Projectively Equivalent Riemannian Spaces as Quasi-bi-Hamiltonian Systems

M. Crampin
Department of Applied Mathematics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, UK.

March 1, 2003

Abstract

The class of Riemannian spaces admitting projectively, or geodesically, equivalent metrics is very closely related to a certain class of spaces for which the Hamilton-Jacobi equation for geodesics is separable. This fact is established, and its consequences explored, by showing that when a Riemannian space has a projectively equivalent metric its geodesic flow is a quasi-bi-Hamiltonian system. The existence of involutive first integrals of the geodesic flow, quadratic in the momenta, follows by a standard type of argument. When these integrals are independent they generate a Stäckel system.

Keywords: Projective equivalence, quasi-bi-Hamiltonian system, Hamilton-Jacobi separability

MCS classification: 53C22, 70H20

1 Introduction

Recent studies [8, 9, 13, 14] of projectively, or as the authors call them geodesically, equivalent Riemannian metrics have thrown up some interesting results which are unexpectedly parallel with results also obtained recently concerning separability of the Hamilton-Jacobi equation for geodesics in Riemannian spaces. The purpose of this paper is to examine these parallels.

My method is to show that the geodesic flow of a space with projectively equivalent metrics is quasi-bi-Hamiltonian in the sense of [10]. Of course the geodesic flow on any Riemannian manifold M, considered as a vector field on T^*M, is Hamiltonian with repect to the standard Poisson structure. However, when there is a second metric on M projectively equivalent to the first the geodesic flow is also quasi-Hamiltonian with respect to a second Poisson structure compatible with the standard one. It follows from this that the flow can be represented as a bi-Hamiltonian system with respect to a certain Poisson pencil on $T^*M \times \mathbb{R}$. The same quasi-bi-Hamiltonian and bi-Hamiltonian structures have been invoked in recent investigations into the separability of the Hamilton-Jacobi equation, initially in [5] and subsequently in [4]. The same constructions turn out to be useful in relation to certain kinds of non-conservative Lagrangian systems on Riemannian manifolds which have also been the subject of recent study (see [3, 7, 11]).

It is well-known that exhibiting a bi-Hamiltonian structure is a sovereign way of demonstrating the existence of involutive first integrals of a flow, and in fact the quadratic integrals of the geodesic flow found with some effort in [8] can be obtained rather more directly by more-or-less standard arguments, as I shall show.

The Riemannian spaces with separable Hamilton-Jacobi equation which are of interest in this context are of a special kind: each admits a tensor field of valence 2 which has vanishing torsion in the sense of Nijenhuis, and which is a conformal Killing tensor of the metric. That the existence of such a tensor implies separability of the Hamilton-Jacobi equation was first recognised in [1]; the properties of such tensors has been developed further in [2], where the name 'special conformal Killing tensor' was coined for them. The fact that the manifold admits a special conformal Killing tensor plays a key role in the construction of the quasi-bi-hamiltonian structure. It also has a number of other interesting consequences, which are described in [2] and summarised later in this paper.

The following results are classical [12]. Two symmetric connections are projectively equivalent if and only if the difference tensor of their connection coefficients takes the form $p_i\delta^k_j + p_j\delta^k_i$ for some covariant vector p_i. In the case in which the two connections are the Levi-Civita connections of metrics g and h, and Γ^k_{ij} are the connection coefficients for g so that those for h are $\Gamma^k_{ij} + p_i\delta^k_j + p_j\delta^k_i$, we have

$$h_{ij|k} = 2p_k h_{ij} + p_i h_{jk} + p_j h_{ik},$$

where the bar denotes covariant differentiation with respect to the Levi-

Civita connection for g. From this it follows that

$$p_i = -\tfrac{1}{2}\frac{\partial}{\partial x^i}\log\mu, \quad \text{where } \mu = \left(\frac{\det g}{\det h}\right)^{1/(n+1)},$$

n being the dimension of the manifold; recall that the ratio of the determinants of two type $(0,2)$ tensors is a scalar. Furthermore, $\frac{1}{2}\mu^2 h_{ij}\dot{x}^i\dot{x}^j$ is a quadratic first integral of the geodesic $x^i = x^i(s)$, where $\dot{x}^i = dx^i/ds$.

Now set

$$L_{ij} = \mu^{-1}\bar{h}_{ij} = \mu^{-1}g_{ik}g_{jl}\bar{h}^{kl}$$

where $\bar{h}^{kl}$ is the inverse of h_{kl}. It is easy to see that

$$L_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik}), \quad \text{where } \alpha_i = -2\mu^{-1}g_{ik}\bar{h}^{jk}p_j.$$

This result is Lemma 6 in [13]. It says that L is a special conformal Killing tensor, in the terminology of [2] (and the numerical factor in the definition of α_i is inserted to ensure notational consistency with that paper). It follows from it that the symmetric tensor L_{ij} is a conformal Killing tensor of g_{ij}:

$$L_{ij|k} + L_{jk|i} + L_{ki|j} = \alpha_i g_{jk} + \alpha_j g_{ki} + \alpha_k g_{ij}.$$

More significantly, so far as this paper is concerned, the torsion, or Nijenhuis tensor, of the corresponding type $(1,1)$ tensor $L^i_j = g^{ik}L_{jk}$ vanishes:

$$L^k_l\left(L^l_{i|j} - L^l_{j|i}\right) = L^l_j L^k_{i|l} - L^l_i L^k_{j|l}.$$

Note that $\mu = \det L$, where L here denotes the type $(1,1)$ tensor.

The conclusion that L is a special conformal Killing tensor was not drawn in [13]; however, several important consequences flow from it, which will be discussed below.

(Throughout the paper, an overbar on a tensor of valence 2 denotes an inverse, but with one exception: the inverse of g_{ij} is denoted by g^{ij} in the usual way. Indices will be raised and lowered only with g. Thus although the metrics g and h are symmetrically related by their assumed projective equivalence, they will be treated unequally, with precedence being afforded to g.)

Conversely, if a metric g admits an invertible special conformal Killing tensor L and $h_{ij} = (\det L)^{-1}\bar{L}_{ij}$ then h is projectively equivalent to g.

2 The quasi-bi-Hamiltonian structure

Consider first the geodesic flow Γ_g of a Riemannian space with metric g as a vector field on the tangent bundle TM with coordinates (x^i, u^i). It is defined by $\Gamma_g \lrcorner \omega_g = -dH_g$ where $\omega_g = d(g_{ij}u^j dx^i)$ is the Cartan 2-form and $H_g = \frac{1}{2}g_{ij}u^i u^j$ the energy associated with g. It is pointed out in [8, 13] that if g and h are projectively equivalent metrics and Φ is the diffeomorphism of TM less the zero section defined by

$$\Phi(x, u) = (x, f(x, u)u) \quad \text{where } f(x, u) = \sqrt{H_g/H_h},$$

then $\Phi^* H_h = H_g$ and $\Phi_* \Gamma_g = f^{-1}\Gamma_h$. The quadratic first integral $\frac{1}{2}\mu^2 h_{ij}u^i u^j$ can be expressed as $\frac{1}{2}(\det L)\bar{L}_{ij}u^i u^j$; I shall denote this by K. Then $f = \kappa \det L$ where $\kappa = \sqrt{H_g/K}$ is a first integral of Γ_g.

The projective equivalence of g and h, which is summarized in the relation $\Phi_*\Gamma_g = f^{-1}\Gamma_h$, can equivalently be expressed as

$$(\kappa \det L)\Gamma_g \lrcorner\, \Phi^* \omega_h = -dH_g.$$

Now

$$\begin{aligned} \Phi^*\omega_h &= d\left((\kappa \det L) h_{ij}u^j dx^i\right) = d\left(\kappa \bar{L}_{ij}u^j dx^i\right) \\ &= \bar{L}_{ij}u^j d\kappa \wedge dx^i + \kappa\omega_{\bar{L}}, \end{aligned}$$

so that

$$(\kappa \det L)\Gamma_g \lrcorner\, \Phi^*\omega_h = -K d\kappa^2 + (\kappa^2 \det L)\Gamma_g \lrcorner\, \omega_{\bar{L}}.$$

It follows that

$$(\kappa^2 \det L)\Gamma_g \lrcorner\, \omega_{\bar{L}} = K d\kappa^2 - dH_g;$$

but $K d\kappa^2 = dH_g - \kappa^2 dK$, whence

$$(\det L)\Gamma_g \lrcorner\, \omega_{\bar{L}} = -dK.$$

Now $\omega_{\bar{L}}$ is a symplectic form on TM; we may use it, together with the Legendre transformation $(x^i, u^i) \mapsto (x^i, g_{ij}u^j)$, to define a Poisson structure on T^*M. This Poisson structure is easily seen to be given, as a map $P_L\colon T^*(T^*M) \to T(T^*M)$, by $P_L = P_0 \circ \tilde{L} = \tilde{L} \circ P_0$ where P_0 is the standard Poisson structure corresponding to the canonical 2-form on T^*M and $\tilde{L}$ is the complete lift of L to T^*M, a type $(1, 1)$ tensor field whose Nijenhuis torsion vanishes because that of L does [15]. The Poisson maps P_L and P_0

are compatible, in the sense that $P_L + sP_0$ is Poisson for all constants s, due to the vanishing of the torsion of L: in fact $P_L + sP_0 = P_{(L+sI)}$. We have here an example of a Poisson-Nijenhuis structure [6].

The geodesic flow, transferred to T^*M by the Legendre transformation, is Hamiltonian with respect to P_0:

$$\Gamma_g = -P_0(dH_g);$$

and quasi-Hamiltonian with respect to P_L:

$$(\det L)\Gamma_g = -P_L(dK),$$

where objects corresponding in an obvious way under the Legendre transformation have been denoted in the same way. Thus Γ_g is quasi-bi-Hamiltonian (actually, Pfaffian quasi-bi-Hamiltonian) in the terminology of [10].

Fairly standard arguments can be applied in this situation to show the existence of quadratic first integrals of the geodesic flow in involution; the following version is based on [3]. Theorem 4 of that paper states that if a vector field Z on a manifold N is quasi-Hamiltonian with respect to a Poisson bivector Π with corresponding Poisson map P, so that $FZ = -P(dH)$ for some functions F and H on N with F non-vanishing, then

$$\hat{\Pi} = \Pi + \left(Z + zF^{-1}P(dF)\right) \wedge \frac{\partial}{\partial z}$$

is a Poisson bivector on $N \times \mathbb{R}$, where z is the coordinate on $\mathbb{R}$, such that $\hat{\Pi}$ reduces to Π and the Hamiltonian vector field $-\hat{P}(dz)$ restricts to Z on $z = 0$. Furthermore, $H + zF$ is a Casimir of $\hat{\Pi}$.

Now if L is a special conformal Killing tensor then so is $L(s) = L + sI$ for any constant s. Moreover, if L is invertible then so is $L(s)$ for s sufficiently close to zero. But then $L(s)$ generates a metric projectively equivalent to g for every s for which it is invertible. It follows from the previous arguments, with L replaced by $L(s)$, that

$$(\det L(s))\Gamma_g = -P_{L(s)}(dH(s)) \quad \text{where } H(s) = \tfrac{1}{2}\det L(s)\overline{L(s)}^{ij}p_ip_j.$$

Thus Γ_g is the restriction to $z = 0$ of the Hamiltonian vector field $-\hat{P}(s)(dz)$ with respect to the Poisson bivector

$$\begin{aligned}\hat{\Pi}(s) &= \Pi_{(L+sI)} + \left(\Gamma_g + zF(s)^{-1}P(dF(s))\right) \wedge \frac{\partial}{\partial z} \\ &= \Pi_L + s\Pi_0 + \left(\Gamma_g + zF(s)^{-1}P(dF(s))\right) \wedge \frac{\partial}{\partial z},\end{aligned}$$

where $F(s) = \det L(s)$; moreover, $H(s) + zF(s)$ is a Casimir of $\hat{\Pi}(s)$. Now the elements of $(\det L(s))L(s)^{-1}$, considered as a matrix, are the cofactors of $L(s) = L + sI$, and are therefore polynomials of degree at most $n-1$ in s. Thus $H(s)$ is a polynomial of degree $n-1$ in s, whose coefficients are quadratic functions of the momenta. Let

$$H(s) = \sum_{m=1}^{n} H_{(m)} s^{n-m} \quad \text{where } H_{(m)} = \tfrac{1}{2} A^{ij}_{(m)} p_i p_j;$$

then $H_{(1)} = H_g$ and $H_{(n)} = K$. Also, let $F(s) = \sum_{m=0}^{n} \sigma_{(m)} s^{n-m}$; thus $\sigma_{(m)}$ is the symmetric polynomial of degree m in the eigenvalues of L, $\sigma_{(0)} = 1$, and $\sigma_{(n)} = \det L$.

I now show that the functions $H_{(m)}$ are first integrals of Γ_g which are in involution with respect to both the standard Poisson bracket on T^*M and the Poisson bracket associated with Π_L. Since $H(s) + zF(s)$ is a Casimir of $\hat{\Pi}(s)$, it is a first integral of any Hamiltonian vector field, for every s. Since Γ_g is the restriction to $z = 0$ of a Hamiltonian vector field, $H(s)$ is a first integral of Γ_g for every s, and therefore $H_{(m)}$ is a first integral for $m = 1, 2, \ldots, n$. It also follows from the fact that $H(s) + zF(s)$ is a Casimir that (taking $z = 0$)

$$P_L(dH(s)) + sP_0(dH(s)) = -F(s)\Gamma_g.$$

Equating coefficients of like powers of s gives

$$P_L(dH_{(m)}) + P_0(dH_{(m+1)}) = -\sigma_{(n-m)}\Gamma_g$$

for $1 \leq m \leq n-1$; the coefficients of s^n and the constant terms give $P_0(dH_{(1)}) = -\Gamma_g$ and $P_L(dH_{(n)}) = -(\det L)\Gamma_g$, which just restates the fact that Γ_g is quasi-bi-Hamiltonian. It follows that

$$\{H_{(l)}, H_{(m)}\}_L + \{H_{(l)}, H_{(m+1)}\} = -\sigma_{(n-m)}\Gamma_g(H_{(l)}) = 0$$

for $1 \leq l \leq n$ and $1 \leq m \leq n-1$, from which it follows by induction that $\{H_{(r)}, H_{(s)}\} = \{H_{(r)}, H_{(s)}\}_L = 0$.

3 Another derivation of first integrals

The argument in [8] leading to the existence of quadratic first integrals of the geodesic flow for projectively equivalent metrics can be expressed in a form which makes its relation with the results outlined above somewhat clearer. It is based on an analysis of the recursion operator ${\omega_g}^{-1} \circ \Phi^* \omega_h$. I shall compute the recursion operator explicitly.

As is often the case in such calculations, it is advantageous to use a basis of 1-forms on TM adapted to the Levi-Civita connection, given by $\{dx^i, \vartheta^i\}$ where

$$\vartheta^i = du^i + \Gamma^i_{jk} u^k dx^j.$$

With respect to this basis ω_h takes the simple form

$$\omega_h = h_{ik|j} u^k dx^i \wedge dx^j + h_{ij} \vartheta^i \wedge dx^j,$$

for any symmetric tensor h_{ij}; in particular, $\omega_g = g_{ij} \vartheta^i \wedge dx^j$. Now $\Phi^* \vartheta^i = f \vartheta^i + u^i df$, and $dH_h = \frac{1}{2} h_{jk|i} u^j u^k dx^i + h_{ij} u^j \vartheta^i$. After some calculation, using these formulae, one finds that $\Phi^* \omega_h$ takes the form

$$\Phi^* \omega_h = R_{ij} \vartheta^i \wedge dx^j,$$

where

$$R_{ij} = f \left(h_{ij} + \left(\frac{g_{ik} u^k}{g_{rs} u^r u^s} - \frac{h_{ik} u^k}{h_{rs} u^r u^s} \right) h_{jl} u^l \right).$$

The recursion operator is given by

$${\omega_g}^{-1} \circ \Phi^* \omega_h = g^{ik} R_{kj} X_i \otimes dx^j + g^{ik} R_{jk} \frac{\partial}{\partial u^i} \otimes \vartheta^j$$

where $\{X_i, \partial/\partial u^i\}$ is the basis of vector fields dual to the basis $\{dx^i, \vartheta^i\}$ of 1-forms.

Following [8], I next consider $\det(R + tI)$ where

$$R^i_j = g^{ik} R_{kj} = f \left(h^i_j + \left(\frac{u^i}{g_{rs} u^r u^s} - \frac{h^i_k u^k}{h_{rs} u^r u^s} \right) h_{jl} u^l \right).$$

Now $R^i_j u^j = f^{-1} u^j$, so $\det(R + tI)$ is divisible by $f^{-1} + t$. The coefficients of the powers t^m, $m = 0, 1, \ldots, n-2$, in the polynomial

$$\delta(t) = \frac{\det(R + tI)}{(f^{-1} + t)}$$

(whose degree is $n - 1$ and whose leading term is t^{n-1}) are first integrals of Γ_g. This may be established as follows. Set $g_{ij} \vartheta^j = \vartheta_i$; then

$$\Phi^* \omega_h + t \omega_g = (R^i_j + t \delta^i_j) \vartheta_i \wedge dx^j,$$

and therefore

$$\wedge^n(\Phi^*\omega_h + t\omega_g) = \det(R + tI) \wedge^n \omega_g,$$

where for any 2-form ω, $\wedge^k\omega$ is the $2k$-form obtained by taking the exterior product of k copies of ω; $\wedge^n\omega$ is a volume form when ω is non-singular. Thus

$$\Gamma_g \lrcorner (\wedge^n(\Phi^*\omega_h + t\omega_g)) = -n\det(R + tI) dH \wedge (\wedge^{n-1}\omega_g).$$

But on the other hand,

$$\begin{aligned}\Gamma_g \lrcorner (\wedge^n(\Phi^*\omega_h + t\omega_g)) &= n(\Gamma_g \lrcorner (\Phi^*\omega_h + t\omega_g)) \wedge (\wedge^{n-1}(\Phi^*\omega_h + t\omega_g)) \\ &= -n(f^{-1} + t) dH \wedge (\wedge^{n-1}(\Phi^*\omega_h + t\omega_g)),\end{aligned}$$

so that

$$\delta(t) dH \wedge (\wedge^{n-1}\omega_g) = dH \wedge (\wedge^{n-1}(\Phi^*\omega_h + t\omega_g)).$$

Now $\mathcal{L}_{\Gamma_g} dH = 0$ and $\mathcal{L}_{\Gamma_g}\omega_g = 0$, while $\mathcal{L}_{\Gamma_g}(\Phi^*\omega_h) = f^{-2} df \wedge dH$. It follows that

$$\Gamma_g(\delta(t)) dH \wedge (\wedge^{n-1}\omega_g) = 0;$$

but $dH \wedge (\wedge^{n-1}\omega_g) = 0$ if and only if H is constant, so $\Gamma_g(\delta(t)) = 0$.

I shall now show how the coefficients of $\delta(t)$ may be expressed in terms of the quadratic first integrals $H_{(m)}$ obtained earlier.

When h is expressed in terms of L, R can be written

$$R^i_j = \kappa \bar{L}^l_j \left(\delta^i_l + \left(\frac{u^i}{g_{rs}u^r u^s} - \frac{\bar{L}^i_k u^k}{\bar{L}_{rs}u^r u^s} \right) g_{lm} u^m \right).$$

Thus R takes the form $R = \kappa(I + v \otimes \theta)L^{-1}$ where the vector v is given by

$$v^i = \frac{u^i}{g_{rs}u^r u^s} - \frac{\bar{L}^i_k u^k}{\bar{L}_{rs}u^r u^s}$$

and the covector θ by $\theta_i = g_{im}u^m$. Thus

$$\begin{aligned}&\det(R + tI) = (\det L)^{-1} \det(\kappa I + tL + \kappa v \otimes \theta) \\ &\quad = (\det L)^{-1}(t^n \det L(\kappa/t) + \kappa t^{n-1} \langle (\det L(\kappa/t)) L(\kappa/t)^{-1} v, \theta \rangle\end{aligned}$$

using the fact that the rank of $v \otimes \theta$ is 1 (recall that $(\det L(\kappa/t))L(\kappa/t)^{-1}$ is the cofactor matrix of $L(\kappa/t) = L + (\kappa/t)I$). Now

$$t^{n-1}(\det L(\kappa/t))L(\kappa/t)^{-1} = \sum_{m=1}^{n} \tfrac{1}{2}\kappa^{n-m} t^{m-1} A_{(m)},$$

while

$$t^n \det L(\kappa/t) = \sum_{m=0}^{n} \sigma_{(m)} \kappa^{n-m} t^m.$$

Thus

$$\begin{aligned} \det(R + tI) &= (\det L)^{-1}\kappa^n \left(1 + \tfrac{1}{2}A_{(1)ij}u^i v^j\right) \\ &\quad + (\det L)^{-1} \sum_{m=1}^{n-1} \left(\sigma_{(m)} + \tfrac{1}{2}A_{(m+1)ij}u^i v^j\right) \kappa^{n-m} t^m + t^n. \end{aligned}$$

Now

$$A_{(m+1)ij}u^i v^j = \frac{A_{(m+1)ij}u^i u^j}{g_{rs}u^r u^s} - \frac{A_{(m+1)ik}\bar{L}^k_j u^i u^j}{\bar{L}_{rs}u^r u^s},$$

whence $A_{(1)ij}u^i v^j = g_{ij}u^i v^j = 0$. It follows from the self-evident identity $((\det L(s)L(s)^{-1})L(s) = \det L(s)I$ that

$$\sigma_{(m)}L^{-1} - \tfrac{1}{2}A_{(m+1)}L^{-1} = \tfrac{1}{2}A_{(m)},$$

whence (recalling that $\kappa = \sqrt{H_{(1)}/H_{(n)}}$ and $f = \kappa \det L$)

$$\begin{aligned} &\det(R + tI) \\ &= (\det L)^{-1}\kappa^n + \tfrac{1}{2}(\det L)^{-1} \sum_{m=1}^{n-1} \left(\frac{H_{(m+1)}}{H_{(1)}} + \det L \frac{H_{(m)}}{H_{(n)}}\right) \kappa^{n-m} t^m + t^n \\ &= \frac{\kappa^{n+1}}{f} + \tfrac{1}{2} \sum_{m=1}^{n-1} \left(\frac{\kappa^{n-m-1}H_{(m+1)}}{fH_{(n)}} + \frac{\kappa^{n-m}H_{(m)}}{H_{(n)}}\right) t^m + t^n \\ &= \tfrac{1}{2}(f^{-1} + t) \sum_{m=1}^{n} \left(\frac{\kappa^{n-m}H_{(m)}}{H_{(n)}}\right) t^{m-1}. \end{aligned}$$

4 Hamilton-Jacobi separability

I now summarize some recent results on separability which, as a consequence of the existence of a special conformal Killing tensor, apply to spaces admitting projectively equivalent metrics; details are to be found in [2].

Consider a Riemannian manifold which admits a special conformal Killing tensor L. Suppose that L has functionally independent eigenfunctions, which is to say that there are n functions u^1, u^2, $\ldots$, u^n, such that at each point x, $u^i(x)$ is an eigenvalue of $L(x)$ for each i, and the Jacobian matrix $(\partial u^i/\partial x^j)$ is everywhere non-singular. The u^i may then be taken as local coordinates; as a consequence of the fact that its torsion vanishes, L takes the form

$$L = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \otimes du^i$$

with respect to them. Moreover they are orthogonal coordinates, so that $g_{ij} = 0$ for $i \neq j$, and it may be shown that the diagonal elements of the metric are given by $g_{ii} = \phi_i(u^i) \prod_{j\neq i}(u^i - u^j)$.

It follows from the fact that the functions $H_{(m)} = \frac{1}{2}A^{ij}_{(m)}p_i p_j$ are involutory first integrals of the geodesic flow that the tensors $A_{(m)}$ are independent commuting Killing tensors; they are moreover simultaneously diagonalizable and have closed common eigenforms. They therefore form a Stäckel system. It follows that the Hamilton-Jacobi equation for geodesics is separable in the orthogonal coordinates u^i.

With respect to coordinates adapted to a special conformal Killing tensor with independent eigenfunctions, $R_{ijik} = 0$ for $j \neq k$; and conversely, provided that a certain technical condition is satisfied, if, with respect to orthogonal separation coordinates, $R_{ijik} = 0$ for $j \neq k$, then the separation coordinates can be derived from a special conformal Killing tensor.

Suppose that a Riemannian space admits two special conformal Killing tensors which are independent in the sense that they have no non-trivial common invariant subspaces. Suppose further that one of them has functionally independent eigenfunctions. Then the space is a space of constant curvature.

So the class of metrics g which have projectively equivalent ones h for which the type $(1,1)$ tensor $\bar{h}^{ik}g_{jk}$ has functionally independent eigenfunctions is identical with a certain subclass of the class of metrics for which the Hamilton-Jacobi equation is separable in orthogonal coordinates, namely those that admit a special conformal Killing tensor with functionally independent eigenfunctions.

5 The case of coincident eigenfunctions

Finally, I shall consider the case in which the special conformal Killing tensor L has coincident eigenfunctions.

Due to the vanishing of the torsion of L, we can find local coordinates $(u_1, u_2, \ldots, u_m)$, where $u_a = (u_a^1, u_a^2, \ldots, u_a^{N_a})$, such that for each $a = 1, 2, \ldots, m$, $\{\partial/\partial u_a^i\}$ is a basis for the ath eigendistribution of L, with eigenfunction λ_a say. I assume that the λ_a are distinct on the coordinate neighbourhood. Each eigenfunction λ_a is a function only of the corresponding coordinates u_a^i; in particular, if the eigendistribution associated with λ_a is 1-dimensional we can take $u_a^1 = \lambda_a$. Then as a type $(1, 1)$ tensor, L is given by

$$L = \sum_{a=1}^{m} \sum_{i=1}^{N_a} \lambda_a \frac{\partial}{\partial u_a^i} \otimes du_a^i.$$

The 1-form α, where $L_{ij|k} = \frac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$, is the exterior derivative of the trace of L; that is,

$$\alpha = d\left(\sum_{a=1}^{m} N_a \lambda_a\right) = \sum_{a=1}^{m} \sum_{i=1}^{N_a} N_a \frac{\partial \lambda_a}{\partial u_a^i} du_a^i$$

since λ_a is a function of the u_a^i alone.

The eigendistributions corresponding to distinct eigenfunctions of L are orthogonal. Let us write i_a for the ith index of the ath set of coordinates. Then for $a \neq b$, $g_{i_a j_b} = 0$ and also $g^{i_a j_b} = 0$. The summation convention holds henceforth only for indices with the same suffix, and summation is over the range determined by that suffix. Thus for example $g_{i_a k_a} g^{j_a k_a} = \delta_{i_a}^{j_a}$, and the implied sum is over $k_a = 1, 2, \ldots, N_a$. Due to the orthogonality of coordinates for distinct a, b, the connection coefficients are given as follows:

$$\begin{aligned}
\Gamma^{i_a}_{j_b k_c} &= 0 \quad \text{when } a,\ b \text{ and } c \text{ are distinct;} \\
\Gamma^{i_a}_{j_a k_b} &= \tfrac{1}{2} g^{i_a l_a} \frac{\partial g_{j_a l_a}}{\partial u_b^k}, \quad \Gamma^{i_a}_{j_b k_b} = -\tfrac{1}{2} g^{i_a l_a} \frac{\partial g_{j_b k_b}}{\partial u_a^l}, \quad \text{for } a \neq b; \\
\Gamma^{i_a}_{j_a k_a} &= \tfrac{1}{2} g^{i_a l_a} \left(\frac{\partial g_{j_a l_a}}{\partial u_a^k} + \frac{\partial g_{k_a l_a}}{\partial u_a^j} - \frac{\partial g_{j_a k_a}}{\partial u_a^l} \right).
\end{aligned}$$

The equations

$$L_{i_a j_b | k_c} = \tfrac{1}{2} \left(\alpha_{i_a} g_{j_b k_c} + \alpha_{j_b} g_{i_a k_c} \right)$$

are identically satisfied except when $a = c$ or $b = c$. When $a = b = c$ they are formally the same as the equations for a special conformal Killing tensor on the N_a-dimensional space with metric $g_{i_a j_a}$; the conformal Killing tensor is however a scalar multiple of the metric, and in such a case the scalar factor must be constant if $N_a > 1$. In fact the equations are

$$\frac{\partial \lambda_a}{\partial u_a^k} g_{i_a j_a} = \tfrac{1}{2} N_a \left(\frac{\partial \lambda_a}{\partial u_a^i} g_{j_a k_a} + \frac{\partial \lambda_a}{\partial u_a^j} g_{i_a k_a} \right);$$

on taking the trace on j_a and k_a one finds that

$$\frac{\partial \lambda_a}{\partial u_a^i} = \tfrac{1}{2} N_a(N_a+1)\frac{\partial \lambda_a}{\partial u_a^i},$$

whence $\partial\lambda_a/\partial u_a^i = 0$ for $N_a > 1$. The equations

$$L_{i_a j_b | k_a} = \tfrac{1}{2}\alpha_{j_b} g_{i_a k_a}, \quad a \neq b$$

become

$$(\lambda_b - \lambda_a)\frac{\partial g_{i_a k_a}}{\partial u_b^j} = \frac{\partial \lambda_b}{\partial u_b^j} g_{i_a k_a}.$$

Thus $g_{i_a j_a}$ is independent of all those u_b^k for which $N_b > 1$, while if $N_b = 1$,

$$(u_b^1 - \lambda_a)\frac{\partial g_{i_a j_a}}{\partial u_b^1} = g_{i_a j_a}.$$

The coordinates u_b^1 for which $N_b = 1$ thus have a special role; accordingly, I shall redefine the coordinates as (u^r, u_a^i), where the coordinates u^r are those corresponding to 1-dimensional eigendistributions, the u_a^i being defined as before except that $N_a > 1$; thus r ranges from 1 to $n - \sum_{a=1}^m N_a$. Then

$$g_{rr} = \phi_i(u^r)\prod_{q\neq r}(u^r - u^q), \quad g_{qr} = 0 \text{ for } q \neq r;$$

while

$$g_{i_a j_a} = \prod_r (u^r - \lambda_a)\hat{g}_{i_a j_a},$$

where $\hat{g}_{i_a j_a}$ is a function of the u_a^k alone. This result is ascribed to Levi-Civita in [8]. It may be summarized by saying that if a Riemannian manifold (M, g) admits a projectively equivalent one then it is formed from Riemannian manifolds (M_0, g_0), (M_1, g_1), ..., (M_m, g_m), where (M_0, g_0) admits a special conformal Killing tensor with functionally independent eigenvalues, and is therefore separable as described above; the other Riemannian manifolds are arbitrary; and g is an orthogonal direct sum $g = g_0 + \sum_{a=1}^m p_a g_a$ where p_a is a function on M_0 given in terms of coordinates u^r adapted to the special conformal Killing tensor by $p_a = \prod_r (u^r - \lambda_a)$, the λ_a being m distinct constants.

References

[1] Benenti, S: Inertia tensors and Stäckel systems in the Euclidean spaces *Rend. del Sem. Mat. Torino* **50** (1992) 1–20

[2] Crampin, M: Conformal Killing tensors with vanishing torsion and the separation of variables in the Hamilton-Jacobi equation *Preprint* The Open University (2001)

[3] Crampin, M. and W. Sarlet: A class of non-conservative Lagrangian systems on Riemannian manifolds *J. Math. Phys.* (2001) to appear

[4] Crampin, M., W. Sarlet and G. Thompson: Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors *J. Phys. A: Math. Gen.* **33** (2000) 8755–8770

[5] Ibort, A., F. Magri and G. Marmo: Bi-hamiltonian structures and Stäckel separability *J. Geom. Phys.* **33** (2000) 210–223

[6] Kosmann-Schwarzbach, Y. and F. Magri: Poisson-Nijenhuis structures *Ann. Inst. Henri Poincaré, Phys. Théor.* **53** (1990) 35–81

[7] Lundmark, H: Higher-dimensional integrable Newton systems with quadratic integrals of motion *Preprint* Linköping University (1999)

[8] Matveev, V.S. and P.Ĭ. Topalov: Trajectory equivalence and corresponding integrals *Regular and Chaotic Dynamics* **3** (1998) 30–45

[9] Matveev, V.S. and P.J. Topalov: Integrability in the theory of geodesically equivalent metrics *J. Phys. A: Math. Gen.* **34** (2001) 2415–2433

[10] Morosi, C. and G. Tondo: Quasi-bi-Hamiltonian systems and separability *J. Phys. A: Math. Gen.* **30** (1997) 2799–2806

[11] Rauch-Wojciechowski, S., K. Marciniak and H. Lundmark: Quasi-Lagrangian systems of Newton equations *J. Math. Phys.* **39** (1998) 5292–5306

[12] Schouten, J.A: *Ricci-calculus* (Springer, 1954) Ch.III §7, Ch.VI §2

[13] Topalov, P: Integrability criterion of geodesical equivalence. Hierarchies *Acta Appl. Math.* **59** (1999) 271–298

[14] Topalov, P: Families of metrics geodesically equivalent to the analogs of the Poisson sphere *J. Math. Phys.* **41** (2000) 7510–7520

[15] Yano, K. and S. Ishihara: *Tangent and Cotangent Bundles* (Marcel Dekker, 1973) p. 242, p. 253