Isotropic and R-flat sprays

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1 Introduction

It is well known that in a differentiable manifold of dimension greater than 2, equipped with a symmetric affine connection (or equivalently an affine spray), the vanishing of the projective curvature tensor is necessary and sufficient for the affinely connected manifold to be projectively flat, that is, for the affine spray to be projectively equivalent to one with vanishing Riemann curvature. The result is due to Weyl; for a classical proof see [3]. It has apparently not been noticed before that there is a directly analoguous result for general sprays. For a general spray, again in dimension greater than 2, the condition analoguous to the vanishing of the projective curvature tensor is that the spray is isotropic. A spray whose Riemann curvature vanishes is said to be R-flat (the terminology is due to Shen [4]). The result in question, which will be proved in this note, is that locally an isotropic spray is projectively equivalent to an R-flat one.

It was pointed out already by Douglas in 1928, in what could be regarded as the foundational paper in the projective geometry of sprays [2], that there is a natural analogue of the projective curvature tensor, which he called the generalized Weyl tensor; that in addition there is another projectively invariant curvature tensor associated with any spray, called the Douglas tensor, which has no affine counterpart; and that in dimension greater than 2 the vanishing of both the Douglas and the Weyl tensors is the necessary and sufficient condition for a spray to be projectively flat. (The term 'tensor' here, and throughout the paper, should strictly speaking be replaced by 'tensor along the tangent bundle projection'.) The argument proceeds by first establishing that the vanishing of the Douglas tensor is necessary and sufficient for a spray to be projectively equivalent to an affine one, and then applying the theorem which introduced this paper to the affine spray. As I shall show below, in dimension greater than 2 a spray is isotropic if and only if its generalized Weyl tensor vanishes.

I mention these familiar results in order to emphasise that something new is being offered in this paper, which I can perhaps best do by observing that the Douglas tensor makes no further appearance: the standard result described above says that a spray whose generalized Weyl tensor vanishes is projectively flat if its Douglas tensor also vanishes; the result to be proved here is that a spray whose generalized Weyl tensor vanishes is projectively R-flat with no further conditions.

2 Definitions, notation and basic results

I denote by $T^\circ M \to M$ the tangent bundle of M with the zero section deleted. Coordinates on $T^\circ M$ will generally be written (x^i, u^i). The Liouville field $u^i\partial/\partial u^i$ is denoted by Δ.

A spray Γ on $T^\circ M$ is a second-order differential equation field

$$u^i\frac{\partial}{\partial x^i} - 2\Gamma^i\frac{\partial}{\partial u^i}$$

such that the coefficients Γ^i are homogeneous of degree 2 in the u^i; if they are quadratic then the spray is affine.

The horizontal distribution associated with a spray is spanned by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j\frac{\partial}{\partial u^j}, \qquad \Gamma_i^j = \frac{\partial \Gamma^j}{\partial u^i};$$

Γ_i^j is of homogeneity degree 1. It will often be convenient to denote the vertical vector field $\partial/\partial u^i$ by V_i.

The bracket of a pair of such horizontal vector fields is given by

$$[H_i, H_j] = -R^l_{kij}u^k V_l = -R^l_{ij}V_l$$

where R^l_{kij} is the Riemann curvature of the spray (that is, the Riemann curvature of the associated Berwald connection):

$$R^l_{kij} = H_i\left(\Gamma^{\,l}_{jk}\right) - H_j\left(\Gamma^{\,l}_{ik}\right) + \Gamma^{\,l}_{im}\Gamma^m_{jk} - \Gamma^{\,l}_{jm}\Gamma^m_{ik}, \qquad \Gamma^{\,i}_{jk} = \frac{\partial \Gamma^i_j}{\partial u^k} = \frac{\partial \Gamma^i_k}{\partial u^j}.$$

In fact R^l_{kij}, $R^l_{ij} = R^l_{kij}u^k$ and $R^l_k = R^l_{ikj}u^iu^j$ all contain essentially the same information, since R^l_{kij} and R^l_{ij} are both completely determined by R^l_k, as follows:

$$R^l_{ij} = \tfrac{1}{3}\left(R^l_{i,j} - R^l_{j,i}\right), \qquad R^l_{kij} = R^l_{ij,k},$$

where the comma denotes the Berwald covariant derivative with respect to a vertical coordinate field, which is just the partial derivative with respect to the corresponding fibre coordinate. See for example [4] for the derivation of these formulae.

Two sprays Γ, $\hat{\Gamma}$ are projectively equivalent if $\hat{\Gamma} - \Gamma = -2\alpha\Delta$, or $\hat{\Gamma}^i = \Gamma^i + \alpha u^i$, where the function α is homogeneous of degree 1 in the u^i. This is the necessary condition for the two sprays to have the same geodesic paths, that is to say, for the base integral curves, with any given initial point and direction, of each to differ only by an orientation-preserving reparametrization.

It can be shown that the tensor P^l_{kij} given by

$$P^l_{kij} = \tilde{R}^l_{kij} - \frac{1}{n^2-1}\left(Q_{jk}\delta^l_i - Q_{ik}\delta^l_j - (Q_{ij} - Q_{ji})\delta^l_k\right),$$

where

$$\tilde{R}^l_{kij} = R^l_{kij} - \frac{1}{n+1}u^l R^m_{mij,k}, \quad Q_{ij} = \tilde{R}^k_{ikj} + n\tilde{R}^k_{jki},$$

is projectively invariant, that is, the same for projectively equivalent sprays. It is the counterpart of the projective curvature tensor of the affine theory, to which it reduces in the affine case. It has the same symmetries as the Riemann curvature, and in addition all of its traces vanish. As I mentioned before, this tensor was introduced, albeit in a somewhat different form, by Douglas [2], and called by him the generalized Weyl tensor; an intrinsic definition was given by del Castillo [1]. However, this tensor is not to be confused with the tensor called the Weyl curvature by Shen [4] and denoted by W^i_j, though this is also projectively invariant; in fact W^i_j bears the same relation to P^i_{jkl} as R^i_j does to R^i_{jkl}. In order to avoid confusion I shall call P^i_{jkl} the generalized projective curvature tensor.

When $\dim M = 2$ there is no non-zero tensor with the properties of P^l_{kij}; this explains in part why attention is restricted henceforth to manifolds M of dimension greater than 2.

3 Isotropic sprays

A spray is isotropic if and only if there is a function ρ and covector τ_i such that

$$R^i_j = \rho\delta^i_j + \tau_j u^i;$$

see for example [4]. Since $R^i_j u^j = 0$, $\rho = -\tau_j u^j$; then by taking the trace

$$\rho = \frac{1}{n-1}R = -\tau_i u^i, \quad R = R^k_k.$$

In fact τ_i is uniquely determined and can be expressed in terms of R^i_j, though we will not need the explicit expression. Note that τ_i is homogeneous of degree 1.

Given the relations between R^i_j, R^i_{jk} and R^i_{jkl} it should clearly be possible to express the condition for a spray to be isotropic equally well in terms of the latter two tensors. This is done in the following lemma.

Lemma A spray is isotropic if and only if

1. there is a covector ξ_i and skew-symmetric tensor η_{ij} such that
$$R^i_{jk} = \xi_k\delta^i_j - \xi_j\delta^i_k + \eta_{jk}u^i;$$

2. there are tensors λ_{ij}, μ_{ij} and ν_{ijk}, with λ_{ij} skew-symmetric and ν_{ijk} skew-symmetric in its second pair of indices, such that
$$R^i_{jkl} = \lambda_{kl}\delta^i_j + \mu_{jl}\delta^i_k - \mu_{jk}\delta^i_l + \nu_{jkl}u^i.$$

Proof Using the formulae expressing R^i_{jk} and R^i_{jkl} in terms of R^i_j given earlier, with $R^i_j = \rho\delta^i_j + \tau_j u^i$, one finds that

$$\begin{aligned} R^i_{jk} &= \tfrac{1}{3}\left((\rho_{,k} - \tau_k)\delta^i_j - (\rho_{,j} - \tau_j)\delta^i_k + (\tau_{j,k} - \tau_{k,j})u^i\right) \\ R^i_{jkl} &= \tfrac{1}{3}\left((\tau_{k,l} - \tau_{l,k})\delta^i_j + (\rho_{,jl} - \tau_{l,j})\delta^i_k - (\rho_{,jk} - \tau_{k,j})\delta^i_l + (\tau_{k,jl} - \tau_{l,jk})u^i\right). \end{aligned}$$

(These formulae are also to be found in [4].)

On the other hand, if $R^i_{jk} = \xi_k\delta^i_j - \xi_j\delta^i_k + \eta_{jk}u^i$ then since $R^i_j = R^i_{jk}u^k$ we find that
$$R^i_j = (\xi_k u^k)\delta^i_j + (\eta_{jk}u^k - \xi_j)u^i;$$
while if $R^i_{jkl} = \lambda_{kl}\delta^i_j + \mu_{jl}\delta^i_k - \mu_{jk}\delta^i_l + \nu_{jkl}u^i$, since $R^i_j = R^i_{kjl}u^k u^l$ we find that
$$R^i_j = (\mu_{kl}u^k u^l)\delta^i_j + ((\lambda_{jk} - \mu_{kj})u^k + \nu_{kjl}u^k u^l)u^i;$$
so the spray is isotropic in both cases. □

Corollary With $\dim M > 2$, if $R^i_{jk} = \xi_k\delta^i_j - \xi_j\delta^i_k + \eta_{jk}u^i$ for some ξ_i and skew-symmetric η_{jk}, then $\eta_{jk} = \xi_{k,j} - \xi_{j,k}$.

Proof It is easy to see that if $\pi_k\delta^i_j - \pi_j\delta^i_k + \sigma_{jk}u^i = 0$ for some π_i and skew-symmetric σ_{ij}, then provided $\dim M > 2$, $\pi_i = 0$ and $\sigma_{jk} = 0$. Thus by the lemma, if $R^i_{jk} = \xi_k\delta^i_j - \xi_j\delta^i_k + \eta_{jk}u^i$ then there are ρ and τ_i such that $\xi_i = \frac{1}{3}(\rho_{,i} - \tau_i)$ and $\eta_{jk} = \frac{1}{3}(\tau_{j,k} - \tau_{k,j}) = \xi_{k,j} - \xi_{j,k}$. □

With the lemma to hand it is easy to show that the property of being isotropic is equivalent to the vanishing of the generalized projective curvature tensor.

Theorem A spray on a manifold of dimension greater than 2 is isotropic if and only if its generalized projective curvature tensor vanishes; in particular, the property of being isotropic is projectively invariant.

Proof Suppose that $R^i_j = \rho\delta^i_j + \tau_j u^i$; then
$$R^i_{jkl} = \tfrac{1}{3}\left((\tau_{k,l} - \tau_{l,k})\delta^i_j + (\rho_{,jl} - \tau_{l,j})\delta^i_k - (\rho_{,jk} - \tau_{k,j})\delta^i_l + (\tau_{k,jl} - \tau_{l,jk})u^i\right).$$

It follows that

$$R^k_{kij} = \tfrac{1}{3}(n+1)(\tau_{i,j} - \tau_{j,i}),$$

whence

$$\tilde{R}^i_{jkl} = \tfrac{1}{3}\left((\tau_{k,l} - \tau_{l,k})\delta^i_j + (\rho_{,jl} - \tau_{l,j})\delta^i_k - (\rho_{,jk} - \tau_{k,j})\delta^i_l\right).$$

Therefore

$$\tilde{R}^k_{ikj} = \tfrac{1}{3}(n-1)\left(\rho_{,ij} + \tau_{i,j} - n\tau_{j,i}\right),$$

so that

$$Q_{ij} = \tfrac{1}{3}(n^2-1)(\rho_{,ij} - \tau_{i,j}).$$

Thus

$$P^i_{jkl} = \tilde{R}^i_{jkl} - \frac{1}{n^2-1}\left(-(Q_{kl} - Q_{lk})\delta^i_j + Q_{lj}\delta^i_k - Q_{kj}\delta^i_l\right) = 0.$$

On the other hand, if $P^i_{jkl} = 0$ then

$$R^i_{jkl} = \frac{1}{n^2-1}\left(-(Q_{kl} - Q_{lk})\delta^i_j + Q_{lj}\delta^i_k - Q_{kj}\delta^i_l\right) + \frac{1}{n+1}u^i R^m_{mkl,j},$$

so the spray is isotropic by the lemma. □

An alternative, intrinsic, proof is to be found in [5].

4 The main theorem

Theorem For $\dim M > 2$, a spray is isotropic if and only if it is locally projectively equivalent to one which is R-flat.

Proof A spray is R-flat if and only if its horizontal distribution is integrable.

Let H_i be the horizontal vector fields for the given isotropic spray; then

$$[H_i, H_j] = -R^k_{ij}V_k = -\xi_j V_i + \xi_i V_j - \eta_{ij}\Delta.$$

After a projective transformation with factor α the new horizontal vector fields $\hat{H}_i$ are given by

$$\hat{H}_i = H_i - \alpha V_i - \alpha_{,i}\Delta;$$

we have

$$[\hat{H}_i, \hat{H}_j] = -R^k_{ij}V_k + (\alpha_{|j} - \alpha\alpha_{,j})V_i - (\alpha_{|i} - \alpha\alpha_{,i})V_j + (\alpha_{|j,i} - \alpha_{|i,j})\Delta,$$

where the bar indicates a Berwald covariant derivative with respect to H_i. In deriving this formula (and elsewhere) it is useful to note that $[V_i, H_j] = -\Gamma^k_{ij}V_k = [V_j, H_i]$. The projectively transformed spray will have $[\hat{H}_i, \hat{H}_j] = 0$ if and only if

$$(\alpha_{|j} - \alpha\alpha_{,j})V_i - (\alpha_{|i} - \alpha\alpha_{,i})V_j + (\alpha_{|j,i} - \alpha_{|i,j})\Delta = R^k_{ij}V_k,$$

or equivalently

$$(\alpha_{|j} - \alpha\alpha_{,j})\delta_i^k - (\alpha_{|i} - \alpha\alpha_{,i})\delta_j^k + (\alpha_{|j,i} - \alpha_{|i,j})u^k = R_{ij}^k.$$

Notice that

$$\alpha_{|j,i} - \alpha_{|i,j} = (\alpha_{|j} - \alpha\alpha_{,j})_{,i} - (\alpha_{|i} - \alpha\alpha_{,i})_{,j}.$$

Thus if $[\hat{H}_i, \hat{H}_j] = 0$ then every projectively equivalent spray is isotropic. On the other hand, if we start with an isotropic spray then we will obtain one for which $[\hat{H}_i, \hat{H}_j] = 0$ if we can find α such that

$$\alpha_{|i} - \alpha\alpha_{,i} = \xi_i,$$

and which is homogeneous of degree 1. It may be more informative to write the conditions as

$$H_i(\alpha) = \alpha V_i(\alpha) + \xi_i, \quad \Delta(\alpha) = \alpha.$$

Thus α must satisfy a system of $n+1$ first-order partial differential equations. In a region in which $u^1 \neq 0$ (say) we can write these equations in the form

$$\frac{\partial\alpha}{\partial x^i} = \cdots, \qquad \frac{\partial\alpha}{\partial u^1} = \cdots,$$

where the only derivatives the right-hand sides contain are those with respect to u^2, u^3, $\ldots$, u^n, $n = \dim M$. A system of first-order partial differential equations which can be solved for certain derivatives in this way is integrable if cross-differentiation yields no new equations. Equivalently, the equations in the original form will be integrable provided that the integrability conditions

$$H_i(\alpha V_j(\alpha) + \xi_j) - H_j(\alpha V_i(\alpha) + \xi_i) + R_{ij}^k V_k(\alpha) = 0$$

and

$$H_i(\alpha) - \Delta(\alpha V_i(\alpha) + \xi_i) = 0$$

hold, by virtue of the original equations. The second of these clearly holds, since

$$\Delta(\alpha V_i(\alpha) + \xi_i) = \alpha V_i(\alpha) + \xi_i$$

(ξ_i is homogeneous of degree 1). For the first, we have from the condition which led to the partial differential equations

$$\begin{aligned} R_{ij}^k V_k(\alpha) &= (H_j(\alpha) - \alpha V_j(\alpha))V_i(\alpha) - (H_i(\alpha) - \alpha V_i(\alpha))V_j(\alpha) \\ &\quad + (V_i(H_j(\alpha)) - V_j(H_i(\alpha))\alpha \\ &= H_j(\alpha)V_i(\alpha) - H_i(\alpha)V_j(\alpha) + (H_j(V_i(\alpha)) - H_i(V_j(\alpha))\alpha \\ &= H_j(\alpha V_i(\alpha)) - H_i(\alpha V_j(\alpha)), \end{aligned}$$

from which it follows that the integrability conditions are satisfied if and only if

$$H_j(\xi_i) - H_i(\xi_j) = 0;$$

since the coefficients Γ^k_{ij} of the Berwald connection are symmetric this can be equivalently written

$$\xi_{i|j} - \xi_{j|i} = 0.$$

Now R^k_{ij} satisfies the Bianchi identity

$$R^l_{ij|k} + R^l_{jk|i} + R^l_{ki|j} = 0,$$

which with $R^k_{ij} = \xi_j \delta^k_i - \xi_i \delta^k_j + \eta_{ij} u^k$ becomes

$$(\xi_{j|k} - \xi_{k|j})\delta^l_i + (\xi_{k|i} - \xi_{i|k})\delta^l_j + (\xi_{i|j} - \xi_{j|i})\delta^l_k + (\eta_{jk|i} + \eta_{ki|j} + \eta_{ij|k})u^l = 0.$$

But $\eta_{ij} = \xi_{j,i} - \xi_{i,j}$, whence

$$\eta_{ij|k} = H_k(V_i(\xi_j)) - \Gamma^l_{kj} V_i(\xi_l) - \Gamma^l_{ki} V_l(\xi_j) - H_k(V_j(\xi_i)) + \Gamma^l_{ki} V_j(\xi_l) + \Gamma^l_{kj} V_l(\xi_i);$$

on taking the cyclic sum the terms involving connection coefficients all cancel, and we obtain

$$\begin{aligned}
&\eta_{jk|i} + \eta_{ki|j} + \eta_{ij|k} \\
&\quad = \; H_i(V_j(\xi_k)) - H_i(V_k(\xi_j)) + H_j(V_k(\xi_i)) - H_j(V_i(\xi_k)) + H_k(V_i(\xi_j)) - H_k(V_j(\xi_i)) \\
&\quad = \; H_j(V_k(\xi_i)) - H_k(V_j(\xi_i)) + H_k(V_i(\xi_j)) - H_i(V_k(\xi_j)) + H_i(V_j(\xi_k)) - H_j(V_i(\xi_k)) \\
&\quad = \; V_k(H_j(\xi_i)) - V_j(H_k(\xi_i)) + V_i(H_k(\xi_j)) - V_k(H_i(\xi_j)) + V_j(H_i(\xi_k)) - V_i(H_j(\xi_k)) \\
&\quad = \; V_i(H_k(\xi_j) - H_j(\xi_k)) + V_j(H_i(\xi_k)) - H_k(\xi_j)) + V_k(H_j(\xi_i) - H_i(\xi_j)) \\
&\quad = \; V_i(\xi_{j|k} - \xi_{k|j}) + V_j(\xi_{k|i} - \xi_{i|k}) + V_k(\xi_{i|j} - \xi_{j|i}).
\end{aligned}$$

Let me set $\xi_{i|j} - \xi_{j|i} = X_{ij}$; note that X_{ij} is skew-symmetric in its indices, and homogeneous of degree 1. The Bianchi identity therefore becomes

$$X_{jk}\delta^l_i + X_{ki}\delta^l_j + X_{ij}\delta^l_k + (X_{jk,i} + X_{ki,j} + X_{ij,k})u^l = 0.$$

It is required to show that $X_{ij} = 0$. Take the trace on k and l, to obtain

$$2X_{ji} + nX_{ij} + (X_{jk,i} + X_{ki,j})u^k + X_{ij} = 0$$

or

$$(n-1)X_{ij} + (X_{jk,i} + X_{ki,j})u^k = 0,$$

using the skew-symmetry and homogeneity of X_{ij}. Now contract with u^j:

$$(n-1)X_{ij}u^j + X_{ki}u^k = (n-2)X_{ij}u^j = 0.$$

Thus for $n > 2$, $X_{ij}u^j = 0$. It follows that

$$X_{jk,i}u^k = V_i(X_{jk})u^k = V_i(X_{jk}u^k) - X_{ji} = X_{ij},$$

so that the earlier equation becomes $(n+1)X_{ij} = 0$, and $X_{ij} = 0$ as required.

Thus the integrability conditions for the equations for α are satisfied, the equations admit a solution locally, and the spray locally determined by a solution is projectively equivalent to the given isotropic spray and is R-flat. $\square$

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