} = \hat{y}(x, y)$. Thus dependent and independent variables are mixed up under the allowed coordinate transformations, and it is this fact that creates the link with the projective differential geometry of sprays in two dimensions. Likewise, our theory of Cartan projective connections in the general case in higher dimensions may be thought of as a method for finding invariants of a system of second-order differential equations

$$\frac{d^2x^i}{dt^2} = f^i\left(t, x^j, \frac{dx^i}{dt}\right)$$

under point transformations $\hat{t} = \hat{t}(t, x^j)$, $\hat{x}^i = \hat{x}^i(t, x^j)$. It is therefore distinct from the theory of invariants of such equations under the restricted class of transformations which preserve the independent variable, that is, those with $\hat{t} = t$. There is a large literature on the latter problem. So far as the study of systems of second-order differential equations under point transformations is concerned, on the other hand, we need to refer to one previous publication only, the paper by Fels [6]. In this paper the invariants of second-order equations under point transformations are found by Cartan's other method, the method of equivalence. Our interests are more geometrical and global than those of Fels, but it may be reassuring to know that we obtain the same invariants as he does.

The paper is organized as follows. In Section 2 we review the projective geometry of sprays, both in general terms and with reference to systems of second-order ordinary differential equations. Section 3 contains the generalization of the concept of a TW-connection, which we call the Berwald-Thomas-Whitehead projective connection, or BTW-connection; it is based on the formulation of the conditions that determine the normal TW-connection in terms of affine sprays that we gave in [4]. The Cartan theory is discussed in Section 4, though the actual calculations leading to the explicit formulæ for the normal projective connection are relegated to an appendix since they are complicated and not especially illuminating. Section 5 deals with the Cartan bundle, and Section 6 with construction of the Cartan connection form on the Cartan bundle from the BTW-connection.

In our previous paper we defined, for a given manifold M, several associated manifolds; these will reappear in the present paper, so we repeat their definitions here for ease of reference.

- The (unoriented) volume bundle $\nu : \mathcal{V}M \to M$ is the set of pairs $[\pm\theta]$ where $\theta \in \bigwedge^m T_x^* M$ is a non-zero volume element at $x \in M$. We use adapted coordinates $(x^0, x^1, \ldots, x^m)$ on $\mathcal{V}M$ such that

  $$\theta = \pm(x^0)^{m+1} \left(dx^1 \wedge \ldots dx^m\right)_x, \quad x^0 > 0.$$

  The volume bundle is a principal bundle over M under the multiplicative action of $\mathbf{R}_+$ given by $\mu_s[\pm\theta] = s^{1/(m+1)}[\pm\theta]$; the corresponding fundamental vector field is $\Upsilon = x^0 \partial/\partial x^0$.

- The Cartan algebroid $\rho : \mathcal{W}M \to TM$ is the quotient of $T(\mathcal{V}M)$ by Υ^{C}, the complete lift of Υ to $T(\mathcal{V}M)$. It is a vector bundle — indeed, a Lie algebroid — over M, with fibre dimension $m+1$, and admits a global section e_0, the image of Υ considered as a section of $T(\mathcal{V}M)$.

- The Cartan projective bundle $\mathrm{P}(\mathcal{W}M)$ is the quotient of the Cartan algebroid by the equivalence relation of non-zero scalar multiplication in the fibres.

- The simplex bundle $\mathcal{S}_{\mathcal{W}}M$ is the quotient of the frame bundle of $\mathcal{W}M$ by the equivalence relation of non-zero scalar multiplication in the fibres. A point of $\mathcal{S}_{\mathcal{W}}M$ over $x \in M$ is a reference $(m+1)$-simplex for the m-dimensional projective space $\mathrm{P}(\mathcal{W}_x M)$; $\mathcal{S}_{\mathcal{W}}M$ is a principal $\mathrm{PGL}(m+1)$-bundle over M.

- The Cartan bundle $\mathcal{C}M \subset \mathcal{S}_{\mathcal{W}}M$ consists of those simplices with first element a multiple of the global section e_0 of ρ. It is a reduction of $\mathcal{S}_{\mathcal{W}}M$ to the group $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$ which is the stabilizer of the point $[1, 0, \ldots, 0]$ of projective space P^m. The Cartan bundle is the carrier of global Cartan projective connection forms in the affine case.

Our dicussion of the Cartan projective connection is based on the account of Cartan's theory of connections given by Sharpe [11]. We repeat Sharpe's definition of a Cartan connection here for convenience. It depends on the previous concept of a Klein geometry. A Klein geometry is a homogeneous space of a Lie group G, that is, a manifold on which G acts effectively and transitively to the left. Let H be the stabilizer of some chosen point of the manifold; then the homogeneous space may be identified with the coset space G/H, and we may refer to the pair (G, H) as the Klein geometry. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H. A Cartan geometry on a manifold M, modelled on a Klein geometry (G, H), is a right principal H-bundle $P \to M$ such that $\dim P = \dim G = \dim H + \dim M$, together with a $\mathfrak{g}$-valued 1-form ω on P, the Cartan connection form, such that

1. for each $p \in P$, $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism;
2. for each $h \in H$, $R_h^*\omega = \mathrm{ad}(h^{-1})\omega$;
3. if $A \in \mathfrak{h}$ then $\langle A^\dagger, \omega\rangle = A$, where $A^\dagger$ is the vertical vector field on P generated by A through the action of H.

A local section κ of $P \to M$ is called a gauge; the local $\mathfrak{g}$-valued form $\kappa^*\omega$ on M is the connection form in that gauge. Given two gauges κ and $\hat{\kappa}$ with overlapping domains, the corresponding local $\mathfrak{g}$-valued forms $\kappa^*\omega$ and $\hat{\kappa}^*\omega$ are related by the transformation rule $\hat{\kappa}^*\omega = \mathrm{ad}(h^{-1})(\kappa^*\omega) + h^*(\theta_H)$, where θ_H is the Maurer-Cartan form of H and h is the local H-valued function on M relating the two gauges.

Any curve $\sigma(t)$ in P determines a curve $\langle\dot{\sigma}, \omega\rangle$ in $\mathfrak{g}$, which can be integrated up to give a curve in G, and then projected onto a curve in G/H; the resulting curve depends only on the projection of σ into M, and is called a development of the curve in M into G/H. Let γ be a gauge on G/H: then we can express any curve in G in the form $R_{h(t)}\gamma(\xi(t))$ where $\xi(t)$ is a curve in G/H and $h(t)$ one in H. Then the development $\xi(t)$ of a curve $x(t)$ in M, when expressed with respect to gauges for both the Cartan and the model geometry, satisfies the differential equation

$$\mathrm{ad}(h^{-1})\langle\dot{\xi}, \gamma^*\theta_G\rangle + \langle\dot{h}, \theta_H\rangle = \langle\dot{x}, \kappa^*\omega\rangle;$$

this comprises $\dim\mathfrak{g}$ equations for $\dim\mathfrak{g}/\mathfrak{h}$ unknowns ξ and $\dim\mathfrak{h}$ unknowns h. If the model geometry contains straight lines a curve in M is a geodesic if its developments are straight lines.

We use the Einstein summation convention for repeated indices. Indices $a, b, \ldots$ range and sum from 1 to m, indices $\alpha, \beta, \ldots$ from 0 to m, and indices $i, j, \ldots$ from 2 to m.

2 Projective differential geometry of sprays

We review here the projective geometry of sprays. A useful reference for this material is Shen's book [12]; however, our approach differs from his in that we put more emphasis on the similarities between the general case and the affine case as described for example in Schouten's 'Ricci-Calculus' [10]. Douglas [5] also covers much of this ground of course.

2.1 Sprays and Berwald connections

We denote by $\tau_M^\circ : T^\circ M \to M$ the slit tangent bundle of M (TM with the zero section deleted). Coordinates on $T^\circ M$ will generally be written (x^a, u^a). The Liouville field $u^a\partial/\partial u^a$ is denoted by Δ.

A spray S on $T^\circ M$ is a second-order differential equation field

$$S = u^a \frac{\partial}{\partial x^a} - 2\Gamma^a \frac{\partial}{\partial u^a}$$

whose coefficients Γ^a are positively homogeneous of degree 2 in the u^a; if they are quadratic in the u^a (so that S is the geodesic field of a symmetric affine connection) then the spray is said to be affine.

Homogeneity occurs frequently and is always with respect to the u^a, so we will just say, for example, that α is of degree 1. Moreover, the distinction between being positively homogeneous and being homogeneous without qualification won't be important in this subsection, so we won't repeat the qualifier 'positively'.

The horizontal distribution associated with a spray is spanned by the vector fields

$$H_a = \frac{\partial}{\partial x^a} - \Gamma_a^b \frac{\partial}{\partial u^b}, \quad \Gamma_a^b = \frac{\partial \Gamma^b}{\partial u^a};$$

Γ_a^b is of degree 1. It will often be convenient to denote the vertical vector field $\partial/\partial u^a$ by V_a.

The Berwald connection (see for example [2]) associated with a spray S is a connection on the pullback bundle $\tau_M^{\circ *}(TM) \to T^\circ M$. We will use tensor calculus methods, so we write sections of $\tau_M^{\circ *}(TM)$ as $X^a\partial/\partial x^a$ where the coefficients X^a are local functions on $T^\circ M$. The Berwald connection can be specified by giving its covariant differentiation operator ∇ operating on $\partial/\partial x^a$ (regarded as a local section of $\tau_M^{\circ *}(TM)$, or vector field along the projection τ_M°), together with the usual rules of covariant differentiation: in fact

$$\nabla_{H_a} \frac{\partial}{\partial x^b} = \Gamma_{ab}^c \frac{\partial}{\partial x^c}, \quad \nabla_{V_a} \frac{\partial}{\partial x^b} = 0,$$

where the connection coefficients are given by

$$\Gamma_{ab}^c = \frac{\partial \Gamma_a^c}{\partial u^b} = \frac{\partial^2 \Gamma^c}{\partial u^a \partial u^b};$$

they are symmetric, of degree 0, and reduce to the usual connection coefficients in the affine case.

Note that covariant differentiation with respect to the vertical vector field V_a of any tensor field along τ_M° amounts simply to partial differentiation of the components of the field with respect to u^a; and that therefore if one takes a tensor field along τ_M° and partially differentiates its components with respect to the u^a one obtains another tensor field, with one more covariant index.

We will use index notation, so that (for example) if T is a type $(1,1)$ tensor along τ_M° and ξ a vector field on $T^\circ M$, $(\nabla_\xi T)_a^b$ is just written $\nabla_\xi T_a^b$.

The so-called total derivative $\mathbf{T}$ is the vector field along τ_M° whose coordinate representation is $u^a\partial/\partial x^a$; its covariant derivative in any horizontal direction vanishes.

The curvature of the connection is defined in the usual way, but can be broken down into various components according to whether the vector field arguments are taken to be horizontal or vertical. First, evidently

$$\left(\nabla_{V_a}\nabla_{V_b} - \nabla_{V_b}\nabla_{V_a} - \nabla_{[V_a,V_b]}\right)\frac{\partial}{\partial x^c} = 0.$$

Next, we have

$$\left(\nabla_{V_a}\nabla_{H_b} - \nabla_{H_b}\nabla_{V_a} - \nabla_{[V_a,H_b]}\right)\frac{\partial}{\partial x^c} = B_{cab}^d\frac{\partial}{\partial x^d},$$

where (since $[V_a, H_b]$ is vertical)

$$B_{cab}^d = \frac{\partial\Gamma_{bc}^d}{\partial u^a} = \frac{\partial^3\Gamma^d}{\partial u^a\partial u^b\partial u^c}.$$

This component of the curvature has no affine counterpart — in fact its vanishing is the necessary and sufficient condition for the spray to be affine. It is completely symmetric in the lower indices, is homogeneous of degree -1, and satisfies $B_{cab}^d u^c = 0$. It is called the Berwald curvature.

Finally,

$$\left(\nabla_{H_a}\nabla_{H_b} - \nabla_{H_b}\nabla_{H_a} - \nabla_{[H_a,H_b]}\right)\frac{\partial}{\partial x^c} = R_{cab}^d\frac{\partial}{\partial x^d},$$

where R_{cab}^d, the counterpart of the usual curvature, is given by

$$R_{cab}^d = H_a\left(\Gamma_{bc}^d\right) - H_b\left(\Gamma_{ac}^d\right) + \Gamma_{ae}^d\Gamma_{bc}^e - \Gamma_{be}^d\Gamma_{ac}^e.$$

It has the usual symmetries, is of degree 0, and reduces to the ordinary curvature tensor when the spray is affine. It is called the Riemann curvature.

We can also express the curvatures conveniently using forms. We write φ^a for the 1-form $du^a + \Gamma_b^a dx^b$, so that $\{dx^a, \varphi^a\}$ is the local basis of 1-forms on $T^\circ M$ dual to the local basis $\{H_a, V_a\}$ of vector fields. Define connection forms $\omega_b^a = \Gamma_{bc}^a dx^c$; if $\Omega_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c$ are the associated curvature forms then

$$\Omega_b^a = \tfrac{1}{2}R_{bcd}^a dx^c \wedge dx^d + B_{bcd}^a\varphi^c \wedge dx^d.$$

By taking traces of the curvatures we obtain tensors

$$B_{ab} = B^c_{cab}, \quad R_{ab} = R^c_{acb}.$$

The first is symmetric. The second is not in general symmetric; moreover, by the cyclic identity

$$R^c_{cab} = -R^c_{abc} - R^c_{bca} = R_{ab} - R_{ba}.$$

By differentiating the formula for the Riemann curvature with respect to u^e one obtains the following relation between the two curvatures:

$$\nabla_{V_e} R^d_{cab} = \nabla_{H_a} B^d_{bce} - \nabla_{H_b} B^d_{ace};$$

this is in fact part of the second Bianchi identity for the curvature taken as a whole. From this formula, by taking a trace one obtains

$$\nabla_{V_c} R_{ab} = \nabla_{H_d} B^d_{abc} - \nabla_{H_b} B_{ac},$$

whence using the symmetry of B^d_{abc}

$$\nabla_{V_c}(R_{ab} - R_{ba}) = \nabla_{H_a} B_{bc} - \nabla_{H_b} B_{ac},$$

which turns out to be useful later. Furthermore, it follows from the last equation but one, again using symmetry, that

$$\nabla_{V_a} R_{bc} = \nabla_{V_b} R_{ac}.$$

We will also be concerned with the associated tensor

$$R^a_b = R^a_{cbd} u^c u^d = 2\frac{\partial \Gamma^a}{\partial x^b} - S(\Gamma^a_b) - \Gamma^a_c \Gamma^c_b.$$

This type $(1,1)$ tensor field is often called the Jacobi endomorphism, because it is the curvature term that appears in the Jacobi equation. It contains the same information as the Riemann tensor, which can be recovered from it by use of the formula

$$R^d_{cab} = \tfrac{1}{3}\left(\nabla_{V_c}\nabla_{V_b} R^d_a - \nabla_{V_c}\nabla_{V_a} R^d_b\right).$$

We denote by R the trace of R^a_b; we have $R = R_{cd}u^c u^d$. It follows from the relationship $\nabla_{V_a} R_{bc} = \nabla_{V_b} R_{ac}$ and the fact that R_{ab} is homogeneous of degree 0 that $u^b \nabla_{V_a} R_{bc} = 0$, whence

$$\frac{\partial R}{\partial u^a} = (R_{ab} + R_{ba})u^b, \quad \frac{\partial^2 R}{\partial u^a \partial u^b} = R_{ab} + R_{ba}.$$

A spray whose Jacobi endomorphism has the property that for any $v \in T_u(T^\circ M)$, $R^a_b v^b$ is a linear combination of u^a and v^a is said to be isotropic. For an isotropic spray R^a_b takes the form $R^a_b = \lambda \delta^a_b + \mu_b u^a$ for some scalar λ and vector μ_b. Since $R^a_b u^b = 0$ we have $\lambda = -\mu_b u^b$, and then by taking the trace we find that $(m-1)\lambda = R$, so for an isotropic spray

$$R^a_b - \frac{1}{m-1} R \delta^a_b = \mu_b u^a,$$

with $\mu_b u^b = -R/(m-1)$.

2.2 Projective equivalence

Two sprays S, $\hat{S}$ are projectively equivalent if $\hat{S} - S = -2\alpha\Delta$, or $\hat{\Gamma}^a = \Gamma^a + \alpha u^a$, where the function α is positively homogeneous of degree 1 in the u^a.

From the basic projective transformation rule it follows that the horizontal vector fields associated with the spray $\hat{S}$ are given by

$$\hat{H}_a = H_a - \alpha V_a - \alpha_a \Delta,$$

where

$$\alpha_a = \frac{\partial \alpha}{\partial u^a};$$

α_a is of degree 0, and $u^a \alpha_a = \alpha$. Furthermore,

$$\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \left(\alpha_{ab} u^c + \alpha_a \delta^c_b + \alpha_b \delta^c_a\right),$$

where

$$\alpha_{ab} = \frac{\partial^2 \alpha}{\partial u^a \partial u^b};$$

α_{ab} is symmetric and of degree -1, and $\alpha_{ab} u^b = 0$.

By taking a trace in the equation for the transformation of the Γ^c_{ab} we obtain (writing Γ_a for Γ^b_{ab})

$$\hat{\Gamma}_a = \Gamma_a + (m+1)\alpha_a,$$

whence the quantity

$$\Pi^c_{ab} = \Gamma^c_{ab} - \frac{1}{m+1}\left(\Gamma_a \delta^c_b + \Gamma_b \delta^c_a + B_{ab} u^c\right)$$

is projectively invariant. Douglas calls it the fundamental invariant and says in effect that every projective invariant is expressible in terms of it and its partial derivatives. However, the Π^c_{ab} are not components of a tensor, nor even of a connection, and this has to be borne in mind when forming projective invariants from it.

Note that $\Pi^b_{ab} = \Pi^b_{ba} = 0$.

It may appear that if we set $\Gamma = \Gamma^d_d$ and take

$$\alpha = -\frac{1}{m+1}\Gamma = -\frac{1}{m+1}\frac{\partial \Gamma^d}{\partial u^d}$$

then the transformed spray has Π^c_{ab} for its connection coefficients. However, Γ is not strictly speaking a function: its transformation law under coordinate transformations of the x^a (and the induced transformations of the u^a) involves the determinant of the Jacobian of the coordinate transformation; however, it transforms as a function under coordinate transformations for which the determinant of the Jacobian is 1.

We will derive the projective transformation formulæ we require entirely tensorially; we will however point out the simplifications that arise when one chooses the spray whose connection coefficients with respect to some coordinates are the $\Pi_{ab}^{\,c}$. We will denote objects calculated in this way by setting their kernel letters in black-letter. Thus it follows from the observation above that the traces of $\Pi_{ab}^{\,c}$ vanish that $\mathfrak{B}_{ab} = 0$.

It follows from the vanishing of the traces of $\Pi_{ab}^{\,c}$ that $\mathfrak{R}^c_{cab} = 0$, and thus that $\mathfrak{R}_{ba} = \mathfrak{R}_{ab}$.

2.3 Projective transformation of the curvatures

An easy calculation leads to the following transformation formula for B^d_{cab}:

$$\hat{B}^d_{cab} = B^d_{cab} + \alpha_{abc}u^d + \alpha_{ab}\delta^d_c + \alpha_{bc}\delta^d_a + \alpha_{ac}\delta^d_b,$$

where α_{abc} denotes a third partial derivative of α; it satisfies $\alpha_{abc}u^c = -\alpha_{ab}$. Then by taking a trace

$$\hat{B}_{ab} = B_{ab} + (m+1)\alpha_{ab},$$

whence

$$D^d_{cab} = B^d_{cab} - \frac{1}{m+1}\left(u^d\nabla_{V_c}B_{ab} + B_{ab}\delta^d_c + B_{bc}\delta^d_a + B_{ac}\delta^d_b\right)$$

is a projectively invariant tensor — the Douglas tensor. It is symmetric in its lower indices, of degree 0, it satisfies $D^d_{cab}u^c = 0$, and all of its traces vanish.

Since $\mathfrak{B}_{ab} = 0$,

$$D^d_{cab} = \mathfrak{B}^d_{cab} = \frac{\partial \Pi_{ab}^{\,d}}{\partial u^c}.$$

The vanishing of the Douglas tensor is the necessary and sufficient condition for a spray to be projectively equivalent to an affine one.

The projective transformation of the Riemann curvature is given by

$$\hat{R}^d_{cab} = R^d_{cab} + \nabla_{H_a}\alpha_{bc}^{\,d} - \nabla_{H_b}\alpha_{ac}^{\,d} + (\alpha\alpha_{bc} + \alpha_b\alpha_c)\delta^d_a - (\alpha\alpha_{ac} + \alpha_a\alpha_c)\delta^d_b,$$

where $\alpha_{ab}^{\,c}$ is the difference tensor of the connection coefficients,

$$\alpha_{ab}^{\,c} = \alpha_{ab}u^c + \alpha_a\delta^c_b + \alpha_b\delta^c_a.$$

Using the fact that $\nabla_{H_a}\mathbf{T} = 0$, so that multiplying by u^a commutes with covariant differentiation with respect to H_a, we obtain

$$\nabla_{H_a}\alpha_{bc}^{\,d} = u^d\nabla_{H_a}\alpha_{bc} + (\nabla_{H_a}\alpha_b)\delta^d_c + (\nabla_{H_a}\alpha_c)\delta^d_b.$$

We know that

$$\alpha_{ab} = \frac{1}{m+1}\left(\hat{B}_{ab} - B_{ab}\right).$$

It can be shown that

$$\nabla_{H_a}\hat{B}_{bc} - \nabla_{H_b}\hat{B}_{ac} = \hat{\nabla}_{\hat{H}_a}\hat{B}_{bc} - \hat{\nabla}_{\hat{H}_b}\hat{B}_{ac},$$

whence the transformation law for R^d_{cab} can be rewritten in the form

$$\hat{S}^d_{cab} = S^d_{cab} - A_{bc}\delta^d_a + A_{ac}\delta^d_b + (A_{ab} - A_{ba})\delta^d_c,$$

where

$$\begin{aligned} S^d_{cab} &= R^d_{cab} - \frac{1}{m+1}u^d\left(\nabla_{H_a}B_{bc} - \nabla_{H_b}B_{ac}\right) \\ &= R^d_{cab} - \frac{1}{m+1}\left(u^d\nabla_{V_c}(R_{ab} - R_{ba})\right), \end{aligned}$$

and

$$A_{ab} = \nabla_{H_a}\alpha_b - \alpha\alpha_{ab} - \alpha_a\alpha_b.$$

The modified Riemann curvature S^d_{cab} has the usual symmetries, is of degree 0, and reduces to the Riemann curvature in the affine case. We set $S_{ab} = S^c_{acb}$; then

$$\hat{S}_{ab} = S_{ab} + A_{ab} - mA_{ba},$$

whence

$$A_{ab} = -\frac{1}{m^2-1}\left(\hat{Q}_{ab} - Q_{ab}\right), \quad Q_{ab} = S_{ab} + mS_{ba}.$$

It follows that

$$P^d_{cab} = S^d_{cab} - \frac{1}{m^2-1}\left(Q_{bc}\delta^d_a - Q_{ac}\delta^d_b - (Q_{ab} - Q_{ba})\delta^d_c\right)$$

is a projectively invariant tensor. It is the counterpart of the projective curvature tensor of the affine theory, to which it reduces in the affine case. It is of degree 0; it has the same symmetries as the Riemann curvature, and in addition all of its traces vanish.

Since $\mathfrak{R}_{ab}$ is symmetric with respect to a canonically parametrized spray, $\mathfrak{S}^d_{cab} = \mathfrak{R}^d_{cab}$, whence $\mathfrak{S}_{ab} = \mathfrak{R}_{ab}$, and $\mathfrak{Q}_{ab} = (m+1)\mathfrak{R}_{ab}$, so that

$$P^d_{cab} = \mathfrak{R}^d_{cab} - \frac{1}{m-1}\left(\mathfrak{R}_{bc}\delta^d_a - \mathfrak{R}_{ac}\delta^d_b\right).$$

The Jacobi endomorphism of a spray S transforms as follows:

$$\hat{R}^a_b = R^a_b + A_b u^a - A\delta^a_b,$$

where the vector A_a and scalar A are given by

$$A_a = 2H_a(\alpha) - \nabla_S\alpha_a - \alpha\alpha_a, \quad A = S(\alpha) - \alpha^2 = u^a A_a;$$

A_a is homogeneous of degree 1. For the trace of the Jacobi endomorphism we have

$$\hat{R} = R - (m-1)A.$$

Using these formulæ one can show that

$$W^a_b = R^a_b - \frac{1}{m-1} R\delta^a_b - \frac{1}{m+1} u^a \nabla_{V_c} \left(R^c_b - \frac{1}{m-1} R\delta^c_b \right)$$

is projectively invariant. It is called the Weyl tensor. It is tracefree and satisfies $W^a_b u^b = 0$. The Weyl tensor bears the same relationship to the projective curvature tensor as the Jacobi endomorphism does to the Riemann curvature: that is to say, $P^a_{cbd} u^c u^d = W^a_b$, and P^a_{bcd} can be expressed in terms of second vertical covariant derivatives of W^a_b.

Recall that an isotropic spray is one for which

$$R^a_b - \frac{1}{m-1} R\delta^a_b = \mu_b u^a.$$

From the transformation laws it is easy to see that the property of being isotropic is projectively invariant. Moreover, by substituting into the expression for W^a_b and using the evident fact that μ_b is homogeneous of degree -1 we see that $W^a_b = 0$ for an isotropic spray; the converse is obvious. Thus a spray is isotropic if and only if $W^a_b = 0$; and equivalently if and only if $P^a_{bcd} = 0$.

We will also need the following result. If we carry out the projective transformation with

$$\alpha = -\frac{1}{m+1}\Gamma$$

we obtain

$$\frac{1}{m-1}\mathfrak{R} = \frac{1}{m-1} R + \frac{1}{m+1} S(\Gamma) + \frac{1}{(m+1)^2}\Gamma^2.$$

When $m > 2$ the vanishing of both the Douglas and the projective curvature tensors is the necessary and sufficient condition for a spray to be projectively flat, that is, projectively equivalent to a spray that can be written $u^a \partial/\partial x^a$ in some coordinates. In dimension 2, however, a tensor with the symmetries of the Riemann tensor is determined by its traces, and if they vanish so does the tensor; so the projective curvature tensor is identically zero in dimension 2. We will therefore assume that $m > 2$ hereafter.

2.4 Systems of differential equations

The base integral curves of a spray are the solutions of the equations

$$\ddot{x}^a + 2\Gamma^a(x, \dot{x}) = 0;$$

all sprays in a projective equivalence class have the same base integral curves up to change of parameter which preserves sense. Thus a projective equivalence class of sprays determines, and in fact is is determined by, a path space, that is, a collection of paths (unparametrized but oriented curves) in M with the property that there is a unique path of the collection through each point in each direction. A choice of spray in a projective equivalence class amounts to a choice of parametrization of the corresponding paths; Douglas calls the parametrization resulting from the choice with $\Gamma^a_{bc} = \Pi^a_{bc}$ the canonical parametrization for the given coordinates.

Since sprays are required to be only positively homogeneous, reversing the initial direction may give a different path. We will be interested in a restricted class of sprays, those having the property that the integral curve through x with initial tangent vector $-u$ is just the integral curve through x with initial tangent vector u traversed in the opposite sense; we call such sprays, and their base integral curves, reversible. Reversible sprays are such that the coefficients Γ^a are homogeneous of degree 2 without qualification, that is, satisfy $\Gamma^a(x^b, \lambda u^b) = \lambda^2 \Gamma^a(x^b, u^b)$ for all non-zero λ. Alternatively, they satisfy $\Gamma^a(x^b, -u^b) = \Gamma^a(x^b, u^b)$ in addition to being positively homogeneous. The corresponding path space has the property that given a point $x \in M$ and a line in T_xM there is a unique path (now an unparametrized and unoriented curve) through x whose tangent line at x is the given line. The set of lines in T_xM is just $\mathrm{P}T_xM$, the projective tangent space at x; thus a path space in this sense determines and is determined by a congruence of paths on $\mathrm{P}TM$, the projective tangent bundle of M (one and only one path of the congruence passes through each point of $\mathrm{P}TM$); the corresponding projective equivalence class of sprays determines and is determined by a line element field on $\mathrm{P}TM$, the tangent line element field of the congruence of paths.

From here on we will deal only with reversible sprays.

In a local coordinate system we can choose to parametrize suitable paths of a projective class of sprays with one of the coordinates, say x^1; with such a parametrization $\dot{x}^1 = 1$, $\ddot{x}^1 = 0$, and the differential equations take the form

$$\frac{d^2x^i}{d(x^1)^2} = f^i\left(x^1, x^j, \frac{dx^j}{dx^1}\right).$$

In other words, there is always locally a member of the projective class for which $\Gamma^1 = 0$; then $f^i(x^a, y^j) = -2\Gamma^i(x^a, 1, y^j)$ where $y^j = u^j/u^1$. Conversely, given a system of $m-1$ second-order differential equations in the variables x^i, with parameter x^1, we can locally recover a spray by setting

$$\Gamma^1 = 0, \quad \Gamma^i(x^a, u^a) = -\tfrac{1}{2}(u^1)^2 f^i(x^a, u^j/u^1).$$

Such a spray is reversible.

If we make a point transformation (a coordinate transformation involving all of the coordinates x^a) the spray corresponding to the new system of differential equations will not be the same as that corresponding to the original one; but it will be projectively equivalent to it. The invariants of the system of second-order ordinary differential equations under point transformations will be the projective invariants of the corresponding projective equivalence class of sprays.

It will be useful to be able to represent the projective quantities in terms of the f^i. We therefore compute the fundamental invariants Π^a_{bc} of the spray

$$u^a\frac{\partial}{\partial x^a} - 2\Gamma^a\frac{\partial}{\partial u^a}, \quad \Gamma^1 = 0, \quad \Gamma^i(x^a, u^a) = -\tfrac{1}{2}(u^1)^2 f^i(x^a, u^j/u^1).$$

We set

$$\gamma^i_j = -\tfrac{1}{2}\frac{\partial f^i}{\partial y^j}, \quad \gamma^{\ i}_{jk} = \frac{\partial \gamma^i_j}{\partial y^k}, \quad \gamma = \gamma^k_k, \quad \gamma_i = \frac{\partial \gamma}{\partial y^i} = \gamma^{\ k}_{ik}$$

and so on, and

$$\Phi^i_j = \frac{\partial f^i}{\partial x^j} + \frac{d}{dx^1}(\gamma^i_j) + \gamma^i_k\gamma^k_j$$

where

$$\frac{d}{dx^1} = \frac{\partial}{\partial x^1} + y^i\frac{\partial}{\partial x^i} + f^i\frac{\partial}{\partial y^i};$$

Φ^i_j is called in the relevant literature, with an unfortunate disagreement over sign, the Jacobi endomorphism of the second-order differential equation field [3]. We will show that for $m > 2$ the Douglas tensor D^a_{bcd} and the Weyl tensor W^a_b are completely determined by the quantities

$$K^i_{jkl} = \gamma^{\ i}_{jkl} - \frac{1}{m+1}\left(\delta^i_j\gamma_{kl} + \delta^i_k\gamma_{jl} + \delta^i_l\gamma_{jk}\right), \quad L^i_j = \Phi^i_j - \frac{1}{(m-1)}\delta^i_j\Phi^k_k.$$

Since the projective curvature tensor P^a_{bcd} determines and is determined by the Weyl tensor, it too is completely determined by these quantities. This is related to a result of Fels [6], who showed, using Cartan's method of equivalence, that K^i_{jkl} and L^i_j are the fundamental invariants of the system of second-order ordinary differential equations under point transformations.

We need to compute several quantities from the Γ^a by differentiating with respect to the u^a and taking traces. The calculations are much simplified by the fact that the quantities involved are homogeneous of various degrees in the u^a. Any function ϕ of degree n is determined by its value at $u^1 = 1$, since $\phi(u^a) = (u^1)^n\phi(1, y^i)$ (this is of course just the principle used to define the spray coefficients). Moreover, we have Euler's theorem at our disposal.

First we have $\Gamma^1_b = 0$, while

$$\Gamma^i_j = u^1\gamma^i_j, \quad \Gamma^i_1 = -(u^1)(f^i + y^l\gamma^i_l).$$

For the Γ^a_{bc} we obtain

$$\Gamma^1_{bc} = 0, \quad \Gamma^i_{jk} = \gamma^{\ i}_{jk}, \quad \Gamma^i_{1j} = \Gamma^i_{j1} = \gamma^i_j - y^l\gamma^{\ i}_{jl}, \quad \Gamma^i_{11} = -f^i - 2y^l\gamma^i_l + y^l y^m\gamma^{\ i}_{lm}.$$

Next, the traces:

$$\Gamma = u^1\gamma; \quad \Gamma_i = \gamma_i, \quad \Gamma_1 = \gamma - y^l\gamma_l;$$

and their derivatives

$$\Gamma_{ij} = (u^1)^{-1}\gamma_{ij}, \quad \Gamma_{1i} = \Gamma_{i1} = -(u^1)^{-1}y^l\gamma_{il} \quad \Gamma_{11} = (u^1)^{-1}y^l y^m\gamma_{lm}.$$

The fundamental invariants are given by

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{m+1}\left(\Gamma_b\delta^a_c + \Gamma_c\delta^a_b + \Gamma_{bc}u^a\right);$$

in terms of f^i and its derivatives we have

$$\begin{aligned}
\Pi^1_{11} &= -\frac{1}{m+1}(2\gamma - 2y^l\gamma_l + y^l y^m\gamma_{lm}) \\
\Pi^1_{1i} &= -\frac{1}{m+1}(\gamma_i - y^l\gamma_{il}) \\
\Pi^1_{ij} &= -\frac{1}{m+1}\gamma_{ij} \\
\Pi^i_{11} &= -f^i - 2y^l\gamma^i_l + y^l y^m\gamma^i_{lm} - \frac{1}{m+1}y^i y^l y^m\gamma_{lm} \\
\Pi^i_{1j} &= \gamma^i_j - y^l\gamma^i_{jl} - \frac{1}{m+1}((\gamma - y^l\gamma_l)\delta^i_j - y^i y^l\gamma_{jl}) \\
\Pi^i_{jk} &= \gamma^i_{jk} - \frac{1}{m+1}(\gamma_j\delta^i_k + \gamma_k\delta^i_j + y^i\gamma_{jk}).
\end{aligned}$$

Thus with $u^1 = 1$,

$$D^i_{jkl} = \gamma^i_{jkl} - \frac{1}{m+1}(\gamma_{jl}\delta^i_k + \gamma_{kl}\delta^i_j + \gamma_{jk}\delta^i_l + y^i\gamma_{jkl}).$$

We differentiate K^i_{jkl} with respect to y^m to obtain

$$\frac{\partial K^i_{jkl}}{\partial y^m} = \gamma^i_{jklm} - \frac{1}{m+1}\left(\delta^i_j\gamma_{klm} + \delta^i_k\gamma_{jlm} + \delta^i_l\gamma_{jkm}\right),$$

then take a trace to get

$$\frac{m-2}{m+1}\gamma_{jkl} = \frac{\partial K^m_{jkl}}{\partial y^m},$$

whence

$$D^i_{jkl} = K^i_{jkl} - \frac{1}{m-2}y^i\frac{\partial K^m_{jkl}}{\partial y^m}$$

Furthermore

$$D^1_{jkl} = -\frac{1}{m+1}\gamma_{jkl} = -\frac{1}{m-2}\frac{\partial K^m_{jkl}}{\partial y^m}.$$

Now $u^d D^a_{bcd} = 0$, whence $D^a_{bc1} = -y^i D^a_{bci}$, so that the remaining components of D^a_{bcd} are determined by those which have already been calculated.

For the Jacobi endomorphism of the spray we have $R^1_a = 0$,

$$R^i_j = -(u^1)^2 \left(\frac{\partial f^i}{\partial x^j} + \frac{d}{dx^1}(\gamma^i_j) + \gamma^i_k \gamma^k_j \right) = -(u^1)^2 \Phi^i_j,$$

so that $R = -(u^1)^2 \Phi^k_k$. Thus

$$R^i_j - \frac{1}{m-1} R \delta^i_j = -(u^1)^2 L^i_j, \quad R^1_j - \frac{1}{m-1} R \delta^1_j = 0.$$

Thus when $u^1 = 1$,

$$W^i_j = -L^i_j + \frac{1}{m+1} y^i \frac{\partial L^k_j}{\partial y^k}, \quad W^1_j = \frac{1}{m+1} \frac{\partial L^k_j}{\partial y^k}.$$

As before, the remaining components of W^a_b are determined by these, since $W^a_b u^b = 0$.

It follows that $L^i_j = 0$ is a necessary and sufficient condition for the spray to be isotropic.

3 *BTW*-connections

In our previous paper [4] we showed that the geometry of a projective equivalence class of affine sprays on TM can be described in terms of a single affine spray on $T(\mathcal{V}M)$, where $\mathcal{V}M$ is the volume bundle (see the Introduction), whose corresponding symmetric affine connection is known as the normal TW-connection. In that paper we followed the historical order of events, by developing the theory of TW-connections first, basing our account on the work of Roberts [8], and subsequently showing that the defining properties of a TW-connection can be specified in terms of its spray. We are now faced with the problem of generalizing these ideas to the case of a projective equivalence class of (not necessarily affine) sprays. The properties formerly used to define a TW-connection do not translate straightforwardly into properties of Berwald connections; however, the equivalent properties for an affine spray carry over almost without change to general sprays. We will therefore approach the definition of the Berwald connection on $T^\circ(\mathcal{V}M)$ which generalizes the normal TW-connection, which we will call the Berwald-Thomas-Whitehead projective connection, or BTW-connection for short (we will deal only with the analogue of the normal TW-connection and therefore need no qualifier), by first proving the existence of a uniquely determined spray on $T^\circ(\mathcal{V}M)$ which carries all the information about a given projective class of sprays on $T^\circ M$. We call this spray the BTW-spray, and define the BTW-connection as the Berwald connection of this spray.

For a spray $\tilde{S}$ on $T^\circ(\mathcal{V}M)$,

$$\tilde{S} = u^\alpha \frac{\partial}{\partial x^\alpha} - 2\tilde{\Gamma}^\alpha \frac{\partial}{\partial u^\alpha},$$

we denote by $\tilde{R}^\alpha_\beta$ its Jacobi endomorphism,

$$\tilde{R}^\alpha_\beta = 2\frac{\partial \tilde{\Gamma}^\alpha}{\partial x^\beta} - \tilde{S}(\tilde{\Gamma}^\alpha_\beta) - \tilde{\Gamma}^\alpha_\gamma \tilde{\Gamma}^\gamma_\beta,$$

and by $\tilde{R}$ the trace of $\tilde{R}^\alpha_\beta$.

There is a well-defined volume form vol on $T^\circ(\mathcal{V}M)$,

$$\mathrm{vol} = (x^0)^{2m} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^m \wedge du^0 \wedge du^1 \wedge \cdots \wedge du^m.$$

In the affine case the affine spray $\tilde{S}$ on $T(\mathcal{V}M)$ whose corresponding symmetric affine connection is the normal TW-connection of a given projective equivalence class of sprays on M is uniquely determined by the following conditions:

- $\mathcal{L}_{\Upsilon^{\mathrm{C}}}\tilde{S} = 0$;
- $\mathcal{L}_{\Upsilon^{\mathrm{V}}}\tilde{S} = \Upsilon^{\mathrm{C}} - 2\tilde{\Delta}$;
- $\mathcal{L}_{\tilde{S}}\mathrm{vol} = 0$;
- $\tilde{R} = 0$.

These conditions apply without change to general sprays on $T^\circ(\mathcal{V}M)$.

We will derive their consequences in terms of coordinates adapted to $\mathcal{V}M$, thus showing that sprays satisfying them exist locally; we postpone the proof of the global existence of such sprays until later.

Locally,

$$\begin{aligned} \mathcal{L}_{\Upsilon^{\mathrm{C}}}\tilde{S} &= \left[x^0\frac{\partial}{\partial x^0} + u^0\frac{\partial}{\partial u^0}, u^\alpha\frac{\partial}{\partial x^\alpha} - 2\tilde{\Gamma}^\alpha\frac{\partial}{\partial u^\alpha}\right] \\ &= -2\left(x^0\frac{\partial\tilde{\Gamma}^\alpha}{\partial x^0} + u^0\frac{\partial\tilde{\Gamma}^\alpha}{\partial u^0}\right)\frac{\partial}{\partial u^\alpha} + 2\tilde{\Gamma}^0\frac{\partial}{\partial u^0}, \end{aligned}$$

while

$$\begin{aligned} \mathcal{L}_{\Upsilon^{\mathrm{V}}}\tilde{S} - \Upsilon^{\mathrm{C}} &= \left[x^0\frac{\partial}{\partial u^0}, u^\alpha\frac{\partial}{\partial x^\alpha} - 2\tilde{\Gamma}^\alpha\frac{\partial}{\partial u^\alpha}\right] - x^0\frac{\partial}{\partial x^0} - u^0\frac{\partial}{\partial u^0} \\ &= -2x^0\frac{\partial\tilde{\Gamma}^\alpha}{\partial u^0}\frac{\partial}{\partial u^\alpha} - 2u^0\frac{\partial}{\partial u^0}. \end{aligned}$$

So in order for $\tilde{S}$ to satisfy the first pair of conditions we must have

$$x^0\frac{\partial\tilde{\Gamma}^0}{\partial x^0} + u^0\frac{\partial\tilde{\Gamma}^0}{\partial u^0} = \tilde{\Gamma}^0, \quad x^0\frac{\partial\tilde{\Gamma}^a}{\partial x^0} + u^0\frac{\partial\tilde{\Gamma}^a}{\partial u^0} = 0, \quad \frac{\partial\tilde{\Gamma}^0}{\partial u^0} = 0, \quad x^0\frac{\partial\tilde{\Gamma}^a}{\partial u^0} = u^a.$$

It follows that

$$x^0\frac{\partial\tilde{\Gamma}^0}{\partial x^0} = \tilde{\Gamma}^0,$$

whence

$$\frac{\partial}{\partial x^0}((x^0)^{-1}\tilde{\Gamma}^0) = 0,$$

so that

$$\tilde{\Gamma}^0 = x^0 G^0$$

say, where G^0 is a function on $T^\circ M$ homogeneous of degree 2. Moreover

$$\frac{\partial \tilde{\Gamma}^a}{\partial x^0} = -(x^0)^{-2} u^0 u^a,$$

so that

$$\tilde{\Gamma}^a = (x^0)^{-1} u^0 u^a + G^a,$$

where again G^a is a function on $T^\circ M$ homogeneous of degree 2. Thus a spray satisfies the first pair of conditions if locally it takes the form

$$\tilde{S} = u^\alpha \frac{\partial}{\partial x^\alpha} - 2(G^a + (x^0)^{-1} u^0 u^a) \frac{\partial}{\partial u^a} - 2x^0 G^0 \frac{\partial}{\partial u^0}.$$

The remaining two conditions impose further restrictions on G^a and G^0. We have

$$\mathcal{L}_{\tilde{S}} \mathrm{vol} = \left(2m \frac{u^0}{x^0} - 2 \left(G^a_a + m \frac{u^0}{x^0} \right) \right) \mathrm{vol},$$

so, setting $G = G^a_a$, we see that $\mathcal{L}_{\tilde{S}} \mathrm{vol} = 0$ if and only if $G = 0$. Now

$$\tilde{\Gamma}^0_0 = 0, \quad \tilde{\Gamma}^0_a = x^0 G^0_a, \quad \tilde{\Gamma}^a_0 = \frac{u^a}{x^0}, \quad \tilde{\Gamma}^a_b = G^a_b + \left(\frac{u^0}{x^0} \right) \delta^a_b,$$

whence $\tilde{\Gamma} = \tilde{\Gamma}^\alpha_\alpha = m u^0 / x^0$, and

$$\tilde{\Gamma}^\alpha_\beta \tilde{\Gamma}^\beta_\alpha = G^a_b G^b_a + 4G^0 + m \left(\frac{u^0}{x^0} \right)^2,$$

using $G = 0$ and the homogeneity of G^0. Thus

$$\begin{aligned} \tilde{R} &= 2 \left(\frac{\partial G^a}{\partial x^a} + G^0 \right) + m \left(\left(\frac{u^0}{x^0} \right)^2 + 2G^0 \right) - G^a_b G^b_a - 4G^0 - m \left(\frac{u^0}{x^0} \right)^2 \\ &= 2 \frac{\partial G^a}{\partial x^a} - G^a_b G^b_a + 2(m-1) G^0, \end{aligned}$$

so that the fourth condition is satisfied (given that the others are) if and only if

$$G^0 = -\frac{1}{2(m-1)} \left(2 \frac{\partial G^a}{\partial x^a} - G^a_b G^b_a \right).$$

Now $\mathcal{L}_{\Upsilon^{\mathrm{C}}} \tilde{S} = 0$ is the necessary and sufficient condition for $\tilde{S}$ to project to a vector field on $\mathcal{W}^\circ M$, say $\tilde{S}_{\mathcal{W}}$. With coordinate $w = u^0/x^0$ we have

$$u^0 \frac{\partial}{\partial x^0} \mapsto -w^2 \frac{\partial}{\partial w}, \quad x^0 \frac{\partial}{\partial u^0} \mapsto \frac{\partial}{\partial w},$$

so that

$$\tilde{S}_{\mathcal{W}} = u^a\frac{\partial}{\partial x^a} - 2(G^a + wu^a)\frac{\partial}{\partial u^a} - (w^2 + 2G^0)\frac{\partial}{\partial w},$$

with G^a, G^0 as above. Now $\rho : \mathcal{W}^\circ M \to T^\circ M$ is a line bundle. It admits global sections. A section σ is homogeneous if $\sigma_*(\Delta) = \Delta_{\mathcal{W}} \circ \sigma$, where Δ is the Liouville field of $T^\circ M$ and $\Delta_{\mathcal{W}}$ is that of the vector bundle $\tau : \mathcal{W}M \to M$ restricted to $\mathcal{W}^\circ M$. For any homogeneous section σ, $\rho_*(\tilde{S}_{\mathcal{W}}|_\sigma)$ is a spray on $T^\circ M$, given locally by

$$u^a\frac{\partial}{\partial x^a} - 2(G^a + \sigma u^a)\frac{\partial}{\partial u^a}.$$

The difference between two homogeneous sections is a homogeneous function on $T^\circ M$, so the corresponding sprays are projectively equivalent. The fundamental invariant of this equivalence class of sprays on $T^\circ M$ is just

$$\Pi^a_{bc} = \frac{\partial^2 G^a}{\partial u^b \partial u^c},$$

whence $G^a = \frac{1}{2}\Pi^a_{bc}u^b u^c$ by homogeneity. Furthermore,

$$G^0 = -\frac{1}{2(m-1)}\left(2\frac{\partial G^a}{\partial x^a} - G^a_b G^b_a\right) = -\frac{1}{2(m-1)}\mathfrak{R}_{cd}u^c u^d.$$

Suppose given a projective equivalence class of sprays on $T^\circ M$; then over each coordinate patch U on M there is a unique spray $\tilde{S}_U$ on $(T^\circ(\mathcal{V}M))|_U$ which satisfies the four conditions given earlier and generates the class by the construction just given. Since the conditions which determine $\tilde{S}_U$ are coordinate independent, and determine it uniquely, the $\tilde{S}_U$ agree on overlaps of coordinate patches, and therefore fit together to give a global spray. This is the BTW-spray of the projective equivalence class.

The Berwald connection coefficients of the BTW-spray of a projective equivalence class are

$$\tilde{\Gamma}^0_{0\alpha} = \tilde{\Gamma}^0_{\alpha 0} = 0, \quad \tilde{\Gamma}^0_{ab} = -\frac{1}{m-1}x^0\mathfrak{R}_{ab}, \quad \tilde{\Gamma}^a_{00} = 0, \quad \tilde{\Gamma}^a_{0b} = \tilde{\Gamma}^a_{b0} = (x^0)^{-1}\delta^a_b, \quad \tilde{\Gamma}^a_{bc} = \Pi^a_{bc};$$

these are of course the connection coefficients of the BTW-connection.

The BTW-spray of a reversible spray is itself reversible.

4 A Cartan connection for a system of second-order ordinary differential equations

We show how to construct a normal Cartan connection associated with a system of second-order ordinary differential equations, using Cartan's method from the second part of his projective connections paper [1], in the framework set out by Sharpe [11].

4.1 Model geometry and choice of gauge

We take as model geometry $\mathrm{P}T(\mathrm{P}^m)$, the projective tangent bundle of m-dimensional real projective space P^m. Each point of $\mathrm{P}T(\mathrm{P}^m)$ consists of a line through the origin in $\mathbf{R}^{m+1}$ and a 2-plane containing the line. The group $\mathrm{PGL}(m+1)$ acts transitively on $\mathrm{P}T(\mathrm{P}^m)$. The stabilizer of the point consisting of the first coordinate axis and the 2-plane containing the first two coordinate axes is the subgroup K_{m+1} of $\mathrm{PGL}(m+1)$ which is the image of the subgroup of $\mathrm{GL}(m+1)$ consisting of matrices with zeros below the main diagonal in the first and second columns; so we can identify $\mathrm{P}T(\mathrm{P}^m)$ with $\mathrm{PGL}(m+1)/\mathrm{K}_{m+1}$. Note the difference between this and the affine case, where the model geometry is $\mathrm{P}^m = \mathrm{PGL}(m+1)/\mathrm{H}_{m+1}$, H_{m+1} being the projective image of the subgroup of $\mathrm{GL}(m+1)$ consisting of matrices with zeros below the diagonal in the first column only. Of course K_{m+1} is a subgroup of H_{m+1}.

We consider a Cartan geometry on the projective tangent bundle $\mathrm{P}TM$ of an m-dimensional manifold M, modelled on $\mathrm{P}T(\mathrm{P}^m) = \mathrm{PGL}(m+1)/\mathrm{K}_{m+1}$, in which the projective tangent bundle structures are compatible in the following sense. First, note that any curve in M has a natural lift to $\mathrm{P}TM$ obtained by adjoining, to each point on it, its tangent line at that point. The compatibility conditions are that the development into $\mathrm{P}T(\mathrm{P}^m)$ of a vertical curve in $\mathrm{P}TM$ is vertical, and the development into $\mathrm{P}T(\mathrm{P}^m)$ of a lifted curve in $\mathrm{P}TM$ is a lifted curve.

We can introduce local coordinates on $\mathrm{P}TM$ by taking local coordinates (x^a) on M and by noting that every equivalence class of tangent vectors

$$u^a \frac{\partial}{\partial x^a}$$

for which $u^1 \neq 0$ has a unique representative of the form

$$\frac{\partial}{\partial x^1} + y^i \frac{\partial}{\partial x^i};$$

then (x^a, y^i) are local coordinates on $\mathrm{P}TM$. We are effectively using affine (jet-bundle-like) coordinates on $\mathrm{P}TM$, in which we identify an open subset of the fibre of $\mathrm{P}TM$ with an affine submanifold (a hyperplane) of the corresponding fibre of TM, by $(y^i) \mapsto (1, y^i)$.

Before proceeding, in order to examine gauge transformations it will be useful to calculate the effect of conjugation by an element of K_{m+1}, particularly on the entries below the main diagonal in the first and second columns in an arbitrary $(m+1)\times(m+1)$ matrix. Let us write an element $k \in \mathrm{K}_{m+1}$ as a matrix

$$k = \begin{pmatrix} k_0^0 & k_1^0 & k_j^0 \\ 0 & k_1^1 & k_j^1 \\ 0 & 0 & k_j^i \end{pmatrix},$$

where the $(m-1)\times(m-1)$ matrix (k^i_j) is non-singular and $|k^0_0 k^1_1 \det(k^i_j)| = 1$. Then

$$k^{-1} = \begin{pmatrix} \bar{k}^0_0 & -\bar{k}^0_0 k^0_1 \bar{k}^1_1 & g_j \\ 0 & \bar{k}^1_1 & -\bar{k}^1_1 k^1_k \bar{k}^k_j \\ 0 & 0 & \bar{k}^i_j \end{pmatrix},$$

where the overbar indicates an inverse, and $g_j = \bar{k}^0_0(k^0_1\bar{k}^1_1 k^1_k - k^0_k)\bar{k}^k_j$. One finds that

$$\text{if} \quad M = \begin{pmatrix} * & * & * \\ u & * & * \\ v^i & w^i & * \end{pmatrix} \quad \text{then} \quad k^{-1}Mk = \begin{pmatrix} * & * & * \\ \hat{u} & * & * \\ \hat{v}^i & \hat{w}^i & * \end{pmatrix}$$

where

$$\hat{u} = k^0_0 \bar{k}^1_1(u - k^1_i \bar{k}^i_j v^j), \quad \hat{v}^i = k^0_0 \bar{k}^i_j v^j, \quad \hat{w}^i = \bar{k}^i_j(k^0_1 v^j + k^1_1 w^j).$$

Note that $(\hat{v}^i)$ depends only on (v^i) and that $(\hat{u}, \hat{v}^i)$ depends only on (u, v^i).

A curve in $\mathrm{P}TM$ is vertical if its tangent vector is annihilated by the dx^a, and a curve in $\mathrm{P}TM$ is a natural lift if its tangent vector is annihilated by the so-called contact forms $dx^i - y^i dx^1$. It is easy to see that

$$(\xi^a, \eta^i) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \xi^1 & 1 & 0 \\ \xi^i & \eta^i & \delta^i_j \end{pmatrix}$$

is a local section of $\mathrm{PGL}(m+1) \to \mathrm{P}T(\mathrm{P}^m)$, and that the corresponding gauged Maurer-Cartan form is

$$\begin{pmatrix} 0 & 0 & 0 \\ d\xi^1 & 0 & 0 \\ d\xi^i - \eta^i d\xi^1 & d\eta^i & 0 \end{pmatrix}.$$

We take some gauge on $\mathrm{P}TM$ and write the connection form in the chosen gauge as

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_1 & \omega^0_j \\ \omega^1_0 & \omega^1_1 & \omega^1_j \\ \omega^i_0 & \omega^i_1 & \omega^i_j \end{pmatrix};$$

note that the component forms are forms on $\mathrm{P}TM$. It follows from the conditions for a Cartan connection that the forms ω^1_0, ω^i_0, ω^i_1 must be linearly independent. The equations for the development of a curve σ in $\mathrm{P}TM$ into $\mathrm{P}T(\mathrm{P}^m)$ give

$$a\dot{\xi}^1 - b_i(\dot{\xi}^i - \eta^i\dot{\xi}^1) = \langle\dot{\sigma}, \omega^1_0\rangle, \quad c^i_j(\dot{\xi}^j - \eta^j\dot{\xi}^1) = \langle\dot{\sigma}, \omega^i_0\rangle$$

for some functions $a(t)$, $b_i(t)$, $c^i_j(t)$. The compatibility conditions therefore require that if σ is vertical $\langle\dot{\sigma}, \omega^1_0\rangle = \langle\dot{\sigma}, \omega^i_0\rangle = 0$, while if σ is a lift $\langle\dot{\sigma}, \omega^i_0\rangle = 0$. It follows that ω^1_0 is a linear combination of the dx^a and each ω^i_0 is a linear combination of the contact forms $dx^j - y^j dx^1$, which we will denote by θ^j. We can set $\omega^1_0 = \alpha_1 dx^1 + \alpha_i \theta^i$ and $\omega^i_0 = \beta^i_j \theta^j$

for some functions α_a and β^i_j on PTM, where $\alpha_1 \neq 0$ and (β^i_j) is non-singular since the ω^a_0 (in particular) must be linearly independent. Then a change of gauge with

$$k^1_1 = k^0_0\alpha_1, \quad k^1_i = k^0_0\alpha_i, \quad k^i_j = k^0_0\beta^i_j,$$

and for m even

$$k^0_0 = (\alpha_1 \det(\beta^i_j))^{-1/(m+1)},$$

while for m odd

$$k^0_0 = \pm|\alpha_1 \det(\beta^i_j)|^{-1/(m+1)},$$

makes $\omega^1_0 = dx^1$ and $\omega^i_0 = \theta^i$.

This does not fix the gauge: the remaining freedom is

$$k = \begin{pmatrix} 1 & k^0_1 & k^0_j \\ 0 & 1 & 0 \\ 0 & 0 & \delta^i_j \end{pmatrix}, \quad \text{with} \quad k^{-1} = \begin{pmatrix} 1 & -k^0_1 & -k^0_j \\ 0 & 1 & 0 \\ 0 & 0 & \delta^i_j \end{pmatrix}.$$

A gauge transformation with such k takes ω^0_0 to $\omega^0_0 - k^0_1 dx^1 - k^0_j\theta^j$. We may therefore choose k^0_a such that ω^0_0 is independent of dx^a, that is, such that $\omega^0_0 = \kappa_i dy^i$ for some functions κ_i. This finally fixes the gauge; we call the resulting gauge the standard gauge for the given coordinates.

Our standard gauge is not the same as Cartan's in the case $m = 2$. In the last step he uses k^0_1 to simplify ω^2_1, and then eliminates θ from $\omega^0_0 - \omega^1_1$; but the first of these moves does not work for $m > 2$, and the choice of gauge made above seems to be preferable because it leads to a coordinate transformation rule between connections in standard gauge which depends only on the general geometry of the spaces concerned, not on the details of the connection forms.

A geodesic of this connection is a curve whose development satisfies $\dot{\xi}^i - \eta^i\dot{\xi}^1 = 0$ and $\dot{\eta}^i = 0$; that is, a geodesic is a curve whose tangents are annihilated by both θ^i and ω^i_1. Now we can write $\omega^i_1 = A^i_j dy^j + B^i dx^1 \bmod(\theta^k)$, where by linear independence (A^i_j) is non singular; so we may equivalently write $\omega^i_1 = A^i_j(dy^j - f^j dx^1) \bmod(\theta^k)$ for some functions $f^i(x^a, y^j)$. Then a geodesic is a solution of the system of $m - 1$ second-order differential equations

$$\frac{d^2x^i}{d(x^1)^2} = f^i\left(x^a, \frac{dx^j}{dx^1}\right).$$

To put it another way, the geodesics are the integral curves of the vector field

$$\frac{d}{dx^1} = \frac{\partial}{\partial x^1} + y^i\frac{\partial}{\partial x^i} + f^i\frac{\partial}{\partial y^i};$$

we call this the second-order differential equation field corresponding to the Cartan connection.

4.2 The normal Cartan connection

We now show that we can fix the connection uniquely by imposing conditions on the curvature; the connection obtained in this way is called the normal connection.

The process of choosing a normal connection is carried out in a coordinate system with the connection in standard gauge; but we want the normal connection to be a global connection, and the conditions imposed on the curvature form when specifying it must respect this. Suppose that ω, $\hat{\omega}$ are the normal connection forms with respect to coordinates (x^a, y^i), $(\hat{x}^a, \hat{y}^i)$ (where the transformation of the y^i is induced from that of the x^a); thus $\hat{\omega}$ is in standard form with respect to the coordinates $\hat{x}^a$. Then there is a unique gauge transformation which puts $\hat{\omega}$ into standard form with respect to the coordinates (x^a), and the gauge transform of $\hat{\omega}$ must be normal, that is, $\hat{\omega} = k^{-1}\omega k + k^{-1}dk$ where k is determined by the transformation rules for dx^1, θ^i and dy^i under a coordinate transformation. Then $\hat{\Omega} = k^{-1}\Omega k$. It will therefore be possible to work out the transformation rules for the components of Ω; and any condition on the components of Ω imposed in the course of making ω normal must hold for the corresponding components of $\hat{\Omega}$.

The following conditions on the curvature satisfy this requirement:

- $\Omega^\alpha_0 = 0$;
- Ω^0_1 is semi-basic;
- Ω^i_1 is semi-basic;
- if we set $\Omega^i_j = K^i_{jkl} dy^k \wedge \theta^l \mod(dx^a \wedge dx^b)$ then $K^k_{kij} = 0$ and $K^k_{ijk} = 0$;
- if we set $\Omega^i_1 = L^i_j dx^1 \wedge \theta^j \mod(\theta^k \wedge \theta^l)$ then $L^k_k = 0$.

(Recall that the Ω^α_β are 2-forms on PTM; semi-basic here means semi-basic with respect to the projection P$TM \to M$.) These conditions uniquely determine the coefficients of the connection in standard gauge to be

$$\begin{aligned}
\omega^0_0 &= 0, \quad \omega^1_0 = dx^1, \quad \omega^i_0 = \theta^i \\
\omega^1_1 &= -\frac{2}{m+1}\gamma dx^1 - \frac{1}{m+1}\gamma_i\theta^i \\
\omega^1_j &= -\frac{1}{m+1}\gamma_j dx^1 - \frac{1}{m+1}\gamma_{jk}\theta^k \\
\omega^i_1 &= dy^i - f^i dx^1 + \left(\gamma^i_j - \frac{1}{m+1}\delta^i_j\gamma\right)\theta^j \\
\omega^i_j &= \left(\gamma^i_j - \frac{1}{m+1}\delta^i_j\gamma\right)dx^1 + \left(\gamma^i_{jk} - \frac{1}{m+1}(\gamma_j\delta^i_k + \gamma_k\delta^i_j)\right)\theta^k \\
\omega^0_1 &= \varrho dx^1 + \varrho_i\theta^i, \quad \omega^0_i = \varrho_i dx^1 + \varrho_{ij}\theta^j
\end{aligned}$$

where

$$\varrho = \frac{1}{m-1}\Phi - \frac{1}{m+1}\frac{d}{dx^1}(\gamma) - \frac{1}{(m+1)^2}\gamma^2$$

and

$$\varrho_j = \tfrac{1}{2}\frac{\partial \varrho}{\partial y^j}, \quad \varrho_{jk} = \frac{\partial \varrho_k}{\partial y^j} = \tfrac{1}{2}\frac{\partial^2 \varrho}{\partial y^j \partial y^k},$$

with

$$\Phi = \Phi^i_i, \quad \Phi^i_j = \frac{\partial f^i}{\partial x^j} + \frac{d}{dx^1}(\gamma^i_j) + \gamma^i_k \gamma^k_j.$$

The components K^i_{jkl} and L^i_j of the curvature which are used to fix the connection are determined by the stated conditions and are given by

$$K^i_{jkl} = \gamma^i_{jkl} - \frac{1}{m+1}\left(\delta^i_j \gamma_{kl} + \delta^i_k \gamma_{jl} + \delta^i_l \gamma_{jk}\right), \quad L^i_j = \Phi^i_j - \frac{1}{(m-1)}\delta^i_j \Phi^k_k;$$

they are the quantities given earlier.

Since in general $\Omega^i_1 \neq 0$, the normal connection does not necessarily have zero torsion.

The calculations that lead to these results (including the demonstration that the conditions on the curvature are gauge invariant) are given in an appendix.

The ω^a_b can be expressed in terms of the fundamental invariants of the corresponding projective equivalence class of sprays, using the formulæ found earlier. These can be solved for the expressions occuring in the connection coefficients: one finds that

$$\begin{aligned}
-\frac{2}{m+1}\gamma &= \Pi^1_{11} + 2y^k \Pi^1_{1k} + y^k y^l \Pi^1_{kl} \\
-\frac{1}{m+1}\gamma_i &= \Pi^1_{i1} + y^k \Pi^1_{ik} \\
\gamma^i_j - \frac{1}{m+1}\delta^i_j \gamma &= \Pi^i_{1j} + y^k \Pi^i_{jk} - y^i \Pi^1_{j1} - y^i y^k \Pi^1_{jk} \\
\gamma^i_{jk} - \frac{1}{m+1}(\gamma_j \delta^i_k + \gamma_k \delta^i_j) &= \Pi^i_{jk} + y^i \Pi^1_{jk} \\
-f^i &= \Pi^i_{11} + 2y^k \Pi^i_{1k} + y^k y^l \Pi^i_{kl}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\omega^1_1 &= (\Pi^1_{1a} + y^k \Pi^1_{ka})dx^a \\
\omega^1_j &= \Pi^1_{ja} dx^a \\
\omega^i_1 &= dy^i + (\Pi^i_{1a} - y^i \Pi^1_{1a} + y^k \Pi^i_{ka} - y^i y^k \Pi^1_{ka})dx^a \\
\omega^i_j &= (\Pi^i_{ja} - y^i \Pi^1_{ja})dx^a.
\end{aligned}$$

We now consider ω^0_1 and ω^0_j. Using results obtained earlier we have

$$\begin{aligned}
\varrho &= \frac{1}{m-1}\Phi - \frac{1}{m+1}\frac{d}{dx^1}(\gamma) - \frac{1}{(m+1)^2}\gamma^2 \\
&= -(u^1)^{-2}\left(\frac{1}{m-1}R + S(\tilde{\Gamma}) + \tilde{\Gamma}^2\right) \\
&= -(u^1)^{-2}\frac{1}{m-1}\mathfrak{R}.
\end{aligned}$$

As a function on $T^\circ M$, $\mathfrak{R}$ is homogeneous of degree 2, so ϱ is homogeneous of degree 0; moreover on $u^1 = 1$, bearing in mind the symmetry of $\mathfrak{R}_{ab}$, we have

$$\varrho = -\frac{1}{m-1}(\mathfrak{R}_{11} + 2\mathfrak{R}_{1i}y^i + \mathfrak{R}_{ij}y^i y^j).$$

Now $\mathfrak{R}^d_{cab}$, while not a tensor, still satisfies the properties of the Riemann curvature in the coordinate patch in which it is defined. In particular, by a result obtained earlier we have

$$u^b \frac{\partial \mathfrak{R}_{bc}}{\partial u^a} = 0,$$

whence on $u^1 = 1$

$$\frac{\partial \mathfrak{R}_{1c}}{\partial y^i} + y^j \frac{\partial \mathfrak{R}_{jc}}{\partial y^i} = 0.$$

It follows that

$$\varrho_i = \tfrac{1}{2}\frac{\partial \varrho}{\partial y^i} = -\frac{1}{m-1}(\mathfrak{R}_{1i} + y^j \mathfrak{R}_{ji}), \quad \varrho_{ij} = \tfrac{1}{2}\frac{\partial^2 \varrho}{\partial y^i \partial y^j} = -\frac{1}{m-1}\mathfrak{R}_{ij},$$

whence

$$\omega^0_1 = -\frac{1}{m-1}(\mathfrak{R}_{1a} + y^i \mathfrak{R}_{ia})dx^a, \quad \omega^0_i = -\frac{1}{m-1}\mathfrak{R}_{ia}dx^a.$$

So the connection form of the normal Cartan connection in standard gauge is

$$\begin{pmatrix} 0 & -\dfrac{1}{m-1}(\mathfrak{R}_{1a} + y^i \mathfrak{R}_{ia})dx^a & -\dfrac{1}{m-1}\mathfrak{R}_{ia}dx^a \\ dx^1 & (\Pi^1_{1a} + y^k \Pi^1_{ka})dx^a & \Pi^1_{ja}dx^a \\ dx^i - y^i dx^1 & dy^i + (\Pi^i_{1a} - y^i \Pi^1_{1a} + y^k \Pi^i_{ka} - y^i y^k \Pi^1_{ka})dx^a & (\Pi^i_{ja} - y^i \Pi^1_{ja})dx^a \end{pmatrix}.$$

This expression may be simplified further, though at the expense of carrying out what in the context of the Cartan theory is an illegitimate gauge transformation. Even so, what we are about to do is perfectly acceptable if regarded as a purely computational device at this stage; we will explain the geometrical significance of the step later.

We denote by Y the locally defined $(m+1) \times (m+1)$-matrix-valued function

$$Y = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -y^2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -y^m & 0 & \cdots & 1 \end{pmatrix},$$

and define a new matrix-valued 1-form $\tilde{\omega}$ by $\tilde{\omega} = Y^{-1}\omega Y + Y^{-1}dY$, where ω is the normal connection form given above. Now

$$Y^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & y^2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & y^m & 0 & \cdots & 1 \end{pmatrix},$$

whence

$$Y^{-1}dY = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & -dy^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -dy^m & 0 & \cdots & 0 \end{pmatrix};$$

and we find that

$$\tilde{\omega} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\mathfrak{R}_{bc}dx^c \\ & \\ dx^a & \Pi^a_{bc}dx^c \end{pmatrix}.$$

Thus $\tilde{\omega}$ is formally identical to the normal Cartan connection form in the affine case, though of course it is defined on PTM, not M. However, it is semi-basic over M, and the coefficients Π^a_{bc} and $\mathfrak{R}_{bc}$ are functions on $T^\circ M$, homogeneous of degree 0. We may therefore think of $\tilde{\omega}$ as the restriction to $u^1 = 1$ of a (locally defined) form on $T^\circ M$. The 'curvature' of $\tilde{\omega}$ is the restriction to $u^1 = 1$ of

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\left(\mathfrak{R}_{b[c|d]}dx^c \wedge dx^d + \dfrac{\partial \mathfrak{R}_{bd}}{\partial u^c}\varphi^c \wedge dx^d\right) \\ & \\ 0 & \frac{1}{2}P^a_{bcd}dx^c \wedge dx^d + D^a_{bcd}\varphi^c \wedge dx^d \end{pmatrix}.$$

We may therefore express the normal Cartan connection form in standard gauge as $Y\tilde{\omega}Y^{-1} - Y^{-1}dY$, and its curvature as $Y\tilde{\Omega}Y^{-1}$, with $\tilde{\omega}$ and $\tilde{\Omega}$ given above.

As we noted above, the normal connection is not necessarily torsion-free. For it to be so we must have $\Omega^i_1 = 0$. A necessary condition for this to hold is that $L^i_j = 0$. We show that it is also sufficient. We know that if $L^i_j = 0$ then $P^a_{bcd} = 0$. Now Ω^i_1 depends only on the components $\tilde{\Omega}^a_b$, and does so linearly; moreover Ω^i_1 is semi-basic. It follows that if $P^a_{bcd} = 0$ then $\Omega^i_1 = 0$. Thus $\Omega^i_1 = 0$ if and only if $L^i_j = 0$, and the normal connection is torsion-free if and only if the sprays of the corresponding projective equivalence class are isotropic.

4.3 Coordinate transformations

Consider a coordinate transformation on M, from (x^a) to $(\hat{x}^a)$, with the induced transformation of the y^i. Let (J^a_b) be the Jacobian matrix of the transformation, J its determinant.

With each coordinate system we have a unique gauged normal connection form, which we can express in standard form with respect to the appropriate coordinates. By hypothesis these two local connection forms are gauged versions of the same global connection form, and are therefore related by a gauge transformation. Let ω, $\hat{\omega}$ be the two gauged normal connection forms, and set $\omega^\alpha_\beta = \bar{k}^\alpha_\gamma \hat{\omega}^\gamma_\delta k^\delta_\beta + \bar{k}^\alpha_\gamma dk^\gamma_\beta$. Then the gauge transformation matrix (k^α_β) is the transition function for the principal bundle on which the global normal connection form lives, with respect to the open sets on which the coordinates are defined.

We can determine the gauge transformation simply by using the same trick as in the previous subsection. Let us write $\hat{Y}$ for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -\hat{y}^2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\hat{y}^m & 0 & \cdots & 1 \end{pmatrix}.$$

Then

$$\hat{Y}^{-1}\hat{\omega}\hat{Y} + \hat{Y}^{-1}d\hat{Y} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\hat{\mathfrak{R}}_{bc}d\hat{x}^c \\ \\ d\hat{x}^a & \hat{\Pi}^a_{bc}dx^c \end{pmatrix} = \tilde{\hat{\omega}}.$$

But just as in the affine case

$$\tilde{\omega} = g^{-1}\tilde{\hat{\omega}}g + g^{-1}dg$$

where g is the projection into $\mathrm{PGL}(m+1)$ of the matrix-valued function

$$G = \begin{pmatrix} 1 & -\dfrac{1}{m+1}\dfrac{\partial \log |J|}{\partial x^b} \\ \\ 0 & J^a_b \end{pmatrix}.$$

It follows that $\omega = k^{-1}\hat{\omega}k + k^{-1}dk$ where $k = \hat{Y}gY^{-1}$. But Y and $\hat{Y}$ are both unimodular, so we can express k as the projection into $\mathrm{PGL}(m+1)$ of $\hat{Y}GY^{-1}$. It is self-evident that this defines an element of H_{m+1}; it is easy to check, though it is not self-evident, that in fact it defines an element of K_{m+1}.

It would be possible to determine k explicitly from this expression, using the fact that

$$\hat{y}^i = \frac{J^i_1 + J^i_j y^j}{J^1_1 + J^1_j y^j},$$

but fortunately we do not need to do so. An alternative expression for k is given in the appendix.

4.4 What happens when the spray is projectively affine

If the second-order differential equation field is projectively equivalent to an affine spray, the connection ω should reduce to a connection on M gauge equivalent to the normal projective connection associated with the affine spray. But what could one mean by 'reduce'? We can obtain a connection on M by pulling ω back by any local section of $\mathrm{P}TM \to M$. Of course, different sections will give different reduced connections; the requirement is that the different reduced connections should all be gauge equivalent, with the gauge transformation being taken from the gauge group appropriate to the affine case, namely H_{m+1}; we call such a gauge transformation a gauge transformation of the first kind. This will be the case if, for every transformation ψ of $\mathrm{P}TM$ fibred over the identity, $\psi^*\omega$ is a gauge transform of ω by a gauge transformation of the first kind. An equivalent condition is that for any vector field V vertical with respect to the projection $\mathrm{P}TM \to M$, $\mathcal{L}_V\omega$ should be infinitesimally gauge equivalent to ω by a gauge transformation of the first kind. That is to say, there should be a function H taking its values in $\mathfrak{h}_{m+1}$, the Lie algebra of H_{m+1}, such that $\mathcal{L}_V\omega = [\omega, H] + dH$ (the equation obtained by differentiating the gauge transformation equation at the identity). Now

$$\mathcal{L}_V\omega = V \lrcorner\, d\omega + d\langle V, \omega\rangle = V \lrcorner\, \Omega + [\omega, \langle V, \omega\rangle] + d\langle V, \omega\rangle,$$

and $\langle V, \omega\rangle$ takes its values in $\mathfrak{h}_{m+1}$. So if $V \lrcorner\, \Omega = 0$ then ω satisfies the requisite condition with $H = \langle V, \omega\rangle$.

Now Y from the previous subsection does define a gauge transformation of the first kind, and so the argument above applies equally as well to $\tilde{\omega}$ as to ω. Thus the condition for the connection to reduce to M can equivalently be expressed as $V \lrcorner\, \tilde{\Omega} = 0$. On the face of it, this amounts to two conditions, namely $D^a_{bcd} = 0$ and $\partial\mathfrak{R}_{bd}/\partial u^c = 0$. The first of these, the vanishing of the Douglas tensor, is just the necessary and sufficient condition for the second-order differential equation field to be projectively equivalent to an affine spray. But then Π^a_{bc} is independent of u^d, and therefore $\mathfrak{R}_{bd}$ is independent of u^c.

We have shown that a necessary and sufficient condition for the second-order differential equation field associated with the normal Cartan connection to be projectively equivalent to an affine spray is that the curvature satisfies $V \lrcorner\, \Omega =$ for all vector fields V on $\mathrm{P}TM$ vertical over M; and that this latter condition is necessary and sufficient for the connection to be reducible to a projective connection of affine type. When the condition $V \lrcorner\, \Omega = 0$ holds we may choose any local section of $\mathrm{P}TM$ over M to obtain the reduced connection form. The obvious choice is $y^i = 0$, and with this choice the reduced connection is the normal Cartan connection associated with the affine spray in standard form.

5 The Cartan bundle

In our investigation of projective equivalence classes of symmetric affine connections [4] we demonstrate how any such class of affine connections gives rise to a normal Cartan projective connection on the Cartan bundle $\mathcal{C}M$ having the same geodesics. In the present paper we have seen how to construct a Cartan connection corresponding to more general families of paths — those generated by an arbitrary reversible spray on $T^\circ M$ rather than by the geodesic spray of an affine connection — and so it is natural to look for a similar geometric interpretation, involving suitable bundles, in this more general situation.

We have expressed the normal Cartan connection in gauged form, but with the implicit understanding that there is a global Cartan connection on a principal K_{m+1}-bundle over $\mathrm{P}TM$ of which the gauged connection is a local representative; and in effect we defined this bundle when we gave its transition functions corresponding to coordinate transformations. In the affine case we carried out a similar procedure to obtain the gauged normal projective connection before giving an explicit description of the principal bundle which is the home of the global connection. This is the Cartan bundle $\mathcal{C}M$, which is a principal H_{m+1}-bundle over M. We wish now to describe explicitly the carrier of the global connection in the general case. As a space it is in fact $\mathcal{C}M$ again — but of course $\mathcal{C}M$ with a different bundle structure. As we now show, $\mathcal{C}M$ is a principal K_{m+1}-bundle over $\mathrm{P}TM$; and by computing the transition functions for it in this new guise we will show that it is the bundle found implicitly through the normal connection calculations.

The Cartan bundle $\mathcal{C}M$ consists of the Cartan simplices of the Cartan algebroid $\mathcal{W}M$, that is, the simplices with first element a multiple of the global vector section e_0 of $\tau : \mathcal{W}M \to M$. Let $[\zeta_\alpha]$ be a Cartan simplex of $\mathcal{W}M$ at some point $x \in M$, so that ζ_0 is a multiple of $(e_0)_x$. Now ζ_1 is an element of $\mathcal{W}_x M$ independent of $(e_0)_x$, and therefore determines a non-zero element of $T_x M$ under $\rho : \mathcal{W}M \to TM$, the anchor map of the Cartan algebroid. By projectivizing we obtain a unique element of $\mathrm{P}T_x M$ corresponding to the simplex element $[\zeta_1]$. Let $\varsigma : \mathcal{C}M \to \mathrm{P}TM$ be the map so defined. We show that ς is the projection map of a principal K_{m+1}-bundle structure on $\mathcal{C}M$. We define a right action of K_{m+1} on $\mathcal{C}M$ as follows. First we consider a transformation of frames (ζ_α) of the form $(\zeta_\alpha) \mapsto (\hat{\zeta}_\alpha)$ where

$$\hat{\zeta}_0 = k_0^0 \zeta_0, \quad \hat{\zeta}_1 = k_1^1 \zeta_1 + k_1^0 \zeta_0, \quad \hat{\zeta}_i = k_i^\alpha \zeta_\alpha$$

where $k_0^0 k_1^1 \det(k_j^i) \neq 0$; we then projectivize. The corresponding transformation of simplices defines an element of K_{m+1}, and $[\hat{\zeta}_\alpha]$ is a Cartan simplex if $[\zeta_\alpha]$ is. We obtain in this way an action of K_{m+1} which is clearly an effective right action. Now $\rho(e_0)$ is the zero section of TM, so the orbit of a point of $\mathcal{C}M$ under the action of K_{m+1} is just a fibre of the projection $\varsigma : \mathcal{C}M \to \mathrm{P}TM$.

We can define local sections of ς as follows. Given coordinates (x^a) on M, and adapted coordinates (x^α) on $\mathcal{V}M$, we obtain the global section e_0 and local sections e_a of $\mathcal{W}M \to M$ as the images of Υ and ∂_a respectively. In a coordinate patch on TM where $u^1 \neq 0$, so that we can use coordinates (x^a, y^i) on PTM, we set

$$\zeta_0 = e_0, \quad \zeta_1 = e_1 + y^i e_i, \quad \zeta_i = e_i;$$

then $[\zeta_\alpha]$ is a local section of ς. Thus $\varsigma : \mathcal{C}M \to \mathrm{P}TM$ is a principal K_{m+1}-bundle. We call $\mathcal{C}M$ with this bundle structure the projective Cartan bundle.

Notice that $\zeta_\alpha = \bar{Y}^\beta_\alpha e_\beta$. We showed in [4] that the transition function for the Cartan bundle $\mathcal{C}M$ relative to local trivializations (over M) of the form $[e_\alpha]$ is given by projectivizing the GL$(m+1)$-valued function G where

$$G = \begin{pmatrix} 1 & -\dfrac{1}{m+1}\dfrac{\partial \log|J|}{\partial x^b} \\ \\ 0 & J^a_b \end{pmatrix}.$$

Thus the transition function for $\mathcal{C}M$ relative to local trivializations (over PTM) of the form $[\zeta_\alpha]$ given above is the projection into PGL$(m+1)$ of $\hat{Y}GY^{-1}$, which is exactly the transition function obtained earlier by consideration of the normal Cartan connection. Thus the global normal Cartan connection of any projective equivalence class of sprays is a connection on the projective Cartan bundle.

We will now give an interpretion of this construction in the light of Cartan's approach.

As we have pointed out, Cartan's study of projective connections [1] covers both the affine and the general cases, although he describes the latter explicitly only when $m = 2$. In the affine case, he envisages a projective space attached to each point of the manifold. Our interpretation of this is that, for each point $x \in M$, we should study the m-dimensional projective space $\mathrm{P}\mathcal{W}_x M$; this space has a distinguished point $[(e_0)_x]$, the point at which the space is 'attached' to M. The geodesics of the connection are the curves in M whose developments into these projective spaces are straight lines.

In the more general case, we can no longer describe the developments of curves into a single projective space at each point. Instead, we have to use a family of projective spaces at each point, with the family parametrized by the set of rays (1-dimensional subspaces of the tangent space) at that point: the projective spaces will therefore need, not just a distinguished point, but also a distinguished ray through that point. This is consistent with Cartan's view in [1], where he takes as a base manifold not M itself, but instead the 'manifold of elements', where an 'element' is a ray at a point.

This suggests that we should consider the pull-back bundle $\tau_M^{\circ *}(\tau) : \tau_M^{\circ *}(\mathcal{W}M) \to T^\circ M$. The canonical global section $e_0 : M \to \mathcal{W}M$ gives rise to a global section of this pull-back bundle which we will continue to denote by e_0. There is now, however, a distinguished 1-dimensional affine sub-bundle $\mathcal{T}_M \subset \tau_M^{\circ *}(\mathcal{W}M)$, defined by specifying that $(v, \zeta) \in \mathcal{T}_M$ whenever $\rho(\zeta) = v$: here we consider the pull-back as a fibre product $\tau_M^{\circ *}(\mathcal{W}M) = T^\circ M \times_M \mathcal{W}M$. Any section of $\mathcal{T}_M \to T^\circ M$ maps, under ρ, to the total derivative section $\mathbf{T}$ of $\tau_M^{\circ *}(TM) \to T^\circ M$, and any two such sections differ by a multiple of e_0.

We now projectivize this construction, both in the fibre and in the base, to give the pull-back bundle $\pi_M^*(\pi) : \pi_M^*(\mathrm{P}\mathcal{W}M) \to \mathrm{P}TM$ where $\pi_M : \mathrm{P}TM \to M$ and $\pi : \mathrm{P}\mathcal{W}M \to M$ are the projective tangent bundle and projective Cartan algebroid respectively. This new bundle also has a global section which we continue to denote by $[e_0]$; thus each projective fibre of $\pi_M^*(\mathrm{P}\mathcal{W}M)$ has a distinguished point, the image of this global section. But now each fibre also has a distinguished line containing that point: we define $\mathrm{P}\mathcal{T}_M \subset \pi_M^*(\mathrm{P}\mathcal{W}M)$ by specifying that $([v], [\zeta]) \in \mathrm{P}\mathcal{T}_M$ if either $[\rho(\zeta)] = [v]$, or else $\rho(\zeta) = 0$ (so that, in the latter case, $[\zeta] = [e_0]_x$, and then $([v], [\zeta])$ is the distinguished point in the fibre at $[v]$). Another way of constructing $\mathrm{P}\mathcal{T}_M$ would be to take the 2-dimensional linear hull of the affine sub-bundle $\mathcal{T}_M$, giving a projective line in each fibre of $\tau_M^{\circ *}(\mathrm{P}\mathcal{W}M) \to T^\circ M$; these lines then map consistently to lines in the fibres of $\pi_M^*(\mathrm{P}\mathcal{W}M) \to \mathrm{P}TM$.

We construct the pull-back bundle $\pi_M^*(\mathcal{S}_\mathcal{W}M) \to \mathrm{P}TM$ in the same way: this is, of course, a principal $\mathrm{PGL}(m+1)$-bundle. Then the projective Cartan bundle $\varsigma : \mathcal{C}M \to \mathrm{P}TM$ is the sub-bundle of $\pi_M^*(\mathcal{S}_\mathcal{W}M) \to \mathrm{P}TM$ containing pairs $([v], [\zeta_\alpha])$ where $[\zeta_0]$ is the distinguished point and $[\zeta_1]$ is some other element of the distinguished line, so that $[\zeta_0] = [e_0]_x$ and $[\rho(\zeta_1)] = [v]$ (here, of course, ζ_0 and ζ_1 must be linearly independent, so the case $\rho(\zeta_1) = 0$ does not arise).

6 The projective connections

In this section we will show how to construct the normal Cartan connection form of a projective equivalence class of sprays from the BTW-connection, at the global level.

Our plan is to use the same general approach as in the affine case [4]. There we start with a TW-connection on the volume bundle $\mathcal{V}M$ giving rise to a $\mathfrak{gl}(m+1)$-valued Ehresmann connection form on the frame bundle $\mathcal{F}(\mathcal{V}M) \to \mathcal{V}M$, and we show how to construct from it a Cartan connection on the Cartan bundle $\mathcal{C}M \to M$, such that the geodesics of the Cartan connection are precisely the geodesics of the projective equivalence class of symmetric affine connections associated with the TW-connection. In particular, the normal TW-connection corresponds under this construction to the normal Cartan projective connection.

In the more general case we will limit our ambitions to generalizing this last step of the affine programme; that is, we will deal only with the normal connections. Now, therefore, we will start with the BTW-connection on the volume bundle, and we will show how to construct from it the normal Cartan connection as a global Cartan connection form on the projective Cartan bundle $\varsigma : \mathcal{C}M \to \mathrm{P}TM$.

6.1 Passing to the quotient

The first step of the process can be described in quite general terms.

Consider a manifold $\mathcal{N}$ with reversible spray S and corresponding Berwald connection ∇, such that there is defined on $\mathcal{N}$ a nowhere-vanishing complete vector field X such that

- $\mathcal{N}$ is fibred over an m-dimensional manifold $\mathcal{M}$ where the fibres are the integral curves of X;
- the complete lift X^{C} of X to $T^\circ\mathcal{N}$ satisfies $\mathcal{L}_{X^{\mathrm{C}}}S = 0$;
- the vertical lift X^{V} of X to $T^\circ\mathcal{N}$ satisfies $\mathcal{L}_{X^{\mathrm{V}}}S = X^{\mathrm{C}} - 2\Delta$.

The Lie derivative conditions are modelled on the first two conditions for a BTW-spray, of course.

Let $\xi : \mathcal{N} \to \mathcal{M}$ be the projection. Note that the vector fields X^{C} and X^{V} define an integrable distribution on $T\mathcal{N}$ whose leaves are the fibres of the projection $\xi_* : T\mathcal{N} \to T\mathcal{M}$. The inverse image of the zero section of $T\mathcal{M}$ under ξ_*, $\xi_*^{-1}(0)$, is the 1-dimensional vector sub-bundle of $T\mathcal{N}$ spanned by X (considered as a section of $\tau_{\mathcal{N}} : T\mathcal{N} \to \mathcal{N}$). Denote by $T^X\mathcal{N}$ the complement of $\xi_*^{-1}(0)$ in $T\mathcal{N}$; it is an open submanifold of $T\mathcal{N}$, fibred over $\mathcal{N}$, contained in $T^\circ\mathcal{N}$. We denote by $\tau_{\mathcal{N}}^X : T^X\mathcal{N} \to \mathcal{N}$ the restriction of τ_N to $T^X\mathcal{N}$.

We denote by ϕ_t the 1-parameter group on $\mathcal{N}$ whose infinitesimal generator is X.

Let $\mathcal{F}\mathcal{N}$ be the frame bundle of $\mathcal{N}$, $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$ its pullback over $T\mathcal{N}$. We define a group structure on $\mathbf{R}^2 \times \mathbf{R}_\circ \times \mathbf{R}_\circ$, where $\mathbf{R}_\circ$ is the multiplicative group of non-zero reals, by $(q, r, s, t) \cdot (q', r', s', t') = (q+q', rs'+r', ss', tt')$. This group acts on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$ to the right by

$$\psi_{(q,r,s,t)} : (x, u, \{e_\alpha\}) \mapsto (\phi_q x, \phi_{q*}(su + rX_x), \{t\phi_{q*}e_\alpha\}).$$

Note that this action is fibred over the action $\bar{\psi}$ of $\mathbf{R}^2 \times \mathbf{R}_\circ$ on $T\mathcal{N}$ given by

$$\bar{\psi}_{(q,r,s)} : (x, u) \mapsto (\phi_q x, \phi_{q*}(su + rX_x)).$$

This action leaves $T^X\mathcal{N}$ invariant, and the quotient of $T^X\mathcal{N}$ by it is $\mathrm{P}T\mathcal{M}$. Furthermore, the ψ action commutes with the right action of $\mathrm{GL}(m+1)$ on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$, and leaves $\tau_{\mathcal{N}}^{X*}(\mathcal{F}\mathcal{N})$ invariant. Let $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ be the quotient of $\tau^{X*}(\mathcal{F}\mathcal{N})$ under the ψ action; it is a principal fibre bundle over $\mathrm{P}T\mathcal{M}$ with group $\mathrm{PGL}(m+1)$, and for any $a \in \mathrm{GL}(m+1)$, $\pi^X \circ R_a = R_{o(a)} \circ \pi^X$ where $\pi^X : \tau_{\mathcal{N}}^{X*}(\mathcal{F}\mathcal{N}) \to \mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ and $o : \mathrm{GL}(m+1) \to \mathrm{PGL}(m+1)$ are the projections.

Introduce local coordinates $(x^\alpha, u^\alpha, x^\alpha_\beta)$ on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$, where for a frame $\{e_\alpha\}$, $e_\alpha = x^\beta_\alpha \partial_\beta$. The infinitesimal generator of the 1-parameter group $\psi_{(q,0,1,1)}$ on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$ is the vector field Ψ where

$$\Psi = X^\alpha \frac{\partial}{\partial x^\alpha} + u^\beta \frac{\partial X^\alpha}{\partial x^\beta} \frac{\partial}{\partial u^\alpha} + x^\gamma_\beta \frac{\partial X^\alpha}{\partial x^\gamma} \frac{\partial}{\partial x^\alpha_\beta}.$$

The generator of $\psi_{(0,r,1,1)}$ is

$$\Xi = X^\alpha \frac{\partial}{\partial u^\alpha},$$

while that of $\psi_{(0,0,e^s,1)}$ is

$$\tilde{\Delta} = u^\alpha \frac{\partial}{\partial u^\alpha};$$

Ξ is formally identical to X^{V}, and $\tilde{\Delta}$ to Δ, but both are vector fields on $\tau^*_{\mathcal{N}}(\mathcal{FN})$. The generator of $\psi_{(0,0,1,e^t)}$ is the vertical vector field on the $\mathrm{GL}(m+1)$-bundle $\tau^*_{\mathcal{N}}(\mathcal{FN})$ corresponding to the identity matrix $I \in \mathfrak{gl}(m+1)$, that is

$$I^\dagger = x^\alpha_\beta \frac{\partial}{\partial x^\alpha_\beta}.$$

The pairwise brackets of the vector fields Ψ, Ξ, $\tilde{\Delta}$ and $I^\dagger$ all vanish except that $[\Xi, \tilde{\Delta}] = \Xi$. These vector fields, when restricted to $\tau^{X*}_{\mathcal{N}}(\mathcal{FN})$, are linearly independent, and span an integrable distribution $\mathcal{D}$ there whose leaves are just the orbits of the $\psi_{(q,r,e^s,e^t)}$ action. The distribution is invariant under $\psi_{(0,0,\pm 1,\pm 1)}$. The leaves of $\mathcal{D}$, quotiented by the action of $\psi_{(0,0,\pm 1,\pm 1)}$, are the fibres of the projection $\pi^X : \tau^{*X}_{\mathcal{N}}(\mathcal{FN}) \to \mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$.

With respect to the Berwald connection ∇, any curve σ in $T^\circ\mathcal{N}$ has a horizontal lift σ^{H} to $\tau^{\circ *}_N(\mathcal{FN})$ starting at a given frame $\{e_\alpha\}$ at $\sigma(0)$, defined as follows: $\sigma^{\mathrm{H}}(t)$ is the frame at $\sigma(t)$ obtained by parallelly transporting $\{e_\alpha\}$ to $\sigma(t)$ along σ; a frame field $\{E_\alpha\}$ along σ is parallel if $\nabla_{\dot\sigma} E_\alpha = 0$. Thus any vector field Z on $T^\circ\mathcal{N}$ has a horizontal lift Z^{H} to $\tau^{\circ *}_N(\mathcal{FN})$; in particular the horizontal lift $(X^{\mathrm{C}})^{\mathrm{H}}$ of X^{C} is given by

$$(X^{\mathrm{C}})^{\mathrm{H}} = X^\alpha \frac{\partial}{\partial x^\alpha} + u^\beta \frac{\partial X^a}{\partial x^\beta} \frac{\partial}{\partial u^\alpha} - x^\gamma_\beta \Gamma^\alpha_{\gamma\delta} X^\delta \frac{\partial}{\partial x^\alpha_\beta},$$

where of course

$$\text{if} \quad S = u^\alpha \frac{\partial}{\partial x^\alpha} - 2\Gamma^\alpha \frac{\partial}{\partial u^\alpha} \quad \text{then} \quad \Gamma^\alpha_{\beta\gamma} = \frac{\partial^2 \Gamma^\alpha}{\partial u^\beta \partial u^\gamma}.$$

Thus

$$\Psi - (X^{\mathrm{C}})^{\mathrm{H}} = x^\gamma_\beta \left(\frac{\partial X^\alpha}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} X^\delta \right) \frac{\partial}{\partial x^\alpha_\beta}.$$

Now for any vector field X on $\mathcal{N}$ and Berwald connection ∇, the condition that $\mathcal{L}_{X^{\mathrm{V}}} S = X^{\mathrm{C}} - 2\Delta$, in coordinates, is

$$\frac{\partial X^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} X^\gamma = \delta^\alpha_\beta;$$

so this condition is equivalent to

$$\Psi - (X^{\mathrm{C}})^{\mathrm{H}} = I^\dagger.$$

We denote by ω the connection form on $\tau^{\circ *}_N(\mathcal{FN})$ corresponding to ∇; in terms of local coordinates the matrix components of ω are given by

$$\omega^\alpha_\beta = \bar{x}^\alpha_\gamma \Gamma^\gamma_{\delta\epsilon} x^\delta_\beta dx^\epsilon + \bar{x}^\alpha_\gamma dx^\gamma_\beta$$

where the matrix $(\bar{x}^\alpha_\beta)$ is the inverse of the matrix (x^α_b). A straightforward calculation shows that the condition $\mathcal{L}_{X^{\mathrm{C}}}S = 0$ entails that $\mathcal{L}_\Psi \omega^\alpha_\beta = 0$ (it would be natural therefore to say that X^{C} is an infinitesimal affine transformation of the Berwald connection). It is also the case that $\mathcal{L}_\Xi \omega^\alpha_b = 0$: we have

$$\mathcal{L}_\Xi \omega^\alpha_\beta = \bar{x}^\alpha_\gamma \left(X^\lambda \frac{\partial \Gamma^\gamma_{\delta\epsilon}}{\partial u^\lambda} \right) x^\delta_\epsilon dx^\epsilon,$$

and it follows from the coordinate form of the condition $\Psi - (X^{\mathrm{C}})^{\mathrm{H}} = I^\dagger$ above, on differentiating with respect to u^λ, that the coefficient vanishes. Furthermore, $\mathcal{L}_{\tilde{\Delta}}\omega = 0$ by homogeneity, and $\mathcal{L}_{I^\dagger}\omega = [I, \omega] = 0$.

We now restrict to $\tau^{X*}_{\mathcal{N}}(\mathcal{FN})$. We can write any vector field in $\mathcal{D}$ in the form $Z = f\Psi + g\Xi + h\tilde{\Delta} + kI^\dagger$, so that

$$\mathcal{L}_Z\omega = (f\mathcal{L}_\Psi + g\mathcal{L}_\Xi + h\mathcal{L}_{\tilde{\Delta}} + k\mathcal{L}_{I^\dagger})\omega + \langle \Psi, \omega\rangle df + \langle \Xi, \omega\rangle dg + \langle \tilde{\Delta}, \omega\rangle dh + \langle I^\dagger, \omega\rangle dk.$$

But $\langle \Xi, \omega\rangle = \langle \tilde{\Delta}, \omega\rangle = 0$, while $\langle \Psi, \omega\rangle = \langle I^\dagger, \omega\rangle = I$. It follows that $\mathcal{L}_Z\omega = I(df + dk)$, that is, for any $Z \in \mathcal{D}$, $\mathcal{L}_Z\omega$ is a multiple of the identity element of $\mathfrak{gl}(m+1)$. Finally, ω is invariant under $\psi_{(0,0,\pm 1,1)}$ because S is reversible by assumption, and under $\psi_{(0,0,1,\pm 1)}$ by inspection.

We can therefore define an $\mathfrak{sl}(m+1)$-valued 1-form $\hat{\omega}$ on $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ as follows: for $Q \in \mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$, $w \in T_Q\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$,

$$\langle w, \hat{\omega}_Q\rangle = \langle v, o_*\omega_P\rangle$$

for any $P \in \tau^{X*}_{\mathcal{N}}(\mathcal{FN})$ such that $\pi^X(P) = Q$, and any $v \in T_P(\tau^{X*}_{\mathcal{N}}(F\mathcal{N}))$ such that $\pi^X_* v = w$, where $o_* : \mathfrak{gl}(m+1) \to \mathfrak{sl}(m+1)$ is the homomorphism of Lie algebras induced by $o : \mathrm{GL}(m+1) \to \mathrm{PGL}(m+1)$; $\hat{\omega}$ is well-defined because $o_*\omega_P(v)$ is unchanged by a change of choices of P and v satisfying the same conditions. We have $\pi^{X*}\hat{\omega} = o_*\omega$, and so for any $a \in \mathrm{GL}(m+1)$,

$$\begin{aligned}
\pi^{X*}(R^*_{o(a)}\hat{\omega}) &= R^*_a(\pi^{X*}\hat{\omega}) = R^*_a(o_*\omega) \\
&= o_*(R^*_a\omega) = o_*(\mathrm{ad}(a^{-1})\omega) = \mathrm{ad}(o(a)^{-1})o_*\omega \\
&= \pi^{X*}(\mathrm{ad}(o(a)^{-1})\hat{\omega}),
\end{aligned}$$

and so since π^X is surjective, $R^*_{o(a)}\hat{\omega} = \mathrm{ad}(o(a)^{-1})\hat{\omega}$. Moreover, for any $A \in \mathfrak{gl}(m+1)$ we have $\pi^X_* A^\dagger = (o_*A)^\dagger$, and therefore

$$\hat{\omega}((o_*(A)^\dagger) = (\pi^X_*\hat{\omega})(A^\dagger) = o_*\omega(A^\dagger) = o_*A.$$

Thus $\hat{\omega}$ is the connection form of an Ehresmann connection on the principal $\mathrm{PGL}(m+1)$-bundle $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$.

We now turn to a further consequence of the condition $\mathcal{L}_{X^{\mathrm{V}}}S = X^{\mathrm{C}} - 2\Delta$. We can regard X, which is a vector field on $\mathcal{N}$, as a section of $\tau^{\circ*}_{\mathcal{N}}T\mathcal{N}$, and therefore calculate its

Berwald covariant differential: using the coordinate form of this condition we find that

$$\nabla X = dx^\alpha \otimes \frac{\partial}{\partial x^\alpha}.$$

We also have for the total derivative $\mathbf{T} = u^\alpha \partial/\partial x^\alpha$

$$\nabla \mathbf{T} = \varphi^\alpha \otimes \frac{\partial}{\partial x^\alpha}$$

where as we explained in Section 2, φ^α is the 1-form $du^\alpha + \Gamma^\alpha_\beta dx^\beta$, so that $\{dx^\alpha, \varphi^\alpha\}$ is the local basis of 1-forms on $T^\circ\mathcal{N}$ dual to the local basis $\{H_\alpha, V_\alpha\}$ of vector fields associated with the Berwald connection. With an eye to the description of the structure of $\mathcal{C}M \to \mathrm{P}TM$ given in the previous section, we seek those vector fields η on $T^\circ\mathcal{N}$ with the properties that $\nabla_\eta(fX) = 0$ for some non-vanishing function f, and $\nabla_\eta(g\mathbf{T}+hX) = 0$ for some functions g and h with g non-vanishing; or equivalently, with the properties that $\nabla_\eta X$ is a multiple of X, and $\nabla_\eta \mathbf{T}$ is a linear combination of X and $\mathbf{T}$. Such η must satisfy $\langle \eta, dx^\alpha \rangle = \lambda X^\alpha$ and $\langle \eta, \varphi^\alpha \rangle = \mu X^\alpha + \nu u^\alpha$. It follows that η must be a linear combination of X^{C}, X^{V} and Δ.

Note that X and $\mathbf{T}$ are linearly independent over $T^X\mathcal{N}$. Let us denote by $\mathcal{F}_X\mathcal{N} \subset \tau^{X*}_{\mathcal{N}}(\mathcal{F}\mathcal{N})$ the sub-bundle consisting of those frames whose first member is a multiple of X and whose second member is a linear combination of $\mathbf{T}$ and X. Then at any point $P \in \mathcal{F}_X\mathcal{N}$, we have $H_P \cap T_P(\mathcal{F}_X\mathcal{N}) = \langle (X^{\mathrm{C}})^{\mathrm{H}}_P, (X^{\mathrm{V}})^{\mathrm{H}}_P, \Delta^{\mathrm{H}}_P \rangle$, that is, the horizontal subspace at P (the kernel of ω_P) intersects the tangent space to $\mathcal{F}_X\mathcal{N}$ at P in the 3-dimensional subspace spanned by the horizontal lifts of X^{C}, X^{V} and Δ to P. But $(X^{\mathrm{V}})^{\mathrm{H}} = \Xi$, $\Delta^{\mathrm{H}} = \tilde{\Delta}$, and $(X^{\mathrm{C}})^{\mathrm{H}} = \Psi - I^\dagger$. Thus $\ker \omega_P \cap T_P(\mathcal{F}_X\mathcal{N}) \subset \mathcal{D}_P$; so when we pass to the quotient, at any point $Q \in \pi^X(\mathcal{F}_X\mathcal{N})$ we have $\ker \hat{\omega}_Q \cap T_Q(\pi^X(\mathcal{F}_X\mathcal{N})) = \{0\}$.

6.2 The connection forms

We can apply the above results with $\mathcal{N} = \mathcal{V}M$, $X = \Upsilon$, S the BTW-spray of a projective equivalence class of sprays. The manifold $\mathcal{M}$ is just M, and $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ is $\pi^*_M(\mathcal{S}_\mathcal{W}M)$, a principal $\mathrm{PGL}(m+1)$-bundle over $\mathrm{P}TM$. Then $\hat{\omega}$ is an Ehresmann connection form on $\pi^*_M(\mathcal{S}_\mathcal{W}M) \to \mathrm{P}TM$. Now the projective Cartan bundle $\mathcal{C}M \to \mathrm{P}TM$ is a sub-bundle of $\pi^*_M(\mathcal{S}_\mathcal{W}M) \to \mathrm{P}TM$. This sub-bundle has codimension $2m-1 = \dim(\mathrm{P}TM)$, and so the restriction ω of $\hat{\omega}$ to $\mathcal{C}M$ will define a Cartan connection if the intersection (in $T(\pi^*_M(\mathcal{S}_\mathcal{W}M))$) of $\ker \hat{\omega}$ and $T\mathcal{C}M$ contains only zero vectors ([11], Proposition A.3.1; [7]). But $\mathcal{C}M$ is the image in $\pi^*_M(\mathcal{S}_\mathcal{W}M)$ of the sub-bundle of $\tau^{\Upsilon*}_{\mathcal{V}M}\mathcal{F}(\mathcal{V}M)$ consisting of those frames with first element a multiple of Υ and second a linear combination of Υ and $\mathbf{T}$, so this follows from the results of the final paragraph of the previous subsection.

The Ehresmann connection form $\tilde{\omega}$ of the BTW-connection in the coordinate gauge (∂_α) is given by

$$\tilde{\omega}_{(\partial_\alpha)} = \begin{pmatrix} 0 & -\dfrac{1}{m-1} x^0 \mathfrak{R}_{bc} dx^c \\ (x^0)^{-1} dx^a & \Pi^a_{bc} dx^c + \delta^a_b (x^0)^{-1} dx^0 \end{pmatrix};$$

this is formally the same as in the affine case, but it must be borne in mind that $\tilde{\omega}_{(\partial_\alpha)}$ is a local matrix-valued 1-form on $T^\circ M$ rather than on M; it is semi-basic over $T^\circ M \to M$. We first change the gauge to (Υ, ∂_a); we obtain

$$\begin{aligned}
\tilde{\omega}_{(\Upsilon,\partial_a)} &= \begin{pmatrix} (x^0)^{-1}dx^0 & -\dfrac{1}{m-1}x^0\mathfrak{R}_{bc}dx^c \\ & \\ dx^a & \Pi^a_{bc}dx^c + \delta^a_b(x^0)^{-1}dx^0 \end{pmatrix} \\
&= ((x^0)^{-1}dx^0)I + \begin{pmatrix} 0 & -\dfrac{1}{m-1}x^0\mathfrak{R}_{bc}dx^c \\ & \\ dx^a & \Pi^a_{bc}dx^c \end{pmatrix}.
\end{aligned}$$

The Ehresmann connection $\hat{\omega}$ on the simplex bundle $\pi_M^*(\mathcal{S}_\mathcal{W}M)$, in the gauge $[e_\alpha]$, is therefore

$$\hat{\omega}_{[e_\alpha]} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}x^0\mathfrak{R}_{bc}dx^c \\ & \\ dx^a & \Pi^a_{bc}dx^c \end{pmatrix}.$$

To obtain the Cartan connection form we need to change the gauge to $[\zeta_\alpha]$ where

$$\zeta_0 = e_0, \quad \zeta_1 = e_1 + y^i e_i, \quad \zeta_i = e_i;$$

the result is

$$\begin{pmatrix} 0 & -\dfrac{1}{m-1}(\mathfrak{R}_{1a} + y^i\mathfrak{R}_{ia})dx^a & -\dfrac{1}{m-1}\mathfrak{R}_{ia}dx^a \\ & & \\ dx^1 & (\Pi^1_{1a} + y^k\Pi^1_{ka})dx^a & \Pi^1_{ja}dx^a \\ & & \\ dx^i - y^i dx^1 & dy^i + (\Pi^i_{1a} - y^i\Pi^1_{1a} + y^k\Pi^i_{ka} - y^i y^k\Pi^1_{ka})dx^a & (\Pi^i_{ja} - y^i\Pi^1_{ja})dx^a \end{pmatrix},$$

the connection form of the normal Cartan connection in standard gauge. Indeed, the last step is just the inverse of the illegitimate gauge transformation of Subsection 4.3, and we see that the simplified connection form introduced there is $\hat{\omega}_{[e_\alpha]}$.

Finally, we review the result of Subsection 4.4 — what happens when the spray is affine — from the present point of view. We have in any case a global Cartan connection form ω on the manifold $\mathcal{C}M$, which satisfies the defining conditions

1. the map $\omega_p : T_p\mathcal{C}M \to \mathfrak{sl}(m+1)$ is an isomorphism for each $p \in \mathcal{C}M$;
2. $R_k^*\omega = \mathrm{ad}(k^{-1})\omega$ for each $k \in \mathrm{K}_{m+1}$; and
3. $\omega(A^\dagger) = A$ for each $A \in \mathfrak{k}_{m+1}$, where $\mathfrak{k}_{m+1}$ is the Lie algebra of K_{m+1} and where $A^\dagger$ is the fundamental vector field corresponding to A.

A Cartan projective connection form in the affine case is a form ϖ on the same manifold, satisfying the same conditions but with H_{m+1} replacing K_{m+1}, and with condition (3) being replaced, explicitly, by

3a. $\varpi(A^\ddagger) = A$ for each $A \in \mathfrak{h}_{m+1}$, where $\mathfrak{h}_{m+1}$ is the Lie algebra of H_{m+1} and where $A^\ddagger$ is the fundamental vector field corresponding to A.

Now ω is the restriction to $\mathcal{C}M$ of the Ehresmann connection form $\hat{\omega}$ on $\pi_M^*(\mathcal{S}_\mathcal{W}M)$; and $\hat{\omega}$, being an $\mathfrak{sl}(m+1)$-valued connection form, satisfies $\hat{\omega}(A^\dagger) = A$ for all $A \in \mathfrak{sl}(m+1)$, and in particular for all $A \in \mathrm{H}_{m+1}$. The submanifold $\mathcal{C}M \subset \pi_M^*(\mathcal{S}_\mathcal{W}M)$ is not invariant under the action of H_{m+1} on $\pi_M^*(\mathcal{S}_\mathcal{W}M)$, and so the restriction of the fundamental vector field $A^\dagger$ to $\mathcal{C}M$ is not tangent to $\mathcal{C}M$ and, in particular, is not the same as $A^\ddagger$. It is, however, easy to check that

$$\hat{\omega}(A^\dagger) = \omega(A^\ddagger)$$

at all points of $\mathcal{C}M$ for any connection form ω arising in this way: thus, in order for ω to be of affine type, it is enough for the condition that $R_h^*\omega = \mathrm{ad}(h^{-1})\omega$ for each $h \in \mathrm{H}_{m+1}$ to be satisfied. The differential version of this condition is that $\mathcal{L}_{A^\ddagger}\omega = [\omega, A]$ for all $A \in \mathrm{H}_{m+1}$; if we express this in terms of the curvature Ω of ω it becomes $A^\ddagger \lrcorner\, \Omega = 0$, so this is the necessary and sufficient condition for ω to be of affine type. Since $\mathfrak{h}_{m+1}/\mathfrak{k}_{m+1}$ parametrizes the vertical subspaces of $\mathrm{P}TM \to M$ at each point, and since necessarily $A^\ddagger \lrcorner\, \Omega = 0$ for $A \in \mathrm{K}_{m+1}$, this condition can be seen to be essentially equivalent to the local one found in Subsection 4.4.

Appendix: determination of the normal Cartan connection

In this appendix we give the detailed calculations leading to the normal Cartan connection using the curvature conditions from Subsection 4.2, which we repeat here for convenience:

- $\Omega^\alpha_0 = 0$;
- Ω^0_1 is semi-basic;
- Ω^i_1 is semi-basic;
- if we set $\Omega^i_j = K^i_{jkl} dy^k \wedge \theta^l \mod(dx^a \wedge dx^b)$ then $K^k_{kij} = 0$ and $K^k_{ijk} = 0$;
- if we set $\Omega^i_1 = L^i_j dx^1 \wedge \theta^j \mod(\theta^k \wedge \theta^l)$ then $L^k_k = 0$.

The connection form in standard gauge is given by

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_1 & \omega^0_j \\ \omega^1_0 & \omega^1_1 & \omega^1_j \\ \omega^i_0 & \omega^i_1 & \omega^i_j \end{pmatrix}$$

where $\omega_0^1 = dx^1$, $\omega_0^i = \theta^i$, $\omega_0^0 = \kappa_i dy^i$, and $\omega_1^i = A_j^i(dy^j - f^j dx^1) \bmod(\theta^k)$ where (A_j^i) is non singular.

We first consider Ω_0^a, leaving Ω_0^0 until later. It is clear from the gauge transformation formulæ that if $\Omega_0^a = 0$, then $\hat{\Omega}_0^a = 0$ also, so making this part of the torsion zero is a valid first step in determining a normal connection.

We now proceed to make the Ω_0^a zero. First,

$$\Omega_0^1 = dx^1 \wedge (\omega_0^0 - \omega_1^1) + \omega_j^1 \wedge \theta^j;$$

in order for this to be zero, it must be the case that

$$\omega_0^0 - \omega_1^1 = \lambda dx^1 + \lambda_j \theta^j, \quad \omega_j^1 = -\lambda_j dx^1 + \lambda_{jk}\theta^k$$

for some functions λ, λ_j and λ_{jk}, with $\lambda_{kj} = \lambda_{jk}$.

Next,

$$\begin{aligned} \Omega_0^i &= d\theta^i + \theta^i \wedge \omega_0^0 + \omega_1^i \wedge dx^1 + \omega_j^i \theta^j \\ &= -(dy^i - \omega_1^i) \wedge dx^1 + (\omega_j^i - \delta_j^i \omega_0^0) \wedge \theta^j. \end{aligned}$$

For this to be zero we must first have that $\omega_1^i = dy^i - f^i dx^1 \bmod \theta^j$ (in other words, $A_j^i = \delta_j^i$), and that $\omega_j^i - \delta_j^i \omega_0^0$ is semi-basic (take the inner product with $\partial/\partial y^i$). (The component forms of the connection, and also of the curvature, are forms on PTM; semi-basic will always mean semi-basic from this point of view). Then if we set $\omega_1^i = dy^i - f^i dx^1 + \mu_j^i \theta^j$ we must have

$$\omega_j^i - \delta_j^i \omega_0^0 = \mu_j^i dx^1 + \nu_{jk}^i \theta^k \quad \text{with} \quad \nu_{kj}^i = \nu_{jk}^i.$$

Now ω takes its values in $\mathfrak{sl}(m+1)$, so

$$0 = \omega_0^0 + \omega_1^1 + \omega_i^i = (m+1)\omega_0^0 - (\lambda dx^1 + \lambda_i \theta^i) + \mu_i^i dx^1 + \nu_{ji}^j \theta^i,$$

and therefore

$$\omega_0^0 = 0, \quad \lambda = \mu_i^i, \quad \lambda_i = \nu_{ji}^j.$$

We must now consider gauge transformations between connections in standard form in more detail. Take a coordinate transformation on M, from (x^a) to $(\hat{x}^a)$, with the induced transformation of the y^i. Let (J_b^a) be the Jacobian matrix of the transformation,

$$J_b^a = \frac{\partial \hat{x}^a}{\partial x^b}, \quad J = \det(J_b^a).$$

Then

$$\begin{aligned} d\hat{x}^1 &= (J_1^1 + J_k^1 y^k) dx^1 + J_k^1 \theta^k \\ d\hat{x}^i &= (J_1^i + J_k^i y^k) dx^1 + J_k^i \theta^k \\ \hat{y}^i &= \frac{J_1^i + J_k^i y^k}{J_1^1 + J_k^1 y^k} \\ \hat{\theta}^i &= \left(J_j^i - \left(\frac{J_1^i + J_k^i y^k}{J_1^1 + J_k^1 y^k} \right) J_j^1 \right) \theta^j. \end{aligned}$$

Suppose that we have two (gauged) connection forms, one associated with each of the two coordinate systems, each in standard form with respect to its coordinates, and both having zero in the top left-hand corner. Suppose further that these are local representatives of the same global connection form, so that they are related by a gauge transformation. Then the gauge transformation relating them is uniquely determined in terms of the coordinate transformation, and is given by

$$k^1_1 = k^0_0(J^1_1 + J^1_k y^k), \quad k^1_j = k^0_0 J^1_j, \quad k^i_j = k^0_0 \beta^i_j, \quad k^0_a dx^a = \bar{k}^0_0 dk^0_0,$$

where

$$k^0_0 = \begin{cases} J^{-1/(m+1)} & m \text{ odd} \\ \pm|J|^{-1/(m+1)} & m \text{ even} \end{cases}$$

with

$$\beta^i_j = J^i_j - \left(\frac{J^i_1 + J^i_k y^k}{J^1_1 + J^1_k y^k}\right) J^1_j.$$

Note in particular that $d\hat{x}^1$ is a linear combination of dx^1 and the θ^i, while $d\hat{y}^i$ depends on all of dy^j, dx^1 and θ^k.

For a curvature form with $\Omega^a_0 = 0$, and with k as described above, we have

$$\begin{aligned} k^{-1}\Omega k &= \begin{pmatrix} \bar{k}^0_0 & * & * \\ 0 & \bar{k}^1_1 & * \\ 0 & 0 & \bar{k}^i_j \end{pmatrix} \begin{pmatrix} \Omega^0_0 & \Omega^0_1 & * \\ 0 & \Omega^1_1 & * \\ 0 & \Omega^i_1 & \Omega^i_j \end{pmatrix} \begin{pmatrix} k^0_0 & * & * \\ 0 & k^1_1 & k^1_j \\ 0 & 0 & k^i_j \end{pmatrix} \\ &= \begin{pmatrix} \Omega^0_0 & \hat{\Omega}^0_1 & * \\ 0 & * & * \\ 0 & k^1_1 \bar{k}^i_k \Omega^k_1 & \bar{k}^i_k k^1_j \Omega^k_1 + \bar{k}^i_k k^l_j \Omega^k_l \end{pmatrix}, \end{aligned}$$

where $\hat{\Omega}^0_1$ is a linear combination of Ω^0_0, Ω^0_1, Ω^1_1 and Ω^i_1. It follows, first of all, that requiring that Ω^i_1 is semi-basic is a coordinate independent condition; our next move is to impose it.

We have

$$\begin{aligned} \Omega^i_1 &= d\omega^i_1 + \theta^i \wedge \omega^0_1 + (\omega^i_k - \delta^i_k \omega^1_1) \wedge \omega^k_1 \\ &= -df^i \wedge dx^1 + d\mu^i_j \wedge \theta^j - \mu^i_j dy^j \wedge dx^1 + \theta^i \wedge \omega^0_1 + (\omega^i_k - \delta^i_k \omega^1_1) \wedge \omega^k_1, \end{aligned}$$

and

$$\omega^i_j - \delta^i_j \omega^1_1 = (\mu^i_j + \delta^i_j \lambda) dx^1 + (\nu^i_{jk} + \delta^i_j \lambda_k)\theta^k.$$

Now we cannot make Ω^i_1 vanish, but we can ensure that it is semi-basic. In fact

$$\frac{\partial}{\partial y_j} \lrcorner\, \Omega^i_1 = -\left(\frac{\partial f^i}{\partial y^j} + 2\mu^i_j + \delta^i_j \lambda\right) dx^1 + \left(\frac{\partial \mu^i_k}{\partial y^j} - \nu^i_{jk} - \delta^i_j \lambda_k - \delta^i_k \omega_{1j}\right)\theta^k,$$

where

$$\omega_{1j} = \left\langle \frac{\partial}{\partial y^j}, \omega^0_1 \right\rangle.$$

So we must take

$$2\mu^i_j + \delta^i_j\lambda = -\frac{\partial f^i}{\partial y^j} = 2\gamma^i_j, \quad \nu^i_{jk} + \delta^i_j\lambda_k + \delta^i_k\omega_{1j} = \frac{\partial \mu^i_k}{\partial y^j}$$

where

$$\gamma^i_j = -\tfrac{1}{2}\frac{\partial f^i}{\partial y^j}.$$

On taking the trace of the first equation, recalling that $\lambda = \mu^i_i$, we find that

$$\lambda = \frac{2}{m+1}\gamma, \quad \gamma = \gamma^i_i,$$

and so

$$\mu^i_j = \gamma^i_j - \frac{1}{m+1}\delta^i_j\gamma.$$

Thus

$$\nu^i_{jk} + \delta^i_j\lambda_k + \delta^i_k\omega_{1j} = \gamma^i_{jk} - \frac{1}{m+1}\delta^i_k\gamma_j,$$

where

$$\gamma^i_{jk} = \frac{\partial \gamma^i_k}{\partial y^j} = \frac{\partial \gamma^i_j}{\partial y^k}, \quad \gamma_k = \gamma^j_{jk}.$$

Taking two traces in the equation for ν^i_{jk}, using its symmetry, and recalling that $\lambda_i = \nu^j_{ji}$, we obtain

$$\begin{aligned} m\lambda_k + \omega_{1k} &= \frac{m}{m+1}\Gamma_k \\ 2\lambda_k + (m-1)\omega_{1k} &= \frac{2}{m+1}\Gamma_k, \end{aligned}$$

whence

$$\lambda_k = \frac{1}{m+1}\gamma_k, \quad \omega_{1k} = 0$$

for $m > 2$, and therefore

$$\nu^i_{jk} = \gamma^i_{jk} - \frac{1}{m+1}(\delta^i_j\gamma_k + \delta^i_k\gamma_j).$$

We now turn to Ω^i_j:

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j + \theta^i \wedge \omega^0_j + \omega^i_1 \wedge \omega^1_j.$$

We wish to calculate the terms in Ω^i_j involving dy^k, which we do as follows. Taking into account the terms we know to be semi-basic we have

$$\frac{\partial}{\partial y^k} \lrcorner\, \Omega^i_j = \frac{\partial}{\partial y^k} \lrcorner\, d\omega^i_j - \tau_{jk}\theta^i + \delta^i_k(-\lambda_j dx^1 + \lambda_{jl}\theta^l), \quad \tau_{jk} = \left\langle \frac{\partial}{\partial y^k}, \omega^0_j \right\rangle.$$

Now

$$\frac{\partial}{\partial y^k} \lrcorner\, d\omega^i_j = \mathcal{L}_{V_k}\omega^i_j = \mathcal{L}_{V_k}\left(\left(\gamma^i_j - \frac{1}{m+1}\delta^i_j\gamma\right)dx^1 + \left(\gamma^i_{jl} - \frac{1}{m+1}\left(\delta^i_j\gamma_l + \delta^i_l\gamma_j\right)\theta^l\right)\right),$$

$V_k = \partial/\partial y^k$. But $\mathcal{L}_{V_k}\theta^l = -\delta^l_k dx^1$, whence

$$\mathcal{L}_{V_k}\omega^i_j = \frac{1}{m+1}\delta^i_k\gamma_j dx^1 + \left(\gamma^i_{jkl} - \frac{1}{m+1}\left(\delta^i_j\gamma_{kl} + \delta^i_l\gamma_{jk}\right)\right)\theta^l, \quad \gamma^i_{jkl} = \frac{\partial\gamma^i_{jk}}{\partial y^l}, \quad \gamma_{kl} = \frac{\partial\gamma_k}{\partial y^l}.$$

So finally

$$\frac{\partial}{\partial y^k}\lrcorner\, \Omega^i_j = \left(\gamma^i_{jkl} - \frac{1}{m+1}\left(\delta^i_j\gamma_{kl} + \delta^i_l\gamma_{jk}\right)\right)\theta^l - \tau_{jk}\theta^i + \delta^i_k\lambda_{jl}\theta^l.$$

This means that

$$\Omega^i_j = K^i_{jkl}dy^k \wedge \theta^l \pmod{dx^a \wedge dx^b}$$

with

$$K^i_{jkl} = \gamma^i_{jkl} - \frac{1}{m+1}\left(\delta^i_j\gamma_{kl} + \delta^i_l\gamma_{jk}\right) + \delta^i_k\lambda_{jl} - \delta^i_l\tau_{jk}.$$

We will now find the transformation rule for K^i_{jkl}. To do so, we need to demonstrate a relationship between $\partial\hat{y}^k/\partial y^l$ and k^i_j. We have $\hat{\theta}^k = k^0_0\bar{k}^k_l\theta^l$, whence, on taking the exterior derivative,

$$\begin{aligned} d\hat{y}^k \wedge d\hat{x}^1 &= \frac{\partial\hat{y}^k}{\partial y^l}\frac{\partial\hat{x}^1}{\partial x^1}dy^l \wedge dx^1 \pmod{dx^a \wedge dx^b} \\ &= k^0_0\bar{k}^k_l dy^l \wedge dx^1 \pmod{dx^a \wedge dx^b}, \end{aligned}$$

so that

$$\frac{\partial\hat{y}^k}{\partial y^l} = c\bar{k}^k_l$$

for some non-vanishing function c. Now

$$\begin{aligned} \hat{K}^i_{jkl}d\hat{y}^k \wedge \hat{\theta}^l &= c\bar{k}^k_m\bar{k}^l_n\hat{K}^i_{jkl}dy^m \wedge \theta^n \pmod{dx^a \wedge dx^b} \\ &= \bar{k}^i_r k^s_j K^r_{skl}dy^k \wedge \theta^l \pmod{dx^a \wedge dx^b} \end{aligned}$$

using the fact that Ω^i_1 is semi-basic. Thus

$$\hat{K}^i_{jkl}d\hat{y}^k \wedge \hat{\theta}^l = c^{-1}\bar{k}^i_r k^s_j k^m_k k^n_l K^r_{smn},$$

from which it follows that $\hat{K}^i_{ikl} = 0$ if and only if $K^i_{ikl} = 0$, and also that $\hat{K}^i_{kil} = 0$ if and only if $K^i_{kil} = 0$. We may therefore validly impose the condition that both of these traces of this component of Ω^i_j vanish. These conditions give

$$\frac{1}{m+1}\gamma_{kl} + \lambda_{kl} - \tau_{lk} = 0, \quad \frac{m-1}{m+1}\gamma_{kl} + (m-1)\lambda_{kl} - \tau_{kl} = 0.$$

It follows that

$$\lambda_{kl} = -\frac{1}{m+1}\gamma_{kl}, \quad \tau_{kl} = 0,$$

and

$$K^i_{jkl} = \gamma^i_{jkl} - \frac{1}{m+1}\left(\delta^i_j\gamma_{kl} + \delta^i_k\gamma_{jl} + \delta^i_l\gamma_{jk}\right).$$

Note that with this choice K^i_{jkl} is completely symmetric in its lower indices and completely trace-free.

In the course of the argument so far we have shown that ω^0_1 and ω^0_i are semi-basic. Since Ω^0_0 is unchanged by a gauge transformation we can require that it is zero: for this to be the case we must have $\omega^0_1 \wedge dx^1 + \omega^0_i \wedge \theta^i = 0$. So if we set $\omega^0_1 = \varrho dx^1 + \varrho_i\theta^i$ then $\omega^0_i = \varrho_i dx^1 + \varrho_{ij}\theta^j$ where $\varrho_{ji} = \varrho_{ij}$.

Since Ω takes its values in $\mathfrak{sl}(m+1)$, and $\Omega^0_0 = 0$, $\Omega^1_1 + \Omega^i_i = 0$. Since $K^i_{ijk} = 0$, Ω^i_i is semi-basic, and therefore Ω^1_1 is semi-basic. Now $\hat{\Omega}^0_1$ is a linear combination of Ω^0_1, Ω^0_0, Ω^1_1 and Ω^i_1, all of which except the first we now know to be semi-basic. It follows that $\hat{\Omega}^0_1$ will be semi-basic if and only Ω^0_1 is, and so we can validly impose the condition that Ω^0_1 is semi-basic. But

$$\Omega^0_1 = d\omega^0_1 + \omega^0_1 \wedge \omega^1_1 + \omega^0_j \wedge \omega^j_1,$$

whence

$$\frac{\partial}{\partial y^i} \lrcorner\, \Omega^0_1 = \mathcal{L}_{V_i}\omega^0_1 - (\varrho_i dx^1 + \varrho_{ij}\theta^j) = \left(\frac{\partial\varrho}{\partial y^i} - 2\varrho_i\right)dx^1 + \left(\frac{\partial\varrho_j}{\partial y^i} - \varrho_{ij}\right)\theta^j,$$

so that Ω^0_1 is semi-basic if and only if

$$\varrho_i = \tfrac{1}{2}\frac{\partial\varrho}{\partial y^i}, \quad \varrho_{ij} = \frac{\partial\varrho_j}{\partial y^i} = \tfrac{1}{2}\frac{\partial^2\varrho}{\partial y^i\partial y^j}.$$

We have now determined the whole of the connection except for ϱ. To fix it we go back to Ω^i_1, which is known to be semi-basic. We can therefore write $\Omega^i_1 = L^i_j dx^1 \wedge \theta^j$ $\mathrm{mod}\,\theta^m \wedge \theta^n$. Then

$$\begin{aligned}
\hat{\Omega}^i_1 &= \hat{L}^i_j d\hat{x}^1 \wedge \hat{\theta}^j \quad (\mathrm{mod}\ \hat{\theta}^m \wedge \hat{\theta}^n)\\
&= k^0_0\bar{k}^j_l\frac{\partial\hat{x}^1}{\partial x^1}\hat{L}^i_j dx^1 \wedge \theta^l \quad (\mathrm{mod}\ \theta^m \wedge \theta^n)\\
&= k^1_1\bar{k}^i_k L^k_j dx^1 \wedge \theta^j \quad (\mathrm{mod}\ \theta^m \wedge \theta^n),
\end{aligned}$$

whence

$$\hat{L}^i_j = c\bar{k}^i_k k^l_j L^k_j$$

where c is a scalar factor. Thus $\hat{L}^k_k = 0$ if and only if $L^k_k = 0$, and we may validly impose the condition that the trace of this component of Ω^i_1 vanishes. Now

$$\frac{d}{dx^1} \lrcorner\, \Omega^i_1 \quad = \quad L^i_j\theta^j$$

$$
\begin{aligned}
&= \left(\frac{\partial f^i}{\partial x^j} + \frac{d}{dx^1}(\gamma^i_j) + \gamma^i_k\gamma^k_j - \delta^i_j\left(\varrho + \frac{1}{m+1}\frac{d}{dx^1}(\gamma) + \frac{1}{(m+1)^2}\gamma^2\right)\right)\theta^j \\
&= (\Phi^i_j - \delta^i_j\tilde{\varrho})\theta^j
\end{aligned}
$$

where Φ^i_j is the Jacobi endomorphism of the second-order differential equation field,

$$
\Phi^i_j = \frac{\partial f^i}{\partial x^j} + \frac{d}{dx^1}(\gamma^i_j) + \gamma^i_k\gamma^k_j,
$$

and we have set

$$
\tilde{\varrho} = \varrho + \frac{1}{m+1}\frac{d}{dx^1}(\gamma) + \frac{1}{(m+1)^2}\gamma^2.
$$

Thus $L^k_k = 0$ if and only if $(m-1)\tilde{\varrho} = \Phi^i_i$, and then

$$
L^i_j = \Phi^i_j - \frac{1}{(m-1)}\delta^i_j\Phi^k_k.
$$

The connection is now completely determined; it is the normal connection corresponding to the given system of second order differential equations.

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