Randers spaces with reversible geodesics

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Abstract

A Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. The conditions for a Randers space to have reversible geodesics are obtained; this leads to a new simple proof of a well-known theorem giving necessary and sufficient conditions for a Randers space to be Berwald.

A geodesic in a Finsler space (where the Finsler function is *positively* homogeneous) should be thought of as an oriented path, that is, an imbedded one-dimensional submanifold with a sense of direction, or an equivalence class of curves determined up to reparametrization with positive derivative. There is in general no reason why a path which coincides with a geodesic as a point set but is traversed in the opposite direction should be a geodesic. If a Finsler space has the property that all of its geodesics remain geodesics when their orientation is reversed I shall say that the space has reversible geodesics. If the space is such that when $t \mapsto x^i(t)$ is a geodesic with constant Finslerian speed then $t \mapsto x^i(-t)$ is also a geodesic with constant Finslerian speed then I shall say that the space has strictly reversible geodesics.

A Riemannian space has strictly reversible geodesics, and so more generally has a Finsler space whose Finsler function is absolutely homogeneous. However, these examples do not by any means exhaust the possibilities for Finsler spaces with reversible geodesics. Consider a Randers space, with Finsler function

$$F(x, y) = \alpha + \beta = \sqrt{a_{ij}y^iy^j} + b_iy^i$$

where $a_{ij}b^ib^j < 1$. If $b_{j|i} = b_{i|j}$, where the $b_{i|j}$ are the components of the covariant differential of b_i with respect to the Levi-Civita connection of the Riemannian metric a_{ij}, then the Randers space has reversible geodesics (it is a Douglas space, and its geodesics, as paths, coincide with the geodesics of the Riemannian metric); if $b_{i|j} = 0$ then its geodesics are strictly reversible (it is a Berwald space, and its geodesics with constant

Finslerian speed coincide wth the affinely parametrized geodesics of the Riemannian metric). In fact the equation for geodesics with constant Finslerian speed is [1]

$$\begin{aligned}
&\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + a^{ij}(b_{j|k} - b_{k|j})\dot{x}^k\alpha(\dot{x}) \\
&\quad = \frac{1}{F}\dot{x}^i\left(a^{jl}b_l(b_{j|k} - b_{k|j})\dot{x}^k\alpha(\dot{x}) - b_{j|k}\dot{x}^j\dot{x}^k\right)
\end{aligned}$$

where Γ^i_{jk} are the Riemannian connection coefficients; it is clear that if $b_{j|i} = b_{i|j}$ then the Finslerian geodesics are projectively equivalent to the Riemannian ones, while if $b_{i|j} = 0$ they are identical. One aim of this note is to prove the converse, namely that if a Randers space has reversible geodesics then $b_{j|i} = b_{i|j}$, and if it has strictly reversible geodesics then $b_{i|j} = 0$. These results generalize in some small way the well-known result that the vanishing of $b_{i|j}$ is the necessary and sufficient condition for a Randers space to be Berwald, and enable one to view that result from a new perspective, as well as providing a simple proof of it, different from the one in [3], that requires practically no calculation (which cannot be said of the derivation of the explicit geodesic spray coefficients of a Randers space quoted above).

I shall discuss the reversibility of geodesics in some generality. In fact the definitions of reversibility and corresponding conditions can be formulated for any spray. Consider a spray

$$\Gamma = y^i\frac{\partial}{\partial x^i} + G^i(x,y)\frac{\partial}{\partial y^i}$$

where the G^i are positively homogeneous of degree 2 in the y^i. A curve $t \mapsto x^i(t)$ is a base integral curve of the spray if and only if it satisfies the equations $\ddot{x}^i = G^i(x,\dot{x})$. The curve $t \mapsto x^i(-t)$ is a base integral curve, up to reparametrization, if for some function $\varphi(t)$, $\ddot{x}^i = G^i(x,-\dot{x}) + \varphi\dot{x}^i$. Thus the spray is reversible, in the sense that the paths defined by its base integral curves remain so when their orientation is reversed, if and only if

$$G^i(x,-y) = G^i(x,y) + \lambda(x,y)y^i$$

for all $y^i \neq 0$, for some function λ, which must clearly be absolutely homogeneous of degree 1 in y^i. The spray is strictly reversible if $\lambda = 0$.

We can express the condition for reversibility in a rather more elegant form, as follows. Denote by ρ the 'reflection map' $(x,y) \mapsto (x,-y)$, and for any spray Γ set $\bar{\Gamma} = -\rho_*\Gamma$. Then

$$\bar{\Gamma} = y^i\frac{\partial}{\partial x^i} + G^i(x,-y)\frac{\partial}{\partial y^i},$$

so it is natural to call $\bar{\Gamma}$ the reverse of Γ. Then Γ is reversible if and only if it is projectively equivalent to its reverse, and strictly reversible if and only if the two are equal.

The concept of reversibility is a projective one; that is to say, if a spray is reversible so are all sprays projectively equivalent to it. In fact a spray is reversible if and only if its projective equivalence class is invariant under the map which takes a spray to its reverse.

Since we have to deal with projectively equivalent sprays, the following simple observations about the geodesic sprays of Finsler spaces will prove very useful. Let F be a Finsler function — by assumption, positively homogeneous, and strongly convex, so that its fundamental tensor g_{ij} is positive-definite; then a spray Γ belongs to the projective equivalence class of geodesic sprays of F if and only if the Euler-Lagrange equation with Lagrangian F,

$$\Gamma\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0,$$

is satisfied. The geodesic spray Γ with constant Finslerian speed is singled out from amongst all those satisifying the Euler-Lagrange equation by the additional condition that $\Gamma(F) = 0$. I shall speak of 'a geodesic spray' when I mean any spray of the projective class of solutions of the Euler-Lagrange equation for F, and 'the geodesic spray' when I mean the one with constant Finslerian speed. With this choice, if $\tilde{\Gamma}$ is any projectively equivalent spray, so that $\tilde{\Gamma} = \Gamma + \lambda\Delta$ where Δ is the Liouville vector field and λ is homogeneous of degree 1 in y^i, then

$$\tilde{\Gamma}(F) = \Gamma(F) + \lambda\Delta(F) = \lambda F,$$

so we have an explicit expression for λ, namely

$$\lambda = \frac{\tilde{\Gamma}(F)}{F}.$$

(Though it may not be immediately obvious, these results are essentially equivalent to those given by Shen in [5], Theorem 12.2.6. See also [6] for an intrinsic formulation of this and equivalent conditions, originally due to Rapcsák [4].)

It follows that a Finsler space has reversible geodesics if and only if

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0,$$

where $\bar{\Gamma}$ is the reverse of a geodesic spray Γ; it will be enough to check reversibility when Γ is the geodesic spray. Furthermore, the geodesics are strictly reversible if and only if, in addition, $\bar{\Gamma}(F) = 0$.

Now if F is any Finsler function, and $\bar{F}$ is defined by $\bar{F}(x, y) = F(x, -y)$ then $\bar{F}$ is also a Finsler function; it is certainly positively homogeneous in y^i, and its fundamental tensor $\bar{g}_{ij}$ is given by $\bar{g}_{ij}(x, y) = g_{ij}(x, -y)$ (where g_{ij} is the fundamental tensor of F), so $\bar{g}_{ij}$, like g_{ij}, is everywhere positive definite. The geodesic spray $\bar{\Gamma}$ of $\bar{F}$ is just the reverse of the geodesic spray of F.

We can now apply these observations to a Randers space, with

$$F = \alpha + \beta = \sqrt{a_{ij}y^i y^j} + b_i y^i,$$

to show that the necessary and sufficient condition for the space to have reversible geodesics is that $b_{k|j} = b_{j|k}$, and strictly reversible geodesics that $b_{j|k} = 0$. Of course, another way of saying that $b_{k|j} = b_{j|k}$ is that the 1-form $b = b_i dx^i$ is closed. Given that b is closed, another way of saying that $b_{j|k} = 0$ is that the function $\beta = b_i y^i$ is a first integral of the geodesic flow of the Riemannian metric a_{ij}. So we may equivalently say that the necessary and sufficient condition for the Randers space to have reversible geodesics is that b is closed, and the necessary and sufficient condition for its geodesics to be strictly reversible is that b is closed and $\beta = b_i y^i$ is a first integral of the Riemannian geodesic flow.

These results about reversibility of geodesics in a Randers space are in fact particular cases (though probably the most interesting ones) of more general, but similar, results concerning Randers changes. Let F_0 be a Finsler function, and $b = b_i dx^i$ a 1-form on the base manifold such that

$$\sup_{F_0(y)=1} |b_i y^i| < 1;$$

then $F(x,y) = F_0(x,y) + b_i(x)y^i$ is again a Finsler function, and the process of transforming F_0 to F is called a Randers change (see, for example, [5] and [6]). Suppose that F_0 is absolutely homogeneous; then the necessary and sufficient condition for F to have reversible geodesics is that b is closed, and the necessary and sufficient condition for the geodesics to be strictly reversible is that b is closed and $\beta = b_i y^i$ is a first integral of the geodesic flow of F_0. I shall devote the rest of this note to proving these assertions.

The necessary and sufficient condition for F to have reversible geodesics is that

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0$$

where $\bar{\Gamma}$ is the reverse of Γ, the geodesic spray of F; moreover, $\bar{\Gamma}$ is the geodesic spray of $\bar{F}$. The necessary and sufficient condition for F to have strictly reversible geodesics is that, in addition, $\bar{\Gamma}(F) = 0$. Now $F = F_0 + \beta$ where F_0 is absolutely homogeneous. Then $\bar{F} = F_0 - \beta$, so $F = \bar{F} + 2\beta$. Since

$$\bar{\Gamma}\left(\frac{\partial \bar{F}}{\partial y^i}\right) - \frac{\partial \bar{F}}{\partial x^i} = 0,$$

we have

$$\begin{aligned}
\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} &= 2\left(\bar{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i}\right) \\
&= 2\left(\bar{\Gamma}(b_i) - \frac{\partial b_j}{\partial x^i} y^j\right) \\
&= 2\left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) y^j.
\end{aligned}$$

It follows that if F is obtained by a Randers change from an absolutely homogeneous Finsler function then it is geodesically reversible if and only the 1-form defining the Randers change is closed.

Notice that for any spray $\tilde{\Gamma}$,

$$\tilde{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i} = \left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) y^j.$$

So for the geodesic spray Γ_0 of the 'reference' Finsler function F_0

$$\Gamma_0\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = \Gamma_0\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i} = \left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) y^j,$$

from which it follows that the closure of b is also the necessary and sufficient condition for Γ_0 to be projectively equivalent to Γ; if b is closed we have $\Gamma = \Gamma_0 - \lambda\Delta$ with $\lambda = \Gamma_0(F)/F = \Gamma_0(\beta)/(F_0+\beta)$, and similarly $\bar{\Gamma} = \Gamma_0 - \mu\Delta$ with $\mu = -\Gamma_0(\beta)/(F_0-\beta)$. Thus given that b is closed, the condition for F to have strictly reversible geodesics, so that $\Gamma = \bar{\Gamma}$, is that $\Gamma_0(\beta)/(F_0+\beta) = -\Gamma_0(\beta)/(F_0-\beta)$, or $\Gamma_0(\beta) = 0$.

In fact, when b is closed the geodesic sprays of both F and $\bar{F}$ are projectively equivalent to the (strictly reversible) geodesic spray of F_0; and when $\Gamma_0(\beta) = 0$ the two geodesic sprays coincide with the geodesic spray of F_0. (Projective equivalence under a Randers change is discussed in [5] and [6]. The condition on b was originally found by Hashiguchi and Ichijyō [2].)

The necessary and sufficient conditions for a Randers space to be Douglas or Berwald are simple corollaries of the results just obtained. Those results apply of course to a Randers space, with F_0 the Riemannian Finsler function. If a Randers space is a Douglas space, so that its geodesic spray is projectively equivalent to an affine spray, then the geodesics of the Randers space must be reversible, so b must be closed. If a Randers space is Berwald, so that its geodesic spray is affine, its geodesics must be strictly reversible, so β must be a first integral of the Riemannian geodesic flow. In each case, the affine spray is the Riemannian geodesic spray.

Finally, I shall point out how an example of Shen's [5] provides a memorable illustration of a Randers space with non-reversible geodesics. We start with the spray Γ on R^2 given by

$$\Gamma = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \alpha\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right), \quad \alpha = \sqrt{u^2+v^2},$$

where now (x, y) are the base coordinates and (u, v) the fibre coordinates. This spray is manifestly non-reversible. Its base integral curves are in fact circles of constant radius 1, traversed in the anti-clockwise sense. To see this, note first that $\Gamma(u^2+v^2) = 0$, which means that $\dot{x}^2 + \dot{y}^2$ is constant on any base integral curve. For a point describing the circle with centre (a, b) and radius 1, with constant speed α in the anti-clockwise sense, we have $(x-a)^2 + (y-b)^2 = 1$; $\dot{x}(x-a) + \dot{y}(y-b) = 0$; $\dot{x} = -\alpha(y-b)$, $\dot{y} = \alpha(x-a)$ with $\alpha = \sqrt{\dot{x}^2+\dot{y}^2}$ constant — note that at $x = a+1$, $y = b$ we have $\dot{x} = 0$, $\dot{y} = \alpha > 0$ as is required for the motion to be anti-clockwise; and finally $\ddot{x} = -\alpha\dot{y}$, $\ddot{y} = \alpha\dot{x}$, so the circle is indeed a base integral curve of Γ. Consider now the function

$$F(x, y, u, v) = \sqrt{u^2+v^2} + \tfrac{1}{2}(yu - xv) = \alpha + \beta.$$

I show that Γ is a geodesic spray of this function, by calculating the Euler-Lagrange expressions, using the fact that (due to rotational symmetry)

$$\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right)(\alpha) = 0;$$

we easily find that

$$\begin{aligned}\Gamma\left(\frac{\partial F}{\partial u}\right)-\frac{\partial F}{\partial x} &= \Gamma\left(\frac{u}{\alpha}+\tfrac{1}{2}y\right)+\tfrac{1}{2}v=-v+\tfrac{1}{2}v+\tfrac{1}{2}v=0\\ \Gamma\left(\frac{\partial F}{\partial v}\right)-\frac{\partial F}{\partial y} &= \Gamma\left(\frac{v}{\alpha}-\tfrac{1}{2}x\right)-\tfrac{1}{2}u=u-\tfrac{1}{2}u-\tfrac{1}{2}u=0.\end{aligned}$$

Now F is a Finsler function on the open disc $x^2+y^2<4$; so we have here an example of a Finsler function whose geodesics are unit circles — but always traversed in the anti-clockwise sense.

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References

[1] D. Bao, S.-S. Chern and Z. Shen *An Introduction to Riemann-Finsler Geometry* (Springer, 2000): Chapter 11, Section 11.3.

[2] M. Hashiguchi and Y. Ichijyō Randers spaces with rectilinear geodesics *Rep. Fac. Sci. Kagoshima Univ.* **13** (1980), 33–40.

[3] S. Kikuchi On the condition that a space with (α,β) metric be locally Minkowskian *Tensor (N.S.)* **33** (1979), 242–246.

[4] A. Rapcsák Über die bahntreuen Abbildungen metrischer Räume *Publ. Math. Debrecen* **8** (1961), 285–290.

[5] Z. Shen *Differential Geometry of Spray and Finsler Spaces* (Kluwer, 2001): Chapter 12.

[6] J. Szilasi and Sz. Vattamány On the Finsler-metrizabilities of spray manifolds *Period. Math. Hungar.* **44** (2002), 81–100.