Structural equations for a special class of conformal Killing tensors of arbitrary valence

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Abstract

A symmetric tensor T on a (pseudo-)Riemannian manifold which satisfies $T_{i_1 i_2 \ldots i_r|j} = S_{(i_1 i_2 \ldots i_{r-1}} g_{i_r)j}$ for some symmetric tensor S is a conformal Killing tensor of a special kind. Such special conformal Killing tensors of valence 1 and 2 have been extensively studied. In this paper special conformal Killing tensors of arbitrary valence, and indeed certain non-metrical generalizations of them, are investigated. In particular, it is shown that the space of special conformal Killing tensors is finite-dimensional, and the maximal dimension is attained (in the (pseudo-)Riemannian case) if and only if the manifold is a space of constant curvature. This result is obtained by constructing a set of structural equations for special conformal Killing tensors.

Keywords: special conformal Killing tensor; structural equations; separation of variables; completely integrable system; projectively equivalent metrics.

1 Introduction

This paper is concerned with symmetric tensors T of valence r on a Riemannian or pseudo-Riemannian manifold which satisfy equations of the form

$$T_{i_1 i_2 \ldots i_r|j} = S_{(i_1 i_2 \ldots i_{r-1}} g_{i_r)j},$$

where the rule signifies covariant differentiation with respect to the Levi-Civita connection, S is a symmetric tensor of valence $r-1$, g is the metric, and the brackets denote symmetrization over the enclosed indices. The tensor S is in fact determined by the equation. Manifolds of dimension 2 are atypical, as is often the case, and so it will be assumed throughout that the dimension of the manifold is at least 3.

A tensor T which satisfies such an equation also satisfies

$$T_{(i_1 i_2 \dots i_r|j)} = S_{(i_1 i_2 \dots i_{r-1}} g_{i_r j)}$$

and is therefore a conformal Killing tensor, albeit of a special kind: I therefore call such tensors special conformal Killing tensors.

Conformal Killing tensors are of course of interest in their own right. However, special conformal Killing tensors with $r = 1$ and $r = 2$ in particular have been extensively studied, for reasons that are not always directly related to the fact that they are conformal Killing tensors.

When $r = 1$ we are dealing with a covector field such that

$$T_{i|j} = S g_{ij}$$

for some function S. Thus T is a conformal Killing (co-)vector. Moreover $T_{j|i} = T_{i|j}$, and so T is a gradient. Such special conformal Killing vectors are of interest in at least two areas of research.

- Conformal Ricci collineations of Riemannian or pseudo-Riemannian manifolds. A conformal Killing field X on a (pseudo-)Riemannian manifold is a conformal Ricci collineation if $\mathcal{L}_X R_{ij} = \tau g_{ij}$ where R_{ij} is the Ricci tensor. A conformal Killing field is a conformal Ricci collineation if and only if the gradient of its conformal factor is a special conformal Killing vector [1].
- Concircular transformations in Riemannian geometry. A transformation of a Riemannian manifold preserves geodesic circles if and only if it is conformal, and the gradient of the conformal factor is a special conformal Killing vector (see for example [2]).

The infinitesimal generators of transformations preserving geodesic circles are called concircular vector fields; the same name is also applied, confusingly, to the solutions of the equation $T_{i|j} = S g_{ij}$. I shall adopt the latter convention.

The solutions of the tensor differential equation

$$T_{ij|k} = \tfrac{1}{2}(S_i g_{jk} + S_j g_{ik})$$

(the case $r = 2$) are sometimes called Benenti tensors in the literature, sometimes just special conformal Killing tensors. To avoid confusion with the general case I shall use the former term. Benenti tensors play an important role in at least three areas of research.

- Separation of the variables in the Hamilton-Jacobi equation. When a Riemannian manifold admits a Benenti tensor whose eigenfunctions are simple and functionally independent, those eigenfunctions are orthogonal separation coordinates for the Hamilton-Jacobi equation for the geodesics of the manifold, or more generally for suitable Hamiltonian systems of mechanical type [3]-[6].

- Completely integrable dynamical systems. Benenti tensors are involved in the definition of certain nonconservative Lagrangian systems, so-called cofactor-pair systems, which provide interesting examples of completely integrable systems [7]-[10]. Furthermore, the Nijenhuis torsion of a Benenti tensor vanishes as a consequence of the defining conditions, and so Benenti tensors may be used to treat suitable systems as bi-Hamiltonian systems of Poisson-Nijenhuis type [5, 6]. Finally, a class of superintegrable Hamiltonian systems constructed using Benenti tensors has recently been receiving attention [11, 12].

- The projective equivalence of Riemannian manifolds. Two Riemannian manifolds are said to be projectively equivalent if they have the same geodesics up to reparametrization. This situation occurs if and only if a certain tensor formed out of the two metric tensors is a Benenti tensor [13]-[16].

Since special conformal Killing tensors have these interesting applications, it seems important to establish the basic properties of the solutions of the special conformal Killing tensor equations. It is known (see [15, 16]) that for $r = 1$ and $r = 2$ the space of solutions of the special conformal Killing tensor equations is a finite-dimensional vector space, of maximal dimension $n + 1$ for $r = 1$ and $\frac{1}{2}n(n + 1)$ for $r = 2$, where n is the dimension of the underlying manifold. Moreover, the maximal dimension is achieved if and only if the Riemannian space is a space of constant curvature. The aim of this paper is to extend these results to the general case.

It turns out that, so far as deriving the main result of this paper is concerned, it is just as easy to write the defining equation in the contravariant form

$$T^{i_1 i_2 \ldots i_r}{}_{|j} = S^{(i_1 i_2 \ldots i_{r-1}} \delta_j^{i_r)}.$$

It is evident that equations of this type may be formulated without appeal to the existence of a metric. The main result will hold therefore for symmetric contravariant tensors on any manifold equipped with an affine connection, which for convenience I take to be symmetric; the results contained in this paper are consequently very considerable generalizations of the known results for concircular vector fields and Benenti tensors. I call tensors satisfying the equation immediately above generalized special conformal Killing tensors (with an apology for the slight air of accompanying paradox).

The method of analysis employed here is to derive a system of structural equations, in the sense of Hauser and Malhiot [17, 18] and Wolf [19], for generalized special conformal Killing tensors. Hauser and Malhiot introduced this method in the study of Killing tensors of valence 2; Wolf extended it to deal with Killing tensors of arbitrary valence. In outline, the method works as follows. Tensorial quantities F^A are found which satisfy a system of equations of the form $F^A_{|i} = \Gamma^A_{Bi} F^B$ (sum over B intended), among which are the generalized special conformal Killing tensor equations (or whichever equations are of actual interest). The equations of this extended set are the structural equations. The F^A consist of the symmetric tensor T and tensors constructed from it and its covariant derivatives; the coefficients Γ^A_{Bi} are tensorial quantities which are independent of the F^A and in fact are built out of the curvature and its covariant derivatives. The structural equations are equivalent to the original generalized special conformal Killing tensor equations, in the sense that given any solution T of the generalized special conformal Killing tensor equations, the corresponding F^A satisfy the structural equations, and conversely given any solution of the structural equations the T component of F^A satisfies the generalized special conformal Killing tensor equations.

The advantage of expressing the problem of finding generalized special conformal Killing tensors in the form of solving the structural equations derives from the distinctive nature of these equations: each covariant derivative $F^A_{|i}$ is a linear combination of the F^A. It follows that given any point x of the underlying manifold, the linear map sending a solution T of the generalized special conformal Killing tensor equations to $F^A(x)$ is injective, so that the largest value the dimension of the solution space can have is the number of variables F^A in the structural equations. Moreover, the integrability conditions of the structural equations are in principle easily found by covariantly differentiating the equations, using the Ricci identities to eliminate second covariant derivatives, and substituting for the first derivatives introduced by using the original equations. The resulting conditions are

$$(\Gamma^A_{Bi|j} - \Gamma^A_{Bj|i} + \Gamma^C_{Bi}\Gamma^A_{Cj} - \Gamma^C_{Bj}\Gamma^A_{Ci} - R^A_{Bij})F^A = 0,$$

where the R^A_{Bij} are appropriate combinations of components of the curvature tensor. When the solution space has maximal dimension the values of the F^A may be chosen arbitrarily at each point of the underlying manifold, so

$$\Gamma^A_{Bi|j} - \Gamma^A_{Bj|i} + \Gamma^C_{Bi}\Gamma^A_{Cj} - \Gamma^C_{Bj}\Gamma^A_{Ci} - R^A_{Bij}$$

must vanish everywhere, and this gives algebraic conditions on the curvature and its covariant derivatives from which the properties of the spaces for which the solution space has maximal dimension can be determined. (In practice it may not be necessary to carry out this integrability analysis in its entirety: short cuts may be available, as is the case here.)

The relevant features of structural equations, including those described above, seem to be treated as common knowledge rather than derived in the literature; I give a brief discussion with proofs in an appendix. As well as obtaining an interesting result about a special class of conformal Killing tensors, this paper provides a quite subtle example of the use of structural equations.

In tensor calculations I follow the sign conventions of Eisenhart [20], so that the Ricci identities are (for example) $K_{i|jk} - K_{i|kj} = R^l_{ijk}K_l$, and the Ricci tensor is given by $R_{ij} = R^k_{ijk}$. The Einstein summation convention is in force almost throughout.

2 Structural equations for generalized special conformal Killing tensors

Let ∇ be the covariant derivative operator of a symmetric affine connection on a manifold M. The tensors under consideration are contravariant and symmetric: I denote the space of valence r symmetric contravariant tensors by $\mathfrak{S}_r$.

I shall consider tensor equations of the following form:

$$\nabla T = S \odot I \tag{1}$$

where I is the identity tensor and $\odot$ is the symmetrised tensor product. This is to be construed as a set of first-order partial differential conditions on the unknown symmetric contravariant tensor T, namely that ∇T takes the indicated form where S is some other symmetric contravariant tensor, whose valence is one less than that of T. In component form, if $T \in \mathfrak{S}_r$,

$$T^{i_1 i_2 \ldots i_r}{}_{|j} = S^{(i_1 i_2 \ldots i_{r-1}}\delta_j^{i_r)}.$$

The brackets indicate symmetrization; since S is by assumption already symmetric, the right-hand side is obtained by taking the cyclic sum over the indices $i_1, i_2, \ldots, i_r$ and dividing by r.

The tensor S in the equation $\nabla T = S \odot I$ can be expressed in terms of T by taking a trace. If we take the trace over i_r and j in the component version we obtain

$$\left(\frac{n+r-1}{r}\right) S^{i_1 i_2 \ldots i_{r-1}} = T^{i_1 i_2 \ldots i_{r-1} j}{}_{|j}.$$

In order to be able to write this more succinctly I shall define a divergence operator $\mathfrak{d} : \mathfrak{S}_r \to \mathfrak{S}_{r-1}$ by

$$(\mathfrak{d}T)^{i_1 i_2 \ldots i_{r-1}} = T^{i_1 i_2 \ldots i_{r-1} j}{}_{|j}.$$

More generally, if U is a tensor of type (r, s) which is symmetric in its contravariant indices I set

$$(\mathfrak{d}U)^{i_1 i_2 \ldots i_{r-1}}_{j_1 j_2 \ldots j_s} = U^{i_1 i_2 \ldots i_{r-1} k}_{j_1 j_2 \ldots j_s}{}_{|k}.$$

Equation (1) can be written

$$\nabla T = \left(\frac{r}{n+r-1}\right) \mathfrak{d}T \odot I.$$

I shall need the commutator of the operators ∇ and $\mathfrak{d}$. Now for $T \in \mathfrak{S}_r$

$$(\nabla\mathfrak{d}T)^{i_1 i_2 \ldots i_{r-1}}{}_j = ((\mathfrak{d}T)^{i_1 i_2 \ldots i_{r-1}})_{|j} = T^{i_1 i_2 \ldots i_{r-1} k}{}_{|kj},$$

while

$$(\mathfrak{d}\nabla T)^{i_1 i_2 \ldots i_{r-1}}{}_j = ((\nabla T)^{i_1 i_2 \ldots i_{r-1} k}{}_j)_{|k} = T^{i_1 i_2 \ldots i_{r-1} k}{}_{|jk};$$

thus

$$([\nabla, \mathfrak{d}]T)^{i_1 i_2 \ldots i_{r-1}}{}_j = T^{i_1 i_2 \ldots i_{r-1} k}{}_{|kj} - T^{i_1 i_2 \ldots i_{r-1} k}{}_{|jk}.$$

Now by the Ricci identity

$$T^{i_1 i_2 \ldots i_{r-1} i_r}{}_{|kj} - T^{i_1 i_2 \ldots i_{r-1} i_r}{}_{|jk} = \sum_{s=1}^{r} R^{i_s}_{ljk} T^{l i_1 i_2 \ldots \hat{\imath}_s \ldots i_r},$$

where the hat indicates that the corresponding index is to be omitted. It follows that

$$([\nabla, \mathfrak{d}]T)^{i_1 i_2 \ldots i_{r-1}}{}_j = \sum_{s=1}^{r-1} R^{i_s}_{ljk} T^{lk i_1 i_2 \ldots \hat{\imath}_s \ldots i_{r-1}} + R_{kj} T^{k i_1 i_2 \ldots i_{r-1}}.$$

Thus $[\nabla, \mathfrak{d}]$ is an algebraic operator $\mathfrak{S}_r \to \mathfrak{S}_{r-1} \otimes T^*M$, which I shall denote by ρ for succinctness. Notice that $\rho(T)$ has zero trace:

$$\begin{aligned} \rho(T)^{i_1 i_2 \ldots i_{r-2} j}{}_j &= \sum_{s=1}^{r-2} R^{i_s}_{ljk} T^{lk i_1 i_2 \ldots \hat{\imath}_s \ldots i_{r-2} j} \\ &\quad + R^{j}_{ljk} T^{lk i_1 i_2 \ldots i_{r-2}} + R_{kj} T^{k i_1 i_2 \ldots i_{r-2} j} \\ &= -R_{lk} T^{lk i_1 i_2 \ldots i_{r-2}} + R_{kj} T^{k i_1 i_2 \ldots i_{r-2} j} = 0, \end{aligned}$$

using the fact that $R^{i_s}_{ljk}$ is skew in j and k.

I now start to derive the structural equations. I shall use index-free notation, but in order to keep track of the valences of the tensors involved, where necessary I shall write $T_{(p)}$ for an element of $\mathfrak{S}_p$, and $U_{(p)}$ for an element of $\mathfrak{S}_p \otimes T^*M$; thus $T_{(0)}$ is a scalar, and it will be convenient to take $T_{(p)} = 0$ for $p < 0$. The structural equations will turn out to have the form

$$\nabla T_{(p)} = U_{(p)} + T_{(p-1)} \odot I, \quad p = r, r-1, \ldots, 0,$$

where $U_{(p)}$ is a linear expression in the tensors $T_{(q)}$ with $q > p$, with coefficients which are components of the curvature and Ricci tensors and their covariant derivatives; $U_{(r)} = 0$; and as a type $(p, 1)$ tensor $U_{(p)}$ is trace-free. Note that it follows from the latter fact that $T_{(p-1)}$ can be expressed as a multiple of $\mathfrak{d}T_{(p)}$, as before.

(A linear expression, as envisaged here, will involve contractions between the indices of the $T_{(q)}$ and the coefficients: $\rho T_{(q)}$ provides a paradigmatic example. Such expressions will have the form $\Gamma^A_{Bi} F^B$ occuring on the right-hand side of tructural equations as they were described in the Introduction: here the F^A stand for components of the collection of tensors $T_{(p)}$.)

The structural equations will be derived recursively, by use of the following lemma.

Lemma 1 Suppose that for some $p \geq 1$ tensors $T_{(p)}$, $T_{(p-1)}$ and $U_{(p)}$ (with the notational conventions described above) satisfy

$$\nabla T_{(p)} = U_{(p)} + T_{(p-1)} \odot I$$

where $U_{(p)}$ is trace-free (no assumption is made here about how it is related to $T_{(p)}$ etc.). Then there are tensors $T_{(p-2)}$ and $U_{(p-1)}$, with $U_{(p-1)}$ trace-free for $p > 1$, such that

$$\nabla T_{(p-1)} = U_{(p-1)} + T_{(p-2)} \odot I.$$

Proof The expression for $T_{(p-1)}$ is obtained by taking a trace of the defining equation, as before; we obtain

$$T_{(p-1)} = \left(\frac{p}{n+p-1}\right) \mathfrak{d} T_{(p)}. \tag{2}$$

Then

$$\begin{aligned} (n+p-1)\nabla T_{(p-1)} &= p\nabla \mathfrak{d} T_{(p)} \\ &= p\left(\mathfrak{d}\nabla T_{(p)} + \rho T_{(p)}\right) \\ &= p\left(\mathfrak{d}(U_{(p)} + T_{(p-1)} \odot I) + \rho T_{(p)}\right). \end{aligned}$$

The next point to note is that

$$p\mathfrak{d}(T_{(p-1)} \odot I) = \nabla T_{(p-1)} + (p-1)(\mathfrak{d} T_{(p-1)} \odot I),$$

as it is easy to see by a calculation in components. Thus

$$(n+p-2)\nabla T_{(p-1)} = p\left(\mathfrak{d} U_{(p)} + \rho T_{(p)}\right) + (p-1)\left(\mathfrak{d} T_{(p-1)} \odot I\right).$$

That is to say

$$\nabla T_{(p-1)} = U_{(p-1)} + T_{(p-2)} \odot I$$

with

$$T_{(p-2)} = \left(\frac{p-1}{n+p-2}\right) \mathfrak{d} T_{(p-1)}$$

and

$$U_{(p-1)} = \left(\frac{p}{n+p-2}\right)\left(\mathfrak{d} U_{(p)} + \rho T_{(p)}\right). \tag{3}$$

It follows either from the fact that $T_{(p-2)}$ has the correct relationship with $\mathfrak{d}T_{(p-1)}$, or from the fact that $\rho T_{(p)}$ is trace-free, that $U_{(p-1)}$ is trace-free. □

Suppose now we have a tensor $T_{(r)}$ such that $\nabla T_{(r)} = T_{(r-1)} \odot I$. Define $T_{(p)}$ and $U_{(p)}$ recursively by equations (2) and (3), with in the latter case $U_{(r)} = 0$; in evaluating $\mathfrak{d}U_{(p)}$, terms involving covariant derivatives of the $T_{(q)}$ will occur and these are to be replaced by the appropriate expressions in $T_{(q-1)}$.

Lemma 2 With $T_{(p)}$ and $U_{(p)}$ defined in this way, $U_{(p)}$ is a linear expression in the $T_{(q)}$ with $q > p$, with coefficients formed from the curvature and Ricci tensors and their covariant derivatives.

Proof Clearly $U_{(r-1)}$ is of this form, since it is just a constant multiple of $\rho T_{(r)}$. Suppose that $U_{(p)}$ is of this form. Now up to numerical factors, $U_{(p-1)}$ is obtained from $U_{(p)}$ by firstly operating on it with $\mathfrak{d}$ and substituting for covariant derivatives of the $T_{(q)}$, and secondly adding $\rho T_{(p)}$. The result is a linear expression in the $T_{(q)}$ with $q > p-1$, with coefficients formed from the curvature and Ricci tensors and their covariant derivatives. □

It will sometimes be convenient to denote by $o(p)$ any collection of terms depending linearly on the $T_{(q)}$ with $q \geq p$.

I shall need later the explicit form of the terms involving $T_{(p+1)}$ in $U_{(p)}$. I claim that these terms are of the form

$$(a_p R_{[kj]} + b_p R_{(kj)}) T^{ki_1 i_2 \ldots i_p} + c_p \left(\sum_{s=1}^{p} R^{i_s}_{ljk} T^{lki_1 i_2 \ldots \hat{\imath}_s \ldots i_p} \right)$$

for some numerical factors a_p, b_p and c_p; $R_{[ij]}$ is the skew part of R_{ij} and $R_{(ij)}$ the symmetric part. The formula above clearly holds for $p = r-1$, with $a_r = b_r = c_r = 1$. Now from equation (3), to obtain the corresponding terms in $U_{(p-1)}$ we must first operate on the expression above with $\mathfrak{d}$, and since we are working to $o(p+1)$ we may ignore those terms which involve the derivatives of the curvature and Ricci tensors. We are left with

$$\begin{aligned}&(a_p R_{[kj]} + b_p R_{(kj)}) T^{ki_1 i_2 \ldots i_{p-1} l}{}_{|l} \\ &\quad + c_p \left(\sum_{s=1}^{p-1} R^{i_s}_{ljk} T^{lki_1 i_2 \ldots \hat{\imath}_s \ldots i_{p-1} m}{}_{|m} + R^{m}_{ljk} T^{lki_1 i_2 \ldots i_{p-1}}{}_{|m} \right).\end{aligned}$$

All the T terms here are divergences except the last, and may be replaced by the appropriate components of $T_{(p)}$ (with the appropriate numerical factor).

For the last we have

$$R^m_{ljk} T^{lki_1 i_2 \ldots i_{p-1}}{}_{|m} = R^m_{ljk} T^{(lki_1 i_2 \ldots i_{p-2}} \delta^{i_{p-1})}_m + o(p+1).$$

Now

$$\begin{aligned} & pR^m_{ljk} T^{(lki_1 i_2 \ldots i_{p-2}} \delta^{i_{p-1})}_m \\ & \quad = \sum_{s=1}^{p-1} R^{i_s}_{ljk} T^{lki_1 i_2 \ldots \hat{\imath}_s \ldots i_{p-1}} + R^l_{ljk} T^{ki_1 i_2 \ldots i_{p-1}} + R^k_{ljk} T^{li_1 i_2 \ldots i_{p-1}}. \end{aligned}$$

The sum is a term of the required form, and $R^k_{ljk} = R_{lj} = R_{[lj]} + R_{(lj)}$. From the cyclic identity we have

$$R^l_{ljk} = -R^l_{jkl} - R^l_{klj} = R_{kj} - R_{jk} = 2R_{[kj]}.$$

So finally

$$\begin{aligned} & pR^m_{ljk} T^{(lki_1 i_2 \ldots i_{p-2}} \delta^{i_{p-1})}_m \\ & \quad = \sum_{s=1}^{p-1} R^{i_s}_{ljk} T^{lki_1 i_2 \ldots \hat{\imath}_s \ldots i_{p-1}} + (3R_{[kj]} + R_{(kj)}) T^{ki_1 i_2 \ldots i_{p-1}}. \end{aligned}$$

Collecting everything together we see that the terms involving $T_{(p)}$ in $U_{(p-1)}$ have the required form. Moreover, all the numerical coefficients occurring in the expressions above are positive, which means that a_p, b_p and c_p are positive for all p.

I can now derive the structural equations.

Theorem 1 The equations

$$\nabla T_{(p)} = U_{(p)} + T_{(p-1)} \odot I, \quad p = r, r-1, \ldots, 1, 0$$

for the unknowns $T_{(p)}$, $p = r, r-1, \ldots, 1, 0$ (with $T_{(p)} \in \mathfrak{S}_p$), where

$$U_{(p-1)} = \left(\frac{p}{n+p-2} \right) \left(\mathfrak{d} U_{(p)} + \rho T_{(p)} \right), \quad U_{(r)} = 0,$$

are structural equations for the equation

$$\nabla T_{(r)} = T_{(r-1)} \odot I.$$

Proof These equations have the right form. If $T_{(p)}$, $p = r, r-1, \ldots, 1, 0$, is a solution then $T_{(r)}$ evidently satisfies the initial equation, and each other $T_{(p)}$ is given by

$$T_{(p)} = \left(\frac{p+1}{n+p}\right)\mathfrak{d}T_{(p+1)} = \frac{(p+1)(p+2)\cdots r}{(n+p)(n+p+1)\cdots(n+r-1)}\mathfrak{d}^{r-p}T_{(r)}. \quad (4)$$

Conversely, if $T_{(r)}$ satisfies the initial equation then the $T_{(p)}$ given by equation (4) satisfy the other structural equations. $\square$

Corollary The set of solutions $T_{(r)}$ of $\nabla T_{(r)} = T_{(r-1)} \odot I$ is a finite-dimensional real vector space whose maximal dimension is the dimension of the space of constant symmetric r-tensors on $\mathbf{R}^{n+1}$.

Proof The dimension is $\sum_{p=0}^{r} \sigma(p, n)$, where $\sigma(p, n)$ is the dimension of the space of constant symmetric p-tensors on $\mathbf{R}^n$. But $\sum_{p=0}^{r} \sigma(p, n) = \sigma(n+1, r)$, as may easily be seen by considering the expression of a homogeneous polynomial of degree r on $\mathbf{R}^{n+1}$ in terms of variables (x^0, x^i) with $i = 1, 2, \ldots, n$. $\square$

The next question is: what can we say about a space when the dimension is maximal?

First, it is easy to see that in $\mathbf{R}^n$ the general solution of the equation $\nabla T_{(r)} = T_{(r-1)} \odot I$ is

$$T^{i_1, i_2 \ldots i_r} = A x^{i_1} x^{i_2} \cdots x^{i_r} + A^{(i_1} x^{i_2} \cdots x^{i_r)} + A^{(i_1 i_2} x^{i_3} \cdots x^{i_r)} + \cdots + A^{i_1 i_2 \ldots i_r}$$

where the As are constant and symmetric; so the dimension in this case is $\sum_{p=0}^{r} \sigma(p, n)$.

Secondly, I consider the effects of a restricted projective transformation of the connection. Recall that a projective transformation of a connection takes the form

$$\Gamma^i_{jk} \mapsto \Gamma^i_{jk} + \psi_j \delta^i_k + \psi_k \delta^i_j = \hat{\Gamma}^i_{jk};$$

the transformation is a restricted projective transformation if ψ_k is a gradient, say $\psi_k = \psi_{|k}$. Under a restricted projective transformation, for any $T \in \mathfrak{S}_r$

$$\hat{\nabla}(e^{-r\psi} T) = e^{-r\psi}(\nabla T + T(\psi) \odot I)$$

where $T(\psi)^{i_1 i_2 \ldots i_{r-1}} = r T^{i_1 i_2 \ldots i_{r-1} j} \psi_{|j}$. Thus the dimensions of the solutions in spaces which are projectively equivalent in the restricted sense are the same. A space is projectively equivalent in the restricted sense to a flat space if and only if it is Ricci-symmetric and has vanishing projective curvature tensor, or equivalently

$$R^i_{ljk} - \frac{1}{n-1}\left(R_{lj}\delta^i_k - R_{lk}\delta^i_j\right) = 0, \quad R_{ij} - R_{ji} = 0. \tag{5}$$

Thus if the curvature satisfies equations (5) the solution space of $\nabla T_{(r)} = T_{(r-1)} \odot I$ has maximal dimension.

I show that this result is sufficient as well as necessary. For this purpose, consider the structural equations with $p = 2$, $p = 1$ and $p = 0$:

$$\begin{aligned} \nabla T_{(2)} &= U_{(2)} + T_{(1)} \odot I \\ \nabla T_{(1)} &= U_{(1)} + T_{(0)} I \\ \nabla T_{(0)} &= U_{(0)} \end{aligned}$$

where $U_{(2)} = o(3)$, and from the calculations of the leading terms in the $U_{(p)}$,

$$\begin{aligned} U_{(1)}{}^i_j &= (a_1 R_{[kj]} + b_1 R_{(kj)}) T^{ki} + c R^i_{ljk} T^{kl} \\ U_{(0)i} &= (a_0 R_{[ki]} + b_0 R_{(ki)}) T^k \end{aligned}$$

where the numerical coefficients are all positive.

Theorem 2 The solution space of $\nabla T_{(r)} = T_{(r-1)} \odot I$ has maximal dimension if and only if the curvature satisfies equations (5), that is, the space is Ricci-symmetric and projectively flat.

Proof It remains to be proved that if the solution space has maximal dimension then the curvature satisfies equations (5). For convenience I shall denote $T_{(2)}$ by T, $T_{(1)}$ by S and $T_{(0)}$ by Q. The equations above read

$$\begin{aligned} T^{ij}_{|k} &= \tfrac{1}{2}(S^i \delta^j_k + S^j \delta^i_k) + o(3) \\ S^i_{|j} &= Q\delta^i_j + (a_1 R_{[kj]} + b_1 R_{(kj)}) T^{ki} + c R^i_{ljk} T^{kl} + o(3) \\ Q_{|i} &= (a_0 R_{[ki]} + b_0 R_{(ki)}) S^k + o(2). \end{aligned}$$

The integrability conditions for the last of these equations are

$$0 = Q_{|ij} - Q_{|ji} = 2a_0 Q R_{[ji]} + o(1).$$

If the dimension is maximal this must hold at any point with an arbitrary choice of values of the variables, and in particular with $Q = 1$, all other variables zero. Thus the space must be Ricci-symmetric.

We may therefore rewrite the second equations as

$$S^i_{|j} = Q\delta^i_j + bR_{kj}T^{ki} + cR^i_{ljk}T^{kl} + o(3)$$

The integrability conditions for these equations are

$$\begin{aligned}
-R^i_{ljk}S^l &= Q_{|k}\delta^i_j - Q_{|j}\delta^i_k + b(R_{lj}T^{li}_{|k} - R_{lk}T^{li}_{|j}) \\
&\quad + c(R^i_{ljm}T^{lm}_{|k} - R^i_{lkm}T^{lm}_{|j}) + o(2) \\
&= b_0(R_{lk}\delta^i_j - R_{lj}\delta^i_k)S^l \\
&\quad + \tfrac{1}{2}b\left(R_{lj}(S^l\delta^i_k + S^i\delta^l_k) - R_{lk}(S^l\delta^i_j + S^i\delta^l_j)\right) \\
&\quad + \tfrac{1}{2}c\left(R^i_{ljm}(S^l\delta^m_k + S^m\delta^l_k) - R^i_{lkm}(S^l\delta^m_j + S^m\delta^l_j)\right) + o(2) \\
&= (b_0 - \tfrac{1}{2}b)(R_{lk}\delta^i_j - R_{lj}\delta^i_k)S^l \\
&\quad + \tfrac{1}{2}c(R^i_{ljk} + R^i_{kjl} - R^i_{lkj} - R^i_{jkl})S^l + o(2) \\
&= (b_0 - \tfrac{1}{2}b)(R_{lk}\delta^i_j - R_{lj}\delta^i_k)S^l + \tfrac{3}{2}cR^i_{ljk}S^l + o(2).
\end{aligned}$$

Thus when the dimension is maximal we have a relation of the form

$$\lambda R^i_{ljk} + \mu(R_{lk}\delta^i_j - R_{lj}\delta^i_k) = 0$$

where $\lambda = 1 + \frac{3}{2}c$ is nonzero. But by taking a trace we obtain

$$\lambda R_{lj} - \mu(n-1)R_{lj} = 0,$$

so that

$$R^i_{ljk} - \frac{1}{n-1}(R_{lj}\delta^i_k - R_{lk}\delta^i_j) = 0$$

(or indeed the space is flat). □

3 Some consequences

If T_1 is a generalized special conformal Killing tensor of valence r_1 and T_2 one of valence r_2 then $T_1 \odot T_2$ is a generalized special conformal Killing tensor of valence $r_1 + r_2$, since

$$\nabla(T_1 \odot T_2) = (\nabla T_1) \odot T_2 + T_1 \odot \nabla T_2.$$

In particular, any symmetrized product of say r generalized special conformal Killing vectors is a generalized special conformal Killing tensor of valence r. Now when equations (5) are satisfied, so that the dimension of the space of generalized special conformal Killing vectors is $n+1$, the space of their symmetrized r-fold products has dimension $\sigma(n+1,r)$, which coincides with the dimension of the space of generalized special conformal Killing tensors of valence r. That is to say, when equations (5) are satisfied, and the dimension of the space of generalized special conformal Killing tensors of each valence is maximal, every generalized special conformal Killing tensor of valence r can be uniquely expressed as a linear combination (in the strict sense of vector space theory) with constant coefficients of symmetrized r-fold products of generalized special conformal Killing vectors.

I now turn my attention briefly to the (pseudo-)Riemannian case. In this case we can write the defining equations as

$$T_{i_1 i_2 \ldots i_r | j} = S_{(i_1 i_2 \ldots i_{r-1}} g_{i_r) j},$$

and the solutions are special conformal Killing tensors. The Ricci tensor is automatically symmetric; the remaining part of equations (5), which states that the projective curvature vanishes, holds if and only if the space is of constant curvature. So on a Riemannian space, the space of special conformal Killing tensors of any given valence has maximal dimension if and only if the Riemannian space is a space of constant curvature. The maximal dimension with valence r is $\sigma(n+1,r)$, as before; and the decomposition property above continues to hold.

Finally, I shall write down explicitly the structural equations in the low dimensional cases of greatest interest. The structural equations for concircular vector fields are

$$T_{i|j} = S g_{ij}, \quad S_{|k} = \frac{1}{n-1} R_{lk} T^l.$$

The equations

$$\begin{aligned}
T_{ij|k} &= \tfrac{1}{2}(S_i g_{jk} + S_j g_{ik}) \\
S_{i|j} &= \frac{1}{n}\left(2R^k_j T_{ik} - 2g^{kl} R^m_{ijk} T_{lm} + g_{ij} Q\right) \\
Q_{|i} &= \frac{2}{n-1}\left(g^{jl}(2R^k_{i|l} - R^k_{l|i}) T_{jk} + (n+1) R^j_i S_j\right)
\end{aligned}$$

are structural equations for Benenti tensors.

Appendix

I derive here the results about structural equations used in the main body of the paper. Such equations may be interpreted as follows. Consider a vector bundle $E \to M$, where M is a manifold with an arbitrary symmetric affine connection; E is supposed to be the Whitney sum of tensor bundles over M, so that the connection induces a connection on E. Take fibre coordinates u^A on E, and denote by Λ^A_{Bi} the connection coefficients for the induced connection with respect to coordinates (x^i, u^A). The structural equations $F^A_{|i} = \Gamma^A_{Bi} F^B$ may be regarded as equations for a section of E, given in coordinates by $u^A = F^A(x^i)$.

My approach is motivated by the case in which the Γ^A_{Bi} all vanish. Then a solution of the equations defines a section of E which is covariantly constant; or if we think of the connection as defining and being defined by a horizontal distribution on E, a section which is horizontal (that is, an n-dimensional submanifold of E, transverse to the fibres, to which the horizontal distribution is everywhere tangent). Given any point $u \in E$, if there is a horizontal section through u it is unique. Now the zero section of any vector bundle equipped with a linear connection is a horizontal section, and is the unique horizontal section through any point $(x^i, 0)$. Thus if $F^A_{|i} = 0$ and $F^A(x^i) = 0$ anywhere then $F^A(x^i) = 0$ everywhere, so that the linear map from solutions of the equations $F^A_{|i} = 0$ to their values at an arbitrary point of M is injective, and the maximum dimension of the solution space is the fibre dimension of E. The necessary and sufficient condition for the equations to be completely integrable, that is for there to be a horizontal section through every point of E, is that the horizontal distribution should be integrable (in the sense of Frobenius), or equivalently that the curvature of the induced connection should vanish.

Theorem Consider a system of structural equations $F^A_{|i} = \Gamma^A_{Bi} F^B$, as interpreted above. Then

1. if, for any $x \in M$, there is a solution with prescribed values $F^A(x)$ it is unique;
2. the space of solutions has maximum dimension equal to the fibre dimension of E;
3. the maximum dimension of the solution space is attained if and only

if the equations are completely integrable;

4. the necessary and sufficient conditions for the system to be completely integrable are that

$$\Gamma^A_{Bi|j} - \Gamma^A_{Bj|i} + \Gamma^C_{Bi}\Gamma^A_{Cj} - \Gamma^C_{Bj}\Gamma^A_{Ci} = R^A_{Bij},$$

where R^A_{Bij} are the components of the curvature of the induced connection on E.

Proof It is only necessary to note that since the coefficients Γ^A_{Bi} are assumed to be tensorial, $\Lambda^A_{Bi} - \Gamma^A_{Bi}$ determines a new connection on E, with respect to which the structural equations are equations for covariantly constant sections. Thus items (1), (2) and (3) follow immediately from the earlier discussion. It is easy to see that the curvature of the new connection is

$$R^A_{Bij} - \left(\Gamma^A_{Bi|j} - \Gamma^A_{Bj|i} + \Gamma^C_{Bi}\Gamma^A_{Cj} - \Gamma^C_{Bj}\Gamma^A_{Ci}\right),$$

whence item (4). □

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