Conformal Killing tensors with vanishing torsion and the separation of variables in the Hamilton-Jacobi equation

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Abstract

If a Riemannian manifold admits a conformal Killing tensor whose torsion, in the sense of Nijenhuis, vanishes, and whose eigenfunctions are independent, then the Hamilton-Jacobi equation for its geodesics is solvable by separation of variables. The paper is devoted to developing a theory of this class of conformal Killing tensors, including an explanation of this result.

This paper is about a special class of conformal Killing tensors, which plays an interesting role in the study of the conditions under which one can find coordinates with respect to which the Hamilton-Jacobi equation for geodesics,

$$\sum_{i,j=1}^{n} g^{ij}\frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^j} = 1,$$

is separable, that is, solvable by additive separation of variables.

The tensors of this class were first mentioned, in recent times at least, by Benenti in [1], one of a series of publications recording that author's extensive researches on the separation of variables problem. Since then they have appeared in a number of papers, such as [3, 4, 7, 10], in which separability, while an important consideration, is perhaps secondary to such topical concerns as bi-Hamiltonian systems and stationary flows of soliton equations. Another field in which tensors of the type under consideration have an important, though not yet properly recognised, role to play is that of integrable systems of hydrodynamic type: see for example [5]. Yet another is geodesic equivalence ([11, 12]).

Despite the interest and importance both of the separability problem itself and of the applications just mentioned, there has not been available until now a coherent theory of conformal Killing tensors of this class, though some of their properties may be found by trawling through the publications quoted above. The present paper attempts to rectify this situation. In part this has been done by bringing together the previously established results; but much of what is to be found below is new.

1 Special conformal Killing tensors

A symmetric type $(0,2)$ tensor field L is a conformal Killing tensor of a metric g if there is a 1-form α such that

$$L_{ij|k} + L_{jk|i} + L_{ki|j} = \alpha_i g_{jk} + \alpha_j g_{ki} + \alpha_k g_{ij},$$

where the bar denotes covaraiant differentiation with repsect to the Levi-Civita connection of g. If α is exact then L is called a conformal Killing tensor of gradient type.

I shall frequently need to consider the other forms of the tensor L obtained by raising its indices, and in particular to the type $(1,1)$ tensor $L^i_j = \sum_k g^{ik} L_{kj}$. I shall refer to all forms of the tensor indiscriminately as L, often leaving it to the reader to work out which is intended. Note that the summation convention is suspended: summation is intended only where indicated.

The particular class of conformal Killing tensors under consideration consists of those that satisfy

$$L_{ij|k} = \tfrac{1}{2}\left(\alpha_i g_{jk} + \alpha_j g_{ik}\right);$$

such tensors clearly are conformal Killing. By taking the trace on i and j one sees that $\alpha = d(\operatorname{tr} L)$, so L is of gradient type. Furthermore the torsion, or Nijenhuis tensor, of the type $(1,1)$ tensor L vanishes:

$$L_l^k\left(L_{i|j}^l - L_{j|i}^l\right) = L_j^l L_{i|l}^k - L_i^l L_{j|l}^k.$$

This has significant consequences for the eigenfunctions of L, when they are functionally independent. At each point x of the manifold Q, $L(x)$ is a linear transformation of the tangent space T_xQ to Q at x, which is symmetric with respect to the scalar product defined by the metric. There is therefore an orthogonal basis of T_xQ with respect to which $L(x)$ is diagonal, the diagonal entries being its eigenvalues. To say that L has functionally independent eigenfunctions is to say that there are n functions u^1, u^2, …, u^n, such that at each point x, $u^i(x)$ is an eigenvalue of $L(x)$ for each i, and the Jacobian matrix $(\partial u^i/\partial x^j)$ is everywhere non-singular. The u^i may then be taken as local coordinates; as a consequence of the fact that its torsion vanishes, L takes the form

$$L = \sum_{i=1}^n u^i \frac{\partial}{\partial u^i} \otimes du^i$$

with respect to them; moreover they are orthogonal coordinates, so that $g_{ij} = 0$ for $i \neq j$.

A tensor L satisfying $L_{ij|k} = \frac{1}{2}\left(\alpha_i g_{jk} + \alpha_j g_{ik}\right)$ will be called a *special conformal Killing tensor*. I shall mostly be concerned with special conformal Killing tensors whose eigenfunctions are functionally independent. I shall further restrict attention to regions in which the u^i are distinct and non-vanishing; in particular, I shall assume without comment that L is non-singular everywhere.

One reason for preferring to deal with tensors with functionally independent eigenfunctions appears in the following proposition.

Proposition 1 A conformal Killing tensor with functionally independent eigenfunctions whose torsion vanishes is a special conformal Killing tensor.

Proof Note first that if L satisfies $L_{(ij|k)} = \alpha_{(i}g_{jk)}$ (where brackets denote symmetrization) and we set

$$T_{ijk} = L_{ij|k} - \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik}),$$

then $T_{(ijk)} = 0$. Moreover, the vanishing of the torsion of L is equivalent to

$$\sum_l (L_i^l T_{jkl} - L_j^l T_{ikl}) + L_k^l (T_{lij} - T_{lji})) = 0.$$

But $\sum_l L^l_k T_{(lij)} = 0$; and from these two equations we obtain

$$2\sum_l L^l_k T_{lji} = \sum_l (L^l_i T_{jkl} - L^l_j T_{ikl} - L^l_k T_{ijl}).$$

The right-hand side is symmetric in j and k; it follows that

$$\sum_l L^l_j T_{lki} = \sum_l L^l_k T_{jli}.$$

Thus at each point and for each fixed i, the symmetric bilinear form T_i with components T_{jki} satisfies

$$T_i(L(\xi), \eta) = T_i(\xi, L(\eta))$$

for every pair of vectors ξ and η. It follows that if ξ and η are eigenvectors corresponding to distinct eigenvalues of L then $T_i(\xi, \eta) = 0$. Now by assumption L has n distinct eigenvalues at each point of an open dense subset of M. Let $\{\xi_{(p)}\}$, $p = 1, 2, \ldots, n$ be an eigenbasis for L at such a point: then $T_i(\xi_{(p)}, \xi_{(q)}) = 0$ for $p \neq q$. Let $\hat{T}_{pqr}$ be the components of T with respect to this basis: then $\hat{T}_{pqr} = 0$ for $p \neq q$, for each r. Now $\hat{T}$ is symmetric and satisfies the cyclic condition. So $\hat{T}_{ppr} + 2\hat{T}_{rpp} = 0$, whence $\hat{T}_{ppr} = 0$ for $p \neq r$; and $3\hat{T}_{ppp} = 0$. Thus $T_{ijk} = 0$. $\square$

2 Separation of variables

I shall next show that if a Riemannian manifold admits a special conformal Killing tensor with functionally independent eigenfunctions then its Hamilton-Jacobi equation is separable in the orthogonal coordinates determined by the eigenfunctions. The proof is a brute calculation, showing that the Levi-Civita criterion for separability is satisfied. The Levi-Civita criterion for the separability of the Hamilton-Jacobi equation for the Hamiltonian $H(x^i, p_j)$ is that

$$\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial x^i \partial x^j} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial x^j}\frac{\partial^2 H}{\partial x^i \partial p_j} - \frac{\partial H}{\partial x^i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i \partial x^j} + \frac{\partial H}{\partial x^i}\frac{\partial H}{\partial x^j}\frac{\partial^2 H}{\partial p_i \partial p_j} = 0$$

for $i \neq j$, $i, j = 1, 2, \ldots, n$ (see for example [8]). When $H = \frac{1}{2}\sum_{i,j} g^{ij} p_i p_j$ and the coordinates are orthogonal, these conditions become

$$\frac{\partial^2 g^{kk}}{\partial x^i \partial x^j} - \frac{\partial \ln g^{jj}}{\partial x^i}\frac{\partial g^{kk}}{\partial x^j} - \frac{\partial \ln g^{ii}}{\partial x^j}\frac{\partial g^k}{\partial x^i} = 0.$$

It will be convenient to rewrite the conditions first in terms of the covariant components g_{ii} of the metric, and secondly in terms of the quantities $\ln g_{ii} = \gamma_i$; they become

$$\frac{\partial^2 \gamma_k}{\partial x^i \partial x^j} - \frac{\partial \gamma_k}{\partial x^i}\frac{\partial \gamma_k}{\partial x^j} + \frac{\partial \gamma_j}{\partial x^i}\frac{\partial \gamma_k}{\partial x^j} + \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_k}{\partial x^i} = 0.$$

I shall also make use of the fact that, when the coordinates are orthogonal,

$$\Gamma^i_{jj} = -\tfrac{1}{2}\frac{g_{jj}}{g_{ii}}\frac{\partial \gamma_j}{\partial u^i}, \quad \Gamma^i_{ij} = \tfrac{1}{2}\frac{\partial \gamma_i}{\partial u^j}, \quad i \neq j.$$

Proposition 2 Suppose that the Riemannian manifold (M, g) admits a special conformal Killing tensor L with functionally independent eigenfunctions u^i, $i = 1, 2, \ldots, n$. Then the Hamilton-Jacobi equation for geodesics is separable in the orthogonal coordinates u^i.

Proof The 1-form α whose components appear in the formula for $L_{ij|k}$ is given by $\alpha = d(\operatorname{tr} L)$, so $\alpha = \sum du^i$. The formula for $L_{ij|k}$ can be re-expressed for convenience in the form

$$L^i_{j|k} = \tfrac{1}{2}\left(\alpha^i g_{jk} + \alpha_j \delta^i_k\right).$$

When i, j and k are all different, and when they are all the same, this equation is identically satisfied. In the other cases it reduces to

$$(u^j - u^i)\Gamma^i_{jj} = \tfrac{1}{2}\frac{g_{jj}}{g_{ii}}, \quad (u^j - u^i)\Gamma^i_{ij} = -\tfrac{1}{2},$$

for $i \neq j$. Taking account of the expressions for the connection coefficients in orthogonal coordinates, we see that these together amount to just the one condition

$$\frac{\partial \gamma_i}{\partial u^j} = \frac{1}{u^j - u^i}.$$

When this expression is substituted into the separability condition we obtain, for i, j and k all different,

$$\begin{aligned}&\frac{\partial^2 \gamma_k}{\partial u^i \partial u^j} - \frac{\partial \gamma_k}{\partial u^i}\frac{\partial \gamma_k}{\partial u^j} + \frac{\partial \gamma_j}{\partial u^i}\frac{\partial \gamma_k}{\partial u^j} + \frac{\partial \gamma_i}{\partial u^j}\frac{\partial \gamma_k}{\partial u^i} \\ &\quad = -\frac{1}{(u^i - u^k)(u^j - u^k)} + \frac{1}{(u^i - u^j)(u^j - u^k)} + \frac{1}{(u^j - u^i)(u^i - u^k)} = 0;\end{aligned}$$

while

$$\frac{\partial^2\gamma_j}{\partial u^i\partial u^j} - \frac{\partial\gamma_j}{\partial u^i}\frac{\partial\gamma_j}{\partial u^j} + \frac{\partial\gamma_j}{\partial u^i}\frac{\partial\gamma_j}{\partial u^j} + \frac{\partial\gamma_i}{\partial u^j}\frac{\partial\gamma_j}{\partial u^i}$$
$$= \frac{\partial}{\partial u^j}\left(\frac{1}{u^i - u^j}\right) + \frac{1}{(u^j - u^i)(u^i - u^j)} = 0.$$

The separability condition is therefore satisfied. □

The formula

$$\frac{\partial\gamma_i}{\partial u^j} = \frac{1}{u^j - u^i}$$

obtained in the proof of this proposition will be useful later. The argument leading to this formula, read backwards, gives the following result.

Proposition 3 If $\partial\gamma_i/\partial u^j = (u^j - u^i)^{-1}$ then $L = \sum u^i\partial/\partial u^i \otimes du^i$ is a special conformal Killing tensor. □

A closely related result, involving the same formula, is given by Benenti in Proposition 2.1 of [1]: it is however expressed in terms of contravariant rather than covariant components.

3 Stäckel systems

In order to understand more clearly why special conformal Killing tensors are associated with separability it is desirable to consider them in relation to Eisenhart's geometrical reformulation of Stäckel's characterization of metrics for which the Hamilton-Jacobi equation is separable in orthogonal coordinates. Eisenhart showed that the necessary and sufficient conditions for there to be orthogonal coordinates with respect to which the Hamilton-Jacobi equation of the Hamiltonian $H = \frac{1}{2}\sum_{i,j} g^{ij}p_ip_j$ separates, is the existence of n linearly independent symmetric type $(0,2)$ tensor fields $A^{(m)}$, $m = 1, 2, \ldots, n$, with $A^{(1)} = g$, such that

1. each $A^{(m)}$ is a Killing tensor;
2. the quadratic functions $H^{(m)} = \frac{1}{2}\sum_{i,j} A^{(m)ij}p_ip_j$ commute pairwise with respect to the standard Poisson bracket;

3. for each $m > 1$, $A^{(m)}$ (considered as a type $(1, 1)$ tensor) has functionally independent eigenfunctions, and the $n \times n$ matrix whose rows are the eigenfunctions of $A^{(m)}$ for $m = 1, 2, \ldots, n$ is everywhere non-singular;

4. the $A^{(m)}$ for $m = 1, 2, \ldots, n$ are simultaneously diagonalizable, and one can find a set of simultaneous eigenforms for them which are closed.

The closed eigenforms determine a coordinate system with respect to which the tensors $A^{(m)}$ are all diagonal; in particular, these coordinates are orthogonal. The condition for an $A^{(m)}$ for $m > 1$ to be a Killing tensor reduces to a set of n simultaneous first-order partial differential equations for its n diagonal entries (its eigenfunctions, in other words). The Levi-Civita conditions for separability, in the version for orthogonal coordinates quoted above, are just the integrability conditions for these equations; they are equivalent to the equations given in the proof of Benenti's Lemma 2.2 in [1] (again, Benenti uses the contravariant components of the metric). The coordinates so defined are therefore separation coordinates.

A set of tensors $A^{(m)}$, $m = 1, 2, \ldots, n$, satisfying the conditions of Eisenhart's theorem is called a Stäckel system.

A special conformal Killing tensor determines a Stäckel system. I shall now explain how. This involves considering its cofactor tensor. The cofactor tensor C of a given type $(1, 1)$ tensor S has for its component $C^i_j(x)$ at a point x the cofactor of $S^i_j(x)$ in $\det S(x)$ (where $S(x)$ is considered as a matrix). That is to say,

$$C^i_j = \frac{1}{(n-1)!} \sum \delta^{i i_2 \ldots i_n}_{j j_2 \ldots j_n} S^{j_2}_{i_2} \cdots S^{j_n}_{i_n}$$

where $\delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}$ is the generalized Kronecker delta (see for example [9]). Note that S has to be of type $(1, 1)$. The cofactor tensor satisfies $\sum_k S^i_k C^k_j = (\det S)\delta^i_j$, or more succinctly $SC = (\det S)I$; and for S non-singular we can define its cofactor tensor by $C = (\det S)S^{-1}$. (Again, in calculating the determinant it must be assumed that S is of type $(1, 1)$ in order for the determinant to be a scalar.) According to these definitions, C is also of type $(1, 1)$; whenever it appears below with its indices in other positions it is to be assumed that they have been raised or lowered with g. I shall denote inverses of type $(1,1)$ tensors by an overbar when I need to use indices. Thus for example $C_{ij} = (\det S)\bar{S}_{ij}$ when S is nonsingular: it is symmetric if S is.

Note that the cofactor tensor of any non-singular tensor S has the same eigenvectors as S, and its eigenvalues are of the form $(\det S)/\lambda$ where λ is an eigenvalue of S.

One of the most remarkable properties of a special conformal Killing tensor is that its cofactor tensor is a Killing tensor. In order to establish this I need the following result relating the derivatives of the determinant and trace of any tensor whose torsion vanishes.

Lemma If a type $(1,1)$ tensor S has vanishing torsion then

$$\sum_j S_i^j \frac{\partial}{\partial x^j}(\det S) = (\det S)\frac{\partial}{\partial x^i}(\operatorname{tr} S).$$

Proof We have

$$\det S = \frac{1}{n!}\sum \delta^{k_1k_2\ldots k_n}_{l_1l_2\ldots l_n} S^{l_1}_{k_1} S^{l_2}_{k_2}\cdots S^{l_n}_{k_n},$$

so that

$$\frac{\partial}{\partial x^j}\det S = \frac{1}{(n-1)!}\sum \delta^{kk_2\ldots k_n}_{ll_2\ldots l_n} S^{l}_{k|j} S^{l_2}_{k_2}\cdots S^{l_n}_{k_n} = \sum_{k,l} S^l_{k|j} C^k_l,$$

where C is the cofactor tensor. Thus

$$\begin{aligned}
\sum_j S_i^j \frac{\partial}{\partial x^j}(\det S) &= \sum_{j,k,l} S^j_i S^l_{k|j} C^k_l \\
&= \sum_{j,k,l} (S^j_k S^l_{i|j} + S^l_j S^j_{k|i} - S^l_j S^j_{i|k}) C^k_l \\
&= (\det S)\sum_j (S^j_{i|j} + S^j_{j|i} - S^j_{i|j}) = (\det S)\frac{\partial}{\partial x^i}(\operatorname{tr} S)
\end{aligned}$$

as asserted. □

Proposition 4 If L is a special conformal Killing tensor and A is its cofactor tensor then A is a Killing tensor.

Proof If one takes the covariant derivative of the equation $\sum_j A_{ij}L^j_l = (\det L)g_{il}$ and uses the result of the previous lemma one finds that

$$A_{ij|k} = (\det L)\left(\bar{L}_{ij}\bar{L}_{kl} - \tfrac{1}{2}\bar{L}_{ik}\bar{L}_{jl} - \tfrac{1}{2}\bar{L}_{il}\bar{L}_{jk}\right)\alpha^l,$$

from which it follows that A is a Killing tensor. $\square$

This generalizes a result proved, in the Euclidean case, by Lundmark in [10].

Now if L is special conformal Killing then so is $L + kI$ for any constant k. Thus the cofactor tensor $A(k)$ of $L+kI$ is a Killing tensor for every k. Now $A(k)$ is a polynomial of degree $(n-1)$ in k, whose coefficients are tensors. Set

$$A(k)_{ij} = \sum_{m=1}^{n} A_{ij}^{(m)} k^{n-m}.$$

Theorem 1 If the special conformal Killing tensor L has functionally independent eigenfunctions then the tensors $A^{(m)}$, $m = 1, 2, \ldots, n$, so defined form a Stäckel system.

Proof The $A^{(m)}$, $m = 1, 2, \ldots, n$, are symmetric type $(0, 2)$ tensor fields, and $A^{(1)} = g$. Each $A^{(m)}$ is a Killing tensor. If the eigenfunctions of L are u^i, $i = 1, 2, \ldots, n$, then those of $L+kI$ are u^i+k; the eigenfunctions of $A(k)$ are therefore $\prod_{j\neq i}(u^j + k)$ for $i = 1, 2, \ldots, n$, and we can write

$$\prod_{j\neq i}(u^j + k) = \sum_{m=1}^{n} \sigma_m^i k^{n-m}$$

where σ_m^i is the symmetric polynomial of degree m in the $(n-1)$ variables u^j, $j \neq i$. Moreover $A(k)$ has the same eigenforms as $L + kI$, which in turn has the same eigenforms as L. If one writes down the eigenvalue equation for $A(k)$ one sees that each $A^{(m)}$ also has these same eigenforms, and the eigenfunctions of $A^{(m)}$ for $m > 1$ are σ_m^i, $i = 1, 2, \ldots, n$. These are functionally independent, and moreover $\det(\sigma_m^i) = \prod_{j>i}(u^i - u^j)$ (see [1] for example). The du^i are eigenforms of L and therefore are simultaneous closed eigenforms for the $A^{(m)}$. It is clear also, from their eigenfunctions, that the $A^{(m)}$ are linearly independent. It follows from the defining equation $A(k)(L + kI) = \det(L + kI)I$ that (as type $(1, 1)$ tensors) the $A^{(m)}$ are defined by the recurrence relation

$$A^{(m+1)} = -A^{(m)}L + \sigma_m I, \qquad A^{(1)} = I,$$

where σ_m is the symmetric polynomial of degree m in the u^i. It is shown in [7] that when the $A^{(m)}$ are so defined, the quadratic functions $H^{(m)} = \frac{1}{2}\sum_{i,j} A^{(m)ij} p_i p_j$ commute pairwise with respect to the standard Poisson bracket. $\square$

There is a source of possible confusion here that I wish to clear up.

In [2] Benenti states that any Stäckel system can be represened by just one of its Killing tensors. In an orthogonal coordinate system the condition for a tensor which is diagonal with respect to the coordinates to be Killing comes down to a system of first order partial differential equations for the diagonal elements of the tensor. If the integrability conditions of this system are satisfied, then the complete solution gives the Stäckel system. But knowing that there is just one solution is enough to show that the integrability conditions are satisfied. Thus if there is one Killing tensor with the right properties then it follows that there is a Stäckel system.

This construction is superficially similar to the one which is the subject of this section, in that it generates a Stäckel system from a single tensor. Note, however, that first of all Benenti's construction uses a Killing tensor, not a conformal Killing tensor. Moreover, constructing the rest of the Stäckel system from the one known Killing tensor involves finding the complete solution of a system of partial differential equations. If one has available a special conformal Killing tensor, on the other hand, generating the full Stäckel system is a purely algebraic process; and the separation coordinates are simply its eigenfunctions.

When there is a special conformal Killing tensor one can take one of the Killing tensors determined by the cofactor tensor method as the single Killing tensor which represents the corresponding Stäckel system. In particular, $(\operatorname{tr} L)I - L$ is a Killing tensor of the Stäckel system: it is $A^{(2)}$ in the notation used above.

There is an interesting connection here with some recent work on so-called systems of hydrodynamic type. In [5], Ferapontov and Fordy discuss what they describe as a 'surprising relationship between separable Hamiltonians and integrable, linearly degenerate systems of hydrodynamic type'. Such systems are first-order partial differential equations in two variables x and t for vector-valued functions q^i, and take the form $q^i_t = v^i(q)q^i_x$ for some functions $v^i(q)$, the subscripts indicating partial differentiation. Ferapontov and Fordy show essentially that the q^i are orthogonal separation coordinates for some Riemannian metric and the v^i are the eigenvalues of a Killing tensor in the corresponding Stäckel system. They consider at some length the case in which $v^i = \sum_{k=1}^n q^k - q^i$, which they describe as 'an important example'; not only do they discuss this important case in general, but all of their particular examples are special cases of it. It will be recognised that this is precisely the case described in the previous paragraph, the q^i being the eigenfunctions of the special conformal Killing tensor L.

Note that there is no implication that every Stäckel system arises in the way explained by Theorem 1. The next two propositions together constitute a test for determining whether a given Stäckel system is generated by a special conformal Killing tensor. In the first, I establish a property of the coordinates adapted to a special conformal Killing tensor (i.e. those for which it takes the diagonal form with the coordinates as the diagonal entries). These coordinates are of course orthogonal. The components R_{ijkl} of the curvature for which i, j, k and l are all different vanish for any orthogonal coordinate system; but in the case of coordinates adapted to a special conformal Killing tensor $R_{ijik} = 0$ also, for $j \neq k$, which is not generally true. Thus the only components of R which do not vanish necessarily are those of the form R_{ijij}, $i \neq j$.

Proposition 5 With respect to coordinates adapted to a special conformal Killing tensor with independent eigenfunctions, $R_{ijik} = 0$ for $j \neq k$.

Proof We have

$$R_{ijik} = \frac{g_{ii}}{4}\left(2\frac{\partial^2 \gamma_i}{\partial x^j \partial x^k} + \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_i}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_j}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^k}\frac{\partial \gamma_k}{\partial x^j}\right)$$

with respect to any orthogonal coordinates x^i. For coordinates adapted to a special conformal Killing tensor, however,

$$\frac{\partial \gamma_i}{\partial u^j} = \frac{1}{u^j - u^i},$$

whence

$$\frac{\partial^2 \gamma_i}{\partial u^j \partial u^k} = 0 \quad \text{for } j \neq k,$$

and so $R_{ijik} = 0$ as claimed. □

This proposition has a converse: if in orthogonal separation coordinates $R_{ijik} = 0$ for $j \neq k$, and a certain technical condition, which will be spelled out later, is satisfied then these separation coordinates can be derived from a special conformal Killing tensor. Notice that such separation coordinates are not uniquely determined: in particular, there remains the freedom of carrying out a coordinate transformation of each coordinate independently. Now the condition for the existence of a special conformal Killing tensor L, according to Proposition 3, is that $\partial\gamma_i/\partial u^j = (u^j - u^i)^{-1}$, where $\gamma_i = \ln g_{ii}$ and the u^i are the eigenfunctions of L. Thus

$$g_{ii} = \phi_i(u^i) \prod_{j \neq i} |u^i - u^j|.$$

(The coordinate patch is restricted to a sector of $\mathbb{R}^n$ in which $u^i - u^j \neq 0$ for all i, j, $i \neq j$; the g_{ii} are C^∞ on any such sector. We can write $g_{ii} = \phi_i(u^i) \prod_{j \neq i}(u^i - u^j)$, with the understanding that the sign of ϕ_i, which must be non-vanishing, is chosen to ensure that g_i is positive.) In the light of the remark about coordinate freedom, to confirm the existence of a special conformal Killing tensor it is enough to show that with respect to some orthogonal coordinates x^i

$$g_{ii} = \phi_i(x^i) \prod_{j \neq i} (\rho_i(x^i) - \rho_j(x^j))$$

with $d\rho_i/dx^i \neq 0$.

Proposition 6 If, with respect to orthogonal separation coordinates, $R_{ijik} = 0$ for $j \neq k$, and a further condition which will be detailed in the proof is satisfied, then the separation coordinates can be derived from a special conformal Killing tensor.

Proof The proof is adapted from the Appendix to Kalnins's book [8].

The condition that $R_{ijik} = 0$ with respect to any orthogonal coordinates is

$$2\frac{\partial^2 \gamma_i}{\partial x^j \partial x^k} + \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_i}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_j}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^k}\frac{\partial \gamma_k}{\partial x^j} = 0,$$

and we require this to hold for all distinct i, j, k. The conditions for x^i to be orthogonal separation coordinates, on the other hand, are that

$$\frac{\partial^2 \gamma_i}{\partial x^j \partial x^k} - \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_i}{\partial x^k} + \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_j}{\partial x^k} + \frac{\partial \gamma_i}{\partial x^k}\frac{\partial \gamma_k}{\partial x^j} = 0;$$

they are to hold for all i, j and k with $j \neq k$, but e.g. $k = i$ is allowed, and in this case we have

$$\frac{\partial^2 \gamma_i}{\partial x^i \partial x^j} = -\frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_j}{\partial x^i} = \frac{\partial^2 \gamma_j}{\partial x^i \partial x^j}.$$

If both of these conditions hold, then for all distinct i, j, k

$$\frac{\partial^2 \gamma_i}{\partial x^j \partial x^k} = 0, \qquad \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_i}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_j}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^k}\frac{\partial \gamma_k}{\partial x^j} = 0.$$

From the first of these,

$$\gamma_i = \sum_{j \neq i} \Psi_{ij}(x^i, x^j).$$

From the condition

$$\frac{\partial^2 \gamma_i}{\partial x^i \partial x^j} = \frac{\partial^2 \gamma_j}{\partial x^i \partial x^j}$$

we find that

$$\frac{\partial^2 \Psi_{ij}}{\partial x^i \partial x^j} = \frac{\partial^2 \Psi_{ji}}{\partial x^i \partial x^j},$$

whence we can write

$$\Psi_{ij}(x^i, x^j) - \Psi_{ji}(x^i, x^j) = \chi_{ij}(x^i) - \chi_{ji}(x^j).$$

We may therefore write

$$\gamma_i = \Phi_i(x^i) + \sum_{j \neq i} \Phi_{ij}(x^i, x^j),$$

where $\Phi_{ij} = \Psi_{ij} - \chi_{ij} = \Phi_{ji}$ and $\Phi_i = \sum_{j \neq i} \chi_{ij}$. The condition

$$\frac{\partial^2 \gamma_i}{\partial x^i \partial x^j} = -\frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_j}{\partial x^i}$$

now becomes

$$\frac{\partial^2 \Phi_{ij}}{\partial x^i \partial x^j} + \frac{\partial \Phi_{ij}}{\partial x^i}\frac{\partial \Phi_{ij}}{\partial x^j} = 0,$$

or if we set $\phi_{ij} = \exp(\Phi_{ij})$,

$$\frac{\partial^2 \phi_{ij}}{\partial x^i \partial x^j} = 0.$$

Thus

$$\phi_{ij}(x^i, x^j) = \sigma_{ij}(x^i) + \sigma_{ji}(x^j).$$

When this is substituted into the equation

$$\frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_i}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^j}\frac{\partial \gamma_j}{\partial x^k} - \frac{\partial \gamma_i}{\partial x^k}\frac{\partial \gamma_k}{\partial x^j} = 0$$

we obtain

$$\sigma'_{ji}\sigma'_{ki}\phi_{jk} - \sigma'_{ji}\sigma'_{kj}\phi_{ki} - \sigma'_{ki}\sigma'_{jk}\phi_{ij} = 0.$$

The prime indicates the derivative with respect to the appropriate single variable. The order of the indices on the ϕ terms is unimportant; the order on the σ terms is important, and the first index shows which coordinate the term is a function of. Two more independent equations may be obtained by cyclically permuting the indices. These three equations may be regarded as homogeneous linear equations for the unknowns ϕ_{ij} etc.; since these quantities, being factors of the components of the metric, must be non-zero, the determinant of the coefficients must vanish. This condition gives

$$\sigma'_{ij}\sigma'_{jk}\sigma'_{ki} + \sigma'_{ji}\sigma'_{ik}\sigma'_{kj} = 0.$$

I now impose the additional condition, that none of these derivatives is zero. Now $\sigma'_{ij}/\sigma'_{ik}$ is a function of x^i alone; it must therefore be constant, so we may write

$$\sigma_{ij} = a_{ij}\sigma_i + b_{ij},$$

where a_{ij} and b_{ij} are constants, none of the a_{ij} being zero, and σ_i is a function of x^i alone (which could be taken to be σ_{i1} when $i \neq 1$, for example). Thus

$$\begin{aligned} \sigma'_{ij} &= a_{ij}\sigma'_i, \\ \phi_{ij} &= a_{ij}\sigma_i + a_{ji}\sigma_j + B_{ij}, \end{aligned}$$

where $B_{ij} = b_{ij} + b_{ji}$ is symmetric in i and j. From the first of these,

$$a_{ij}a_{jk}a_{ki} + a_{ji}a_{ik}a_{kj} = 0.$$

The equation satisfied by the ϕ_{ij} reduces to

$$a_{ji}a_{ki}B_{jk} = a_{ki}a_{jk}B_{ij} + a_{ji}a_{kj}B_{ik}.$$

Now $a_{ij}a_{jk}a_{ki} + a_{ji}a_{ik}a_{kj} = 0$ is the necessary and sufficient condition for the existence of non-zero constants λ_i such that $\lambda_i a_{ij} + \lambda_j a_{ji} = 0$. (It is clearly necessary. For sufficiency, take $\lambda_1 = 1$ and $\lambda_i = -a_{1i}/a_{i1}$ for $i \geq 2$.) Moreover, given this, $a_{ji}a_{ki}B_{jk} = a_{ki}a_{jk}B_{ij} + a_{ji}a_{kj}B_{ik}$ is the necessary and sufficient condition for there to be constants μ_i such that $B_{ij} = \mu_i a_{ij} + u_j a_{ji}$. (It is easy to see that it is necessary, by substitution. For sufficiency, notice that we can write the required equation for B_{ij} as

$$\frac{B_{ij}}{\lambda_i a_{ij}} = \frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j},$$

and that $C_{ij} = B_{ij}/(\lambda_i a_{ij})$ is skew-symmetric. The B condition becomes $C_{jk} = C_{ji} + C_{ik}$, which is the necessary and sufficient condition for the skew-symmetric array C_{ij} to be of the form $C_{ij} = \nu_i - \nu_j$.) Now set $\rho_i = \lambda_i \sigma_i - \mu_i$: then

$$\phi_{ij} = a_{ij}(\lambda_i \rho_i - \mu_i) + a_{ji}(\lambda_j \rho_j - \mu_j) + B_{ij} = a_{ij}\lambda_i(\rho_i - \rho_j),$$

so that $g_i = \phi_i(x^i) \prod_{j \neq i}(\rho_i(x^i) - \rho_j(x^j))$, as required (the constant factors having been absorbed into ϕ_i). $\square$

Corollary If with respect to coordinates adapted to a Stäckel system (that is, orthogonal coordinates with respect to which all of the Killing tensors are diagonal) $R_{ijik} = 0$ for $j \neq k$, and $\sigma'_{ij} \neq 0$ (where σ_{ij} is defined in the proof above), then the Stäckel system comes from a special conformal Killing tensor. $\square$

Riemannian manifolds which admit orthogonal coordinates such that $R_{ijik} = 0$ for $j \neq k$ have been called spaces of diagonal curvature by Zakharov ([13]). In the context of systems of hydrodynamic type they correspond to the so-called semi-Hamiltonian systems.

4 Spaces of constant curvature

Since $R_{ijik} = 0$ for a space of constant curvature (positive, negative or zero) in orthogonal coordinates, separation coordinates for such a space are derivable from a special conformal Killing tensor. There are clearly spaces which admit a special conformal Killing tensor, on the other hand, which are not of constant curvature. I shall next show that what distinguishes the spaces of constant curvature from the others in this regard is that they admit a multiplicity of independent special conformal Killing tensors.

The result depends on an 'integrability condition' for the equation $L^i_{j|k} = \frac{1}{2}(\alpha^i g_{jk} + \alpha_j \delta^i_k)$, which may be of more general interest.

Proposition 7 A special conformal Killing tensor L satisfies the cyclic identity

$$\sum_m \left(L_{mj} R^m_{ikl} + L_{mk} R^m_{ilj} + L_{ml} R^m_{ijk} \right) = 0.$$

Proof We have

$$\begin{aligned} L_{ij|kl} - L_{ij|lk} &= -\sum_m \left(L_{mj} R^m_{ikl} + L_{im} R^m_{jkl} \right) \\ &= \tfrac{1}{2} \left(\alpha_{i|l} g_{jk} + \alpha_{j|l} g_{ik} - \alpha_{i|k} g_{jl} - \alpha_{j|k} g_{il} \right). \end{aligned}$$

Now $\alpha_{i|j} = \alpha_{j|i}$, because α is exact. Take the cyclic sum over j, k, l: the α terms cancel in pairs, using this symmetry; the second set of terms involving R gives zero by the cyclic identity for the curvature. □

The result to be proved concerns manifolds which admit two or more two special conformal Killing tensors which are in an appropriate sense independent of each other. By 'independent' I mean that they have no non-trivial common invariant subspaces. Consider the type $(1,1)$ tensors corresponding to L and M. At each point x of the manifold Q, $L(x)$ and $M(x)$ are linear transformations of T_xQ. A subspace V of T_xQ is a common invariant subspace of L and M if $L(x)V \subset V$ and $M(x)V \subset V$. The tensors L and M are independent if their only common invariant subspaces at any point x are $\{0\}$ and T_xQ.

Theorem 2 Suppose that a Riemannian manifold (Q, g) admits two special conformal Killing tensors L, M which are independent in the sense just described. Suppose further that L has functionally independent eigenfunctions. Then (Q, g) is a space of constant curvature.

Proof Use coordinates adapted to L. Consider the cyclic identity, which M must satisfy by Proposition 7. Since $R_{ijik} = 0$ for $j \neq k$, the identity is trivially satisfied if i, j, k and l are all different. When two of them are the same we obtain

$$M_{kj} R^k_{iik} + M_{jk} R^j_{iji} = 0$$

when i, j and k are all different. But $M_{jk} = M_{kj}$; so if $M_{jk} \neq 0$, $R^j_{iij} = R^k_{iik}$ for all i different from both j and k. So if we write

$$K_{ij} = \frac{R_{ijij}}{g_{ii} g_{jj}},$$

then for every j, k, $j \neq k$, for which $M_{jk} \neq 0$, $K_{ij} = K_{ik}$ for $i \neq j$, $i \neq k$. I shall show that if L and M are independent then there are sufficiently many non-zero off-diagonal terms of M for this to imply that $K_{ij} = K_{kl}$ for all $i \neq j$, $k \neq l$, whence $R_{ijij} = K g_{ii} g_{jj}$ for some scalar function K. It then follows from Schur's theorem [6] that the manifold is a space of constant curvature.

It remains to show that if L and M are independent then $K_{ij} = K_{kl}$ for all $i \neq j$, $k \neq l$. In fact it is enough to show that $K_{ij} = K_{ik}$ for all distinct i, j and k: for then $K_{ij} = K_{ik} = K_{ki} = K_{kl}$.

For any pair of distinct indices j, k, I shall say that k is accessible from j if there are indices $l_1, l_2, \ldots, l_p$, with $j, l_1, l_2, \ldots, l_p, k$ all distinct, such that $M_{jl_1}, M_{l_1 l_2}, \ldots, M_{l_p k}$ all non-zero. This includes the case $p = 0$, when M_{jk} is itself non-zero. Then if L and M are independent, every index k is accessible from every distinct index j. Suppose that L and M are independent. Recall that the coordinate vectors ∂_i form a basis of eigenvectors of L. For any j, let V be the linear span of those ∂_k such that k is accessible from j. Then V is invariant under L. For any $\partial_k \in V$, $M(\partial_k) = \sum_l g^{ll} M_{kl} \partial_l$; that is to say, $M(\partial_k)$ is a linear combination of those ∂_l for which $M_{kl} \neq 0$. But if k is accessible from j and $M_{kl} \neq 0$ then l is accessible from j. Thus $M(\partial_k) \in V$, and so $M(V) \subset V$. Thus either $V = \{0\}$ or $V = T_x Q$. But $V \neq \{0\}$, since if it were then M would be diagonal, and each one-dimansional subspace $\langle \partial_i \rangle$ would be a common invariant subspace of L and M. So $V = T_x Q$, and every k is accessible from j.

Finally, I show that $K_{ij} = K_{ik}$ for all distinct i, j and k. We know that $K_{ij} = K_{ik}$ if $M_{jk} \neq 0$, so suppose that $M_{jk} = 0$. Then there are $l_1, l_2, \ldots, l_p$ such that $M_{jl_1}, M_{l_1 l_2}, \ldots, M_{l_p k}$ all non-zero. If $i \notin \{l_1, l_2, \ldots, l_p\}$ then $K_{ij} = K_{il_1} = K_{il_2} = \cdots = K_{il_p} = K_{ik}$. If on the other hand $i \in \{l_1, l_2, \ldots, l_p\}$, say $i = l_r$, then $K_{ij} = K_{il_{r-1}} = K_{l_{r-1} l_r}$ and $K_{ik} = K_{il_{r+1}} = K_{l_{r+1} l_r}$. But $M_{l_{r-1} l_r} \neq 0$, so $K_{l_{r+1} l_r} = K_{l_{r+1} l_{r-1}}$; and $M_{l_r l_{r+1}} \neq 0$, so $K_{l_{r-1} l_r} = K_{l_{r-1} l_{r+1}}$. So $K_{ij} = K_{ik}$ in this case too. $\square$

In fact the condition that every index is accessible from every other one is equivalent to the independence of L and M.

For completeness' sake I show that 'non-independence' of two special conformal Killing tensors means what one expects it to mean.

Proposition 8 A tensor M that is diagonal with respect to the coordinates adapted to L is a special conformal Killing tensor if and only if it is a constant multiple of L modulo a constant multiple of the identity.

Proof Assume that M is diagonal. The equation

$$M^i_{j|k} = \frac{\partial M^i_j}{\partial u^k} + \sum_l \Gamma^i_{lk} M^l_j - \sum_l \Gamma^l_{jk} M^i_l = \tfrac{1}{2}\left(\beta^i g_{jk} + \beta_j \delta^i_k\right)$$

reduces to $0 = 0$ when i, j and k are all different; when $i = j \neq k$ it becomes

$$\frac{\partial M^i_i}{\partial u^k} + \Gamma^i_{ik} M^i_i - \Gamma^i_{ik} M^i_i = 0;$$

when $i = k \neq j$ it becomes

$$\Gamma^i_{ji}(M^j_j - M^i_i) = \tfrac{1}{2}\beta_j;$$

when $j = k \neq i$ it becomes

$$\Gamma^i_{jj}(M^j_j - M^i_i) = \tfrac{1}{2} g_j \beta^i;$$

and when $i = j = k$ it becomes

$$\frac{\partial M^i_i}{\partial u^i} + \Gamma^i_{ii}(M^i_i - M^i_i) = \beta_i.$$

From the first of these, M^i_i is a function of u^i alone. The second then becomes

$$-\frac{1}{(u^j - u^i)}(M^j_j - M^i_i) = \frac{dM^j_j}{du^j},$$

from which it follows that $dM^j_j/du^j = dM^i_i/du^i$, and thence that $M^i_i = au^i + b$ for some constants a and b. The remaining equations are easily seen to be satisfied (as they must be, since $M = aL + bI$). □

5 Special conformal Killing tensors on Euclidean space

These results can be nicely illustrated by considering the special conformal Killing tensors on Euclidean space E^n, and their restrictions to its codimension-1 submanifolds.

The special conformal Killing tensors on Euclidean space are easily found: they are given in Cartesian coordinates x^i by

$$L_{ij} = \lambda x_i x_j + \mu_i x_j + \mu_j x_i + \nu_{ij},$$

with λ, μ and ν constant, ν symmetric; I have written $\delta_{ij}x^j = x_i$. (See for example Benenti [1], who calls such tensors 'planar inertia tensors', and Lundmark [10], who calls them 'elliptical coordinates matrices'.) In fact L defines elliptical coordinates if $\lambda \neq 0$ and ν has distinct eigenvalues. In such a case, by a Euclidean change of coordinates and by absorbing a constant into L we may write, in an obvious notation,

$$L = x^T x + \mathrm{diag}(c_1, c_2, \ldots, c_n);$$

I assume for convenience that the eigenvalues c_i of ν are arranged in increasing order. The required elliptical coordinates are the eigenfunctions of L. The characteristic polynomial $\chi(z)$ of L is easily seen to be

$$\chi(z) = (z-c_1)(z-c_2)\cdots(z-c_n) - \sum_k (z-c_1)(z-c_2)\cdots(z-c_{k-1})(z-c_{k+1})\cdots(z-c_n)x_k^2.$$

Thus $\chi(c_i)$ alternates in sign, with $\chi(c_n) < 0$; it follows that the eigenvalues u^i of L are distinct, and satisfy

$$c_1 < u^1 < c_2 < u^2 < \cdots < c_n < u^n.$$

By setting $z = c_i$ in $\chi(z) = (z - u^1)(z - u^2)\cdots(z - u^n)$ we obtain the usual formulae for the Cartesian coordinates x^i in terms of the elliptical coordinates u^i (as given in [8] for example):

$$(x^i)^2 = \frac{(u^1 - c_i)(u^2 - c_i)\cdots(u^n - c_i)}{(c_1 - c_i)(c_2 - c_i)\cdots(c_{i-1} - c_i)(c_{i+1} - c_i)\cdots(c_n - c_i)}.$$

When $\lambda = 0$ parabolic coordinates are obtained. By a Euclidean coordinate transformation L may be brought to the form

$$L = e_1^T x + x^T e_1 + \mathrm{diag}(0, c_1, c_2, \ldots, c_{n-1}).$$

The characteristic polynomial of this matrix is

$$\begin{aligned}\chi(z) \quad &= \quad (z - 2x_1)(z - c_1)(z - c_2)\cdots(z - c_{n-1})\\ &\qquad - \sum_{k=2}^{n}(z - c_1)(z - c_2)\cdots(z - c_{k-2})(z - c_k)\cdots(z - c_{n-1})x_{k-1}^2.\end{aligned}$$

As before, $\chi(c_i)$ alternates in sign, but now with $\chi(c_{n-1}) < 0$; the eigenvalues of L are distinct and satisfy

$$u^1 < c_1 < u^2 < c_2 < u^3 < \cdots < c_{n-1} < u^n.$$

With $\chi(z) = (z - u^1)(z - u^2)\cdots(z - u^n)$, by comparing coefficients of z^{n-1} and then by setting $z = c_i$ we obtain the following formulae for the Cartesian coordinates x^i in terms of the parabolic coordinates u^i:

$$\begin{aligned} x^1 &= \tfrac{1}{2}(u^1 + u^2 + \cdots + u^n - c_1 - c_2 - \cdots - c_{n-1}) \\ (x^i)^2 &= \frac{(u^1 - c_{i-1})(u^2 - c_{i-1})\cdots(u^n - c_{i-1})}{(c_1 - c_{i-1})(c_2 - c_{i-1})\cdots(c_{i-2} - c_{i-1})(c_i - c_{i-1})\cdots(c_{n-1} - c_{i-1})} \qquad i \geq 2. \end{aligned}$$

I now turn my attention to codimension-1 submanifolds of Euclidean space. If Q is a submanifold of a manifold P and L is a type $(0, 2)$ tensor field on P, then L defines a tensor field on Q by restriction. I shall investigate conditions under which a codimension-1 submanifold Q of Euclidean space E^n can be equipped with a special conformal Killing tensor by restriction of one from E^n.

The condition for a tensor L to be a special conformal Killing tensor, when written in coordinate-free form, is

$$(\nabla_Z L)(X, Y) = \tfrac{1}{2}\left(\langle X, \alpha\rangle g(Y, Z) + \langle Y, \alpha\rangle g(X, Z)\right).$$

Let $\bar{\nabla}$ be the Levi-Civita connection of Q with repsect to the metric induced from E^n: then for any vectorfields X, Z tangent to Q,

$$\bar{\nabla}_Z X = \nabla_Z X - n(Z, X)N$$

where ∇ is the standard connection on E^n, N is a unit normal field over Q and n is the corresponding second fundamental form of Q. Thus

$$\begin{aligned} (\bar{\nabla}_Z L)(X, Y) &= (\nabla_Z L)(X, Y) + n(Z, X)L(N, Y) + n(Z, Y)L(N, X) \\ &= \tfrac{1}{2}\left(\langle X, \alpha\rangle g(Y, Z) + \langle Y, \alpha\rangle g(X, Z)\right) + n(Z, X)L(N, Y) + n(Z, Y)L(N, X). \end{aligned}$$

There are two obvious ways in which the right-hand side can be of the form

$$\tfrac{1}{2}\left(\langle X, \beta\rangle g(Y, Z) + \langle Y, \beta\rangle g(X, Z)\right)$$

where β is a 1-form on Q: when n is proportional to g; and when there is some L on E^n whose restriction to Q satisfies $L(N, \cdot) = 0$. In the first case, every point of Q is an umbilic, that is, Q is either a hyperplane or a hypersphere: then every special conformal Killing tensor on E^n induces one on the submanifold.

For the other possibility, suppose that Q is given, with respect to Cartesian coordinates, by $f(x^1, x^2, \ldots, x^n) = 1$. Then in order for a special conformal Killing tensor L on E^n, given by $L_{ij} = \lambda x_i x_j + \mu_i x_j + \mu_j x_i + \nu_{ij}$, to induce one on Q, the equations

$$\sum_j (\lambda x^i (x^j f_j) + \mu^i (x^j f_j) + x^i (\mu^j f_j) + \nu^{ij} f_j) = 0$$

must be identically satisfied by virtue of $f(x^1, x^2, \ldots, x^n) = 1$ (indices are raised with δ^{ij}, and a subscript on f indicates partial differentiation with respect to a Cartesian coordinate). It will rarely be possible to choose constants λ, μ and ν for which this holds. One case in which it does, however, is that of a quadric. Let us take

$$f(x^1, x^2, \ldots, x^n) = \tfrac{1}{2} \sum_{i,j} A_{ij} x^i x^j$$

(a central quadric): then taking account of homogeneity the condition above becomes

$$2(\lambda x^i + \mu^i) + \sum_{j,k} (x^i \mu^j + \nu^{ij}) A_{jk} x^k = 0,$$

which is satisfied when $\mu^i = 0$, $\sum_k \nu^{ik} A_{jk} = -2\lambda \delta^i_j$ (assuming that the quadric is non-degenerate). That is, we can take (for example)

$$L_{ij} = \tfrac{1}{2} x_i x_j - \bar{A}_{ij}.$$

References

[1] S. Benenti Inertia tensors and Stäckel systems in the Euclidean spaces *Rend. del Sem. Mat. Torino* **50** (1992) 1–20

[2] S. Benenti Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation *J. Math. Phys.* **38** (1997) 6578–6602

[3] M. Crampin and W. Sarlet A class of non-conservative Lagrangian systems on Riemannian manifolds *Preprint* The Open University (2000)

[4] M. Crampin, W. Sarlet and G. Thompson Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors *J. Phys. A: Math. Gen.* **33** (2000) 8755–8770

[5] E. V. Ferapontov and A. P. Fordy Separable Hamiltonians and integrable systems of hydrodynamic type *J. Geom. Phys.* **21** (1997) 169–182

[6] L. P. Eisenhart *Riemannian Geometry* (Princeton, 1926) page 83

[7] A. Ibort, F. Magri and G. Marmo Bi-hamiltonian structures and Stäckel separability *J. Geom. Phys.* **33** (2000) 210–223

[8] E. G. Kalnins *Separation of Variables for Riemannian Spaces of Constant Curvature* (Longman, 1986)

[9] D. Lovelock and H. Rund *Tensors, Differential Forms and Variational Principles* (Wiley, 1975) Section 4.2

[10] H. Lundmark Higher-dimensional integrable Newton systems with quadratic integrals of motion *Preprint* Linköping University (1999)

[11] V. S. Matveev and P. Ĭ. Topalov Trajectory equivalence and corresponding integrals *Regular and Chaotic Dynamics* **3** (1998) 30–45

[12] P. Topalov Families of metrics geodesically equivalent to the analogs of the Poisson sphere *J. Math. Phys.* **41** (2000) 7510–7520

[13] V. E. Zhakharov Description of the n-orthogonal curvilinear coordinate systems and Hamiltonian integrable systems of hydrodynamic type, I *Duke Math. J.* **94** (1998) 103–139