Preface

The title of this volume refers to a Colloquium on Applied Differential Geometry and Mechanics, which was held at the Department of Mathematical Physics and Astronomy, Ghent University, on November 1 and 2, 2002. This colloquium was organised in honour of Michael Crampin, on the occasion of his 60th birthday.

Mike Crampin was born on 16 March 1942. He graduated from Oxford University in 1963 and obtained his PhD in Mathematics at King's College, University of London in 1967. He obtained another BA at the age of fifty-five, this time in Philosophy, at Birkbeck College, University of London. Mike spent most of his professional career at the Open University, Milton Keynes, UK. He took early retirement in 2002 and is currently holding honorary part-time research positions at Ghent University and the University of London.

Most of the contributors to this volume have come to know Mike Crampin through the annual Workshop on Differential Geometric Methods in Theoretical Mechanics. Mike was one of the main instigators of this Workshop, which was held for the first time in Ghent in 1986 and has now been running for seventeen consecutive years, meeting in Australia, Belgium, Hungary, Italy, Poland, Spain, UK and USA.

On many occasions, the annual Workshop has been organised around a limited number of themes, selected a year ahead of the meeting, and with one or two people being responsible for each subject. Mike has often volunteered to be one of these session organisers. His own exposés in such sessions have always excelled in clarity of presentation and depth of insight. These, indeed, are also the trade marks of his research papers.

Many of the participants at the Colloquium have had the privilege to collaborate with Mike Crampin in the broad field of applied differential geometry and mechanics. The Editors feel confident that they can speak for all participants in saying that we all have benefited from these collaborations, both scientifically and in the sense of human interactions. The variety of topics presented on this occasion reflects only part of the range of subjects to which Mike has contributed and, with a few exceptions, the articles contained in this volume are faithful accounts of these presentations.

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Contents

Alternative Lie Algebroid structures and Bi-Differential Calculi

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Abstract

The existence of alternative Lie algebroid structures with the same underlying vector bundle is shown to provide explicit examples of the bi-differential calculi introduced by Dimakis and Müller–Hoissen and the the existence of bi-Hamiltonian structures of Poisson–Nijenhuis type introduced by Crampin, Sarlet and Thompson are discussed from this new perspective of the deformation of a Lie algebroid structure.

1 Introduction

In a recent paper [11] Dimakis and Müller–Hoissen have shown how it is possible to generate conservation laws (non-local charges) [16,2] in completely integrable systems, by making use of a bi-differential calculus. More recently, Crampin *et al.* proved that the approach of Dimakis and Müller–Hoissen was related with the standard approach of bi-Hamiltonian structures of Poisson–Nijenhuis type [9], and the results were extended in [10], where the Poisson–Nijenhuis case was discussed in a detailed way. Some interesting remarks on the so called gauged bi-differential calculus by Dimakis and Müller–Hoissen were also given.

The objects behind the structure studied by Dimakis and Müller–Hoissen, and also in our case, are the so called non-local charges. These objects had been introduced in [16] and can be described as a natural generalization of the standard charges in relativistic quantum field theory. The non-local conserved currents have later received much attention because of its relation with different and interesting field theories [1–3,13].

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Our aim in this paper is to show how the recent structures of Lie algebroid and bi-algebroid are the geometric ingredient for a better understanding of this bi-differential calculus.

The organization of the paper is as follows: In Section 2 we summarize the fundamental results of the bi-differential algebra and the corresponding bicomplex linear equation whose solutions provide us sequences of $\mathcal{D}$-closed s-cochains. Some simple ideas of cohomology of Lie algebras which show us how to obtain an alternative Lie algebra structure in the underlying linear space of a Lie algebra are reviewed in Section 3. The main properties of the structure of Lie algebroid are recalled in Section 4, in which some examples illustrating the importance of the concept of Lie algebroid, as the Lie algebroid structure defined in the tangent bundle by means of a Nijenhuis operator N, are given. Section 5 is devoted to introduce the exterior differential operator in a Lie algebroid, and, in particular, to study the relation of the differential operator corresponding to the Nijenhuis tensor N with the de Rham differential d, particular examples of this structure being the vertical endomorphism and a complex structure. The specific example of a bi-differential calculus in Poisson–Nijenhuis manifolds is reviewed in Section 6, the generalization for a general Lie algebroid being given in Section 7. Finally, Lie bialgebroids are reexamined in Section 8.

2 Bicomplex linear equation

Let us begin by recalling some facts about a bi-differential structure, also called bi-differential algebra or double complex [11].

Definition 2.1 *A bicomplex is a triple* $(\mathcal{M}, D, \mathcal{D})$*, where*

$$\mathcal{M} = \bigoplus_{r\geq 0} \mathcal{M}^r$$

is a $\mathbb{N}_0$*-graded linear space and* D, $\mathcal{D}$ *are two linear maps of degree 1, i.e.* $D, \mathcal{D} : \mathcal{M}^r \to \mathcal{M}^{r+1}$*, such that*

$$D^2 = 0 \ , \qquad \mathcal{D}^2 = 0 \ , \qquad D\circ\mathcal{D} + \mathcal{D}\circ D = 0 \ .$$

Note that the condition $D\circ\mathcal{D} + \mathcal{D}\circ D = 0$ means that, for any $\lambda \in \mathbb{R}$, the linear map $D_\lambda = D + \lambda\,\mathcal{D}$ is also such that $D_\lambda^2 = 0$.

Given a bicomplex, we will call “generalized conserved densities” to the $\mathcal{D}$-closed elements of the bicomplex. The remarkable point is that there is an iterative construction of such “generalized conserved densities”, as follows: let assume that for an integer number $s > 0$ there exists a non-vanishing $\chi^{(0)} \in \mathcal{M}^{s-1}$ such that $DJ^{(0)} = 0$, with $J^{(0)} = \mathcal{D}\chi^{(0)}$. Then, $J^{(1)} \in \mathcal{M}^s$ defined by $J^{(1)} = D\chi^{(0)}$ is such that $\mathcal{D}J^{(1)} = -D\mathcal{D}\chi^{(0)} = 0$. If the $\mathcal{D}$-closed element $J^{(1)}$ is $\mathcal{D}$-exact, then there exists $\chi^{(1)}$, not uniquely defined, such that $J^{(1)} = \mathcal{D}\chi^{(1)}$.

As $J^{(1)}$ is $\mathcal{D}$-closed we can iterate the process when each $\mathcal{D}$-closed form is exact, by defining a $J^{(2)} \in \mathcal{M}^s$ by $J^{(2)} = DJ^{(1)}$, and in this way we obtain a sequence of $\mathcal{D}$-closed elements of degree s, $\{J^{(k)} \mid k = 0, 1, \ldots\}$, satisfying

$$J^{(m+1)} = D\chi^{(m)} = \mathcal{D}\chi^{(m+1)} \ . \tag{2.1}$$

The preceding construction can be expressed in terms of the following diagram:

$$\begin{tikzcd}
0 & J^0 \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] & \chi^{(0)} \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] & & & \\
& 0 & J^{(1)} \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] & \chi^{(1)} \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] & & \\
& & 0 & J^{(2)} \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] & \chi^{(2)} \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] & \\
& & & 0 & J^{(3)} \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] & \chi^{(3)} \arrow[l, "\mathcal{D}"'] \arrow[d, "D"'] \\
& & & & 0 & J^{(4)} \arrow[l, "\mathcal{D}"'] \arrow[dr, dotted, no head] \\
& & & & & & {}
\end{tikzcd}$$

Introducing the formal series in the parameter λ

$$\chi = \sum_{m \geq 0} \lambda^m \, \chi^{(m)} \ ,$$

this satisfies the so called *bicomplex linear structure*:

$$\mathcal{D}(\chi - \chi^{(0)}) = \lambda \, D\chi \ .$$

Very often we start with a $\mathcal{D}$-closed $\chi^{(0)} \in \mathcal{M}^{s-1}$ for which $J^{(0)} = 0$, and then the *bicomplex linear structure* becomes:

$$\mathcal{D}\chi = \lambda \, D\chi \ .$$

The remarkable fact is that in the case we are considering this equation is equivalent to the set of the preceding equations, but the key point is, however, that in the more general case in which the $\mathcal{D}$-cohomology is not trivial, $H^{\mathcal{D}} \neq 0$, any solution of the *bicomplex linear structure* also provides us a sequence of elements $\chi^{(m)} \in \mathcal{M}^{s-1}$, and therefore the corresponding sequence $J^{(m)} \in \mathcal{M}^s$.

It is also frequent to deal with a particular example, that of a graded algebra

$$\Omega(\mathcal{A}) = \bigoplus_{r\geq 0} \Omega^r(\mathcal{A})$$

over an associative algebra with a unit $\mathcal{A}$, which extends to a unit on $\Omega(\mathcal{A})$, and for which D and $\mathcal{D}$ are assumed to be derivations of degree 1.

In particular, the case for which $\mathcal{A} = C^\infty(B)$, $\Omega(\mathcal{A}) = \bigwedge(B)$, and $\mathcal{D}$ is the usual exterior differential, is a really interesting example which has recently been studied by Crampin *et al.* [9], who pointed out that according to Frölicher-Nijenhuis theory [12], D is a derivation of type d_* and there must be a (1,1)-tensor field R such that D is of the form $D = d_R$ and the Nijenhuis torsion of R must be zero. It is determined by

$$d_R f = R^*(df) = df \circ R \ .$$

3 The cohomology of Lie algebras

Let us recall some simple notions of the cohomology of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ a $\mathfrak{g}$-module. That means that $\mathfrak{a}$ is a module that is the carrier space for a linear representation Ψ of $\mathfrak{g}$, i.e. the map $\Psi\colon \mathfrak{g} \to \operatorname{End}\mathfrak{a}$ satisfies

$$\Psi(a)\Psi(b) - \Psi(b)\Psi(a) = \Psi([a,b]) \ .$$

A n-cochain is a n-linear alternating mapping from $\mathfrak{g}\times\cdots\times\mathfrak{g}$ (n times) into $\mathfrak{a}$. We will denote by $C^n(\mathfrak{g},\mathfrak{a})$ the linear space of n-cochains.

For every $n\in\mathbb{N}$, we can define a linear map [7] $\delta_n : C^n(\mathfrak{g},\mathfrak{a}) \to C^{n+1}(\mathfrak{g},\mathfrak{a})$ by

$$\begin{aligned}(\delta_n\alpha)(a_1,\ldots,a_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i+1}\Psi(a_i)\alpha(a_1,\ldots,\widehat{a}_i,\ldots,a_{n+1}) + \\ &+\sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j],a_1,\ldots,\widehat{a}_i,\ldots,\widehat{a}_j,\ldots,a_{n+1}) \ ,\end{aligned}$$

where $\widehat{a}_i$ denotes, as usual, that the element a_i is omitted.

Such linear maps δ_n satisfy $\delta_{n+1}\circ\delta_n = 0$. Then, the linear operator δ on $C(\mathfrak{g},\mathfrak{a}) = \bigoplus_{n=0}^\infty C^n(\mathfrak{g},\mathfrak{a})$ whose restriction to each $C^n(\mathfrak{g},\mathfrak{a})$ is δ_n, satisfies the nilpotency condition $\delta^2 = 0$, and, consequently, we can introduce the usual cohomological notions. We will then denote

$$\begin{aligned}B^n(\mathfrak{g},\mathfrak{a}) &= \{\alpha\in C^n(\mathfrak{g},\mathfrak{a}) \mid \exists\beta\in C^{n-1}(\mathfrak{g},\mathfrak{a}) \text{ such that } \alpha = \delta\beta\} = \operatorname{Im}\delta_{n-1} \ ,\\ Z^n(\mathfrak{g},\mathfrak{a}) &= \{\alpha\in C^n(\mathfrak{g},\mathfrak{a}) \mid \delta\alpha = 0\} = \ker\delta_n \ .\end{aligned}$$

The n-th cohomology group $H^n(\mathfrak{g},\mathfrak{a})$ is defined as

$$H^n(\mathfrak{g},\mathfrak{a}) = \frac{Z^n(\mathfrak{g},\mathfrak{a})}{B^n(\mathfrak{g},\mathfrak{a})} \ ,$$

and we will define $B^0(\mathfrak{g}, \mathfrak{a}) = 0$, by convention.

For each $a \in \mathfrak{g}$ we can define the map

$$i_a : C^k(\mathfrak{g}, \mathfrak{a}) \to C^{k-1}(\mathfrak{g}, \mathfrak{a}) \ ,$$

given by

$$(i_a\alpha)(a_1, \ldots, a_{k-1}) = \alpha(a, a_1, \ldots, a_{k-1}) \ .$$

Particular examples in which $\mathfrak{a}$ is either the set of vector fields or forms of a manifold on which a Lie group G with Lie algebra $\mathfrak{g}$ acts, have been shown to be of relevance in the theory of symmetry groups in classical mechanics [5]. The case we are concerned here is when $\mathfrak{g} = \mathfrak{a}$ and we consider the Lie algebra $\mathfrak{g}$ as a $\mathfrak{g}$-module by means of the adjoint action, i.e. $\Psi : \mathfrak{g} \to \operatorname{End}\mathfrak{g}$ is given by $\Psi(a)b = [a, b]$.

Then, a 1-cochain is a linear map $A : \mathfrak{g} \to \mathfrak{g}$. The coboundary of such 1-cochain is

$$\delta A(a_1, a_2) = [a_1, A(a_2)] - [a_2, A(a_1)] - A([a_1, a_2]) \ .$$

Note that the linear map A is a derivation of the Lie algebra $\mathfrak{g}$ if and only if $\delta A = 0$.

The coboundary of a 2-cochain $\zeta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is

$$\begin{aligned}\delta\zeta(a_1, a_2, a_3) = {} & [a_1, \zeta(a_2, a_3)] - \zeta([a_1, a_2], a_3) + [a_2, \zeta(a_3, a_1)] \\ & - \zeta([a_2, a_3], a_1) + [a_3, \zeta(a_1, a_2)] - \zeta([a_3, a_1], a_2) \ .\end{aligned}$$

Then, the Jacobi identity in the Lie algebra can be written $\delta\zeta = 0$ where ζ is just the bilinear map defining the composition law in the Lie algebra, $\zeta(a, b) = [a, b]$.

For any linear map A, δA defines a skew-symmetric bilinear map $[a_1, a_2]_A$ by

$$[a_1, a_2]_A = \delta A(a_1, a_2) = [A(a_1), a_2] + [a_1, A(a_2)] - A([a_1, a_2]) \ .$$

We will analyze next under what conditions the skew-symmetric composition law $[\cdot, \cdot]_A$ defines another Lie algebra structure in $\mathfrak{g}$, i.e. under what conditions the bracket $[\cdot, \cdot]_A$ satisfies the Jacobi identity.

The Nijenhuis torsion of A, $T(A)$, is defined by

$$T(A)(a_1, a_2) = A([a_1, A(a_2)] - [a_2, A(a_1)] - A([a_1, a_2])) - [A(a_1), A(a_2)] \ ,$$

or, using the definition of $[\cdot, \cdot]_A$,

$$T(A)(a_1, a_2) = A([a_1, a_2]_A) - [A(a_1), A(a_2)] \ .$$

Then, A is said to be a Nijenhuis map if its torsion vanishes, $T(A) = 0$.

As far as the Jacobi identity for $[\cdot,\cdot]_A$ is concerned, note that for any three elements, $a_1, a_2, a_3 \in \mathfrak{g}$,

$$\begin{aligned}
&[a_1,[a_2,a_3]_A]_A + [a_3,[a_1,a_2]_A]_A + [a_2,[a_3,a_1]_A]_A = [A(a_1),[A(a_2),a_3]] \\
&\quad +[A(a_1),[a_2,A(a_3)]] - [A(a_1),A([a_2,a_3])] + [A(a_3),[A(a_1),a_2]] \\
&\quad +[A(a_3),[a_1,A(a_2)]] - [A(a_3),A([a_1,a_2])] + [A(a_2),[A(a_3),a_1]] \\
&\quad +[A(a_2),[a_3,A(a_1)]] - [A(a_2),A([a_3,a_1])] + [a_1,A([A(a_2),a_3])) \\
&\quad +A([a_2,A(a_3)]) - A^2([a_2,a_3])] + [a_3,A([A(a_1),a_2])] + A([a_1,A(a_2)]) \\
&\quad -A^2([a_1,a_2])] + [a_2,A([A(a_3),a_1])] + A([a_3,A(a_1)]) - A^2([a_3,a_1])]
\end{aligned}$$

and using the Jacobi identity for $[\cdot,\cdot]$, we finally get

$$\begin{aligned}
[a_1,[a_2,a_3]_A]_A + [a_3,[a_1,a_2]_A]_A + [a_2,[a_3,a_1]_A]_A &= [a_3,T(A)(a_1,a_2)] \\
&+ [a_2,T(A)(a_3,a_1)] + [a_1,T(A)(a_2,a_3)] \ .
\end{aligned}$$

Therefore, we see that if A is a Nijenhuis map, $T(A)=0$, then $[\cdot,\cdot]_A$ satisfies Jacobi identity and therefore it defines a new Lie algebra bracket. This is a sufficient, but not necessary, condition for δA to define a new Lie algebra bracket.

We also remark that the vanishing of the Nijenhuis torsion of A, $T(A)=0$ also implies that $A:(\mathfrak{g},[\cdot,\cdot]_A)\to(\mathfrak{g},[\cdot,\cdot])$ is a Lie algebra homomorphism, because

$$A([a_1,a_2]_A) - [A(a_1),A(a_2)] = T(A)(a_1,a_2) = 0 \ .$$

In summary, the knowledge of a Nijenhuis map A allows us to define a new Lie algebra structure on $\mathfrak{g}$, such that the map A is a homomorphism of Lie algebras $A:(\mathfrak{g},[\cdot,\cdot]_A)\to(\mathfrak{g},[\cdot,\cdot])$.

A particularly important case is that of $\mathfrak{g}=\mathfrak{X}(B)$. Then the linear maps are given by $(1,1)$-tensor fields in B . Given a $(1,1)$-tensor field N, the Nijenhuis torsion of N is defined by

$$T(N)(X,Y) = N([N(X),Y]+[X,N(Y)]) - N^2([X,Y]) - [N(X),N(Y)] \ ,$$

for any pair of vector fields $X,Y\in\mathfrak{X}(B)$.

A Nijenhuis structure on B is a $(1,1)$-tensor field N with vanishing Nijenhuis torsion,

$$T(N)(X,Y) = 0 \ .$$

Such a Nijenhuis structure allows us to define an alternative Lie algebra structure on $\mathfrak{X}(B)$ with the new Lie algebra bracket

$$[X,Y]_N = [N(X),Y] + [X,N(Y)] - N([X,Y]) \ .$$

Moreover, as a consequence of the vanishing of $T(N)$, the linear map

$$N:(\mathfrak{X}(B),[\cdot,\cdot]_N)\to(\mathfrak{X}(B),[\cdot,\cdot])$$

is a Lie algebra homomorphism.

4 Lie algebroids

Lie groups and Lie algebras have been shown to be very efficient tools in the development of physical theories during the last fifty years. But the generalization of such concepts, Lie groupoids and Lie algebroids, have only been incorporated in the physics literature during the very recent years. We will see that the concept of Lie algebroid is useful for a better understanding of bi-differential calculus. Moreover, as Lie algebroids are directly related with the theory of Poisson structures, they should play a relevant rôle in Physics. The main point to be remarked here is that they are endowed with a nilpotent differential operator providing us a generalized exterior differential calculus. Furthermore, as it has been shown recently, it is possible to develop a generalized Lagrangian mechanics in Lie algebroids in full similarity with the usual geometrical approach [19,17,6]. Finally, the structure of Lie algebroid is also related to that of super-manifolds endowed with a special homological super-vector field.

Let us first recall the definition and some properties of Lie algebroids. The concept of Lie algebroid, which was introduced by Pradines [18], not only generalizes the concept of Lie algebra but also that of tangent bundle of a manifold B. We recall that such tangent bundle, $\tau : TB \to B$ is a vector bundle in which the set of its sections, the vector fields, $\Gamma(\tau) = \mathfrak{X}(B)$, is endowed with a Lie algebra structure. Moreover, the sections of the bundle act as derivations on the associative and commutative algebra of functions in the base manifold B. Both properties, together with a compatibility condition, are the essential ingredients of a Lie algebroid structure: given a function $\varphi \in C^\infty(B)$ and two sections $X, Y \in \Gamma(\tau)$, the following relation holds:

$$[X, \varphi\, Y] = \varphi\, [X, Y] + (X\varphi)\, Y\ .$$

Two other properties which will be generalized also to the case of Lie algebroids are that there exists a (regular) Poisson structure on the dual bundle, in our case the cotangent bundle T^*B, and that there is a graded exterior differential operator which is a derivation of degree one in the graded algebra of forms, $d : \Omega^r(B) \to \Omega^{r+1}(B)$, such that $d^2 = 0$. Here $\Omega^r(B)$ denotes $\Omega^r(B) = \Gamma(T^*B \wedge \overset{r}{\cdots} \wedge T^*B)$.

Definition 4.1 *A Lie algebroid with base B is a vector bundle $\tau_E : E \to B$, together with a Lie algebra structure in the space of its sections given by a Lie product $[\cdot, \cdot]_E$, and a vector bundle map over the identity in the base, called anchor, $\rho : E \to TB$, inducing a map between the corresponding spaces of sections, to be denoted with the same name and symbol, such that:*

1. $\rho : \Gamma(\tau_E) \to \mathfrak{X}(B)$ is a Lie algebra homomorphism

$$[\rho(X), \rho(Y)] = \rho([X, Y]_E) \ .$$

2. For any pair of sections for τ_E, X, Y, and each differentiable function φ defined in B,

$$[X, \varphi\, Y]_E = \varphi\, [X, Y]_E + (\rho(X)\varphi) Y \ .$$

Let $\{x^i \mid i = 1, \dots, n\}$ be local coordinates in a chart on an open set $U \subset B$, and let $\{e_\alpha \mid \alpha = 1, \dots, r\}$ be a basis of local sections of the bundle $U_E = \tau_E^{-1}(U) \to B$. Each local section V_U is written $V = y^\alpha\, e_\alpha$. The local coordinates of $p \in U_E$ are $p = (x^i, y^\alpha)$.

The local expressions for the Lie product and the anchor map are (summation on repeated indices is understood):

$$[e_\alpha, e_\beta]_E = C_{\alpha\beta}{}^\gamma\, e_\gamma \,, \qquad\qquad \rho(e_\alpha) = \rho^i{}_\alpha \, \frac{\partial}{\partial x^i} \,, \tag{4.2}$$

where $\alpha, \beta, \gamma = 1, \dots, r$, and $i = 1, \dots, n$. The functions $C_{\alpha\beta}{}^\gamma \in C^\infty(U)$ and $\rho^i{}_\alpha \in C^\infty(U)$ are called structure functions of the Lie algebroid. The conditions for ρ to be a Lie algebra homomorphism are

$$\sum_{\mathrm{cycl}(\alpha,\beta,\gamma)} \left(\rho^i{}_\alpha \, \frac{\partial C_{\beta\,\gamma}{}^\mu}{\partial x^i} + C_{\alpha\,\nu}{}^\mu \, C_{\beta\,\gamma}{}^\nu \right) = 0 \ ,$$

and the compatibility conditions between ρ and $[\cdot, \cdot]$ are

$$\rho^j{}_\alpha \, \frac{\partial \rho^i{}_\beta}{\partial x^j} - \rho^j{}_\beta \, \frac{\partial \rho^i{}_\alpha}{\partial x^j} = \rho^i{}_\gamma \, C_{\alpha\,\beta}{}^\gamma \ .$$

These equations are called structure equations.

Examples of Lie algebroids are the tangent bundle of a manifold B, with the identity as anchor map and the usual bracket of vector fields, or any integrable subbundle of it, and also a finite-dimensional Lie algebra $\mathfrak{g}$, considered as a vector bundle over a point, for which the anchor vanishes identically and the bracket is that of $\mathfrak{g}$. In the first case, with the usual choice of coordinates (q^i, v^i) in TB induced from coordinates (q^i) in the base B, the structure functions are

$$c_{ij}{}^k = 0 \ , \qquad \rho^i{}_j = \delta^i_j \ ,$$

but in arbitrary coordinates the structure functions, in general, do not vanish. For the case of the Lie algebra $\mathfrak{g}$ as a vector bundle over a point, B reduces to a single point, $TB = \{0\}$, and $E = \mathfrak{g}$. Then $E = \mathfrak{g}$ can be seen as a Lie algebroid for which $\rho = 0$, the sections are the elements of $\mathfrak{g}$ and

$$[V, W]_E = [V, W]_{\mathfrak{g}} \ .$$

The structure functions are the structure constants of the Lie algebra, ${C_{\alpha\beta}}^{\gamma}$.

As another example of interest, consider a Poisson manifold (B, P). Let us recall that the Poisson bi-vector P, i.e. such that $[P, P]_{\rm sch} = 0$, where $[\cdot,\cdot]_{\rm sch}$ denotes the Schouten bracket, allows us to define the Poisson bracket of two functions in B by $\{f, g\} = P(df, dg)$. We can also use the Poisson bi-vector P to define a map $\widehat{P} : T^*B \to TB$, which is a vector bundle morphism, by contraction of P with the corresponding covectors, i.e.

$$\langle\beta, \widehat{P}(\alpha)\rangle = P(\alpha, \beta) ,$$

for each pair of covectors α and β. This map induces another one between the spaces of sections of both vector bundles, also denoted $\widehat{P}$, $\widehat{P} : \Omega^1(B) \to \mathfrak{X}(B)$, by the same expression, but where now α and β are 1-forms.

We call Hamiltonian vector field corresponding to the function $f \in C^\infty(B)$ to the vector field given by $X_f = -\widehat{P}(df)$, and it can be shown that $[X_f, X_g] = -X_{\{f,g\}}$.

Moreover (see e.g. [4] and references therein), Fuchsteiner and Koszul, independently, showed that the set of 1-forms $\Omega^1(B)$ can be endowed with a Lie algebra structure by defining the following bracket:

$$[\alpha, \beta]^P = \mathcal{L}_{\widehat{P}(\alpha)}\beta - \mathcal{L}_{\widehat{P}(\beta)}\alpha - d(P(\alpha, \beta)) , \tag{4.3}$$

where $\mathcal{L}_X$ denotes Lie derivative with respect to X. This Lie product is such that

$$[df, dg]^P = d\{f, g\} ,$$

and, therefore,

$$\widehat{P}([df, dg]^P) = \widehat{P}(d\{f, g\}) = -X_{\{f,g\}} = [X_f, X_g] = [\widehat{P}(df), \widehat{P}(dg)] ,$$

i.e. $\widehat{P} : (\Omega^1(B), [\cdot,\cdot]^P) \to (\mathfrak{X}(B), [\cdot,\cdot])$ is a Lie algebra homomorphism.

Using

$$\mathcal{L}_{fX}\alpha = f\,\mathcal{L}_X\alpha + (i(X)\alpha)\,df ,$$

for any function f in B, we arrive at

$$[\alpha, f\,\beta]^P = (\widehat{P}(\alpha)f)\ \beta + f\,[\alpha, \beta]^P , \tag{4.4}$$

and, therefore, we can endow the vector bundle $\pi : T^*B \to B$ with a Lie algebroid structure where the Lie bracket of 1-forms is given by (4.3) and the anchor map by $\rho = \widehat{P}$, because $\widehat{P}$ is a vector bundle morphism $\widehat{P} : T^*B \to TB$ such that $\widehat{P}([\alpha, \beta]^P) = [\widehat{P}(\alpha), \widehat{P}(\beta)]$, and (4.4) provides the compatibility condition between the anchor and the Lie bracket.

Finally, the most relevant example of Lie algebroid for understanding bi-differential calculus is the Lie algebroid structure defined by a Nijenhuis tensor in a manifold B. In fact, as we showed before, a (1,1) Nijenhuis tensor in B

allows us to introduce a different Lie algebra bracket in the set of sections for $\tau : TB \to B$ as follows:

$$[X,Y]_N = [N(X),Y] + [X,N(Y)] - N\left([X,Y]\right) \ .$$

As N is a linear map $N : (\mathfrak{X}(B),[\cdot,\cdot]_N) \to (\mathfrak{X}(B),[\cdot,\cdot])$, this suggests us to introduce a new Lie algebroid structure in the vector bundle $\tau : TB \to B$ by means of the Lie bracket $[\cdot,\cdot]_N$ and the anchor map $\rho = N : TB \to TB$.

In fact, we have pointed out before that N being a Nijenhuis tensor, the bracket $[\cdot,\cdot]_N$ satisfies Jacobi identity, then $(\mathfrak{X}(B),[\cdot,\cdot]_N)$ is a Lie algebra, and $N : (\mathfrak{X}(B),[\cdot,\cdot]_N) \to (\mathfrak{X}(B),[\cdot,\cdot])$ is an homomorphism of Lie algebras.

Furthermore, for any function $f \in C^\infty(B)$, and any pair of fields $X,Y \in \mathfrak{X}(B)$, we have that

$$\begin{aligned}
[X,f\,Y]_N &= [N(X),f\,Y] + [X,f\,N(Y)] - N\left([X,f\,Y]\right) \\
&= (N(X)f)\,Y + f[N(X),Y] + (Xf)\,N(Y) \\
&\quad + f[X,N(Y)] - (Xf)\,N(Y) - f\,N\left([X,Y]\right) \\
&= (N(X)f)\,Y + f\,[X,Y]_N \ ,
\end{aligned}$$

and therefore,

$$[X,f\,Y]_N = f\,[X,Y]_N + (N(X)f)\,Y \ ,$$

which is the compatibility condition of the anchor N with the Lie bracket $[\cdot,\cdot]_N$.

5 Exterior differential of a Lie algebroid.

Given a Lie algebroid, $(E,\rho,[\cdot,\cdot]_E)$, the sections of τ_E will play the rôle of vector fields, and will be called E-vector fields, and the sections of the dual bundle $\pi_E : E^* \to B$ that of 1-forms, and will be called E-1-forms. Similarly, we can consider the bundles $E^* \wedge \cdots \wedge E^*$, sections for the projections from $E^* \wedge \cdots \wedge E^*$ onto B, which allows us to construct the exterior algebra $\bigwedge^\bullet E^*$ of the dual of E. The sections of $\bigwedge^\bullet E^*$ are called E-forms. The set of them, $\Gamma(\bigwedge^\bullet E^*) = \Omega(E)$, is a $C^\infty(B)$-module. An E-k-form is a E-form such that $\theta \in \Gamma(\bigwedge^k E^*)$. Here, by convention $\Gamma(\bigwedge^0 E^*) = C^\infty(B)$.

The exterior differential giving rise to de Rham cohomology can also be generalized to this more general framework and we can define a differential operator d_E which maps, in a linear way, each E-k-form into a E-$(k+1)$-form, $d_E : \Gamma(\bigwedge^k E^*) \to \Gamma(\bigwedge^{k+1} E^*)$, as follows:

$$\begin{aligned}
d_E\theta(V_1,\ldots,V_{k+1}) &= \sum_i (-1)^{i+1}\rho(V_i)\theta(V_1,\ldots,\widehat{V}_i,\ldots,V_{k+1}) + \\
&+ \sum_{i<j}(-1)^{i+j} \quad \theta([V_i,V_j]_E,V_1,\ldots,\widehat{V}_i,\ldots,\widehat{V}_j,\ldots V_{k+1}) \ ,
\end{aligned}$$

for $V_1, \dots, V_{k+1} \in \Gamma(\tau_E)$, where $\widehat{V}_i$ denotes, as usual, that the element V_i is omitted.

The Lie algebroid axioms imply the following properties:

(i) If $f \in C^\infty(B)$, then $\langle d_E f, V\rangle = \rho(V) f$.
(ii) $d_E^2 = 0$.
(iii) d_E is a super-derivation of degree 1, i.e. when θ is homogeneous of degree $|\theta|$, then
$$d_E(\theta \wedge \zeta) = d_E\theta \wedge \zeta + (-1)^{|\theta|}\theta \wedge d_E\zeta \ .$$

Moreover, the exterior differential d_E is fully characterized by these properties, because if $\delta : \Gamma(\bigwedge^k E^*) \to \Gamma(\bigwedge^{k+1} E^*)$ satisfies these properties, then $\delta = d_E$.

Observe that an exterior derivation d_E satisfying $d_E^2 = 0$ on $\Gamma(\bigwedge^\bullet E^*)$ is equivalent to the Lie algebroid structure on E, because both ρ and $[\cdot,\cdot]$ can be recovered from the expressions

$$\rho(V)f := d_E f(V)\,, \quad \theta([V,W]) := \rho(V)\theta(W) - \rho(W)\theta(V) - d_E\theta(V,W)\ ,$$

for $V, W \in \Gamma(\tau_E)$, $f \in C^\infty(B)$, $\theta \in \bigwedge^1(E)$.

This implies that, given two Lie algebroid structures on the same vector bundle $\tau_E : E \to B$, with associated differential operators $d_E^{(1)}$ and $d_E^{(2)}$, then the condition $d_E^{(1)} \circ d_E^{(2)} = d_E^{(2)} \circ d_E^{(1)}$ means that, for any real number λ, $d_E^{(\lambda)} = d_E^{(1)} + \lambda\, d_E^{(2)}$ is also a derivation of degree one such that $d_E^{(\lambda)} \circ d_E^{(\lambda)} = 0$ and, therefore, $d_E^{(\lambda)}$ defines a new Lie algebroid structure in $\tau_E : E \to B$. The new Lie bracket in the linear space of sections for τ_E is given by $[\cdot,\cdot]_\lambda = [\cdot,\cdot]_1 + \lambda\,[\cdot,\cdot]_2$ and the anchor map by $\rho_\lambda = \rho_1 + \lambda\,\rho_2$.

In local coordinates of E^* as indicated above, d_E is determined by

$$d_E x^i = \rho^i{}_\alpha\, e^\alpha\,, \qquad d_E e^\gamma = C_{\alpha\beta}{}^\gamma\, e^\alpha \wedge e^\beta\ ,$$

where $\{e^\alpha \mid \alpha = 1, \dots, r\}$ is the dual basis of $\{e_\alpha \mid \alpha = 1, \dots, r\}$.

The conditions $d_E^2 x^i = 0$ and $d_E^2 e^\alpha = 0$ are equivalent to the structure equations:

$$\rho^j{}_\alpha \frac{\partial \rho^i{}_\beta}{\partial x^j} - \rho^j{}_\beta \frac{\partial \rho^i{}_\alpha}{\partial x^j} = \rho^i{}_\gamma\, C_{\alpha\beta}{}^\gamma\ ,$$
$$\sum_{\mathrm{cyclic}(\alpha\beta\gamma)} \left[\rho^i{}_\alpha \frac{\partial C_{\beta\gamma}{}^\mu}{\partial x^i} + C_{\alpha\nu}{}^\mu\, C_{\beta\gamma}{}^\nu\right] = 0\,.$$

In the particular case of $E = TB$, the anchor is the identity and the commutator of vector fields is the product $[\cdot,\cdot]_E$, then the exterior operator is

$$d_E\theta(V_1, \dots, V_i, \dots, V_{k+1}) = \sum_i (-1)^{i+1} V_i \theta(V_1, \dots, \widehat{V}_i, \dots, V_{k+1})$$
$$+ \sum_{i<j} (-1)^{i+j}\theta([V_i, V_j]_E, V_1, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots V_{k+1})\ ,$$

which is the de Rham operator, and its associated cohomology is the usual de Rham cohomology

In the other example in which $E=\mathfrak{g}$, $\rho=0$ and $[e_i,e_j]_E=c_{\alpha\beta}{}^{\gamma}\,e_\gamma$, with $c_{\alpha\beta}{}^{\gamma}\in\mathbb{R}$, then the differential operator is defined by:

$$d_E\theta(V_1,\dots,V_{k+1})=\sum_{i<j}(-1)^{i+j}\theta([V_i,V_j]_E,V_1,\dots,\widehat{V}_i,\dots,\widehat{V}_j,\dots,V_{k+1})\ ,$$

i.e. in this case this operator is the Chevalley (Chevalley–Eilenberg), and generate the Lie algebra cohomology of $\mathfrak{g}$.

If N is a Nijenhuis structure in B and we consider the corresponding Lie algebroid structure in TB, then the differential operator d_N on $\Gamma(\bigwedge^\bullet T^*B)$ turns out to be such that

$$d_N=[i_N,d]_s=i_N\circ d-d\circ i_N\ , \tag{5.5}$$

where d is the de Rham differential and i_N is a derivation of degree 0, defined as

$$i_N\,\alpha\,(X_1,\dots,X_n)=\sum_{i=1}^{n}\alpha(X_1,\dots,N(X_i),\dots,X_n)\ ,$$

for any differential n-form α, with $n\geq 1$. For a function $f\in C^\infty(B)$, $i_Nf=0$, by convention. Here $[D_1,D_2]_s$ denotes the supercommutator (or graded commutator) of graded derivations

$$[D_1,D_2]_s:=D_1\circ D_2-(-1)^{|D_1|\,|D_2|}D_2\circ D_1\ ,$$

where $|D_i|$ denotes the degree of the derivation D_i, for $i=1,2$. In fact, d_N is a derivation of degree one which is defined by its action on functions and on 1-forms. Let us now consider the action of d_N on functions and on 1-forms and we will find that the relation (5.5) holds.

a) If f is a function,

$$d_Nf(V)=N(V)f\ ,$$

while $i_N(df)=N^*(df)$, $d(i_Nf)=0$, and, therefore,

$$d_Nf=i_N(df)-d(i_Nf)\ .$$

b) If $\alpha\in\Gamma(\bigwedge^1(T^*B))$, then

$$(d_N\alpha)(V_1,V_2)=N(V_1)\alpha(V_2)-N(V_2)\alpha(V_1)-\alpha([V_1,V_2]_N)\ ,$$

while

$$\begin{aligned}i_N(d\alpha)(V_1,V_2)&=d\alpha(N(V_1),V_2)+d\alpha(V_1,N(V_2))\\&=N(V_1)\alpha(V_2)-V_2\alpha(N(V_1))-\alpha([N(V_1),V_2])\\&\quad+V_1\alpha(N(V_2))-N(V_2)\alpha(V_1)-\alpha([V_1,N(V_2)])\,,\end{aligned}$$

and

$$d(i_N\alpha)(V_1,V_2)=V_1\alpha(N(V_2))-V_2\alpha(N(V_1))-\alpha(N([V_1,V_2]))\ ,$$

and, therefore,

$$\begin{aligned}(i_N(d\alpha)\,-\,d(i_N\alpha))(V_1,V_2)&=N(V_1)\alpha(V_2)-N(V_2)\alpha(V_1)\\&+\,\alpha(N([V_1,V_2]))-\alpha([N(V_1),V_2])-\alpha([V_1,N(V_2)])\ .\end{aligned}$$

Having in mind that

$$[V_1,V_2]_N=[N(V_1),V_2]+[V_1,N(V_2)]-N([V_1,V_2])\ ,$$

it shows that

$$i_N(d\alpha)-d(i_N\alpha)=d_N\alpha\ .$$

The two exterior differential operators d and d_N acting over $\Gamma(\bigwedge^\bullet T^*B)$, are derivations of degree 1, such that satisfy

$$d^2=0\ ,\quad d_N^2=0\ , \tag{5.6}$$

and, as $d\circ d_N=d\circ i_N\circ d$, $d_N\circ d=-d\circ i_N\circ d$,

$$[d,d_N]_{\rm s}=d\circ d_N+d_N\circ d=0\ . \tag{5.7}$$

Consequently, the pair (d,d_N) gives rise to the classical theory of the bi-differential calculus of Frölicher–Nijenhuis.

As a first example, let S be the vertical endomorphism S of the tangent bundle [8]. This is a (1,1) tensor such that $\operatorname{Im}S=\ker S$ and is integrable in the sense that its Nijenhuis torsion vanishes. It allows us to endow the tangent bundle $\tau_{TB}:T(TB)\to TB$ with an alternative Lie algebroid structure, for which the Lie bracket in $\Gamma(\tau_{TB})$ is

$$[X,Y]_S=[S(X),Y]+[X,S(Y)]-S([X,Y])\ .$$

and the anchor map is S itself.

The corresponding differential operator is given by:

$$\begin{aligned}d_S\theta(X_1,\dots,X_{k+1})=&\sum_{i01}^{k+1}(-1)^{i+1}S(X_i)\theta(X_1,\dots,\widehat{X_i},\dots,X_{k+1})\\&+\sum_{i<j}(-1)^{i+j}\theta([X_i,X_j]_S,X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots X_{k+1})\ ,\end{aligned}$$

and it satisfies

$$d_S=[i_S,d]_s\ .$$

As indicated above, $d\circ d_S=-d_S\circ d=-d\circ i_S\circ d$.

Remark that $S^2 = 0$, and that $d_S\alpha = 0$ does not necessarily implies that there exists a 1-form β such that $\alpha = d_S\beta$. This differential operator is used to define the Cartan 1-form, $\theta_L = d_S L$, and the Lagrangian 2-form defined by a Lagrange function in TB is $\Omega_L = -d(d_S L) = d_S(dL)$. Then $\Omega(TB)$ is endowed with a bi-differential structure $(\Omega(TB), d, d_S)$.

Another interesting example is that given by the tangent bundle of a complex manifold. Let (B, J) be a manifold of real dimension $\dim_{\mathbb{R}} B = 2m$, endowed with an almost complex structure, given by a $(1,1)$-tensor field J, i.e. such that $J^2 = -\mathbb{I}$ and the Nijenhuis torsion $T(J)$ vanishes

$$T(J)(X, Y) = 0 \ , \quad \forall X, Y \in \mathfrak{X}(B) \ .$$

Then, TB can be endowed with a new Lie algebroid structure given by the bracket

$$[X, Y]_J = [J(X), Y] + [X, J(Y)] - J\left([X, Y]\right) \ ,$$

and with anchor $\rho = J$, and, therefore,

$$J\left([X, Y]_J\right) = [J(X), J(Y)] \ .$$

The differential operator d_J acts on $\Gamma(\bigwedge^\bullet T^*B)$ and is such that

$$d_J = [i_J, d]_s \ .$$

So, the vector bundle $\tau : TB \to B$ has an alternative Lie algebroid structure given by a deformation of the usual Lie bracket by J and with an anchor map $\rho = J$. Once again, $d \circ d_J = -d_J \circ d = -d \circ i_J \circ d$, and the pair (d, d_J) gives rise to a bi-differential calculus.

6 Bi-differential calculi and Poisson–Nijenhuis structures

The use of a bi-differential calculus for generating conservation laws developed by Dimakis and Müller–Hoissen has recently been related for $s = 1$ with the standard approach using bi-Hamiltonian structures of the Poisson–Nijenhuis type for systems with a finite number of degrees of freedom [9,10]. Crampin *et al.* assumed that a differentiable manifold B is endowed with a symplectic structure ω_0 and a (1,1)-tensor field R such that

$$\omega_0(R(X), Y) = \omega_0(X, R(Y)) \ , \tag{6.8}$$

so that ω_1 defined by $\omega_1(X, Y) = \omega_0(R(X), Y)$ is a 2-form. If, moreover, ω_1 so defined is closed, then we can define

$$\{f, g\}_1 = \omega_1(X_f, X_g) \ ,$$

where X_f and X_g are the Hamiltonian vector fields (with respect to the symplectic structure ω_0) corresponding to f and g, respectively. The conditions $T(R) = 0$ and $d\omega_1 = 0$ imply that $\{\cdot,\cdot\}_1$ is a Poisson bracket compatible with the first one, and R is the recursion operator of the structure, which is $R = \widehat{\omega}_0^{-1} \circ \widehat{\omega}_1$, where $\widehat{\omega}_i : \mathfrak{X}(B) \to \bigwedge^1(B)$, $i = 0, 1$, is defined by contraction, i.e. $\langle \widehat{\omega}_i(X), Y\rangle = \omega_i(X,Y)$, for all pairs $X, Y \in \mathfrak{X}(B)$.

Now, since

$$\{f, g\}_1 = \omega_0(X_f, R(X_g)) = -R(X_g)f = -d_R f(X_g) \ ,$$

we see that

$$\{\chi^{(m)}, g\}_1 = -d_R\chi^{(m)}(X_g) = d\chi^{(m+1)}(X_g) = \{\chi^{(m+1)}, g\} \ .$$

The (1,1)-tensor field can also be seen as a $C^\infty(B)$-linear map from $\Omega^1(B)$ into $\Omega^1(B)$ and it will be denoted R^T. With this notation, the condition (6.8) is written as $\widehat{\omega}_0 \circ R = R^T \circ \widehat{\omega}_1$, because $\langle (\widehat{\omega}_0 \circ R)(X), Y\rangle = \omega_0(R(X), Y)$ and $\langle (R^T \circ \widehat{\omega}_1)(X), Y\rangle = \langle \widehat{\omega}_1(X), R(Y)\rangle = \omega_0(X, R(Y))$.

When $\widehat{\omega}_0$ is invertible, (6.8) reduces to

$$R \circ \widehat{P} = \widehat{P} \circ R^T \ , \tag{6.9}$$

with P being the Poisson bi-vector field corresponding to ω_0. This is the connection with the theory of Poisson–Nijenhuis manifolds.

We first remark that if P is a Poisson tensor and N a Nijenhuis in B satisfying condition (6.9) for $R = N$, then we can define a new Poisson bi-vector P^N by

$$P^N(\alpha, \beta) = \langle \beta, N(\widehat{P}(\alpha))\rangle \ ,$$

which, obviously, is a bi-vector, because

$$P^N(\beta, \alpha) = \langle \alpha, N(\widehat{P}(\beta))\rangle = \langle \alpha, \widehat{P}(N^T\beta)\rangle = -P(\alpha, N^T(\beta))$$

and

$$P^N(\alpha, \beta) = \langle \beta, N(\widehat{P}(\alpha))\rangle = \langle N^T(\beta), \widehat{P}(\alpha)\rangle = P(\alpha, N^T(\beta)) \ ,$$

from which $P^N(\alpha,\beta) = -P^N(\beta,\alpha)$. It can be shown that the fact that N is a Nijenhuis tensor implies that P^N is also a Poisson tensor, i.e. $[P^N, P^N]_{\rm sch} = 0$.

The coexistence of two different structures always leads to the study of compatibility conditions between them. A Poisson–Nijenhuis (P–N) structure, as introduced by Magri and Morosi and later studied by Kosmann–Schwarzbach and Magri, is made up by a pair of a Poisson and a Nijenhuis structures in a manifold B, (N, P), satisfying

$$N \circ \widehat{P} = \widehat{P} \circ N^T \ ,$$

i.e. $\widehat{P}$ intertwines N and its transpose N^T, and with the following compatibility condition:

$$[\alpha,\beta]^{P^N} - \left([N^T(\alpha),\beta]^P + [\alpha, N^T(\beta)]^P - N^T([\alpha,\beta]^P)\right) = 0 \ .$$

In a recent paper, Crampin *et al.* [10] have shown for $s = 1$ that, if $(\Omega(B), d, d_N)$ is the bi-differential calculus defined by a Nijenhuis tensor N in a Poisson–Nijenhuis manifolds (B, N, P), and the function $\chi^{(0)}$ satisfies $dd_N\chi^{(0)} = 0$, then the functions of the sequence $\{\chi^{(m)} \mid m = 0, 1, \ldots\}$ defined by $d\chi^{(m+1)} = d_N\chi^{(m)}$ satisfy

$$\{\chi^{(m)}, \chi^{(n)}\} = P\left(d\chi^{(m)}, d\chi^{(n)}\right) = 0, \quad \text{for all } m, n \geq 0 \, .$$

These Poisson–Nijenhuis structures correspond in the framework of Lie algebroids to a particular case of the so called Lie bialgebroids [15]. We once again recall that when (B, P) is a Poisson manifold, we have simultaneously algebroid structures for both the tangent and the cotangent bundles, and correspondingly, not only the de Rham differential operator d acting on the set $\Omega(B)$ of forms, but also a differential operator d_P acting on the set of multivector fields, which, as indicated above, corresponds to consider the Schouten bracket with the bi-vector field P. For instance, the integrability condition $[P, P] = 0$ is written $d_P P = 0$. Then, if X and Y are vector fields in B,

$$d_P([X, Y]_{\rm sch}) = [P, [X, Y]_{\rm sch}]_{\rm sch} = [[P, X]_{\rm sch}, Y]_{\rm sch} + [X, [P, Y]_{\rm sch}]_{\rm sch} \ ,$$

where use has been made of the graded Jacobi identity satisfied by the Schouten bracket. In other words, d_P is a derivation

$$d_P[X, Y]_{\rm sch} = [d_P(X), Y]_{\rm sch} + [X, d_P(Y)]_{\rm sch} \ .$$

What happens in the case of a Poisson–Nijenhuis manifold is something similar, but for the new structure of Lie algebroid in TB which is obtained from N, $(TB, [\cdot,\cdot]_N, N)$, and the Poisson structure P^N.

7 The case of a general Lie algebroid

In much the same way as we did before with the sections of τ_E in a Lie algebroid $(\tau_E : E \to B, \rho, [\cdot,\cdot]_E)$, which were a generalization of vector fields, we can consider the exterior algebra, which is a graded algebra whose elements will be called E-multi-vector fields. There is also a graded Lie bracket on the linear space $\Gamma(\bigwedge^\bullet E)$ of sections of $\bigwedge^\bullet E$, which constitutes with the associative and graded commutative product $\wedge$ the so called Gerstenhaber algebra of the Lie algebroid. Then, we can define a E-Poisson structure by a E-bivector field

Π_E such that $[\Pi_E\,,\,\Pi_E]_{(E)}=0$, where $[\cdot,\cdot]_{(E)}$ is the Gerstenhaber bracket defined on $\Gamma(\bigwedge^\bullet E)$.

A section of $E\otimes E^*$ will be called a E-$(1,1)$-tensor. Given a E-$(1,1)$-tensor field, i.e. $N\in\Gamma(E\otimes E^*)$, we can associate with it a map i_N which is a derivation in the tensor algebras of sections of τ_E and π_E, such that

$$\begin{aligned} i_N X &= N(X) \quad \text{for } X\in\Gamma(E)\,,\\ i_N\mu &= N^T(\mu) \quad \text{for } \mu\in\Gamma(E^*)\,. \end{aligned}$$

We will say that N is a Nijenhuis structure for the Lie algebroid E if the torsion $T_E(N)(X,Y)$ vanishes for all E-vector fields X,Y, where $T_E(N)(X,Y)$ is defined as

$$T_E(N)(X,Y)=N\left([N(X),Y]_E+[X,N(Y)]_E-N([X,Y])_E\right)-[N(X),N(Y)]_E\,.$$

An E-Nijenhuis tensor N induces a new Lie algebroid structure on E defined by the bracket

$$[X,Y]_{EN}=[N(X),Y]_E+[X,N(Y)]_E-N([X,Y]_E)\ ,$$

and anchor map $\rho^N=\rho\circ N$ where ρ is the original anchor map of E.

Note that the torsion being zero is equivalent to $N:(E,[\cdot,\cdot]_{EN})\to(E,[\cdot,\cdot]_E)$ to be a homomorphism:

$$N([X,Y]_{EN})=[N(X),N(Y)]_E\ .$$

The exterior differential operator d_{EN} in $\Gamma(\bigwedge^\bullet E^*)$ turns out to be such that

$$d_{EN}=[i_N,d_E]=i_N\circ d_E-d_E\circ i_N\ .$$

Once again we find a bi-differential structure associated in this case with the E-Nijenhuis tensor field N which is defined by the pair of differential operators (d_E,d_{EN}), where both differential operators are of degree 1 and satisfy

$$d_E^2=d_{EN}^2=0\ ,\quad [d_E,d_{EN}]_s=d_E\circ d_{EN}+d_{EN}\circ d_E=0\ ,$$

because

$$d_E\circ d_{EN}=d_E\circ i_E\circ d_E\ ,\quad d_{EN}\circ d_E=-d_E\circ i_E\circ d_E\ .$$

Of course, when $E=TM$ we recover the theory of bi-differential calculus of Frölicher–Nijenhuis.

An interesting example of this E-Nijenhuis tensor field which may be useful in the development of Lagrangian mechanics in Lie algebroid is the vertical endomorphism in the extended Lie algebroid introduced by Martínez [17,6] as an appropriate counterpart of the tangent bundle of the tangent bundle of a manifold.

As indicated above, the concept of Poisson structure can also be generalized to the framework of Lie algebroids, using the Gerstenhaber bracket of E-bi-vector fields instead of the Schouten bracket of bi-vector fields. So, an E-Poisson structure Π_E in a Lie algebroid has associated a vector bundle map $\widehat{\Pi}_E : E^* \to E$ when it is evaluated on E-1-covectors, and allows us to define a Lie algebroid structure on E^* by means of a Lie bracket on E-1-forms given by

$$[\alpha, \beta]^{\Pi_E} = d_{\widehat{\Pi}_E(\alpha)}\beta - d_{\widehat{\Pi}_E(\beta)}\alpha - d(\Pi_E(\alpha, \beta)) \ . \tag{7.10}$$

and with an anchor map given by $\rho^* = \rho \circ \widehat{\Pi}_E$. Here, if X is an E-vector field, d_X denotes $d_X = i_X \circ d_E + d_E \circ i_X$.

The corresponding E^*-exterior differential operator turns out to be $d_{E^*} = d_{\Pi_E} = [\Pi_E, \cdot]_{(E)}$.

A concept of E-Poisson–Nijenhuis can also be introduced as a pair (N, Π_E) of a E-Nijenhuis structure and a E-Poisson tensor such that $N \circ \widehat{\Pi}_E = \widehat{\Pi}_E \circ N^T$ satisfying an additional compatiblity condition, which we do not write here in an explicit way.

8 Lie bialgebroids

Kosmann–Schwarzbach, using the notion of Lie bialgebroid structure of Mackenzie and Xu, presented the compatibility condition for P–N structures in a simpler way. First of all, we recall that a Lie bialgebroid is a pair $((E, \rho), (E^*, \rho^*))$ of Lie algebroids, where E^* is the dual bundle of E, such that the differential d_E of E is a derivation of the graded Lie algebra $(\Gamma(\bigwedge^\bullet E^*), [\cdot, \cdot]_{E^*})$, and the differential d_{E^*} of E^* is a derivation of the graded Lie algebra $(\Gamma(\bigwedge^\bullet E), [\cdot, \cdot]_{E^*})$.

We mentioned an example of Lie bialgebroid: when (B, P) is a Poisson manifold, $\big((TB, [\cdot, \cdot], \tau), (T^*B, [\cdot, \cdot]^P, \pi_B)\big)$ is a Lie bialgebroid. The result obtained by Kosmann–Schwarzbach [15] is that (B, P, N) is a Poisson–Nijenhuis manifold if and only if $\big((TB, [\cdot, \cdot]_N, N), (T^*B, [\cdot, \cdot]^P, \widehat{P})\big)$ is a Lie bi-algebroid. Moreover, she also proved that given a Lie bialgebroid (E, E^*) there is an associated Poisson structure on the base manifold B given by (see [14])

$$\{f, g\}_{(E, E^*)} = \langle d_E f, d_{E^*} g\rangle \ \ ,$$

where $\langle \cdot, \cdot\rangle$ denotes the pairing of elements of E^* with those of E. When (B, P) is a Poisson manifold, $E = TB$ and $E^* = T^*B$, endowed with the Lie algebroid structure (4.3), then the Lie bialgebroid structure (TB, T^*B) is such that $d_{E^*} = d_P = [P, \cdot]$, so $\{f, g\}_{(E,E^*)} = P(df, dg)$, which is the original Poisson structure of B.

The generalization for E-Poisson–Nijenhuis structures is also possible, giving rise to Lie bialgebroids and different bi-differential calculi.

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Tulczyjew's triples and lagrangian submanifolds in classical field theories

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Abstract

In this paper the notion of Tulczyjew's triples in classical mechanics is extended to classical field theories, using the so-called multisymplectic formalism, and a convenient notion of lagrangian submanifold in multisymplectic geometry. Accordingly, the dynamical equations are interpreted as the local equations defining these lagrangian submanifolds.

1 Introduction

In middle seventies, W.M. Tulczyjew [23,24] introduced the notion of special symplectic manifold, which is a symplectic manifold symplectomorphic to a cotangent bundle. Using this notion, Tulczyjew gave a nice interpretation of lagrangian and hamiltonian dynamics as lagrangian submanifolds of convenient special symplectic manifolds.

The other ingredients in the theory were two canonical diffeomorphisms $\alpha : TT^*Q \longrightarrow T^*TQ$ and $\beta : TT^*Q \longrightarrow T^*T^*Q$. β is nothing but the mapping obtained by contraction with the canonical symplectic form ω_Q, but the definition of α is more complicated, and requires the use of the canonical involution of the double tangent bundle TTQ.

The theory was extended to higher order mechanics by several authors (see for instance [2,3,6,8,12]). But the extension to classical field theories has not been achieved up to now. There is a good approach by Kijowski and Tulczyjew [11], and in fact, the present approach is strongly inspired in that monograph.

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The key point is a better understanding of the geometry of lagrangian submanifolds in the multisymplectic setting. A systematic study of the geometry of multisymplectic manifolds was started by Cantrijn *et al* at the beginning of the nineties [7], followed by a pair of papers which clarify that geometry [4,5] (a more detailed study [18] is in preparation).

A multisymplectic manifold is a manifold equipped with a closed form which is non-degenerate in some sense. The canonical examples are the bundles of forms on an arbitrary manifold, providing thus a nice extension of the notion of symplectic manifold. However, this definition is too general for practical purposes. Indeed, in order to have a Darboux theorem which would permit us to introduce canonical coordinates, we need additional properties. In other words, if we want to deal with multisymplectic manifolds which locally behave as the geometric models we need to consider multisymplectic manifolds $(\mathcal{P}, \Omega)$ with additional structure, given by a 1-isotropic foliation $\mathcal{W}$ satisfying some dimensionality condition, or, even a "generalised foliation" $\mathcal{E}$ defined roughly speaking on the space of leaves determined by $\mathcal{W}$.

The tangent and cotangent functors are now substituted by the jet prolongation functor and the exterior power functor, respectively, so that we obtain canonical diffeomorphisms $\tilde{\alpha} : \widetilde{J^1Z^*} \longrightarrow \Lambda_2^{n+1}Z$ and $\tilde{\beta} : \widetilde{J^1Z^*} \longrightarrow \Lambda_2^{n+1}Z^*$, where Z is the 1-jet prolongation of the fibred manifold $Y \longrightarrow X$, X being the space-time n-dimensional manifold, and Z^* is the dual affine bundle of Z. Here a tilde over a manifold of jets means that we are taking a quotient manifold in order to have only those elements with the same divergence.

Using a convenient formulation of the field equations with Ehresmann connections, we construct the corresponding lagrangian submanifolds which encode the dynamics. Indeed, we present a compact form for the De Donder and field equations as follows. From the lagrangian density $\mathbb{L} = L\eta$ (η is a volume form on X), we construct the Poincaré-Cartan $(n+1)$-form Ω_L on Z; then the extremals for $\mathbb{L}$ coincide with the horizontal sections of any Ehresmann connection $\mathbf{h}$ in the fibred manifold $Z \longrightarrow X$ satisfying the equation

$$i_{\mathbf{h}}\,\Omega_L = (n-1)\Omega_L.$$

Since a connection in $Z \longrightarrow X$ can be interpreted as a section of the 1-jet prolongation $J^1Z \longrightarrow Z$, we have all the ingredients we need. In fact, the Euler-Lagrange equations are just the local equations defined by a k-lagrangian submanifold of $\widetilde{J^1Z^*}$, the latter being a multisymplectic manifold equipped with the multisymplectic form Ω_α dragged via $\tilde{\alpha}$ from the canonical one on $\Lambda_2^{n+1}Z$.

A similar procedure can be developed in the hamiltonian setting, but in this case we would need to choose a convenient hamiltonian form. This hamiltonian form is obtained through the corresponding Legendre transformation $Leg_L : Z \longrightarrow Z^*$. Finally, both sides are related.

2 Lagrangian submanifolds and classical mechanics

2.1 Some prelimaries

Let $(\mathcal{V}, \omega)$ a finite dimensional symplectic vector space with symplectic form ω. This means that ω is a 2-form on a vector space V which is non-degenerate in the sense that the linear mapping

$$v \in \mathcal{V} \mapsto i_v \, \omega \in V^*$$

is injective (and hence it is a linear isomorphism).

Therefore, $\mathcal{V}$ has even dimension, say $2n$, and the non-degeneracy is equivalent to the condition $\omega^n \neq 0$.

A linear isomorphism $\phi : (\mathcal{V}_1, \omega_1) \longrightarrow (\mathcal{V}_2, \omega_2)$ is called a symplectomorphism if ϕ preserves the symplectic forms, say $\phi^* \omega_2 = \omega_1$.

Take a subspace $E \subset \mathcal{V}$, and define the ω-complement of E as follows:

$$E^\perp = \{ v \in \mathcal{V} \mid i_{v \wedge e} \, \omega = 0, \text{ for all } e \in E \}.$$

The subspace E is said to be isotropic (resp. coisotropic, lagrangian, symplectic) if $E \subset E^\perp$ (resp. $E^\perp \subset E$, $E = E^\perp$, $E \cap E^\perp = \{0\}$).

An useful characterization of a lagrangian subspace E, is that it is a maximally isotropic subspace or, equivalently, on a finite dimensional symplectic vector space, it is isotropic and $\dim E = \dfrac{1}{2} \dim \mathcal{V}$.

The algebraic model for a symplectic vector space is the following. Given an arbitrary vector space V we construct $\mathcal{V}_V = V \oplus V^*$ equipped with the symplectic form ω_V defined by

$$\omega_V((v_1, \gamma_1), (v_2, \gamma_2)) = \gamma_1(v_2) - \gamma_2(v_1),$$

for all $(v_1, \gamma_1), (v_2, \gamma_2) \in \mathcal{V}_V$.

We know that V and V^* are lagrangian subspaces of $(\mathcal{V}_V, \omega_V)$. Moreover, every symplectic vector space $(\mathcal{V}, \omega)$ is symplectomorphic to $(\mathcal{V}_\mathcal{L}, \omega_\mathcal{L})$ for any lagrangian subspace $\mathcal{L}$ of $(\mathcal{V}, \omega)$.

In addition we can prove that a linear isomorphism $\phi : (\mathcal{V}_1, \omega_1) \longrightarrow (\mathcal{V}_2, \omega_2)$ is a symplectomorphism if and only if its graph $\{(v, \phi(v)) \,|\, v \in \mathcal{V}_1\} \subset \mathcal{V}_1 \times \mathcal{V}_2$ is a lagrangian subspace of the symplectic manifold $(\mathcal{V}_1 \times \mathcal{V}_2, \omega_1 \ominus \omega_2)$, where $\omega_1 \ominus \omega_2 = \pi_1^* \omega_1 - \pi_2^* \omega_2$, $\pi_1 : \mathcal{V}_1, \times \mathcal{V}_2 \longrightarrow \mathcal{V}_1$ and $\pi_2 : \mathcal{V}_1, \times \mathcal{V}_2 \longrightarrow \mathcal{V}_2$ being the canonical projections.

A symplectic manifold is a pair $(\mathcal{P}, \omega)$, where ω is a closed 2-form such that the pair $(T_x\mathcal{P}, \omega_x)$ is a symplectic vector space for any $x \in \mathcal{P}$. Thus, $\mathcal{P}$ has even dimension, say $2n$.

Therefore, given a function $f : \mathcal{P} \longrightarrow \mathbb{R}$ there exists a unique vector field (the hamiltonian vector field X_f with hamiltonian energy f) such that

$$i_{X_f}\,\omega = df.$$

Let now $\pi_Q : T^*Q \longrightarrow Q$ be the cotangent bundle of an arbitrary manifold Q. There exists a canonical 1-form θ_Q on T^*Q defined by

$$\theta_Q(\gamma)(X) = \langle \gamma, T\pi_Q(X) \rangle$$

for all $X \in T_\gamma(T^*Q)$ and for all $\gamma \in T^*Q$. θ_Q is the Liouville 1-form, and in bundle coordinates (q, p) we have

$$\theta_Q = pdq.$$

So, $\omega_Q = -d\theta_Q$ is a canonical symplectic form on T^*Q such that $\omega_Q = dq \wedge dp$.

As is well known, Darboux theorem states that any symplectic manifold is locally symplectomorphic to a cotangent bundle. More precisely, one can find local coordinates around each point of a symplectic manifold $(\mathcal{P}, \omega)$ such that

$$\omega = dq \wedge dp.$$

The following results are the main examples of lagrangian submanifolds.

Theorem 2.1

(i) *The image of a hamiltonian vector field X_f on a symplectic manifold $(\mathcal{P}, \omega)$ is a lagrangian submanifold of the tangent lift symplectic manifold $(T\mathcal{P}, \omega^T)$.*
(ii) *The fibres of T^*Q are lagrangian submanifolds of (T^*Q, ω_Q).*
(iii) *The image of a 1-form γ on a manifold Q is a lagrangian submanifold of (T^*Q, ω_Q) if and only if γ is closed.*
(iv) *Given a diffeomorphism $\phi : (\mathcal{P}_1, \omega_1) \longrightarrow (\mathcal{P}_2, \omega_2)$ between two symplectic manifolds then ϕ is a symplectomorphism if and only if its graph is a lagrangian submanifold in the symplectic manifold $(\mathcal{P}_1 \times \mathcal{P}_2, \omega_1 \ominus \omega_2)$.*

There is an important theorem due to A. Weinstein which gives the normal form for a lagrangian submanifold $\mathcal{L}$ in a symplectic manifold $(\mathcal{P}, \omega)$.

Theorem 2.2 *Let $(\mathcal{P}, \omega)$ be a symplectic manifold, and let $\mathcal{L}$ be a lagrangian submanifold. Then there exists a tubular neighbourhod U of $\mathcal{L}$ in $\mathcal{P}$, and a diffeomorphism $\phi : U \longrightarrow V = \phi(U) \subset T^*\mathcal{L}$ into an open neighbourhood V of the zero cross-section in $T^*\mathcal{L}$ such that $\phi^*\omega_{\mathcal{L}} = \omega_{|U}$, where $\omega_{\mathcal{L}}$ is the canonical symplectic form on $T^*\mathcal{L}$.*

2.2 Lagrangian and hamiltonian dynamics

We shall recall the main results, more details can be found in [19].

Let $L : TQ \longrightarrow \mathbb{R}$ be a lagrangian function. We construct a 2-form ω_L by putting

$$\omega_L = -d\theta_L$$

where $\theta_L = S^*(dL)$. Here S^* is the adjoint operator of the canonical vertical endomorphism $S = dq \otimes \dfrac{\partial}{\partial \dot{q}}$. We have omitted the indices of the coordinates, and used the notation $(q, \dot{q})$ for the bundle coordinates on the tangent bundle $\tau_Q : TQ \longrightarrow Q$.

The energy function is defined by

$$E_L = \Delta(L) - L$$

where $\Delta = \dot{q}\dfrac{\partial}{\partial \dot{q}}$ is the Liouville or dilation vector field.

In local coordinates we have

$$\omega_L = dq \wedge d\hat{p}, \qquad E_{\mathcal{L}} = \dot{q}\hat{p} - L,$$

where $\hat{p} = \dfrac{\partial L}{\partial \dot{q}}$. The lagrangian is regular if and only if the hessian matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$$

is non-singular, where $i, j = 1, \ldots, n = \dim Q$.

We have that L is regular if and only if ω_L is symplectic. In such case, there is a unique vector field ξ_L satisfying the equation

$$i_{\xi_L} \omega_L = dE_L. \tag{2.1}$$

ξ_L is a second order differential equation on TQ such that its solutions (the curves in Q whose lifts to TQ are integral curves of ξ_L) are just the solutions of the Euler-Lagrange equations for L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{2.2}$$

Let now $H : T^*Q \longrightarrow \mathbb{R}$ be a hamiltonian function. We denote by X_H the corresponding hamiltonian vector field with respect to ω_Q. In bundle coordinates we have

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

Therefore, the integral curves $(q(t), p(t))$ of X_H satisfy the Hamilton equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

The lagrangian and hamiltonian formalisms are connected through the Legendre transformation. More precisely, given a lagrangian function $L : TQ \longrightarrow \mathbb{R}$ we define a fibred mapping $Leg_L : TQ \longrightarrow T^*Q$ over Q by

$$Leg_L(q, \dot{q}) = (q, \frac{\partial L}{\partial \dot{q}}).$$

We know that L is regular if and only if Leg_L is a local diffeomorphism. For simplicity, we will assume that $\mathcal{L}$ is hyperregular, which means that Leg_L is a diffeomorphism. In such case, Leg_L is in fact a symplectomorphism and, therefore, ξ_L and X_H are Leg_L-related, when $H = E_L \circ {Leg_L}^{-1}$. As a consequence, the Euler-Lagrange equations are translated into the Hamilton equations via Leg_L.

2.3 Dynamics as lagrangian submanifolds

In [23,24] W.M. Tulczyjew defined two canonical diffeomorphisms

$$\alpha : TT^*Q \longrightarrow T^*TQ$$
$$\beta : TT^*Q \longrightarrow T^*T^*Q$$

locally given by

$$\alpha(q, p, \dot{q}, \dot{p}) = (q, \dot{q}, \dot{p}, p)$$
$$\beta(q, p, \dot{q}, \dot{p}) = (q, p, -\dot{p}, \dot{q})$$

with the obvious notations, where we have omitted the indices for the sake of simplicity.

The second diffeomorphism is nothing but the contraction with the canonical symplectic form ω_Q on T^*Q. The intrinsic definition of α is more involved, and we remit to [23] for details. We have the following commutative diagram which justifies the name of Tulczyjew' s triple for the above construction:

$$\begin{tikzcd}
T^*TQ \arrow[dr, "\pi_{TQ}"'] & & TT^*Q \arrow[ll, "\alpha"'] \arrow[rr, "\beta"] \arrow[dl, "T\pi_Q"'] \arrow[dr, "\tau_{T^*Q}"] & & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"] \\
& TQ & & T^*Q &
\end{tikzcd}$$

The manifold TT^*Q is endowed with two symplectic structures, in principle different. Indeed, they are $\omega_\alpha = \alpha^*\omega_{TQ}$ and $\omega_\beta = \beta^*\omega_{T^*Q}$. A direct computation shows that both coincide up to the sign (say $\omega_\alpha + \omega_\beta = 0$), and, in addition, that the symplectic form ω_α is nothing but the complete or tangent lift ω_Q^T of ω_Q to TT^*Q.

We denote by $\theta_\alpha = \alpha^*\theta_{TQ}$ and $\theta_\beta = \beta^*\theta_{T^*Q}$, such that $\omega_\alpha = -d\theta_\alpha$ and $\omega_\beta = -d\theta_\beta$. In local coordinates we have

$$
\begin{aligned}
\theta_\alpha &= \dot{p}dq + pd\dot{q} \\
\theta_\beta &= -\dot{p}dq + \dot{q}dp
\end{aligned}
$$

In fact, TT^*Q, equipped with the symplectic form $\omega_\alpha = -\omega_\beta = \omega_Q^T$ is an example of special symplectic manifold according to the definition introduced by Tulczyjew in [23].

Definition 2.3 *A* special symplectic manifold *is a symplectic manifold* $(\mathcal{P}, \omega)$ *which is symplectomorphic to a cotangent bundle. More precisely, there exists a fibration* $\pi : \mathcal{P} \longrightarrow M$*, and a 1-form* θ *on* $\mathcal{P}$*, such that* $\omega = -d\theta$*, and* $\alpha : \mathcal{P} \longrightarrow T^*M$ *is a diffeomorphism such that* $\pi_M \circ \alpha = \pi$ *and* $\alpha^*\theta_M = \theta$*.*

The following is an important result for our discussion.

Theorem 2.4 *Let* $(\mathcal{P}, \omega = -d\theta)$ *an special symplectic manifold, let* $f : M \longrightarrow \mathbb{R}$ *be a function, and denote by* N_f *the submanifold of* $\mathcal{P}$ *where* df *and* θ *coincide. Then* N_f *is a lagrangian submanifold of* $(\mathcal{P}, \omega)$ *and* f *is a generating function.*

Theorem 2.4 applies to the particular case of Mechanics. Indeed, if we consider a lagrangian function $L : TQ \longrightarrow \mathbb{R}$ we obtain a lagrangian submanifold N_L of the symplectic manifold (TT^*Q, ω_α) with generating function L.

Now, assume that $H : T^*Q \longrightarrow \mathbb{R}$ is a hamiltonian function, with hamiltonian vector field X_H.

We have the following results.

Theorem 2.5

(i) The image of X_H *is a lagrangian submanifold of* (TT^*Q, ω_α)*.*
(ii) The image of dH *is a lagrangian submanifold of* (T^*T^*Q, ω_{T^*Q})*.*
(iii) $\beta(Im\, X_H) = Im\, dH$*.*

Finally, we relate both lagrangian submanifolds N_L and Im X_H.

Theorem 2.6 *Let* H *be the hamiltonian function corresponding to the hyperregular lagrangian function* L*, say* $H = E_L \circ Leg_L^{-1}$*. Then we have* $N_L = Im\, X_H$*.*

3 Multisymplectic manifolds and their lagrangian submanifolds

3.1 Multisymplectic vector spaces

Definition 3.1 *Let Ω be a $(k+1)$-form on a vector pace $\mathcal{V}$. The pair $(\mathcal{V}, \Omega)$ is called a* multisymplectic vector space *if the form Ω is non-degenerate, that is, the linear mapping*

$$v \in \mathcal{V} \mapsto i_v \Omega \in \Lambda^k \mathcal{V}^*$$

is injective. The form Ω is called multisymplectic*.*

Let $(\mathcal{V}_1, \Omega_1)$ and $(\mathcal{V}_2, \Omega_2)$ be two multisymplectic vector spaces (of the same order $(k+1)$) and let $\phi : (\mathcal{V}_1, \Omega_1) \longrightarrow (\mathcal{V}_2, \Omega_2)$ be a linear isomorphism.

Definition 3.2 *ϕ is called a* multisymplectomorphism *if it preserves the multisymplectic forms, i.e. $\phi^* \Omega_2 = \Omega_1$.*

Example 3.3 Let V be an arbitrary vector space and consider the direct product $\mathcal{V}_V = V \times \Lambda^k V^*$. Define a k-form Ω_V on $\mathcal{V}_V$ as follows:

$$\Omega_V((v_1, \gamma_1), \ldots, (v_{k+1}, \gamma_{k+1})) = \sum_{i=1}^{k} (-1)^i \gamma_i(v_1, \ldots, \check{v}_i, \ldots, v_{k+1}),$$

for all $(v_i, \gamma_i) \in \mathcal{V}_V$, $i = 1, \ldots, k+1$, where a check accent over a letter means that it is omitted. A direct computation shows that Ω_V is indeed multisymplectic.

If E is a vector subspace of V, we consider the subspace $\mathcal{V}_V^r = V \times \Lambda_r^k V^*$, where $\Lambda_r^k V^*$ denotes the space of k-forms on V vanishing when applied to at least r of their arguments from E. Of course, $\mathcal{V}_V^r$ equipped with the restriction Ω_V^r of Ω_V to $\mathcal{V}_V^r$ is a multisymplectic vector space. If $E = \{0\}$ we recover $\mathcal{V}_V$.

Let $(\mathcal{V}, \Omega)$ be a multisymplectic vector space of order $k+1$, and $\mathcal{W} \subset \mathcal{V}$ a vector subspace. We define

$$\mathcal{W}^{\perp, l} = \{v \in \mathcal{V} \mid i_{v \wedge w_1 \wedge \cdots \wedge w_l} \Omega = 0, \text{for all } w_1, \ldots, w_l \in \mathcal{W}\}.$$

Definition 3.4 *$\mathcal{W}$ is said to be*

(i) l-isotropic *if $\mathcal{W} \subset \mathcal{W}^{\perp, l}$;*
(ii) l-coisotropic *if $\mathcal{W}^{\perp, l} \subset \mathcal{W}$;*
(iii) l-lagrangian *if $\mathcal{W} = \mathcal{W}^{\perp, l}$;*
(iv) multisymplectic *if $\mathcal{W} \cap \mathcal{W}^{\perp, k} = \{0\}$;*

Proposition 3.5 *A subspace $\mathcal{W}$ is l-lagrangian if and if it is l-isotropic and maximal.*

Proposition 3.6 *Let V an arbitrary vector space. Then:*

(i) V is a k-lagrangian subspace of $\mathcal{V}_V$ and $\mathcal{V}_V^r$, for all r;
(ii) $\Lambda^k V^*$ (resp. $\Lambda_r^k V^*$) is a 1-isotropic subspace of $\mathcal{V}_V$ (resp. $\mathcal{V}_V^r$).

Proof (i) A direct computation shows that

$$V^{\perp,k} = \{(x,\gamma) \mid \Omega_V((x,\gamma),(x_1,0),\ldots,(x_k,0)) = 0, \text{ for all } x_1,\ldots,x_k\}$$

which is equivalent to the condition $\gamma(x_1,\ldots,x_k) = 0$ for all $x_1,\ldots,x_k \in V$, and therefore $\gamma = 0$. Hence $V^{\perp,k} = V$.

The same proof holds for $\mathcal{V}_V^r$.

(ii) We have to prove that

$$\Lambda^k V^* \subset (\Lambda^k V^*)^{\perp,1}$$

which is obvious because

$$i_{(0,\gamma_1)\wedge(0,\gamma_2)}\,\Omega_V = 0.$$

The same argument works for $\mathcal{V}_V^r$. □

Remark 3.7 In addition, notice that

$$(\Lambda^k V^*)^{\perp,1} = \Lambda^k V^*$$

which implies that $\Lambda^k V^*$ is in fact 1-lagrangian.

Theorem 3.8 *[20,21] Let $(\mathcal{V},\Omega)$ be a multisymplectic vector space and $\mathcal{W} \subset \mathcal{V}$ a 1-isotropic subspace such that* $\dim \mathcal{W} = \dim \Lambda^k(\mathcal{V}/\mathcal{W})^*$ *and* $\dim \mathcal{V}/\mathcal{W} > k$*. Then there exists a k-lagrangian subspace V of $\mathcal{V}$ which is transversal to $\mathcal{W}$ (i.e. $V \cap \mathcal{W} = \{0\}$) such that $(\mathcal{V},\Omega)$ is multisymplectomorphic to the model $(\mathcal{V}_V,\Omega_V)$.*

Proof First step: Define the mapping

$$\begin{array}{rcl} \iota : \mathcal{W} & \longrightarrow & \Lambda^k(\mathcal{V}/\mathcal{W})^* \\ v & \mapsto & \iota(v) = \widetilde{i_v\Omega} \end{array}$$

where $\widetilde{i_v\Omega}$ is the induced linear form from $i_v\Omega \in \Lambda^k\mathcal{V}^*$. Notice that $\widetilde{i_v\Omega}$ is well-defined because the isotropic character of $\mathcal{W}$. In addition, ι is a linear isomorphism because of the regularity of Ω.

Second step: Such a subspace $\mathcal{W}$ is unique. First of all, we shall prove that if $u, v \in \mathcal{V}$ are linearly independent vectors satisfying $i_{u\wedge v}\,\Omega = 0$, it follows that $\operatorname{span}(u,v) \cap \mathcal{W} \neq \{0\}$. Otherwise, we could choose $v_1,\ldots,v_{k-2} \in \mathcal{V}$ with $v_i \notin \mathcal{W}$ such that $\{u,v,v_1,\ldots,v_{k-2}\}$ are linearly independent and $\operatorname{span}(u,v,v_1,\ldots,v_{k-2}) \cap \mathcal{W} = \{0\}$, because the codimension of $\mathcal{W}$ is at least k. But for any $w \in \mathcal{W}$ we would have $i_{w\wedge u\wedge v\wedge v_1\wedge\cdots\wedge v_{k-2}}\,\Omega = 0$ which contradicts the fact that $\iota : \mathcal{W} \longrightarrow \Lambda^k(\mathcal{V}/\mathcal{W})^*$ is an isomorphism.

Next, let $\mathcal{W}$ and $\mathcal{W}'$ be two subspaces of $\mathcal{V}$ satisfying the hypothesis of the theorem. Assume that $\mathcal{W} \neq \mathcal{W}'$; then, there exists $v \in \mathcal{W}'$ such that $v \notin \mathcal{W}$. Using the argument above, we deduce that $\mathcal{W} \cap \mathcal{W}'$ has dimension at least 1. Consider the subspace $Z = \pi(v) \wedge \Lambda_{k-1}(\mathcal{V}/\mathcal{W})$ of $\Lambda_k(\mathcal{V}/\mathcal{W})$, where $\Lambda_r\mathcal{V}$ is the space of r-vectors on $\mathcal{V}$, and $\pi : \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{W}$ is the canonical projection. Of course, $\dim Z > 1$, and we have $\iota(w)(z) = 0$ for any $w \in \mathcal{W} \cap \mathcal{W}'$ and for any $z \in Z$. Hence we would have $w \in \ker \iota$.

<u>Third step:</u> There exists a k-lagrangian subspace V such that $\mathcal{V} = \mathcal{W} \oplus V$. Obviously, there are k-isotropic subspaces U such that $U \cap \mathcal{W} = \{0\}$. To show this last assertion, one could take a vector $v \in \mathcal{V}$ such that $u \notin \mathcal{W}$. It is obvious that span (u) is k-isotropic.

Assume that $U \oplus \mathcal{W} = \mathcal{V}$. Then $\mathcal{W} \cap U^{\perp,k} \subset \ker \iota$ and hence $\mathcal{W} \cap U^{\perp,k} = \{0\}$. Therefore $U = U^{\perp,k}$, and U is k-lagrangian.

Suppose now that $U \oplus \mathcal{W} \neq \mathcal{V}$, then $U \neq U^{\perp,k}$; indeed, if $U = U^{\perp,k}$ (that is, if U were k-lagrangian) then there would be a vector $x \in \mathcal{V}$ such that $x \notin U \oplus \mathcal{W}$, and then $U \oplus \text{span}\,(x)$ would be k-isotropic in contradiction with the maximality of U. Therefore, there is a vector $v \in U^{\perp,k}$ such that $v \notin U \cup \mathcal{W}$, and we would have a k-isotropic subspace $U' = U \oplus \text{span}\,(u)$ such that $U' \cap \mathcal{W} = \{0\}$. If $U' \oplus \mathcal{W} \neq \mathcal{V}$, we can repeat the argument and will eventually arrive at a k-isotropic subspace V which is complementary to $\mathcal{W}$. And using the argument above, we conclude that V is in fact k-lagrangian.

<u>Fourth step:</u> Define a linear mapping

$$\begin{aligned} \phi : \mathcal{W} &\longrightarrow \Lambda^k V^* \\ \phi(w) &= -\frac{1}{k+1}(i_w\Omega)_{|V} \end{aligned}$$

A direct computation shows that ϕ is an isomorphism. Next, we define

$$\begin{aligned} \psi : \mathcal{V} &\longrightarrow V \times \Lambda^k V^* \\ \psi(v,w) &= (v, \phi(w)) \end{aligned}$$

which is also an isomorphism such that $\psi^*\Omega_V = \Omega$. □

Remark 3.9 A direct application of Theorem 3.8 shows that there exists a basis (a Darboux basis) $\{e_1, \ldots, e_n, f_{\alpha_1\ldots\alpha_k}\}$ such that $\{e_i\}$ is a basis of V and $\{f_{\alpha_1\ldots\alpha_k}\}$ is a basis of $\mathcal{W}$ satisfying the relations

$$i_{f_{\alpha_1\ldots\alpha_k}}\Omega = e^*_{\alpha_1} \wedge \cdots \wedge e^*_{\alpha_k}$$

where $\{e^*_1, \ldots, e^*_n\}$ denotes the dual basis of $\{e_1, \ldots e_n\}$. Therefore we have

$$\Omega = \sum_\alpha f^*_{\alpha_1\ldots\alpha_k} \wedge e^*_{\alpha_1} \wedge \cdots \wedge e^*_{\alpha_k} \tag{3.3}$$

where $\{f^*_{\alpha_1\ldots\alpha_k}\}$ is the dual basis of $\{f_{\alpha_1\ldots\alpha_k}\}$.

Definition 3.10 *A triple $(\mathcal{V}, \Omega, \mathcal{W})$ satisfying the hypothesis in Theorem 3.8 will be called a* multisymplectic vector space of type $(k+1, 0)$.

Theorem 3.11 *Let $(\mathcal{V}, \Omega)$ be a multisymplectic vector space and $\mathcal{W} \subset \mathcal{V}$ a 1-isotropic subspace. Assume that $\mathcal{E} \subset \mathcal{V}/\mathcal{W}$ is a vector subspace of the quotient vector space $\mathcal{V}/\mathcal{W}$. Let us denote by $\pi : \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{W}$ the canonical projection. Assume that*

(i) $i_{v_1\wedge\cdots\wedge v_r}\,\Omega = 0$ *if* $\pi(v_i) \in \mathcal{E}$*, for all* $i = 1, \ldots, r$*;*
(ii) $\dim \mathcal{W} = \dim \Lambda^k_r(\mathcal{V}/\mathcal{W})^*$*, where the horizontal forms are considered with respect to the subspace* $\mathcal{E}$*;*
(iii) $\dim(\mathcal{V}/\mathcal{W}) > k$.

Then there exists a k-lagrangian subspace V of $\mathcal{V}$ which is transversal to $\mathcal{W}$ (i.e., $V \cap \mathcal{W} = \{0\}$) such that $(\mathcal{V}, \Omega)$ is multisymplectomorphic to the model $(\mathcal{V}^r_V, \Omega^r_V)$.

Proof First, we define the linear isomorphism

$$\begin{aligned} \iota : \mathcal{W} &\longrightarrow \Lambda^k_r(\mathcal{V}/\mathcal{W})^* \\ w &\mapsto \iota(w) = \widetilde{i_w\Omega} \end{aligned}$$

where $\widetilde{i_w\Omega}$ is the induced k-form using that $\mathcal{W}$ is isotropic and that Ω satisfies the first condition above.

Next, one follows the arguments given in the proof of Theorem 3.8. □

Remark 3.12 A direct application of Theorem 3.11 shows that the multisymplectic form Ω can be written as the canonical multisymplectic form Ω^r_V on $\mathcal{V}^r_V$ by choosing a convenient Darboux basis.

Definition 3.13 *A triple $(\mathcal{V}, \Omega, \mathcal{W}, \mathcal{E})$ satisfying the hypothesis in Theorem 3.11 will be called a* multisymplectic vector space of type $(k+1, r)$.

Let $(\mathcal{V}_1, \Omega_1)$ and $(\mathcal{V}_2, \Omega_2)$ be two multisymplectic vector spaces of order $k+1$. Take the direct product $\mathcal{V}_1 \times \mathcal{V}_2$ endowed with the $(k+1)$-form $\Omega_1 \ominus \Omega_2 = \pi_1^*\Omega_1 - \pi_2^*\Omega_2$, where $\pi_1 : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathcal{V}_1$ and $\pi_2 : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathcal{V}_2$ are the canonical projections. Then $(\mathcal{V}_1 \times \mathcal{V}_2, \Omega_1 \ominus \Omega_2)$ is a multisymplectic vector space.

Proposition 3.14 *Let $(\mathcal{V}_1, \Omega_1)$ and $(\mathcal{V}_2, \Omega_2)$ be two multisymplectic vector spaces of order $(k+1)$ and $\phi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ a linear isomorphism. Then ϕ is a multisymplectomorphism if and only if its graph is a k-lagrangian subspace of the multisymplectic vector space $(\mathcal{V}_1 \times \mathcal{V}_2, \Omega_1 \ominus \Omega_2)$.*

Proof We recall that

$$\begin{aligned} (\operatorname{graph}\phi)^{\perp,k} = \{&(x, y) \in \mathcal{V}_1 \times \mathcal{V}_2 \mid \\ &(\Omega_1 \ominus \Omega_2)((x, y), (x_1, \phi(x_1)), \ldots, (x_k, \phi(x_k)) = 0, \forall x_1, \ldots, x_k \in \mathcal{V}_1\} \end{aligned}$$

Assume that $\phi^*\Omega_2 = \Omega_1$, then if $(x, \phi(x)) \in \operatorname{graph}\phi$, we have

$$\begin{aligned}
&(\Omega_1 \ominus \Omega_2)((x, \phi(x)), (x_1, \phi(x_1)), \ldots, (x_k, \phi(x_k))) \\
&= \Omega_1(x, x_1, \ldots, x_k) - \Omega_2(\phi(x), \phi(x_1), \ldots, \phi(x_k)) \\
&= \Omega_1(x, x_1, \ldots, x_k) - \phi^*\Omega_2(x, x_1, \ldots, x_k) \\
&= 0
\end{aligned}$$

which implies that $\operatorname{graph}\phi \subset (\operatorname{graph}\phi)^{\perp,k}$.

Conversely, if $\operatorname{graph}\phi$ is k-isotropic, we have $(x, \phi(x)) \in (\operatorname{graph}\phi)^{\perp,k}$ for all $x \in \mathcal{V}_1$, and therefore $\phi^*\Omega_2 = \Omega_1$.

In addition, if $\operatorname{graph}\phi$ is k-isotropic, it is also k-lagrangian. In fact, if $(x, y) \in (\operatorname{graph}\phi)^{\perp,k}$ then we have

$$\Omega_2(\phi(x) - y, \phi(x_1), \ldots, \phi(x_k)) = 0$$

for all $x_1, \ldots, x_k \in \mathcal{V}_1$ and therefore $y = \phi(x)$ because of the regularity of the multisymplectic form Ω_2 and the fact that ϕ is an isomorphism. □

3.2 Multisymplectic manifolds

Definition 3.15 *A* multisymplectic manifold $(\mathcal{P}, \Omega)$ *is a pair consisting of a manifold* $\mathcal{P}$ *equipped with a closed* $(k+1)$*-form* Ω *such that the pair* $(T_x\mathcal{P}, \Omega_x)$ *is a multisymplectic vector space for all* $x \in \mathcal{P}$*. The form* Ω *is called* multisymplectic.

Example 3.16 Let $\Lambda^k M$ be the space of k-forms on an arbitrary manifold M, and denote by $\rho : \Lambda^k M \longrightarrow M$ the canonical projection. We define a canonical k-form Θ^k_M on $\Lambda^k M$ as follows:

$$\Theta^k_M(\gamma)(X_1, \ldots, X_k) = \gamma(T\rho X_1, \ldots, T\rho X_k),$$

for all $X_1, \ldots, X_k \in T_\gamma(\Lambda^k M)$ and for all $\gamma \in \Lambda^k M$.

A direct computation shows that $(\Lambda^k M, \Omega^k_M = -d\Theta^k_M)$ is a multisymplectic manifold (of order $k+1$).

Assume now that M is a fibred manifold over a manifold N, say $\pi : M \longrightarrow N$ is a fibration. Consider the bundle $\Lambda^k_r M$ of k-forms on M which are r-horizontal with respect to the fibration $\pi : M \longrightarrow N$, that is, those k-forms γ on M such that $i_{X_1\wedge\cdots\wedge X_r}\gamma = 0$ when $X_1, \ldots, X_r$ are π-vertical. The space $\Lambda^k_r M$ is a submanifold of $\Lambda^k M$, and hence we have the restriction $(\Theta_M)^k_r$ of Θ^k_M to $\Lambda^k_r M$. A simple computation shows that the pair $(\Lambda^k_r M, (\Omega_M)^k_r = -d(\Theta_M)^k_r)$ is also a multisymplectic manifold. Of course, we have $(\Omega^k_M)_{|\Lambda^k_r M} = (\Omega_M)^k_r$. The canonical projection will be denoted by $\rho_r : \Lambda^k_r M \longrightarrow M$.

Following the notion of special symplectic manifold introduced by Tulczyjew we can give the following definition.

Definition 3.17 *A* special multisymplectic manifold $(\mathcal{P}, \Omega)$ *is a multisymplectic manifold which is multisymplectomorphic to a bundle of forms. More precisely,* $\Omega = -d\Theta$*, and there exists a diffeomorphism* $\alpha : \mathcal{P} \longrightarrow \Lambda^k M$ *(or* $\alpha : \mathcal{P} \longrightarrow \Lambda^k_r M$*), and a fibration* $\pi : \mathcal{P} \longrightarrow M$ *such that* $\rho \circ \alpha = \pi$ *(resp.* $\rho_r \circ \alpha = \pi$*) and* $\Theta = \alpha^* \Theta^k_M$ *(resp.* $\Theta = \alpha^* (\Theta_M)^k_r$*).*

Definition 3.18 *Let* $\mathcal{N}$ *be a submanifold of a multisymplectic manifold* $(\mathcal{P}, \Omega)$ *of order* $k+1$*.* $\mathcal{N}$ *is said to be* l-isotropic *(resp.* l-coisotropic, l-lagrangian, multisymplectic*) if* $T_x\mathcal{N}$ *is a* l*-isotropic (resp.* l*-coisotropic,* l*-lagrangian, multisymplectic) vector subspace of the multisymplectic vector space* $(T_x\mathcal{P}, \Omega_x)$ *for all* $x \in \mathcal{N}$*.*

Proposition 3.19

(i) The fibres of $\rho : \Lambda^k M \longrightarrow M$ *(and of* $\rho_r : \Lambda^k_r M \longrightarrow M$*) are 1-isotropic.*
(ii) The image of a k*-form* γ *on* M *(resp. a* r*-horizontal* k*-form) is* k*-lagrangian if and only if* γ *is closed.*

Proof It follows from Proposition 3.6. □

If γ is a (r-horizontal) closed k-form on M, then $(-d(\Theta_M)^k_r)_{|\mathrm{Im}\gamma} = 0$ which implies that $((\Theta_M)^k_r)_{|\mathrm{Im}\gamma}$ is locally closed, say

$$((\Theta_M)^k_r)_{|\mathrm{Im}\gamma} = d\theta,$$

and θ is called a *generating* k*-form*.

Definition 3.20 *A triple* $(\mathcal{P}, \Omega, \mathcal{W})$*, where* $\mathcal{W}$ *is a 1-isotropic involutive distribution on* $(\mathcal{P}, \Omega)$ *such that the triple* $(T_x\mathcal{P}, \Omega_x, \mathcal{W}(x))$ *is a multisymplectic vector space of type* $(k+1, 0)$*, for all* $x \in \mathcal{P}$*, will be called a* multisymplectic manifold of type $(k+1, 0)$.

Remark 3.21 Along the paper, the distribution $\mathcal{W}$ and the corresponding vector bundle $\pi_0 : \mathcal{W} \longrightarrow \mathcal{P}$ over $\mathcal{P}$ will be denoted by the same letter.

Theorem 3.22 *[21] Let* $(\mathcal{P}, \Omega, \mathcal{W})$ *be a multisymplectic manifold of type* $(k+1, 0)$*. Let* $\mathcal{L}$ *be a* k*-lagrangian submanifold such that* $T\mathcal{L} \cap \mathcal{W}_{|\mathcal{L}} = \{0\}$*. Then there exists a tubular neighbourhood* U *of* $\mathcal{L}$ *in* $\mathcal{P}$*, a manifold* $\mathcal{N}$ *and a diffeomorphism* $\Phi : U \longrightarrow V = \Phi(U) \subset \Lambda^k\mathcal{N}$ *into an open neighbourhood* V *of the zero cross-section in* $\Lambda^k\mathcal{N}$ *such that* $\Phi : \mathcal{L} \longrightarrow \mathcal{N}$ *is an immersion and* $\Phi^*((\Omega^k_{\mathcal{N}})_{|V}) = \Omega_{|U}$*, where* $\Omega^k_{\mathcal{N}}$ *is the canonical multisymplectic* $(k+1)$*-form on* $\Lambda^k\mathcal{N}$*.*

Proof The proof is a direct consequence of Lemmas 3.24 and 3.25. □

First of all, we recall the relative Poincaré lemma, which will be very useful in what follows.

Lemma 3.23 (**Relative Poincaré lemma**) *Let N be a submanifold of a differentiable submanifold M, and let U be a tubular neigbourhood of N with bundle map $\pi_0 : U \longrightarrow N$. Notice that $\pi_0 : U \longrightarrow N$ is a vector bundle. Denote by Δ the dilation vector field of this vector bundle, and let φ_t be the multiplication by t. If we define an integral operator on forms on U as follows*

$$I(\Omega)_p = \int_0^1 i_{\Delta_t}\, \varphi_t^* \Omega_p dt$$

where $\Delta_t = \frac{1}{t}\Delta$, and $p \in U$, then we have

$$I(d\Omega) + d(I\Omega) = \Omega - \pi_0^*(\Omega_{|N})$$

where $\Omega_{|N}$ is the form on N obtained by restricting Ω pointwise to TN (observe that U can be taken as a normal bundle of TN in M).

Next, we shall prove the following result.

Lemma 3.24 *Let $(\mathcal{P}, \Omega, \mathcal{W})$ be a multisymplectic manifold of type $(k+1, 0)$. Let $\mathcal{L}$ be a k-lagrangian submanifold of $\mathcal{P}$ which is complementary to $\mathcal{W}$ (that is, $T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}} = T\mathcal{P}_{|\mathcal{L}}$). Then there is a tubular neighbourhood U of $\mathcal{L}$ and a diffeomorphism $\Phi : U \longrightarrow V \subset \Lambda^k\mathcal{L}$ where V is an neighbourhood of the zero section, such that $\Phi_{|\mathcal{L}}$ is the standard identification of $\mathcal{L}$ with the zero section of $\Lambda^k\mathcal{L}$, and*

$$\Phi^*((\Omega^k_{\mathcal{L}})_{|V}) = \Omega_{|U}.$$

Proof Firstly, we define a vector bundle morphism over the identity of $\mathcal{L}$ by

$$\phi(w) = -\frac{1}{k+1}\, i_w\, \Omega.$$

$$\begin{tikzcd}
\mathcal{W} \arrow[rr, "\phi"] \arrow[dr, "\pi_0"'] & & \Lambda^k\mathcal{L} \arrow[dl, "\rho"] \\
& \mathcal{L} &
\end{tikzcd}$$

Obviously ϕ is injective, and since the dimensionality assumptions, we deduce that ϕ is in fact a vector bundle isomorphism (see the diagram).

Since $T\mathcal{P}_{|\mathcal{L}} = T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}}$, then ϕ induces a diffeomorphism on a tubular neighbourhood defined by $\mathcal{W}$ onto a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$ (as usual, the latter embedding is understood as the identification of $\mathcal{L}$ with the zero section). We shall denote the restriction of ϕ to this tubular neigbourhood by f. Notice that the restriction of f to $\mathcal{L}$ is just the identity, so that Tf is also the identity on $T\mathcal{L}$; on the other hand, Tf restricted to $\mathcal{W}$ coincides with ϕ because it is fiberwise linear. Using the identifications $T\mathcal{P}_{|\mathcal{L}} = T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}}$ and

$T\Lambda^k\mathcal{L}_{|\mathcal{L}} = T\mathcal{L} \oplus \Lambda^k\mathcal{L}$, we have

$$\begin{aligned} f^*\Omega^k_{\mathcal{L}}((v_1,w_1),\ldots,(v_{k+1},w_{k+1})) &= \Omega^k_{\mathcal{L}}((v_1,\phi(w_1),\ldots,(v_{k+1},\phi(w_{k+1})) \\ &= \sum_{i=1}^{k+1}(-1)^i\,\phi(w_i)(v_1,\ldots,\check{v}_i,\ldots,v_{k+1}) \\ &= \sum_{i=1}^{k+1}\frac{1}{k+1}\,\Omega(v_1,\ldots,w_i,\ldots,v_{k+1}) \\ &= \Omega((v_1,w_1),\ldots,(v_{k+1},w_{k+1})) \end{aligned}$$

which implies $f^*\Omega^k_{\mathcal{L}} = \Omega$ on $\mathcal{L}$.

Next, we use f to pushforward Ω to obtain a $k+1$-form Ω_1 in a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$. Using Lemma 3.23 we deduce that $\Omega_1 = d\Theta_1$, where $\Theta_1 = I(\Omega_1)$. Recall that $\Omega^k_{\mathcal{L}} = -d\Theta^k_{\mathcal{L}}$, and

$$(\Theta^k_{\mathcal{L}})_{|\mathcal{L}} = (\Theta_1)_{|\mathcal{L}} = 0 \tag{3.4}$$

because of the definition of I. Define

$$\Omega_t = \Omega^k_{\mathcal{L}} + t(\Omega_1 - \Omega^k_{\mathcal{L}}), \qquad t \in [0,1].$$

Since

$$(\Omega_t)_{|\mathcal{L}} = (\Omega^k_{\mathcal{L}})_{|\mathcal{L}} = (\Omega_1)_{|\mathcal{L}}$$

is non-singular, and this is an "open condition", we can find a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$ on which all Ω_t are non-singular for all $t \in [0,1]$. In addition, $\mathcal{W}_{\mathcal{L}} = \ker\{T\rho : T\Lambda^k\mathcal{L} \longrightarrow T\mathcal{L}\}$ is 1-isotropic for all Ω_t, in such a way that $(\Lambda^k\mathcal{L}, \Omega_t, \mathcal{W}_{\mathcal{L}})$ is a multisymplectic manifold of type $(k+1,0)$, for all t. Notice that $\Omega_1 - \Omega^k_{\mathcal{L}} = d(\Theta_1 + \Theta^k_{\mathcal{L}})$.

From (3.4) we deduce that there is a unique time-dependent vector field X_t taking values in $\mathcal{W}_{\mathcal{L}}$ (in other words, ρ-vertical) such that

$$i_{X_t}\,\Omega_t = -\Theta^k_{\mathcal{L}} + \Theta_1.$$

Since the vector field X_t vanishes on $\mathcal{L}$, we can find a neighbourhood of $\mathcal{L}$ in $\Lambda^k\mathcal{L}$ such that the flow φ_t of X_t is defined at least for all $t \leq 1$. Therefore we have

$$\begin{aligned} \frac{d}{dt}(\varphi_t^*\Omega_t) &= \varphi_t^*(L_{X_t}\,\Omega_t) + \varphi_t^*(\frac{d\Omega_t}{dt}) \\ &= \varphi_t^*(di_{X_t}\Omega_t) + \varphi_t^*(\Omega_1 - \Omega^k_{\mathcal{L}}) \\ &= \varphi_t^*(-d(\Theta_1 - \Theta^k_{\mathcal{L}}) + \Omega_1 - \Omega^k_{\mathcal{L}}) = 0. \end{aligned}$$

Then we have

$$\varphi_1^*\Omega_1 = \varphi_0^*\Omega^k_{\mathcal{L}} = \Omega^k_{\mathcal{L}}.$$

But $(X_t)_{|\mathcal{L}} = 0$ implies $(\varphi_t)_{|\mathcal{L}} = id_{|\mathcal{L}}$, and then we deduce that $\varphi_1 \circ f$ gives the desired local diffeomorphism. □

Lemma 3.25 *Let $(\mathcal{P}, \Omega, \mathcal{W})$ be a multisymplectic manifold of type $(k+1, 0)$. Let $\mathcal{L}'$ be a k-isotropic submanifold of $\mathcal{P}$ which is transversal to $\mathcal{W}$ (that is, $T\mathcal{L}' \cap \mathcal{W}_{|\mathcal{L}'} = \{0\}$). Then there is a k-lagrangian submanifold $\mathcal{L}$ of $\mathcal{P}$ which is complementary to $\mathcal{W}$ and contains $\mathcal{L}'$.*

Proof Since $\mathcal{L}'$ is transversal to $\mathcal{W}$ we can choose a submanifold $\mathcal{L}''$ of U' such that $\mathcal{L}'$ is a deformation retract of $\mathcal{L}''$, and $\mathcal{L}''$ is complementary to $\mathcal{W}$. As in the theorem above, since $T\mathcal{P}_{|\mathcal{L}''} = T\mathcal{L}'' \oplus \mathcal{W}_{|\mathcal{L}''}$, then $\mathcal{W}$ induces a tubular neighbourhood of $\mathcal{L}''$ in the usual way: $\pi_1 : U' \longrightarrow \mathcal{L}''$.

Next, we apply the relative Poincaré lemma to the restricted form Ω to this tubular neigborhood. Therefore, there is a k-form μ on U' such that

$$d\mu = \Omega - \pi_1^*(\Omega_{|\mathcal{L}''})$$

(indeed, $\mu = I(\Omega)$).

Now, we can repeat the construction developed in the proof of Lemma 3.24 for the $k+1$-form $d\mu$. In fact, the mapping $\psi : \mathcal{W} \longrightarrow \Lambda^k \mathcal{L}''$ defined by $\psi(u) = -\dfrac{1}{k+1}(i_u\, d\mu)$ is a vector isomorphism, and it induces a local diffeomorphism $g : U'' \subset U' \longrightarrow g(U'') \subset \Lambda^k \mathcal{L}''$; g restricted to $\mathcal{L}''$ is the identity, and ψ on the fibers. Again we can prove

$$g^* \Omega^k_{\mathcal{L}''} = d\mu$$

since $(d\mu)_{|\mathcal{L}''} = 0$. Proceeding as in the proof of Lemma 3.24 we can find a local diffeomorphism Ψ from a tubular neigbourhood V of $\mathcal{L}''$ onto a neighbourhood of the zero section of $\Lambda^k \mathcal{L}''$ which maps $\mathcal{L}''$ onto the zero section, and such that

$$\Psi^* \Omega^k_{\mathcal{L}''} = \Omega$$

on V.

Now, if $j : \mathcal{L}' \longrightarrow \mathcal{L}''$ is the natural inclusion, we know that j induces an isomorphism in cohomology. Therefore $j^*(\Omega_{|\mathcal{L}''}) = \Omega_{|\mathcal{L}'} = 0$ implies $[\Omega_{|\mathcal{L}''}]_{DR} = 0$, and we deduce that $\Omega_{|\mathcal{L}''} = d\nu$, for some k-form ν on $\mathcal{L}''$. A direct computation shows now that

$$\mathcal{L} = \Psi^{-1} \circ (-\nu)(\mathcal{L}'')$$

is a k-lagrangian submanifold in $(\mathcal{P}, \Omega)$, and in addition $T\mathcal{P}_{|\mathcal{L}} = T\mathcal{L} \oplus \mathcal{W}_{|\mathcal{L}}$. □

Corollary 3.26 *A multisymplectic manifold $(\mathcal{P}, \Omega, \mathcal{W})$ of type $(k+1, 0)$ is locally multisymplectomorphic to a canonical multisymplectic manifold $\Lambda^k M$ for some manifold M. Therefore, there are Darboux coordinates around each point of $\mathcal{P}$.*

Proof We only need to choose a point in Lemma 3.25, and then apply Theorem 3.22. □

Definition 3.27 *Let $(\mathcal{P}, \Omega)$ be a multisymplectic manifold of order $k+1$. Assume that $\mathcal{W}$ is a 1-isotropic foliation of $(\mathcal{P}, \Omega)$, and $\mathcal{E}$ is a "generalised distribution" on $\mathcal{P}$ in the sense that $\mathcal{E}(x) \subset T_x\mathcal{P}/\mathcal{W}(x)$ is a vector subspace for all $x \in \mathcal{P}$. Assume that the quadruple $(T_x\mathcal{P}, \Omega_x, \mathcal{W}(x), \mathcal{E}(x))$ is a multisymplectic vector space of type $(k+1, r)$, for all $x \in \mathcal{P}$. A quadruple $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ satisfying these conditions will be called a* multisymplectic manifold of type *$(k+1, r)$.*

Theorem 3.28 *Let $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ be a multisymplectic manifold of type $(k+1, r)$. Let $\mathcal{L}$ be a k-lagrangian submanifold such that $T\mathcal{L} \cap \mathcal{W}_{\mathcal{L}} = \{0\}$. Then there exists a tubular neighbourhood U of $\mathcal{L}$ in $\mathcal{P}$, a manifold $\mathcal{N}$, and a diffeomorphism $\Phi : U \longrightarrow V = \Phi(U) \subset \Lambda^k_r\mathcal{N}$ into an open neighbourhood V of the zero cross-section in $\Lambda^k\mathcal{N}$ such that $\Phi : \mathcal{L} \longrightarrow \mathcal{N}$ is an immersion, and $\Phi^*(((\Omega_{\mathcal{N}})^k_r)_{|V}) = \Omega_{|U}$, where $(\Omega_{\mathcal{N}})^k_r$ is the canonical multisymplectic $(k+1)$-form on $\Lambda^k_r\mathcal{N}$.*

Proof The proof is a consequence of the following two lemmas, which are proved in a similar way to Lemma 3.24 and Lemma 3.25. □

Lemma 3.29 *Let $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ be a multisymplectic manifold of type $(k+1, r)$. Let $\mathcal{L}$ be a k-lagrangian submanifold of $\mathcal{P}$ which is complementary to $\mathcal{W}$. Then there is a tubular neighbourhood U of $\mathcal{L}$ and a diffeomorphism $\Psi : U \longrightarrow V \subset \Lambda^k_r\mathcal{L}$, where V is an neighbourhood of the zero section, such that $\Psi_{|\mathcal{L}}$ is the standard identification of $\mathcal{L}$ with the zero section of $\Lambda^k_r\mathcal{L}$, and*

$$\Psi^*(((\Omega_{\mathcal{L}})^k_r)_{|V}) = \Omega_{|U}.$$

Lemma 3.30 *Let $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ be a multisymplectic manifold of type $(k+1, r)$. Let $\mathcal{L}'$ be a k-isotropic submanifold of $\mathcal{P}$ which is transversal to $\mathcal{W}$. Then there is a k-lagrangian submanifold $\mathcal{L}$ of $\mathcal{P}$ which is complementary to $\mathcal{W}$ and contains $\mathcal{L}'$.*

Corollary 3.31 *A multisymplectic manifold $(\mathcal{P}, \Omega, \mathcal{W}, \mathcal{E})$ of type $(k+1, r)$ is locally multisymplectomorphic to a canonical multisymplectic manifold $\Lambda^k_r M$ for some fibration $M \longrightarrow N$. Therefore, there are Darboux coordinates around each point of $\mathcal{P}$.*

Proof We only need to choose a point in Lemma 3.30, and then apply Theorem 3.28. □

4 Lagrangian and hamiltonian settings for classical field theories

We remit to [1,9,10,13–17,22] for more details.

4.1 Lagrangian formalism

Let $\pi_{XY} : Y \longrightarrow X$ be a fibred manifold, where X is an oriented n-dimensional manifold with volume form η. We choose fibred coordinates (x^μ, y^i) on Y such that

$$\eta = d^n x = dx^1 \wedge \cdots \wedge dx^n, \qquad \pi_{XY}(x^\mu, y^i) = (x^\mu),$$

where $\mu = 1, \ldots, n$, $i = 1, \ldots, m$, and $\dim Y = n + m$. The notation

$$d^{n-1}x^\mu = i_{\frac{\partial}{\partial x^\mu}}\, d^n x$$

will be very useful, since $dx^\mu \wedge d^{n-1}x^\mu = d^n x$.

Let $\mathbb{L} : Z \longrightarrow \Lambda^n X$ be a lagrangian density, that is, $\mathbb{L}$ is an n-form on Z along the canonical projection $\pi_{XZ} : Z \longrightarrow X$. Therefore, $\mathbb{L} = L\eta$, where $L : Z \longrightarrow \mathbb{R}$ is a function on Z, and η equally denotes the volume form on X and its lifts to the different bundles over X.

One constructs an n-form Θ_L on Z locally given by

$$\Theta_L = (L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}) d^n x + \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x^\mu.$$

The $(n+1)$-form $\Omega_L = -d\Theta_L$ is called the Poincaré-Cartan form.

The de Donder equation is

$$i_{\mathbf{h}}\, \Omega_L = (n-1)\Omega_L \tag{4.5}$$

where $\mathbf{h}$ is a connection in the fibred manifold $\pi_{XZ} : Z \longrightarrow X$.

Indeed, if σ is a horizontal section of a solution $\mathbf{h}$ of (4.5) then σ is a critical section of the variational problem determined by L.

If L is regular (that is, the hessian matrix

$$\left(\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right)$$

is regular) then such a section σ is necessarily a 1-jet prolongation, say $\sigma = j^1\tau$, where τ is a section of the fibred manifold $\pi_{XY} : Y \longrightarrow X$.

If $\mathbf{h}$ is a solution of equation (4.5) and

$$\mathbf{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + y^i_\mu \frac{\partial}{\partial y^i} + z^i_{\nu\mu} \frac{\partial}{\partial z^i_\nu}$$

then we have

$$i_{\mathbf{h}}\,\Omega_L = (n-1)\Omega_L \tag{4.6}$$

if and only if

$$(y^j_\nu - z^j_\nu)\,\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} = 0 \tag{4.7}$$

$$\frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\mu \partial z^i_\mu} - y^j_\mu \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - z^j_{\mu\nu}\frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + (y^j_\nu - z^j_\nu)\frac{\partial^2 L}{\partial y^i \partial z^j_\nu} = 0 \tag{4.8}$$

If L is regular, then Eq. (4.7) implies $y^j_\nu = z^j_\nu$ and Eq. (4.8) becomes

$$\frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\mu \partial z^i_\mu} - z^j_\mu \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - z^j_{\mu\nu}\frac{\partial^L}{\partial z^j_\mu \partial z^i_\nu} = 0 \tag{4.9}$$

If $\mathbf{h}$ is flat (that is, the horizontal distribution is integrable) and $\sigma : X \longrightarrow Z$ is an integral section, then $\sigma = j^1(\pi_{YZ}\circ\sigma)$, and (4.9) are nothing but the Euler-Lagrange equations for L:

$$\frac{\partial L}{\partial y^i} - \sum_{\mu=1}^{n} \frac{d}{dx^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right) = 0. \tag{4.10}$$

4.2 Hamiltonian formalism

Denote by $\Lambda^n Y$ the vector bundle over Y of n-forms on Y, and by $\Lambda^n_r Y$ its vector subbundle consisting of those n-forms on Y which vanish contracted with at least r vertical arguments.

We have the short exact sequence of vector bundles over Y

$$0 \longrightarrow \Lambda^n_1 Y \longrightarrow \Lambda^n_2 Y \longrightarrow Z^* = \Lambda^n_2 Y/\Lambda^n_1 Y \longrightarrow 0$$

We choose coordinates as follows:

$$\begin{aligned} \Lambda^n_1 Y &: (x^\mu, y^i, p) \\ \Lambda^n_2 Y &: (x^\mu, y^i, p, p^\mu_i) \\ Z^* &: (x^\mu, y^i, p^\mu_i) \end{aligned}$$

since the generic elements in $\Lambda^n_1 Y$ (resp. $\Lambda^n_2 Y$) have the form $p\,d^n x$ (resp. $p\,d^n x + p^\mu_i\,dy^i \wedge d^{n-1}x^\mu$).

In order to have a dynamical evolution in the hamiltonian setting one need to choose a hamiltonian form h on Z^*, that is, a section $h : Z^* \longrightarrow \Lambda^n_2 Y$ of the canonical fibration $pr : \Lambda^n_2 Y \longrightarrow Z^*$.

The canonical multisymplectic form $(\Omega_Y)^n_2$ on $\Lambda^n_2 Y$ induces a multisymplectic form (of the same type)

$$\Omega_h = h^*(\Omega_Y)^n_2.$$

If $\Theta_h = h^*(\Theta_Y)^n_2$ then $\Omega_h = -d\Theta_h$.

Since

$$(\Omega_Y)^n_2 = -dp \wedge d^n x - dp^\mu_i \wedge dy^i \wedge d^{n-1}x^\mu$$

and

$$h(x^\mu, y^i, p^\mu_i) = (x^\mu, y^i, p = -H(x^\mu, y^i, p^\mu_i), p^\mu_i)$$

(in other words, $h = -H d^n x + p^\mu_i dy^i \wedge d^{n-1}x^\mu$) we obtain

$$\Omega_h = dH \wedge d^n x - dp^\mu_i \wedge dy^i \wedge d^{n-1}x^\mu \tag{4.11}$$

Consider a connection $\mathbf{h}^*$ in the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$, and assume that

$$\mathbf{h}^*(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + y^i_\mu \frac{\partial}{\partial y^i} + p^\nu_{j\mu} \frac{\partial}{\partial p^\nu_j}.$$

Then

$$i_{\mathbf{h}^*} \Omega_h = (n-1)\Omega_h \tag{4.12}$$

if and only if

$$y^i_\mu = \frac{\partial H}{\partial p^\mu_i} \tag{4.13}$$

$$\sum_\mu p^\mu_{i\mu} = -\frac{\partial H}{\partial y^i} \tag{4.14}$$

If $\tau : X \longrightarrow Z^*$ is an integral section of $\mathbf{h}^*$, and $\tau(x^\mu) = (x^\mu, y^i(x), p^\mu_i)$, then it satisfies the Hamilton equations

$$\frac{\partial y^i}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_i} \tag{4.15}$$

$$\sum_\mu \frac{\partial p^\mu_i}{\partial x^\mu} = -\frac{\partial H}{\partial y^i} \tag{4.16}$$

4.3 The Legendre transformation

Let L be a lagrangian. We define the extended Legendre transformation

$$leg_L : Z \longrightarrow \Lambda^n_2 Y$$

by

$$leg_L(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i, L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}, \frac{\partial L}{\partial z^i_\mu}),$$

and the Legendre transformation

$$Leg_L : Z \longrightarrow Z^*$$

by $Leg_L = pr \circ leg_L$. A direct computation shows that L is regular if and only if Leg_L is a local diffeomorphism. L is said to be hyperregular if Leg_L is a global diffeomorphism. In such case, $h = leg_L \circ Leg_L^{-1}$ is a hamiltonian form on Z^*.

Since the next diagram

$$\begin{tikzcd} Z \arrow[rr, "Leg_L"] \arrow[dr, "\pi_{YZ}"'] & & Z^* \arrow[dl, "\pi_{YZ^*}"] \\ & Y & \end{tikzcd}$$

is commutative and $Leg_L^*(\Theta_h) = \Theta_L$, we deduce that Equations (4.6) and (4.12) are equivalent. This means that the solutions of both equations are related by the Legendre transformation.

5 The multisymplectomorphism $\tilde{\alpha}$

Consider the vector bundle $\Lambda_2^{n+1}Z$ with generic elements of the form

$$a_i dy^i \wedge d^n x + b_i^\mu dz_\mu^i \wedge d^n x$$

This allows us to introduce local coordinates $(x^\mu, y^i, z_\mu^i, a_i, b_i^\mu)$ in the manifold $\Lambda_2^{n+1}Z$.

On the other hand, we shall denote by J^1Z^* the manifold of 1-jets of local sections of the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$. We have a canonical projection

$$j^1\pi_{YZ^*} : J^1Z^* \longrightarrow Z$$

Denote by $(x^\mu, y^i, p_i^\mu, y_\nu^i, p_{i\nu}^\mu)$ the induced coordinates on J^1Z^* respect to $\pi_{XZ^*} : Z^* \longrightarrow X$, such that

$$j^1\pi_{YZ^*}(x^\mu, y^i, p_i^\mu, y_\nu^i, p_{i\nu}^\mu) = (x^\mu, y^i, y_\mu^i).$$

Define a mapping

$$\alpha : J^1Z^* \longrightarrow \Lambda_2^{n+1}Z$$

by

$$\alpha(x^\mu, y^i, p_i^\mu, y_\nu^i, p_{i\nu}^\mu) = (x^\mu, y^i, y_\mu^i, \sum_\mu p_{i\mu}^\mu, p_i^\mu).$$

The mapping α is a surjective submersion, or in other words, $\alpha : J^1Z^* \longrightarrow \Lambda_2^{n+1}Z$ is a fibred manifold. In order to obtain a diffeomorphism, we need to "reduce" the manifold J^1Z^*. To do that, we introduce the following equivalence relation:

$$j^1_x\sigma_1 \equiv j^1_x\sigma_2 \text{ if and only if they have the same divergence,}$$

which in local coordinates $(x^\mu, y^i, p^\mu_i, y^i_\nu, p^\mu_{i\nu})$ and $(x^\mu, \bar{y}^i, \bar{p}^\mu_i, \bar{y}^i_\nu, \bar{p}^\mu_{i\nu})$ means

$$\bar{y}^i = y^i, \quad \bar{p}^\mu_i = p^\mu_i, \quad \bar{y}^i_\nu = y^i_\nu, \quad \sum_\mu \bar{p}^\mu_{i\mu} = \sum_\mu p^\mu_{i\mu}.$$

The corresponding quotient manifold will be denoted by $\widetilde{J^1Z^*}$, and we have a fibration $\tilde{pr} : J^1Z^* \longrightarrow \widetilde{J^1Z^*}$. The induced mapping

$$\tilde{\alpha} : \widetilde{J^1Z^*} \longrightarrow \Lambda^{n+1}_2 Z$$

is a diffeomorphism, and we have an induced projection

$$\widetilde{j^1\pi_{YZ^*}} : \widetilde{J^1Z^*} \longrightarrow Z$$

Therefore, we can transport the canonical multisymplectic $(n+2)$-form $(\Omega_Z)^{n+1}_2 = -d(\Theta_Z)^{n+1}_2$ on $\Lambda^{n+1}_2 Z$ to $\widetilde{J^1Z^*}$ such that $(\widetilde{J^1Z^*}, \Omega_\alpha)$ is a multisymplectic manifold, where $\Omega_\alpha = \tilde{\alpha}^*((\Omega_Z)^{n+1}_2)$.

Remark 5.1 Following the terminology introduced by W.M. Tulczyjew in the symplectic context, and accordingly to Definition 3.17, we could call $(\widetilde{J^1Z^*}, \Omega_\alpha)$ a special multisymplectic manifold, since it is multisymplectomorphic to a bundle of forms, and the multisymplectic $(n+2)$-form is $\Omega_\alpha = -d\Theta_\alpha$ (where $\Theta_\alpha = \tilde{\alpha}^*((\Theta_Z)^{n+1}_2)$. In addition, the following diagram is commutative:

$$\begin{tikzcd}
\widetilde{J^1Z^*} \arrow[rr, "\tilde{\alpha}"] \arrow[dr, swap, "\widetilde{j^1\pi_{YZ^*}}"] & & \Lambda^{n+1}_2 Z \arrow[dl, "\pi_{Z\Lambda^{n+1}_2 Z}"] \\
& Z &
\end{tikzcd}$$

Let $\mathbb{L} : Z \longrightarrow \Lambda^n X$ be a lagrangian density, that is, $\mathbb{L}$ is an n-form on Z along the projection $\pi_{XZ} : Z \longrightarrow X$.

We put

$$\mathcal{N}_{\mathbb{L}} = \{u \in \widetilde{J^1Z^*} | \left(\widetilde{j^1\pi_{XZ^*}}\right)^* (d\mathbb{L})_u = (\Theta_\alpha)_u\}$$

Theorem 5.2 *$\mathcal{N}_{\mathbb{L}}$ is a $(n+1)$-lagrangian submanifold of the multisymplectic manifold $(\widetilde{J^1Z^*}, \Omega_\alpha)$. In addition, the local equations defining $\mathcal{N}_{\mathbb{L}}$ are just the Euler-Lagrange equations for L, where $\mathbb{L} = L\eta$.*

Proof From the definition it follows that

$$\tilde{\alpha}(\mathcal{N}_{\mathbb{L}}) = \operatorname{im} d\mathbb{L},$$

In addition, we have

$$\begin{aligned}(\Theta_Z)_2^{n+1} &= a_i dy^i \wedge d^n x + b_i^\mu dz_\mu^i \wedge d^n x \\ \alpha^*((\Theta_Z)_2^{n+1}) &= p_{i\mu}^\mu dy^i \wedge d^n x + p_i^\mu dy_\mu^i \wedge d^n x \\ d\mathbb{L} &= \frac{\partial L}{\partial y^i} dy^i \wedge d^n x + \frac{\partial L}{\partial z_\mu^i} dy_\mu^i \wedge d^n x.\end{aligned}$$

Since

$$(\widetilde{j^1\pi_{XZ^*}})^*(d\mathbb{L}) = \Theta_\alpha$$

if and only if

$$\tilde{pr}^*(\widetilde{j^1\pi_{XZ^*}}^*(d\mathbb{L}) - \Theta_\alpha) = 0$$

which is in turn equivalent to

$$(j^1\pi_{XZ^*})^*(d\mathbb{L}) = \alpha^*(\Theta_Z)_2^n,$$

we deduce that $\mathcal{N}_{\mathbb{L}}$ is locally defined by

$$\sum_\mu p_{i\mu}^\mu = \frac{\partial L}{\partial y^i} \tag{5.17}$$

$$p_i^\mu = \frac{\partial L}{\partial z_\mu^i} \tag{5.18}$$

Equations (5.17) imply that $\tilde{\alpha}(\mathcal{N}_{\mathbb{L}}) = \operatorname{Im} d\mathbb{L}$, and hence $\mathcal{N}_{\mathbb{L}}$ is a $(n+1)$-lagrangian submanifold of $(\widetilde{J^1Z^*}, \Omega_\alpha)$.

Furthermore, we have

$$\sum_\mu p_{i\mu}^\mu = \sum_\mu \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial z_\mu^i}\right) = \frac{\partial L}{\partial y^i}$$

which are just the Euler-Lagrange equations for L. □

6 The multisymplectomorphism $\tilde{\beta}$

Recall that there exists a one-to-one correspondence between connections in the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$ and sections of the 1-jet prolongation $\pi_{Z^*J^1Z^*} : J^1Z^* \longrightarrow Z^*$. (At a pointwise level we have a one-to-one correspondence between horizontal subspaces in the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$ and 1-jets in J^1Z^*.)

Define a mapping

$$\beta : J^1Z^* \longrightarrow \Lambda_2^{n+1}Z^*$$

as follows: given a connection $\mathbf{h}^*$ in the fibred manifold $\pi_{XZ^*} : Z^* \longrightarrow X$, we take the $(n+1)$-form

$$\beta(\mathbf{h}^*) = i_{\mathbf{h}^*}\,\Omega_h - (n-1)\Omega_h.$$

An arbitrary $(n+1)$-form in $\Lambda_2^{n+1}Z^*$ is written as

$$A_i dy^i \wedge d^n x + B^i_\mu dp^\mu_i \wedge d^n x$$

so that we can introduce local coordinates $(x^\mu, y^i, p^\mu_i, A_i, B^i_\mu)$ on $\Lambda_2^{n+1}Z^*$.

If we put

$$\mathbf{h}^*(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + y^i_\mu \frac{\partial}{\partial y^i} + p^\nu_{j\mu} \frac{\partial}{\partial p^\nu_j}$$

or, equivalently,

$$\mathbf{h}^*(x^\mu, y^i, p^\mu_i) = (x^\mu, y^i, p^\mu_i, y^i_\mu, p^\nu_{j\mu})$$

(when $\mathbf{h}^*$ is considered as a section of $J^1Z^* \longrightarrow Z^*$), then a straightforward computation shows that

$$\beta(x^\mu, y^i, p^\mu_i, y^i_\mu, p^\nu_{i\mu}) = (x^\mu, y^i, p^\mu_i, \sum_\mu p^\mu_{i\mu} + \frac{\partial H}{\partial y^i}, -y^i_\mu + \frac{\partial H}{\partial p^\mu_i}).$$

The mapping β is a surjective submersion. Thus, in order to have a diffeomorphism we consider the induced mapping $\tilde{\beta} : \widetilde{J^1Z^*} \longrightarrow \Lambda_2^{n+1}Z^*$. Therefore we obtain a commutative diagram

$$\begin{tikzcd}
\widetilde{J^1Z^*} \arrow[rr, "\tilde{\beta}"] \arrow[dr, "\tilde{\rho}"'] & & \Lambda_2^{n+1}Z^* \arrow[dl, "\pi_{Z^*\Lambda_2^{n+1}Z^*}"] \\
& Z^* &
\end{tikzcd}$$

where $\tilde{\rho} : \widetilde{J^1Z^*} \longrightarrow Z^*$ is the induced projection from the canonical one $\rho : J^1Z^* \longrightarrow Z^*$.

Define a $(n+1)$-form Θ_β on $\widetilde{J^1Z^*}$ as $\Theta_\beta = \tilde{\beta}^*((\Theta_{Z^*})_2^{n+1})$. Therefore, the pair $(\widetilde{J^1Z^*}, \Omega_\beta)$, $\Omega_\beta = -d\Theta_\beta$, is a multisymplectic manifold of type $(n+2, 2)$.

Remark 6.1 It should be noticed that pair $(\widetilde{J^1Z^*}, \Omega_\beta)$ is a special multisymplectic manifold.

Theorem 6.2 *Let $\mathbf{h}^*$ be a solution of the de Donder equation. Then the projection $\mathcal{N}_h$ of the image of $\mathbf{h}^*$ by $\tilde{pr}$ is a $(n+1)$-lagrangian submanifold of the multisymplectic manifold $(\widetilde{J^1Z^*}, \Omega_\beta)$. In addition, the local equations defining $\mathcal{N}_h$ are just the Hamilton equations for h.*

Proof Since

$$(\Theta_{Z^*})_2^{n+1} = A_i dy^i \wedge d^n x + B^i_\mu dp^\mu_i \wedge d^n x$$

we have

$$\beta^*((\Theta_{Z^*})_2^{n+1}) = (p^\mu_{i\mu} + \frac{\partial H}{\partial y^i})dy^i \wedge d^n x + (-y^i_\mu + \frac{\partial H}{\partial p^\mu_i})dp^\mu_i \wedge d^n x.$$

Therefore, the projection $\mathcal{N}_h$ of the image of $\mathbf{h}^*$ by $\tilde{pr}$ is just the inverse image of the zero-cross section of $\Lambda_2^{n+1}Z^*$, and hence it is a $(n+1)$-lagrangian submanifold of $(\widetilde{J^1Z^*}, \Omega_\beta)$.

The second part of the theorem follows directly from the preceding discussion. □

7 Relating $\tilde{\alpha}$ and $\tilde{\beta}$

The above constructions are collected in the following diagram:

$$\begin{tikzcd}
\Lambda_2^{n+1}Z \arrow[dr, "\pi_{Z\Lambda_2^{n+1}Z}"'] & & \widetilde{J^1Z^*} \arrow[ll, "\tilde{\alpha}"'] \arrow[rr, "\tilde{\beta}"] \arrow[dl, "\widetilde{j^1\pi_{YZ^*}}"] \arrow[dr, "\tilde{\rho}"] & & \Lambda_2^{n+1}Z^* \arrow[dl, "\pi_{Z^*\Lambda_2^{n+1}Z^*}"] \\
& Z & & Z^* &
\end{tikzcd}$$

Since

$$\begin{aligned}
\tilde{pr}^*(\Theta_\alpha) &= p^\mu_{i\mu} dy^i \wedge d^n x + p^\mu_i dy^i_\mu \wedge d^n x \\
\tilde{pr}^*(\Theta_\beta) &= (p^\mu_{i\mu} + \frac{\partial H}{\partial y^i})dy^i \wedge d^n x + (-y^i_\mu + \frac{\partial H}{\partial p^\mu_i})dp^\mu_i \wedge d^n x
\end{aligned}$$

we deduce that

$$\begin{aligned}
\tilde{pr}^*(\Theta_\alpha - \Theta_\beta) &= dh - \left(y^i_\mu dp^\mu_i + p^\mu_i dy^i_\mu\right) \wedge d^n x \\
&= dh - d(p^\mu_i y^i_\mu) \wedge d^n x \\
&= d\left(h - (p^\mu_i y^i_\mu) \wedge d^n x\right)
\end{aligned}$$

which implies that $\Omega_\alpha = \Omega_\beta$.

Theorem 7.1 *Let L be a regular lagrangian, and assume that $h = leg_L \circ (Leg_L)^{-1}$. Then $N_{\mathbb{L}} = N_h$.*

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Degenerate metrics and singular geodesics

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Abstract

In this paper we will approach the study of degenerate metrics from the point of view of their geodesics. Thus the geodesic equation for a degenerate metric is derived by means of an optimal control formulation of the problem of determining the shortest path joining two points. The equation of geodesics will in general be implicit and the study of their solutions will require the application of the presymplectic constraints algorithm to the presymplectic problem equivalent to Pontriaguine's principle. Along the application of the algorithm we will observe that the stability of the secondary constraints is equivalent to the integrability of the characteristic distribution of the degenerate metric, a condition that was derived in [7] by different means. The existence of geodesics passing through a given point is related to the completeness of the space. In particular it is shown that the integrable points for the equation of geodesics must necessarily be contained in the b-completion of the space of regular points. Finally, a simple example is discussed exhibiting some of the main features of the previous constructions.

1 Introduction: Degeneracy versus singularities

The celebration of Mike Crampin 60th birthday gives us the extraordinary occasion to look back to some of the ideas that so masterly he taught us. One the very first problems studied by Mike was the geometrical properties of a differentiable manifold M with a degenerate metric g [7].

He proved then that there exists a symmetric torsionless metric connection for a constant rank degenerate metric provided that the first structure function of the orthonormal frame bundle defined by the degenerate metric will vanish, the condition being also necessary. Such condition was also shown to be equivalent to the fact that the null vector fields of the metric were Killing,

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which implied in addition that the null distribution N of the metric were integrable, defining a foliation $\mathcal{N}$. Thus, if we are in the neighborhood of a regular leaf of the foliation defined by N, then the leaf space $M/\mathcal{N}$ inherits a smooth structure around such point and the degenerate metric descends to a nondegenerate one $\tilde{g}$ defined on such neighborhood. The geodesics for the metric $\tilde{g}$ constructed locally in the quotient space $M/\mathcal{N}$, can be lifted up to the original manifold constituting the geodesics of the degenerate metric, more properly called degenerate or singular geodesics. These ideas were recast in the setting of reduction of singular lagrangians in [5] and it was shown that the existence of such connection was equivalent to the fact that the singular kinetic energy Lagrangian defined by the metric was of type II.

It would be our purpose here to continue the study of such geodesics for degenerate metrics not necessarily of constant rank, and connect this problem to other problems of interest in the geometry and topology of pseudoriemannian manifolds. For that we will look at it from the point of view of studying the existence and uniqueness of solutions of the equation of geodesics for a degenerate metric.

The equation of geodesics for a metric degenerate or not, that will be derived in Section 2 as an easy application of a problem in optimal control theory, becomes an implicit equation when the metric is degenerate . To solve it we will apply the recursive constraint algorithm invented to extract the integrable part of an implicit differential equation [17], [15], i.e., the set of points through which pass at least a solution of the equation. The obstruction in the first step of the algorithm to the existence of such points will be shown to be equivalent under the appropriate conditions to Crampin's theorem on the vanishing of the first structure function of the orthonormal bundle defined by the metric.

At the time that Mike was writing his paper on degenerate metrics, the study of singularities of space-time was reaching a climax with the publication of various articles by S. Hawking and R. Penrose [12], [13] showing the existence of singularities under appropriate conditions (see [14] for a masterly exposition of the subject). Even though by that time there was not a completely clear understanding of what a singularity of a space-time geometry was (see [4] for a detailed history of the subject) it was clear that some sort of degeneracy of the metric could be responsible for such phenomena. In fact a singularity in a broad sense can be thought as points where Einstein's equations doesn't make sense because some components of the curvature tensor are not defined. In this sense the study of degenerate metrics was a natural problem in order to understand how such singularities emerged [1] .

This point of view was not pursued further. However more recently there have

[1] Even though this was not the true reason why Mike was interested in the problem as I was told by him but rather trying to understand some conformal properties of asymptotically flat spaces.

been emerging alternative foundations of General Relativity where degenerate metrics where not excluded. For instance Ashtekar's formulation of General Relativiy allows degenerate metrics as solutions of the equations of the theory [2], [3]. Even more recently, other ideas like signature change and topology change in General Relativity have appeared where degenerate metrics play an important role.

We will review in Section 3 the geometrical theory of singularities of connections as established by Schmidt [18] and developed further by Friederich [10], Dodson [9], and a number of researchers (see [4] and references therein) until more recently [1]. Then we will apply it to the regular manifold obtained by removing the points where the metric degenerates, obtaining in this way the b-completion of a degenerate metric with respect to the Levi-Civita connection it defines in the regular open submanifold. It will be clear from the previous results that the integrable points of the geodesic equation according to the theory described previously, will be in the b-boundary of the regular manifold. We will conjecture that both sets agree. The difficulty of the actual computation of the b-completion of a metric has been a serious drawback of the theory, that apart from other difficulties like the, in general non Hausdorff character of the b-completion, has caused its abandon by the researchers in the field. These notes will offer an alternative way to approach the problem that apart from providing a new source of examples, it will also give explicit computations of b-completions and will bring more light on the subject.

2 Singular geodesics for degenerate metrics

2.1 Geodesics equation for degenerate metrics: an optimal control theory point of view

We shall consider a differentiable manifold Q with local coordinates x^i and a symmetric bilinear form g defined on it (possibly degenerate) that will be called a metric or a degenerate metric if we want to emphasize the fact that is not necessarily nondegenerate. For that metric g it makes sense to consider the problem of determining the "shortest" path joining two given points, i.e., determining the minima of the functional

$$S(\gamma) = \int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt,$$

on the space of piecewise C^1 curves $\gamma\colon [0,1] \to Q$. This problem can be equivalently formulated as a problem in optimal control theory as follows:

Determine the piecewise C^1 curves $\sigma\colon [0,1] \to TQ$, $\sigma(t) = (\gamma(t), v(t))$, such that the dynamical equation

$$\dot{\gamma}(t) = v(t),$$

is satisfied and that minimizes the objective functional

$$S(\sigma) = \int_0^1 g_{\gamma(t)}(v(t), v(t))dt,$$

together with the endpoint conditions:

$$\gamma(0) = x_0, \qquad \gamma(1) = x_1.$$

Natural local coordinates on the tangent bundle TQ of the manifold Q will be denoted by (x^i, v^i) and by (x^i, p_i) on the cotangent bundle $T * Q$.

It is well-known that the solution to this problem is given by Pontriaguine's maximum principle [16], that states that a solution $(\sigma(t) = (\gamma(t), v(t))$ to the previous problem exists if there exists a lifting $\xi(t) = (\gamma(t), p(t))$ of the curve $\gamma(t)$ to the cotangent bundle T^*Q, such that the curve $(\gamma(t), p(t), v(t))$ verifies the equations,

$$\dot{x}^i(t) = \frac{\partial H_0}{\partial p_i}(\gamma(t), p(t), v(t)), \qquad \dot{p}_i(t) = -\frac{\partial H_0}{\partial x^i}(\gamma(t), p(t), v(t)),$$

where H_0 denotes Pontriaguine's Hamiltonian function:

$$H_0(x, p, v) = p_i v^i - L(x, v),$$

where L denotes the Lagrangian density defining the objective functional S, the endpoint conditions and the maximum condition:

$$H_0(\gamma(t), v(t), p(t)) = \max_{\tilde{v}} H_0(\gamma(t), p(t), \tilde{v}).$$

Hence the solutions of the optimal control problem piecewise C^1 will be found among the paths satisfying the extremal equations:

$$\dot{x}^i = \frac{\partial H_0}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_0}{\partial x^i}, \qquad \frac{\partial H_0}{\partial v^i} = 0. \tag{1}$$

The previous equations are readily seen to be given in local coordinates by

$$\dot{x}^i = v^i, \qquad \dot{p}_k = -\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} v^i v^j, \qquad p_i = g_{ij} v^j,$$

thus they are equivalent to the set of implicit second order differential equations on Q given by

$$g_{ij}\ddot{x}^i + \Gamma_{ijk}\dot{x}^j\dot{x}^k = 0, \tag{2}$$

where Γ_{ijk} denote the Christoffel symbols of first class. Equations (2) will be called the geodesic equation of the metric g. A solution of such equation will be called a geodesic of g.

A point $x \in Q$ will be called regular if $\det g_{ij}(x) \neq 0$ and singular (or degenerate) otherwise. The open dense set of regular points of Q will be called the regular submanifold of Q and denoted by $Q_{\rm reg}$. The set of all degenerate points of g will be called the degenerate locus of g and denoted by $\Sigma(Q)$. Notice that if x is a regular point, then the geodesic equation can be written in normal form in an open neighborhood of x, hence there will be a local solution of eq. (2) passing through x, that will be an ordinary geodesic of the metric g restricted to an open neighborhood of x where it is nondegenerate. However if $x \in \Sigma(Q)$, then in general there is no reason for the existence nor uniqueness of solutions passing through it.

A solution of the geodesic equation passing through a degenerate point will be called a degenerate or singular geodesic. Our purpose is to characterize the set of points in Q such that there exists a geodesic or a singular geodesic passing through them. Such points will be called integrable and denoted by $Q_{\rm int}$. It is clear from the previous observations that $Q_{\rm reg} \subset Q_{\rm int}$.

2.2 A geometrical description

Before entering the delicate problem of studying the solutions of the equation of geodesics, we will rephrase the previous discussion in a more palatable geometrical setting.

We shall denote by $M_0 = T^*Q \times_Q TQ$ the fibrered product over Q of the cotangent and tangent bundles over Q. We shall also denote by $p_0 \colon M_0 \to T^*Q$ and by $q_0 \colon M_0 \to TQ$ the corresponding canonical projections. We shall denote by $\Omega_0 = p_0^*\omega_0$ the pull-back to M_0 of the canonical symplectic form on T^*Q.

Clearly Ω_0 is degenerate with a rank n characteristic distribution $K = \ker \Omega_0$, spanned by the "vertical" vectors $\partial/\partial v^i$, i.e., $K = \ker Tp_0$. Pontriaguine's Hamiltonian H_0 is well defined on M_0 an given by,

$$H_0(x, p, v) = \theta_0(x, p)(v) - L(x, v),$$

where θ_0 denotes the canonical Liouville 1-form on T^*Q. We will define in this way the presymplectic Hamiltonian system (M_0, Ω_0, H_0) with dynamical vector field Γ defined by the equation:

$$i_\Gamma \Omega_0 = dH_0. \tag{3}$$

Clearly there will exists a vector field Γ satisfying the previous equations if and only if $i_Z dH_0 = 0$ for all $Z \in K$, i.e., if and only if the following equations are satisfied

$$\varphi_i^{(1)} := \frac{\partial H_0}{\partial v^i} = p_i - g_{ij} v^j = 0, \qquad i = 1, \ldots, n. \tag{4}$$

For generic metrics g the subset defined by the previous conditions (4), that can be called primary constraints of the presymplectic system above, is a smooth submanifold of M_0 that will be denoted by M_1. Hence restricted to M_1 the dynamical vector field Γ defined by the presymplectic system takes the form:

$$\Gamma_C = v^i \frac{\partial}{\partial x^i} - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} v^i v^j \frac{\partial}{\partial v^i} + C^i \frac{\partial}{\partial v^i}, \tag{5}$$

with C^i undetermined functions. Equation (5) provides the geometrical description of Pontriaguine's equations (1).

2.3 Solving the degenerate geodesics equation

It is clear from the previous argument that the geodesic equation (2) will have a solution lying on M_1 or, equivalently, there will exists a vector field Γ_C whose integral curves will be contained in M_1 if and only if there exists C^i's such that the corresponding vector field Γ_C is tangent to M_1, this is, if:

$$\Gamma_C(\varphi_i^{(1)}) = 0, \qquad \text{on } M_1.$$

In other words if the equation,

$$g_{ij} C^i = -\Gamma_{ijk} v^j v^k, \tag{6}$$

has solution on M_1. If g is nondegenerate (6) has a unique solution and Γ_C is uniquely determined on M_1. There is an alternative way of looking at this fact that will be useful later on.

Denoting by p_1 the restriction of p_0 to M_1 we will see immediately that $\Omega_1 = p_1^* \omega_0$. Thus Ω_1 is symplectic if and only if p_1 is a local diffeomorphism. Moreover notice that $\ker(p_1)_* = K \cap TM_1 = TN^v$ where N^v denotes the vertical lifting of the null distribution of g. Thus Ω_1 will be nondegenerate iff g is nondegenerate and hence there will exists a unique solution for Γ given by the equation:

$$i_\Gamma \Omega_1 = dH_1.$$

Notice that the regular submanifold of M_1, i.e., those points where Ω_1 is nondegenerate, is precisely the set $p_1^{-1}(T^*_{Q_{\mathrm{reg}}} Q)$ and the singular points of M_1, that is the points where Ω_1 fails to be nondegenerate, are the critical points of p_0, i.e., the points in M_1 where Tp_0 is not submersive. Let us denote such points by Σ_1.

We must study then the points in Σ_1. A necessary and sufficient condition for the equation (6) to have solution at a singular point is that

$$\eta^i \Gamma_{ijk} = 0, \qquad \forall Z = \eta^i \frac{\partial}{\partial x^i} \in N, \tag{7}$$

in other words, denoting by α_1 the 1-form on TQ along the projection τ_Q given by $\Gamma_{ijk} v^j v^k dx^i$, the previous condition is equivalent to

$$i_Z \alpha_1 = 0, \forall Z \in N.$$

Let us recall that a metric g is of type II if $2 \dim V(\ker \omega_g) = \dim \ker \omega_g$ where ω_g is the Cartan 2–form defined by the kinetic energy Lagrangian given by g, i.e., $\omega_g = -dS^*(dK_g)$, with $K_g = 1/2(g_{ij} v^i v^j)$. If g is of type II, then N is integrable [5]. Moreover if the vector fields $Z \in N$ are Killing vector fields, then g is of type II and it has a global dynamics.

In general however (7) will not be satisfied and we will have secondary constraints:

$$\varphi_a^{(2)} := i_{Z_a} \alpha_1, \qquad a = 1, \ldots, \dim N,$$

where Z_a form a basis of N.

We are clearly in the setting of Dirac's constraints algorithm [8] (see [15], [17] and references therein for a modern perspective). Then, stability of the secondary constraints

$$\Gamma_C(\varphi_a^{(2)}) = 0,$$

on M_1 will define tertiary constraints and so on. Eventually we will obtain a final constraint submanifold M_∞ where the dynamical equation

$$i_{\Gamma_\infty} \Omega_\infty = dH_\infty,$$

could always be solved. The projection of the integral curves of the vector field Γ_∞ to Q will be solutions of the equation of geodesics for g. Thus the projection of the final constraint submanifold M_∞ to Q will define a subset $Q_{\text{int}} \subset Q$ where the equation of geodesics could be solved. Unfortunately so far we do not have a geometrical interpretation of the higher order constraints eventually arising in the theory so we cannot say in advance when the presymplectic constraint algorithm will stop.

3 Singularities of connections and b-completion

3.1 What is a singularity?

We will follow here the discussion in [14] about the nature of singularities in space-time manifolds. However the arguments with a physical origin are much weaker here as we plan to apply them to manifolds which are not necessarily Lorentzian. In any case if we are considering a manifold with a metric defined on it, it would be reasonable to think that the given space has a singularity if it could not be extended to another space without considering points where the metric tensor or some of the other tensors derived from it were undefined.

We could in principle, simply remove these points, but we will be left with the question if in the remaining manifold we could determine if some singularities have been cut off along this process. This is if the manifold without the singular points still "detects" its shadow presence. One way to detect them would be by means of studying if the manifold we are considering is incomplete in some sense.

There are various forms of incompleteness. The simplest and more fundamental one is the notion of m-completeness, or Cauchy completeness, that can be defined whenever we can equip our manifold with the structure of a metric space. Of course, if g is positive definite, there is a natural distance defined on the manifold M as $d(x,y) = \inf_{\gamma\in\Omega(x,y)} l(\gamma)$ with $\Omega(x,y)$ the space of piecewise C^1 curves on M joining x and y, and

$$l(\gamma) = \int_0^1 \parallel \dot{\gamma}(t) \parallel dt.$$

The manifold (M,d) will be said to be m-complete if every Cauchy sequence is convergent. An alternative characterization of m-complete spaces is that any curve of finite length has an endpoint, where a point p is called an endpoint for the curve γ if for every neighborhood U of p there exists T such that $\gamma(t) \in U$ for all $t > T$.

When the metric we are considering is not definite it is not possible to introduce a metric as above, however we can still speak of g-completeness. We will say that (M,g) is g-complete if every geodesic can be extended to arbitrary values of its natural parameter. For a definite metric g, both notions agree.

In Lorentzian manifolds, the notion of m-completeness makes no sense, but it is clear that timelike geodesic incompleteness is a clear exponent of the existence of a singularity. However Geroch [11] constructed a geodesically complete space-time which contains an inextensible time-like curve of bounded acceleration and finite length. For making sense of the latest statements we need to define a natural parameter for all curves and then define a notion of completeness by requiring that any piecewise C^1 curve of finite length as measured by such parameter has an endpoint. This can be done as follows.

Let $\gamma(t)$ an arbitrary curve and $\gamma(t_0) = p$ a point on it. Let us consider now a frame $r = (p, e_1, \ldots, e_n)$ at it. We shall parallel propagate the frame r along the curve by solving the linear equation $\nabla_{\dot\gamma(t)} e_k(t) = 0$ with initial condition $e_k(t_0) = e_k$. With respect to the parallel frame thus constructed we can write $\dot\gamma(t) = v^k(t) e_k(t)$. Then define the natural parameter as $s = \int_{t_0}^{t} \sqrt{v^k(t) v^k(t)}\, dt$. We will say that M is b-complete with respect to the connection ∇ if any curve of finite length with respect to a natural parameter has an endpoint.

It is clear that b-completeness implies g-completeness, but we can already pointed out that the converse is not true. We will say that a space with connection is singularity-free if it is b-complete and singular otherwise.

3.2 The b-completion of a singular space

If we have a singular space (M, g), i.e., M is not b-complete with respect to the Levi-Civita connection defined by g, we would like to know whether it is possible to complete it and to understand where the finite length curves end. The role played by the added points should be a matter of further analysis in the context of the problem studied. There is a natural and elegant way to b-complete M due to Schmidt [18] and described carefully in [1] and [6]. We shall sketch the construction.

Let $\pi\colon F(M) \to M$ the frame bundle of M and ϕ the canonical $\mathbb{R}^n$-valued 1-form on it. If $r = (m; e_k) \in F(M)$ is a frame, we denote by $\hat r\colon \mathbb{R}^n \to T_m M$ the natural isomorphism $\hat r(x) = x^k e_k \in T_m M$. Then $\phi_r(X) = \hat r^{-1}(\pi_*(r)(X))$ for every $X \in T_r F(M)$. We shall consider now a principal connection 1-form A on $F(M)$ and the metric g_A defined on $F(M)$ associated to A given by

$$g_A = \langle A \otimes A \rangle + \langle \phi \otimes \phi \rangle,$$

or acting on vectors, $g_A(X, Y) = \langle A(X), A(Y) \rangle + \langle \phi(X), \phi(Y) \rangle$, for any scalar product $\langle \cdot, \cdot \rangle$ defined on $\mathbb{R}^n$ and the corresponding one defined on the Lie algebra of $GL(n, \mathbb{R})$. It can be proved that $(F(M), g_A)$ is m-complete if and only if (M, g) is b-complete. Thus, the non-completeness of $F(M)$ indicates the non b-completeness of M. Thus it makes sense to consider the m-completion of $F(M)$ and then passing to the quotient under the induced action of $GL(n, \mathbb{R})$. More precisely, consider the m-completion $\overline{F(M)}^m$ of $F(M)$. The right action of $GL(n, \mathbb{R})$ on $F(M)$ can be extended uniquely to a continuous action of $GL(n, \mathbb{R})$ in $\overline{F(M)}^m$. We consider now the quotient space $\overline{F(M)}^m / GL(n, \mathbb{R})$ with the induced topology. Such space contains M and will be called the b-completion of M and denoted by $\overline{M}^b$. The b-boundary of M will be denoted by $\partial_b M$ and is by definition

$$\partial_b M = \overline{M}^b - M.$$

The points in $\partial_b M$ are the endpoints of b-incomplete curves in M. Among the various difficulties arising with this notion we must mention that the b-completion of M needs not to be Hausdorff and the points in $\partial_b M$ need not to be separated from the points in M. In fact, in general M is not an open subset of $\overline{M}^b$.

These negative results, apart from the difficulty of its actual computation, have prevented the notion of b-completion, even if completely general and mathematically sound, to be fully accepted as a correct completeness notion for physical General Relativity.

4 A conjecture

The previous discussion clearly brings together two different aspects of the theory of spaces with metrics (degenerate or not): its b-completeness and the existence of geodesics (degenerate or not). Thus, denoting again by (Q, g) a differentiable manifold with a smooth metric g, possibly degenerate, and denoting by Q_{reg} as before the set of regular points of such space, then it is clear that

$$Q_{\mathrm{int}} \subset \overline{Q}^b_{\mathrm{reg}},$$

because, if $x \in Q_{\mathrm{int}}$ there is a geodesic passing through it, then it is the endpoint of a curve. Alternatively, we can think that the geodesic γ, together with a parallel frame along it, defines a lifting to a horizontal curve on the frame bundle $F(Q_{\mathrm{reg}})$. Such lifting can be completed and projects on the endpoint of the curve γ.

Conversely, as points in $\partial_b Q_{\mathrm{reg}}$ correspond to projections of endpoints of finite length curves in $F(Q_{\mathrm{reg}})$, we can approximate, finite length curves by polygonal geodesics and thus construct approximations of horizontal curves in the frame bundle $F(Q_{\mathrm{reg}})$. Thus if a point r is a limit point of a finite length curve, it is a limit point of limit points of horizontal curves defined by geodesics, hence it is reasonable to think that there is a limit geodesic passing through the projection of the limit point, hence the point of the b-boundary is in the integrable set of the degenerate geodesic equation. Hence we establish our conjecture:

$$Q_{\mathrm{int}} = \overline{Q}^b_{\mathrm{reg}}.$$

If the previous conjecture would be true, this would imply, among other things, that the b-completion of a space with singularities such that the metric can be extended to a degenerate metric is Hausdorff and can be explicitly computed by using the recursive algorithm for implicit differential equations described in section 2.3.

5 An example: the fold

Let Q be the subset of $\mathbb{R}^2$ defined by the equation $\varphi(x,y) = x - y^2 = 0$ and by $p\colon Q \to \mathbb{R}$ the projection onto the first factor, i.e., $p(x,y) = x$. Let g be the metric on Q defined by pulling back the euclidean metric on $\mathbb{R}$ along p, i.e., $g = p^* g_0$, with $g_0 = dx^2$. Notice that a global chart of Q is provided by the projection along the y-axis. In these coordinates we have $g = 4y^2 dy^2$. The metric g degenerates at $(0,0) \in Q$ and $N_0 = \mathbb{R} = T_0 Q$. Moreover $Q_{\mathrm{reg}} = \mathbb{R} - \{0\}$. The degenerate geodesic equation is given by (in y coordinates):

$$y^2 \ddot{y} + \dot{y}^2 = 0.$$

Introducing the optimal control approach of section 2, we see that the total space M is the space $T^*Q \times_Q TQ \cong \mathbb{R}^3$ with (global) coordinates (y,p,v). The lagrangian density is $L(y,v) = 2y^2v^2$. The Hamiltonian is $H(y,p,v) = pv - 2y^2v^2$ and the presymplectic structure $\Omega_0 = dy \wedge dp$. Pontriaguine's equations are given by

$$\dot{y} = v, \qquad \dot{p} = 4yv^2, \qquad p - 4y^2 v = 0.$$

The manifold M_1 is defined by $\varphi_1(y,p,v) = p - 4y^2v = 0$ and the presymplectic form in coordinates (y,v) is given by $\Omega_0 = 4y^2 dy \wedge dv$. Notice that $K = 0$ at any point different from $(0,0,v)$ and, $K_{(0,0,v)} = \mathbb{R}^2 = TM_1$.

We shall consider now the compatibility of the dynamical equation (3) on the submanifold M_1. For that we compute $\dot{\varphi}_1$ to obtain (on M_1),

$$0 = \dot{p} - 8y\dot{y}v - 4y^2\dot{v} = -4yv^2 - 4y^2 C,$$

where $C = \dot{v}$, hence either $y = 0$ or if $y \neq 0$, then $C = -v^2/y$. Thus the dynamics is completely determined in the regular set of Q, the set of points with $y \neq 0$, and the points of the form $(0,0,v)$ are singular on M_1. Notice that the stability of the singular set determined by the secondary constraint $\varphi^{(2)} = y$, implies that $\dot{y} = 0$, thus the consistency of implicit equation demands that $v = 0$. Hence there is a solution passing through the point $(0,0,0)$, the constant solution. All other points in the vertical axis are nonintegrable for the presymplectic system (M, Ω_0, H_0).

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The Inverse Problem for Quantum Systems

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1 Introduction

The inverse problem in the calculus of variations goes back to Helmholtz [11]. In the physics community this problem was raised by Wigner [17] in the framework of quantum mechanics, hoping that different commutation relations might introduce a cut-off at high frequencies and thus eliminate some of the divergencies in field theories [14].

The paper by Wigner was the starting point for Green to introduce parastatistics [10].

Here we shall not take up these problems but we will share the point of view of Dirac [5]:

> *"Classical mechanics must be a limiting case of quantum mechanics. We should thus expect to find that important concepts in classical mechanics correspond to important concepts in quantum mechanics and, from an understanding of the general nature of the analogy between classical and quantum mechanics, we may hope to get laws and theorems in quantum mechanics appearing as simple generalizations of well known results in classical mechanics".*

We will try to find out which are the quantum counterpart of bi-Hamiltonian descriptions for classical systems. To examine these aspects we shall consider Weyl systems and $*$-products in this setting.

The further scheme of the paper is the following.

Section 2. Reviewing the Feynman's problem
Section 3. Weyl systems and $\star$-products
Section 4. Weyl systems associated with alternative Lagrangian descriptions

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Section 5. The inverse problem for quantum systems
Section 6. The classical limit of some alternative quantum descriptions
Section 7. Comments and conclusions

2 Reviewing the Feynman problem

Dyson has reported on an unpublished result of Feynman showing that the only interaction for a quantum particle compatible with localizability and the canonical relation $[q^j, p_k] = i\hbar\delta^j_k$, is the Lorentz force law of electrodynamics [6].

After the presentation of this result, modifications of Feynman's procedure have been used to "prove" several other interactions. They include the descriptions of relativistic particles, spinning particles in an electromagnetic field, particles with isospin moving in a Yang–Mills field, and particles in a gravitational field [13].

According to Dyson, Feynman's real interest was not simply recovering the standard minimal coupling for electrodynamics, but rather in finding new kinds of descriptions, possibly not equivalent to the Lagrangian ones. In this respect, Feynman's result was a no-go theorem, showing that unless some well founded physical conditions are removed, the new descriptions are reducible to the Lagrangian ones.

Here, in order to avoid unnecessary complications due to operator ordering we shall discuss only the classical analogue of the Feynman procedure.

Taking the point of view that classical mechanics should be considered as a limit of quantum mechanics, the appropriate setting for the description of classical systems would be in terms of Poisson brackets and Hamiltonian function. The former is regarded as the classical limit of the commutator brackets of quantum observables, and the latter is the classical limit of the quantum Hamiltonian operator.

Upon taking the classical limit of the quantum system, the carrier space could inherit a non trivial topology. (For example, for the case of a spinning particle belonging to an irreducible representation of SU(2), the classical phase space is a 2-dimensional sphere). The classical limit of a quantum system is then said to be a Hamiltonian system defined on a Poisson manifold, i.e. a carrier space M equipped with a Poisson bracket.

More precisely, let $\mathcal{F} = \mathcal{F}(M)$ be the algebra of classical observables on the manifold M, $\mathcal{F}(M)$ may be obtained as a certain commutative limit ($\hbar \to 0$) of the non-commutative algebra of quantum observables $O_\hbar$ of a quantum system. The Poisson bracket $\{\cdot\,,\,\cdot\}$ on M is defined by $\{F\,,\,G\} = \lim_{\hbar\to 0} \frac{1}{i\hbar}[F, G]$, where $F = \lim_{\hbar\to 0} F_\hbar$, and $\{\cdot\,,\,\cdot\}$ is a skew-symmetric bilinear map on $\mathcal{F}$ which satisfies the Jacobi identity and the Leibnitz rule:

$$\{F,\,\{G,\,H\}\} = \{\{F,G\},\,H\} + \{G,\,\{F,\,H\}\}$$
$$\{F,\,GH\} = \{F,G\},\,H + G,\,\{F,\,H\}\,.$$

This constitutes the formal definition of a Poisson structure on the classical carrier space M. We notice that both properties express the Leibnitz rule with respect to the two bilinear products on $\mathcal{F}$, the Poisson bracket and the pointwise product. These derivation properties are very important and, thanks to them, we may associate with any Poisson bracket a bivector field, the so called Poisson tensor. As usual, this tensorial character of the brackets allows us to perform computations in any coordinate system while preserving their general significance.

In a set of local coordinates, ξ^a, for M we may write a dynamical vector field in the form $\Gamma = \Gamma^a \dfrac{\partial}{\partial \xi^a}$ and a Poisson bracket in the form $\{\xi^a,\,\xi^b\} = \Lambda^{ab}$ with associated Poisson tensor $\Lambda = \Lambda^{ab}\,\dfrac{\partial}{\partial \xi^a} \wedge \dfrac{\partial}{\partial \xi^b}$. The derivation property allows to write

$$\{F,G\} = \frac{\partial F}{\partial \xi^a}\{\xi^a,\xi^b\}\frac{\partial G}{\partial \xi^b} = \Lambda(dF,\,dG)\,.$$

Any function $H \in \mathcal{F}$ defines a dynamical system on M by the formula $\dfrac{dF}{dt} = \{H,\,F\}$ or, in local coordinates, $\dfrac{d\xi^a}{dt} = \Lambda^{ab}\,\dfrac{\partial H}{\partial \xi^b}$. The corresponding Hamiltonian vector field X_H is written as $X_H = -\Lambda(dH)$.

3 The inverse problem for Poisson dynamics

Starting with a second order dynamics on the configuration space Q, say $\ddot{x}_j = f_j(x, \bar{x})$, we must look for a bundle over Q, a Poisson bracket $\{\;,\;\}$ and a function H, such that

$$\dot{x}_j = \{H,\,x_j\}\,, \qquad \ddot{x}_j = \{H,\,\{H,\,x_j\}\} = f_j(x,\dot{x})\,.$$

It is clear that we may use as a bundle over Q just the tangent bundle TQ. Other choices would be possible, in any case our system would be related to one on TQ. Thus, we may always consider the Hamiltonian inverse problem for a second order dynamics in terms of a first order dynamics on TQ.

The way we have formulated our problem, i.e. to solve the problem in terms of the pair $(\{\;,\;\}, H)$, makes it highly non trivial.

Feynman introduced a simplification by requiring, on physical grounds, that $\{x_j,\,x_k\} = 0$. This innocent looking simplification allows us to transform the problem into a linear one. More specifically to an inverse problem in terms of a Lagrangian function, the unknown pair $(\{\,,\,\},\, H)$ is replaced by an unknown function $\mathcal{L}$.

Let us formulate first the inverse problem in the Lagrangian setting.

We consider the usual formulation of Euler-Lagrange equations on Q:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_j} - \frac{\partial \mathcal{L}}{\partial x_j} = 0$$

and expand it into

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial \dot{x}_k}\frac{d\dot{x}_k}{dt} + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial x_k}\frac{dx_k}{dt} - \frac{\partial \mathcal{L}}{\partial x_j} = 0\,.$$

Now we replace $\dfrac{d\dot{x}_k}{dt}$ with $f_k(x,\, \dot{x})$ and we transform the equation for the trajectories into an equation for the Lagrangian

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{x}_k} f_k + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial x_k}\dot{x}_k - \frac{\partial \mathcal{L}}{\partial x_j} = 0\,.$$

This is a linear equation for a function $\mathcal{L}$, if we are able to transform the nonlinear equation for the pair $(\{\,,\,\},\, H)$ into one for $\mathcal{L}$, we have been able to linearize our problem.

We refer to [3] for a general analysis of this problem. Here we give a very simple proof at the expenses of adding a further condition, namely that translations in the velocity space are canonical transformations, i.e. $\dfrac{\partial}{\partial \dot{x}_j}\{\,,\,\} = 0$. Starting with $\dot{x}_k = \{H,\, x_j\}$ we take the derivative with respect to $\dfrac{\partial}{\partial \dot{x}_j}$ to find

$$\delta_{jk} = \frac{\partial H}{\partial \dot{x}_j\,\partial \dot{x}_m}\{\dot{x}_m,\, x_j\}\,.$$

Therefore, the bracket is invertible and the Hessian of H is not degenerate. We may use a Legendre-type transformation to go from H to $\mathcal{L}$ and thus linearize the problem.

4 Solving the inverse problem for linear systems

When the first order dynamics is linear, i.e. $\Gamma = \xi^l A_l^j \partial/\partial\xi^j$, $A_l^j \in \mathbb{R}$, the problem is tractable and reduces to a problem in linear algebra [8]. The unknown Poisson tensor Λ is given by $\Lambda^{jk} = \{\xi^j,\, \xi^k\}$, the unknown Hamiltonian function $H = \dfrac{1}{2}\xi^k H_{ks}\xi^s$ and the inverse problem becomes $\xi^k A_k^j \dfrac{\partial}{\partial \xi^j} = \Lambda^{nk}\dfrac{\partial H}{\partial \xi^n}\dfrac{\partial}{\partial \xi^k}$, or, in more compact notation, $A = \Lambda \cdot H$.

Thus, all decompositions of A into a skew-symmetric matrix Λ times a symmetric matrix H will provide us with alternative Hamiltonian descriptions for our dynamics.

Taking into account the properties of Λ and H (skew-symmetric and symmetric, respectively) we find several interesting consequences:

(1) Dynamical systems associated with odd powers of A represent different linear Hamiltonian systems with respect to the same Poisson bracket and Hamiltonian functions

$$H^{(3)} = \frac{1}{2}\xi^r (H\Lambda H\Lambda H) r_s \xi^s \,.$$

These Hamiltonian functions are in involution, i.e.

$$\{H^{2k+1}\,,\; H^{2j+1}\} = 0$$

for any pair of exponents.

(2) If T represents any linear invertible transformation, we have

$$T^{-1}AT = T^{-1}\Lambda HT = T^{-1}\Lambda (T^t)^{-1}T^t HT \,.$$

Therefore, when $T^{-1}AT = A$, we get that any symmetry transformation for A provides us with a new Hamiltonian description for Γ, provided that $T\Lambda T^t \neq \Lambda$, i.e., if T is not a canonical transformation. It is not difficult to see that all even powers of A represent noncanonical transformations (if they are invertible, otherwise we may consider $\exp A^2$). Thus it is possible to find always alternative Hamiltonian descriptions for linear Hamiltonian systems.

It should be noticed that even if the vector field is linear we may find alternative Hamiltonian descriptions with nonconstant symplectic (Poisson) descriptions and nonquadratic Hamiltonians. An interesting example is provided by the harmonic oscillator.

On $M = \mathbb{R}^{2n}$, we consider

$$\Gamma = \sum_{k=1}^{n} \lambda_k \left(p_k \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial p_k} \right)$$

with $\lambda_k \in \mathbb{R}$.

For any constant of the motion $F(p,q)$ we construct

$$\omega_F = \sum_k \mu_k d\left(\frac{\partial F}{\partial p_k}\right) \wedge d\left(\frac{\partial F}{\partial q_k}\right) \,,$$

with $\mu_k \in \mathbb{R}$ We find easily, because of

$$\left[\Gamma, \frac{\partial}{\partial p_k}\right] = -\lambda_k \frac{\partial}{\partial q_k}, \qquad \left[\Gamma, \frac{\partial}{\partial q_k}\right] = \lambda_k \frac{\partial}{\partial p_k},$$

that $L_\Gamma \omega_F = 0$.

A different way to generate alternative invariant two-forms is provided by the $(1-1)$-tensor field $J = \sum_k \mu_k \left(dp_k \otimes \frac{\partial}{\partial q_k} - dq_k \otimes \frac{\partial}{\partial p_k} \right)$.

It is immediate that $L_\Gamma J = 0$, thus $dd_J f = \omega_f$ determines invariant two-forms for any constant of the motion f. When ω_f is not degenerate, we find alternative descriptions.

A particular example, for a two-dimensional isotropic oscillator ($\lambda_1 = \lambda_2 = 1$), is provided by

$$F_1 = (p_1^2 + q_1^2) \pm (p_2^2 + q_2^2), \quad F_2 = p_1 q_2 - p_2 q_1, \quad F_3 = p_1 p_2 + q_1 q_2$$

all of them will give constant symplectic structures. The function $F = (p^2 + q^2)(1 + f(p^2 + q^2))^2$ provides the most general invariant two-form for the one-dimensional harmonic oscillator. Specifying $F = \exp(p^2 + q^2)\frac{\lambda}{2}$ we get $\omega_F = dP \wedge dQ$, with

$$P = \lambda p \exp\left(\frac{\lambda}{2}(p^2 + q^2)\right)$$
$$Q = \lambda q \exp\left(\frac{\lambda}{2}(p^2 + q^2)\right)$$

with

$$\{P, Q\} = \lambda^2 \exp(\lambda(p^2 + q^2))[1 + \lambda(p^2 + q^2)]\,\{p, q\}$$

providing the Poisson bracket for the new variables in terms of the old ones.

Equations of motion will be again linear in the new variables

$$\frac{d}{dt}P = -Q, \quad \frac{d}{dt}Q = P.$$

It may be useful to notice few facts:

(1) In each canonical coordinate system, say (p, q) and (P, Q), we have the following tensor fields:

$$a) \quad \Delta = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}, \quad \omega = dp \wedge dq, \quad s = dp \otimes dp + dq \otimes dq$$

$$b) \quad \Delta' = P\frac{\partial}{\partial P} + Q\frac{\partial}{\partial Q}, \quad \omega' = dP \wedge dQ, \quad s' = dP \otimes dP + dQ \otimes dQ$$

all these tensor fields are preserved by dynamical evolution. In each set of coordinates we have a realization of the symplectic group (preserving ω) and the rotation group (preserving s), our dynamics preserves both structures, therefore it is an element of the unitary group.

(2) The two different realizations of these groups are linear, but they are not linearly related. This is explained by the following example: the vector $\begin{pmatrix} p \\ 0 \end{pmatrix}$ and the vector $\begin{pmatrix} 0 \\ q \end{pmatrix}$ add to the vector $\begin{pmatrix} p \\ q \end{pmatrix}$, their images are $\begin{pmatrix} \lambda p \exp \frac{\lambda}{2} p^2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \lambda q \exp \frac{\lambda}{2} q^2 \end{pmatrix}$ and $\begin{pmatrix} \lambda p \exp \frac{\lambda}{2}(p^2+q^2) \\ \lambda q \exp \frac{\lambda}{2}(p^2+q^2) \end{pmatrix}$ respectively, it is thus clear that the sum of two vectors (in the (p,q) coordinates) is not mapped into the sum of the images of the two vectors taken separately.

5 Weyl systems

Given a symplectic vector space (E,ω), a Weyl map is a strongly continuous map from E to unitary operators on some Hilbert space $\mathcal{H}$:

$$W \,:\, E \to \mathcal{U}(\mathcal{H})$$

satisfying the condition

$$W(e_1)W(e_2)W^\dagger(e_1+e_2) = \mathbf{1}e^{\frac{i}{2}\omega(e_1,e_2)} \,.$$

It is a projective unitary representation of the Abelian vector group associated with E [16].

A theorem due to von Neumann says that there exists such a map for any finite dimensional symplectic vector space [15]. Indeed, the Hilbert space $\mathcal{H}$ can be realized as the space of square integrable functions on any Lagrangian subspace of E. By using a Lagrangian subspace L it is possible to decompose E into $E = L \oplus L^* = T^*L = L^* \oplus (L^*)^* = T^*(L^*)$.

The Lebesgue measure is a translational invariant measure on L and we have a specific realization of W. We define $U = W|_{L^*}$, $V = W|_L$ and their action on $\mathcal{L}^2(L, d^n x)$ is given by

$$(V(y)\psi)(x) = \psi(x+y) \,, \quad (U(\alpha)\psi)(x) = e^{i\alpha(x)}\psi(x) \,,$$

for $x, y \in L$, $\alpha \in L^*$, $\psi \in \mathcal{L}^2(L, d^n x)$.

The strong continuity requirement in the definition of W allows to use Stone theorem to get

$$W(v) = e^{iR(v)}\,, \quad \forall\, v \in E\,,$$

with $R(v)$ the infinitesimal generator of the one parameter unitary group $W(tv)$, $t \in \mathbb{R}$, depending linearly on v.

When we select a complex structure J, $J : E \to E$, $J^2 = -\mathbf{1}$ it is possible to define "creation" and "annihilation" operators

$$\begin{aligned} a(v) &= \frac{1}{\sqrt{2}}(R(v) + iR(Jv)) \\ a^{\dagger}(v) &= \frac{1}{\sqrt{2}}(R(v) - iR(Jv))\,. \end{aligned}$$

With this complex structure we also associate an inner product on E by setting $\langle v_1, v_2\rangle = \omega(Jv_1, v_2) - i\omega(v_1, v_2)$.

The Weyl map allows to associate automorphisms on the space of operators with elements of the symplectic linear group on E, by setting $\nu_s(W(v)) = W(Sv) = U_S^{\dagger} W(v) U_S$, at the level of the infinitesimal generators of the unitary group, we have

$$U_S^{\dagger} R(v) U_S = R(Sv)\,.$$

The Wigner map can be defined for functions on $T^*L = E$ in the following way, where, for simplicity, we introduce (q, p) coordinates in E.

We define the Fourier transform of $f \in \mathcal{F}(E)$

$$f(q, p) = \int e^{i(\alpha q + xp)} \tilde{f}(\alpha, x) d\alpha dx$$

and we associate with it

$$\hat{A}_f = \int \tilde{f}(\alpha, x) e^{i(\alpha \hat{Q} + x\hat{P})} d\alpha dx\,.$$

Of course some qualifications are necessary for these formulae to make sense, we shall not worry about these points. We assume throughout that various formulae have meaning when the operators and symbols appearing in them are chosen from appropriate spaces (very often operators are assumed to be Hilbert-Schmidt and functions square integrable, but this can be generalized).

We simply notice that other correspondences are also used, for instance

$$\hat{A}_f = \int d\alpha dx \hat{f}(\alpha, x) e^{i\alpha\hat{Q}} e^{ix\hat{P}}$$

or

$$\hat{A}_f = \int d\alpha dx \hat{f}(\alpha, x) e^{(\alpha + ix)a^{\dagger}} e^{(\alpha - ix)a}\,.$$

Various maps are associated with different orderings (symmetric, normal, anti-normal and mixed) [1].

The map associating the function f, with the operator $\hat{A}_f$ in integral form, is often called the Weyl map. With any operator A acting on $\mathcal{H}$ we associate a function f_A on the symplectic space E by setting

$$f_A(v) = \mathrm{Tr}\,(AW(v))\,,$$

this map is called the Wigner map.

This way of writing allows us to easily derive several properties we are interested in, even though at a formal level.

We introduce a deformed product $*$, defined as

$$(f_{\hat{A}} * f_{\hat{B}})(v) = \mathrm{Tr}\,ABW(v)\,,$$

thus we find an associative product on $\mathcal{F}(E)$, which is not commutative. The dynamics on $\mathcal{F}(E)$ can now be written as

$$i\hbar\frac{d}{dt}\,f_{\hat{A}} = f_{\hat{H}} * f_{\hat{A}} - f_{\hat{A}} * f_{\hat{H}}\,.$$

We have obtained a way to write both classical and quantum mechanics on the same vector space of functions $\mathcal{F}(E)$, the difference being in the product we use to multiply functions.

In this approach, we should expect that

$$\frac{d}{dt}f_{\hat{A}} = -\frac{i}{\hbar}(f_{\hat{H}} * f_{\hat{A}} - f_{\hat{A}} * f_{\hat{H}})\,,$$

in the limit $\hbar \to 0$, should reproduce the classical Poisson bracket, this is indeed the case.

To consider this limit, it is convenient to use either an explicit form in terms of bidifferential operators or an integral form. We have, by denoting coordinates for E with (x, y),

$$(f * g)(x,y) = f(x,y)e^{i\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial y} - \frac{\overleftarrow{\partial}}{\partial y}\frac{\overrightarrow{\partial}}{\partial x}\right)}\,g(x,y)\,,$$

where a standard notation for physicists has been used, i.e. $\dfrac{\overleftarrow{\partial}}{\partial x}$ and $\dfrac{\overrightarrow{\partial}}{\partial y}$ mean that the operators act on the left or on the right, respectively.

As $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ commute, we can rewrite the $*$-product in the following form

$$
\begin{aligned}
(f * g)(x, y) &= f\left(x' + \frac{i}{2}\hbar\frac{\partial}{\partial y},\quad y' + i\frac{\hbar}{2}\frac{\partial}{\partial x}\right) g(x, y)\bigg|_{x'=x,\, y'=y} \\
&= f(x, y) g\left(x' + i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y},\quad y' + i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x}\right)\bigg|_{x'=x,\, y'=y} .
\end{aligned}
$$

The expression in terms of bidifferential operators is very convenient, when either one of f or g is a polynomial.

An integral formula gives the product by the following expression

$$
(f * g)(x, y) =
$$

$$
\int \tfrac{dx'dy'dx''dy''}{\hbar^2\pi^2} f(x', y') g(x'', y'') \exp\left[-\tfrac{2i}{\hbar}(y(x' - x'') + y'(x'' - x) + y''(x - x'))\right] .
$$

The cyclic expression in the exponential represents twice the area of the phase space triangle with vertices (x, y), (x', y'), (x'', y'').

Summarizing what we have seen up to now, we have:

(1) alternative quantum descriptions at the level of Hilbert spaces (Schrödinger picture) are provided by different Hermitean products on the same set of states, preserved by the dynamical evolution;
(2) alternative quantum descriptions within the Heisenberg picture are provided by different associative products on the space of observables. These products are such that the linear map, associated with the dynamics, becomes an inner derivation.

6 Weyl systems associated with alternative Lagrangian descriptions

In this section we would like to show how to construct Weyl systems out of Lagrangian descriptions.

We consider the tangent bundle TE of an affine space E. A Lagrangian function $\mathcal{L}$ will define a two-form $\omega_{\mathcal{L}} = d\left(\dfrac{\partial\mathcal{L}}{\partial v_j}\right) \wedge dq^j$. We assume that $\omega_{\mathcal{L}}$ is not degenerate.

We now define a vector space structure on TE adapted to $\omega_{\mathcal{L}}$.

We consider vector fields X, Y given by

$$
i_{X_j}\omega_{\mathcal{L}} = -d\left(\frac{\partial\mathcal{L}}{\partial v_j}\right) , \qquad i_{Y_j}\omega_{\mathcal{L}} = dq_j .
$$

It is easy to see that $[X_j, Y_k] = 0$, $[X_j, X_k] = 0$, $[Y_j, Y_k] = 0$, i.e. we have defined on TE the infinitesimal action of an Abelian Lie group.

We now make the assumption that this actually integrates to an action of $\mathbb{R}^{2n}$ ($\dim E = n$) which is free and transitive. By selecting any fiducial point as the origin, we induce a vector space structure on TE adapted to the Lagrangian function $\mathcal{L}$.

We are now able to use our previous construction to associate with $\omega_{\mathcal{L}}$ a Weyl system and a $*$-product.

By defining $E_{\mathcal{L}} = v_j \dfrac{\partial \mathcal{L}}{\partial v_j} - \mathcal{L}$, we may write the "quantum equations of motion" in the form $i\hbar \dfrac{d}{dt} f = E_{\mathcal{L}} *_{\mathcal{L}} f - f *_{\mathcal{L}} E_{\mathcal{L}}$.

We notice that alternative Lagrangians will define alternative $*$-products and alternative realizations of the equations of motion, i.e.

$$E_{\mathcal{L}_1} *_{\mathcal{L}_1} - f *_{\mathcal{L}_1} E_{\mathcal{L}_1} \equiv E_{\mathcal{L}_2} *_{\mathcal{L}_2} f - f *_{\mathcal{L}_2} E_{\mathcal{L}_2} \,.$$

Clearly, the use of von Neumann theorem will require the realization of the Hilbert space $\mathcal{H}$ on different spaces which are Lagrangians with respect to $\omega_{\mathcal{L}_1}$ or $\omega_{\mathcal{L}_2}$. The two Lagrangian subspaces need not to be linearly related.

The spectrum of the vector field will not depend on the specific realization we use.

We shall now formulate the inverse problem for quantum systems independently of the descriptions via the Weyl map.

7 The inverse problem for Quantum Systems

a) The Schrödinger Picture

The carrier space is the space of states. Because of the superposition principle for states, arising from interference phenomena, the space of states is usually required to be a vector space $\mathcal{H}$. The realization of states in terms of wave functions and the interpretation of them as probability amplitudes, justifies the requirement that the carrier space must be identified with an Hilbert space [5]. The equations of motion on this carrier space are given by a linear vector field

$$\Gamma = -\frac{i}{\hbar} H$$

with associated differential equations

$$\frac{d\psi}{dt} = -\frac{i}{\hbar} H\psi \,.$$

The requirement that the evolution should be compatible with the probabilistic interpretation is satisfied by assuming that the associated one-parameter group is a group of unitary transformations. By Stone–von Neumann theorem one gets that H should be an essentially self-adjoint operator in the Hilbert space $\mathcal{H}$.

Remark: The vector field $\Gamma : \mathcal{H} \to T\mathcal{H} \equiv \mathcal{H} \oplus \mathcal{H}$ is more appropriately described by $\Gamma(\psi) = (\psi, \frac{i}{\hbar}H\psi)$, but with abuse of notation we identify it with the second component.

The self-adjoint character of H implies the preservation by Γ of the inner product in the Hilbert space. We may write

$$L_\Gamma\langle\psi \,|\, \varphi\rangle = \langle L_\Gamma\psi \,|\varphi\rangle + \langle\psi \,|\, L_\Gamma\varphi\rangle\,.$$

If we decompose the Hermitean structure associated with the inner product into its real and imaginary part, we obtain $\langle\cdot\,|\,\cdot\rangle = s(\cdot\,,\,\cdot) + i\omega(\cdot\,,\cdot)$, where the real scalar product $s(\cdot\,,\cdot)$ is the real part of $\langle\cdot\,,\,\cdot\rangle$ and the imaginary part $\omega(\cdot\,,\,\cdot)$ defines the symplectic structure on $\mathcal{H}$.

The invariance of the Hermitean structure under the flow of Γ imposes the separate invariance of s and ω. In particular, if we introduce $f_H(\psi) = \frac{1}{2}\langle\psi \,|\, H\psi\rangle$, we have $i_\Gamma\omega = df_H$, therefore quantum evolution requires the vector field to be Hamiltonian.

Thus, the inverse problem for quantum systems, in the Schrödinger picture, is formulated by the equation $L_\Gamma\langle\psi \,|\, \varphi\rangle = \langle L_\Gamma\psi \,|\, \varphi\rangle + \langle\psi \,|\, L_\Gamma\varphi\rangle$, where the unknown is the scalar product $\langle\,|\,\rangle$.

To avoid domain problems and topology problems arising from operators acting on infinite dimensional vector spaces, we shall make few considerations by restricting our analysis to finite level quantum systems, i.e. Hilbert spaces are represented by $\mathbf{C}^n$.

Let us start with a one-level quantum system.
The Schrödinger equation

$$\frac{d}{dt}\psi = -\frac{i}{\hbar}H\psi$$

can be written in real form by using a two-component vector

$$\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{1}{\hbar}\begin{vmatrix}0 & -H\\ H & 0\end{vmatrix}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}.$$

By introducing $\omega = H/\hbar$ our equations become

$$\ddot{\psi}_1 + \omega\psi_1 = 0 \qquad \ddot{\psi}_2 + \omega\psi_2 = 0\,.$$

This situation extends to many dimensions.

As a matter of fact, because the flow is unitary, for any finite level quantum system the vector field is always equivalent (connected by similarity transformations) with the vector field of a "classical harmonic oscillator", with finite number of degrees of freedom. Assuming our equation of motion is given on the carrier space $\mathbf{C}^n$ (or $\mathbb{R}^{2n}$ with a complex structure $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $J^2 = -\mathbf{1}$), say

$$\frac{d}{dt}\psi = -\frac{i}{\hbar}H\psi$$

or

$$\frac{d}{dt}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix} = \begin{vmatrix}0 & -H/\hbar\\ H/\hbar & 0\end{vmatrix}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}$$

we find

$$\ddot{\psi}_1 + \left(\frac{H}{\hbar}\right)^2\psi_1 = 0\,, \qquad \ddot{\psi}_2 + \left(\frac{H}{\hbar}\right)^2\psi_2 = 0\,,$$

where $\psi_1,\, \psi_2 \in \mathbb{R}^n$.

If we keep fixed the complex structure on $\mathbb{R}^{2n}$, the quest for alternative quantum descriptions is now reduced to searching for Euclidean products on $\mathbb{R}^n$, whose representative matrix, say $S = \|S_{ij}\|$, satisfies the relation $SH + H^tS = 0$ with H. We denote it by s_1 and we find on $\mathbb{R}^{2n}$

$$\langle\psi\,|\,\psi\rangle = s_1(\psi_1\,,\,\psi_1) + s_1(\psi_2\,,\,\psi_2)\,,$$

while the symplectic structure is given by

$$\omega_1((\psi_1\,,\,\psi_2),\,(\varphi_1\,,\,\varphi_2)) = s_1(\psi_1\,,\,\varphi_2) - s_1(\varphi_1\,,\,\psi_2)\,.$$

The associated Hermitean structure will be

$$\langle\psi\,|\,\varphi\rangle_1 = s_1(\psi_1\,,\,\varphi_1) + s_1(\psi_2\,,\,\varphi_2) + i(s_1(\psi_1\,,\varphi_2) - s_1(\varphi_1\,,\,\psi_2))\,.$$

It is known that, in finite dimensions, any two Hermitean structures can simultaneously be brought to diagonal form. In particular, this means that we may express one in terms of the other, namely the product

$$(\psi\,|\,\varphi)_A = \langle\psi\,|\,\hat{A}\varphi\rangle$$

defines $(\cdot\,|\,\cdot)_A$ out of the initial metric and a positive matrix $\hat{A}$. For the invariance of this new product with the dynamics we have

$$\begin{aligned}\frac{d}{dt}(\psi\,|\,\varphi)_A &= \left(\frac{d\psi}{dt}\,|\varphi\right)_A + \left(\psi\,|\,\frac{d\varphi}{dt}\right)_A\\ &= \langle\frac{\hat{H}}{\hbar}\psi\,|\,\hat{A}\varphi\rangle + \langle\psi\,|\,\hat{A}\left(-\frac{\hat{H}}{\hbar}\varphi\right)\rangle\\ &= \langle\psi\,|\,[\hat{H}\,,\,\hat{A}]\varphi\rangle = 0\end{aligned}$$

if and only if $[\hat{H}, \hat{A}] = 0$. This observation allows us to find a large family of alternative Hermitean products on the space of states by using appropriate elements in the commutant of $\hat{H}$. With appropriate care, due to domain problems, we may define in this way alternative Hermitean structures also on infinite-dimensional Hilbert spaces.

b) The Heisenberg Picture

In this picture the dynamics takes place on the algebra of observables and is usually written in the form

$$i\hbar \frac{d}{dt} B = B \cdot H - H \cdot B = [B, H] \,.$$

As we have already seen in the section dealing with Weyl systems and $*$-products, the space of observables has a linear structure and a product, in formulating the inverse problem in this picture, we start with a linear map D acting on the space of observables and write an equation of motion in the form

$$i\hbar \frac{d}{dt} A = D(A) \,.$$

The inverse problem now consists of searching for all associative products which turn the linear space into an algebra and make the linear map D into an inner derivation.

If we denote by $\beta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, the associative product in the algebra $\mathcal{A}$, we are imposing that β satisfies the relation $\beta(H, A) - \beta(A, H) = D(A)$. Therefore, given D, the unknowns are β and H, very much like in the inverse problem for the Poisson dynamics.

Remark: Very much like in the Poisson case, we could have searched just for alternative Lie algebras structures on the space of observables with the requirement $[A, B \cdot C] = [A, B] \cdot C + B \cdot [A, C]$. However a theorem by Dirac [5], [9] shows that the Lie algebra structure, compatible with the associative products in the sense of previous formula, is necessarily of the form $[A, B] = \lambda(A \cdot B - B \cdot C)$, therefore in the inverse problem we have to search for alternative associative products on the space of observables.

A general approach to the search for alternative associative products requires the use of the Hochschild cohomology, we refer to [2] for general considerations. Here we only consider a simple instance of alternative associative products.

Given a generic operator k, we define a product out of the starting one by setting

$$A \underset{k}{\cdot} B = A \, e^{\lambda k} B \,.$$

With this new product we have a new Lie bracket

$$[A, B]_k = A \underset{k}{\cdot} B - B \underset{k}{\cdot} A \,,$$

and this is compatible with the dynamics $D(A) = [H, A]$, if, and only if, $[H,\, K] = 0$.

It is interesting to consider the classical limit of this new associative product, this can be done with the help of the Wigner map.

8 The classical limit of some alternative quantum decriptions

By using the correspondence

$$\mathrm{Tr}\, \hat{A}\hat{B}W(v) = (f_A * f_B)(v)$$

we find that for $\hbar \to 0$ we obtain

$$f_A * f_B \to f_A \cdot f_B$$

and

$$\frac{f_A * f_B - f_B * f_A}{\hbar} \to \{f_A, f_B\}\,.$$

We may therefore consider the deformed product

$$\mathrm{Tr}\, AkBW(v) = f_A *_k f_B(v) = (f_A * f_k * f_B)(v)$$

and look for the limit when $\hbar \to 0$.
We find

$$f_A *_k f_B \to f_A \cdot f_k \cdot f_B$$

and

$$\frac{f_A *_k f_B - f_B *_k f_A}{\hbar} \to f_A\{f_k, f_B\} + f_B\{f_A, f_k\} + \{f_A, f_B\}f_k\,.$$

Denoting by X_k the Hamiltonian vector field associated with f_k, we get

$$\{f_A, f_B\}_k = f_A L_{X_k} f_B - f_B L_{X_k} f_A + f_k\{f_A, f_B\}$$

i.e. the limit is a Jacobi Bracket rather than a Poisson Bracket.

What is more relevant, however, is that all these brackets are compatible among them, therefore these deformations of the associative product on the space of operators do not provide us with the alternative Poisson Structures as those arising in the study of complete integrability for bi-Hamiltonian systems.

It is this result which obliges us to look for more general deformations of the associative product of operators [2].

We shall now consider an example, by applying our general considerations to the Harmonic Oscillator. The Schrödinger equation for the one-dimensional oscillator

$$i\hbar\frac{d}{dt}\psi = \hat{H}\psi = \left(\frac{\hat{p}^2}{2m} + kq^2\right)\psi$$

defines a vector field which is self-adjoint with respect to the Hilbert space of square integrable functions on $\mathbb{R}$.

To deform the Hermitean structure on the Hilbert space we may set

$$\langle \psi \mid \varphi \rangle_\lambda = \int (\psi^* e^{-\lambda \hat{H}} \varphi) dx \,.$$

This new product defines a new symplectic structure and therefore a new Poisson Bracket on the Hilbert space, compatible with the dynamical evolution. By using the eigenstates of $\hat{H}$, we set $\mathbf{1} = \sum_n |n\rangle\langle n|$ and get

$$\langle \psi \mid \varphi \rangle = \sum_n \langle \psi \mid e^{-\lambda \hat{H}} \mid n \rangle \langle n \mid \varphi \rangle = \sum_n e^{-\lambda\left(n+\frac{1}{2}\right)} \langle \psi \mid n \rangle \langle n \mid \varphi \rangle \,.$$

Let us consider the Heisenberg picture. We introduce the usual creation and annihilation operators a, $a^\dagger$ obeying the bosonic commutation relations

$$[a, a^\dagger] = \mathbf{1} \,.$$

The number operator

$$\hat{n} = a^\dagger a$$

satisfies the commutation relations

$$[a, \hat{n}] = a \;; \qquad [a^\dagger, n] = -a^\dagger \,.$$

The realization of these elements as operators on some Hilbert space allows to construct the vacuum state $|0\rangle$, which obeys the equation

$$a \mid 0 \rangle = 0 \,.$$

Excited states are built by setting

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

and they are eigenstates of the number operator $\hat{n}$

$$\hat{n} \mid n \rangle = n \mid n \rangle \quad , \qquad n \in \mathbb{Z}^\dagger \,.$$

Equations of motion, for $H = \dfrac{a^\dagger a + a a^\dagger}{2}$, are

$$\dot{a} = i[H, a] = -ia$$

$$\dot{a}^\dagger = -i[H, a^\dagger] = ia^\dagger \,.$$

The vector field on the space of operators is given by

$$\frac{d}{dt}\begin{vmatrix} a \\ \\ a^\dagger \end{vmatrix} = \begin{vmatrix} -i & 0 \\ \\ 0 & +i \end{vmatrix}\begin{vmatrix} a \\ \\ a^\dagger \end{vmatrix}.$$

Now we notice that $A = af(\hat{n})$, $A^\dagger = f(\hat{n})a^\dagger$ satisfy the same equations of motion

$$\frac{d}{dt}\begin{vmatrix} A \\ \\ A^\dagger \end{vmatrix} = \begin{vmatrix} -i & 0 \\ \\ 0 & +i \end{vmatrix}\begin{vmatrix} A \\ \\ A^\dagger \end{vmatrix}.$$

It is obvious that

$$A\,|\,0\rangle = 0\,,$$

therefore we may construct excited states by setting

$$|N\rangle = \frac{(A^\dagger)^n}{\sqrt{n!}}\,|\,0\rangle\,.$$

We may define two different Hermitean structures on $\mathcal{H}$ by using either

$$\langle n\,|\,m\rangle = \delta_{nm} \quad \text{or} \quad (N\,|\,M) = \delta_{N,M}\,.$$

With the new excited states we have constructed a new scalar product. Let us denote the two scalar product by

$$\langle n\,|\,m\rangle = \delta_{n,m} \quad \text{and} \quad (N\,|\,M) = \delta_{n,m}\,.$$

The operators will have the form

$$a = \sum_n |n-1\rangle\sqrt{n}\langle n|$$
$$A = \sum_n |n-1\rangle f(n)\sqrt{n}\langle n|\,.$$

Having two different metrics it is now clear that the notion of adjoint will change, while A and $A^\dagger$ are adjoint of each other in the old metric, the adjoint of $A^\dagger$ in the new constructed scalar product (described by the round bracket) will be the operator $B = a\frac{1}{f(\hat{n})}$.

The commutation relations give

$$[B, A^\dagger] = a\frac{1}{f(\hat{n})}f(\hat{n})a^\dagger - f(\hat{n})a^\dagger a\frac{1}{f(n)} = \mathbf{1}$$

i.e. B and $A^\dagger$ provide a new realization of the Heisenberg algebra and in terms of these variables we have again

$$\frac{d}{dt}\begin{vmatrix} B \\ \\ A^\dagger \end{vmatrix} = \begin{vmatrix} -i & 0 \\ \\ 0 & +i \end{vmatrix}\begin{vmatrix} B \\ \\ A^\dagger \end{vmatrix} .$$

The introduced deformation gives an alternative description in terms of A, $A^\dagger$ with alternative commutation relations $AA^\dagger - A^\dagger A = (n+1)f^2(n+1) - nf^2(n)$ compatible with the dynamical evolution. We may also think of it as taking place on a new realization of the Heisenberg–Weyl algebra acting on the space of states with a new Hilbert space structure.

9 Comments and conclusions

All our analysis on the classical equations of motion seems to imply that all physical aspects of the dynamics are completely captured by the equations of motion, the Lagrangian and the Hamiltonian descriptions seem to be purely instrumental to connect symmetries and conservation laws, or to write in some economical and covariant way interacting terms when considering composite systems. As a matter of fact, this is consistent with the way equations were written out of the experimental evidence, for instance Newton's equations, Maxwell's equations for electrodynamics, Einstein's equations for general relativity, Schrödinger's and Dirac's equations for quantum particles all were written without a Lagrangian or Hamiltonian descriptions, these came only later. A particular instance where the Hamiltonian seems to play a relevant role is in the computation of the free energy via the partition function in statistical mechanics. However, recently it has been shown that in many cases the free energy also turns out to be independent of the particular Hamiltonian and symplectic structure one uses to define the partition function for a given dynamical system [7].

In this paper we have shown that these conclusions seem to apply also to the quantum equations of motion.

A further comment is in order, if there are many commutation relations compatible with a given dynamical evolution, it seems reasonable that, in agreement with the general Einsteinian point of view, the commutation relations should be dynamically determined [4], i.e. we should have some field equations for them very much as we have field equations for the metric tensor on the space time.

Most of the material presented here has been obtained by the author in several collaborations, it may be useful to list some of them [12].

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Parallel transport and decoupling

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Abstract

The problem of decoupling second-order differential equations by coordinate transformations is studied in terms of parallel distributions with respect to the linear connection associated to that differential equation.

1 Introduction

One of the most fruitful tools in the analysis of qualitative properties of system of second-order differential equations (SODE) during the last decades has been the nonlinear connection [2] associated to such SODE and the corresponding linear connection [5], also known as Berwald connection. For instance, it is very useful in the analysis of the Inverse Problem of Lagrangian Mechanics [11], it is fundamental to characterize those SODEs which can be transformed to linear SODEs by a coordinate transformation [9,5] and also to characterize those systems of second-order differential equations which decouples under a coordinate transformation in two subsystems of independent second-order differential equations [8,1], to mention a few problems in which it has been successfully applied.

In [6] Crampin provides a geometrical interpretation of the parallel transport map associated to the linear connection D along horizontal and vertical curves, which moreover characterizes such connection. The interpretation of the parallel transport along integral curves of a projectable horizontal vectorfield is given in terms of Lie transport under the flow of this vectorfield. This fact strongly suggests that a parallel distribution must satisfy some property of invariance under some flows, and therefore there should be a connection with the integrability of such distribution.

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The aim of this paper is to explore this relation and to apply this results to the problem of decoupling of second-order differential equations, providing a simplified alternative derivation of the results in [1]. In this sense many of the results in this paper are already in the literature, and its merit is mainly to provide simpler and more readable proofs of such results.

The paper is organized as follows. In section 2 the notation and preliminary results that will be needed in the rest of the paper are introduced. In section 3 we recall the construction of the linear connection associated to the nonlinear connection defined by a SODE. In section 4 we recall Crampin's interpretation of parallel transport for such connection, and in section 5 we show that a parallel subbundle is necessarily integrable. In section 6 we characterize those SODEs which can be decoupled by a coordinate change and finally in section 7 the case of a complete decoupling into scalar second-order differential equations is analyzed.

2 Preliminaries

Let $\pi\colon E \to \mathbb{R}$ be a fibre bundle with fibre dimension n, and let $\pi_1\colon J^1\pi \to \mathbb{R}$ be its first jet bundle. The vertical bundle with respect to the bundle projection π is denoted by $\mathrm{Ver}(\pi)$, whereas the vertical bundle with respect to the projection $\pi_{10}\colon J^1\pi \to E$ will be denoted by $\mathrm{Ver}(\pi_{10})$, i.e. $\mathrm{Ver}(\pi) = \mathrm{Ker}(T\pi)$ and $\mathrm{Ver}(\pi_{10}) = \mathrm{Ker}(T\pi_{10})$.

We consider the canonical coordinate t on $\mathbb{R}$ and natural bundle coordinates (t, x^i) on E and (t, x^i, v^i) on $J^1\pi$. Any time-preserving coordinate transformation $(t, x^i) \to (t, \bar{x}^i)$, where $\bar{x}^i = \bar{x}^i(t, x)$ leads to the following formulas for the coordinate transformation on $J^1\pi$,

$$t = t, \qquad \bar{x}^i = \bar{x}^i(t, x), \qquad \bar{v}^i = \frac{\partial \bar{x}^i}{\partial t} + \frac{\partial \bar{x}^i}{\partial x^j} v^j,$$

from where we can clearly see the affine character of the bundle $J^1\pi$, whose associated vector bundle is $\mathrm{Ver}(\pi)$. The fibre over an element $m \in E$ can be considered as an affine hyperplane of the tangent space at m, and therefore we have the following sequence of vector spaces $0 \to \mathrm{Ver}(\pi)_m \to T_mE \to \mathbb{R} \to 0$, where the map in the right consists in taking the t-component $v \mapsto \langle dt, v\rangle$. Elements of $J^1\pi$ can be identified with tangent vectors to E which projects onto $\partial/\partial t$. This identification may be regarded as defining a map $\boldsymbol{T}\colon J^1\pi \to TE$, given by $\boldsymbol{T}(j^1_t\gamma) = \dot{\gamma}(t)$. We therefore have the following commutative diagram of vector bundles over E, where the row is an exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ver}(\pi) \arrow[r, "\mathbf{i}"] & TE \arrow[r, "j"] & E \times \mathbb{R} \arrow[r] & 0. \\ & & J^1\pi \arrow[u, "\boldsymbol{T}"] \arrow[ur, "(\pi_{10},1)"'] & & \end{tikzcd}$$

A section of $\pi_{10}^*(TE)$ is said to be a vectorfield along π_{10}. Alternatively, it can be considered as a map $X\colon J^1\pi \to TE$ such that $\tau_E \circ X = \pi_{10}$, that is $X(p) \in T_m E$ for every $p \in (J^1\pi)_m$. The section associated to this map is just $p \mapsto (p, X(p))$. For instance, the map $\boldsymbol{T}$ above can be considered in a natural way as a vector field along π_{10}, called the total time derivative, and which in coordinates reads

$$\boldsymbol{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}.$$

Any vectorfield Y on E gives rise to a vectorfield along π_{10} by composition with the projection π_{10}. Explicitly the associated section of the pullback bundle is $p \mapsto (p, Y(m))$ where $m = \pi_{10}(p)$. The vectorfields along π_{10} which arise in this way are called basic.

A SODE on E is a vectorfield $\Gamma \in \mathfrak{X}(J^1\pi)$ which projects onto $\boldsymbol{T}$. In coordinates it is of the form

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial v^i}$$

where $f^i = f^i(t, x^j, v^j)$. Therefore the system of differential equations for the integral curves of Γ is the non-autonomous second-order system of differential equations, written as a first order system,

$$\dot{x}^i = v^i \qquad \dot{v}^i = f^i(t, x, v).$$

or in second-order form $\ddot{x}^i = f^i(t, x, \dot{x})$.

After a time-dependent coordinate transformation the coordinate expression for Γ becomes

$$\Gamma = \frac{\partial}{\partial t} + \bar{v}^i \frac{\partial}{\partial \bar{y}^i} + \bar{f}^i \frac{\partial}{\partial \bar{v}^i}$$

where

$$\bar{f}^i = \frac{\partial \bar{x}^i}{\partial x^j} f^j + \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} v^j v^k + 2 \frac{\partial^2 \bar{x}^i}{\partial x^j \partial t} v^j + \frac{\partial^2 \bar{x}^i}{\partial t^2}.$$

a formula which remind us how the fiber coordinates v^i enter into the transformation law.

The problem we are faced to is to find a coordinate transformation such that the system of second-order differential equations is transformed to the so-called submersive form

$$\begin{cases} \ddot{x}^i = f^i(t, x^j, \dot{x}^j) & i, j = 1, \ldots, d \\ \ddot{x}^A = f^A(t, x^j, x^B, \dot{x}^j, \dot{x}^B) & A, B = d+1, \ldots, n \end{cases}$$

or to a separated or decoupled form

$$\begin{cases} \ddot{x}^i = f^i(t, x^j, \dot{x}^j) & i, j = 1, \dots, d \\ \ddot{x}^A = f^A(t, x^B, \dot{x}^B) & A, B = d+1, \dots, n. \end{cases}$$

3 The linear connection associated to a sode

The fibration $\pi_{10}\colon J^1\pi \to E$ determines an exact sequence of vector bundles over $J^1\pi$

$$0 \longrightarrow \pi_{10}^*\mathrm{Ver}(\pi) \xrightarrow{\xi^V} T(J^1\pi) \xrightarrow{j} \pi_{10}^*(TE) \longrightarrow 0\,,$$

where ξ^V is the vertical lifting map (given by $\xi^V(p, v) = \frac{d}{dt}|_{t=0}(p + tv))$ and j is the projection $j(V_p) = (p, T\pi_{10}(V_p))$.

A (nonlinear) connection on $J^1\pi$ is a splitting of such sequence, that is, a linear bundle map $h\colon \pi_{10}^*(TE) \to T(J^1\pi)$ such that $j \circ h = \mathrm{id}$. In other words, it is equivalent to a vector subbundle $\mathrm{Hor}(\pi_{10})$ of $T(J^1\pi)$ which is complementary to the vertical subbundle $\mathrm{Ver}(\pi_{10})$, and is therefore called horizontal.

A sode Γ on $J^1\pi$ determines a connection on $J^1\pi$. The projector onto the horizontal bundle is given in terms of the vertical endomorphism $S = (dx^i - v^i dt) \otimes (\partial/\partial v^i)$ by the formula

$$P_H = \frac{1}{2}\,(I + \Gamma \otimes dt - \mathcal{L}_\Gamma S)\,.$$

The coordinate expressions for a basis $\{H_0, H_1, \dots, H_n\}$ of horizontal vector fields are

$$H_0 = \frac{\partial}{\partial t} - \Gamma_0^j \frac{\partial}{\partial v^j} \qquad \text{and} \qquad H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial v^j}$$

where

$$\Gamma_0^j = -f^i + \frac{1}{2} v^j \frac{\partial f^i}{\partial v^j} \qquad \text{and} \qquad \Gamma_i^j = -\frac{1}{2}\frac{\partial f^j}{\partial v^i}.$$

Note that, in particular, the sode Γ is itself horizontal; it is the horizontal lift of the canonical vectorfield $\boldsymbol{T}$.

By a kind of linearization of the given nonlinear connection, we can define a linear connection on the pullback bundle $\mathrm{pr}_1\colon \pi_{10}^* TE \to J^1\pi$. The associated covariant derivative is given by

$$D_Z X = \kappa([P_H Z, Y^V]) + j([P_V Z, Y^H]) + P_H(Z)\langle Y, dt\rangle \boldsymbol{T},$$

where Z is a vectorfield on $J^1\pi$, Y is a vectorfield along π_{10} (i.e. a section of $\pi_{10}^* TE$), P_H and P_V are the horizontal and vertical projectors of the nonlinear connection, Y^H and Y^V are the horizontal and vertical lifting of Y, and κ is the connection map $\kappa\colon T(J^1\pi) \to \pi_{10}^*(\mathrm{Ver}(\pi))$, defined by the relation $\xi^V \circ \kappa = P_V$. See [5,4] for the details.

The curvature of this connection was also studied in [5]. In particular, most of the components of the curvature tensor are determined in terms of the so called Jacobi endomorphism, Φ defined by

$$\Phi(X) = R(\boldsymbol{T}, X),$$

where R is the curvature of the nonlinear connection. Its coordinate expression is $\Phi = \Phi^i_j\,(dx^j - v^j dt) \otimes \frac{\partial}{\partial q^i}$ with

$$\Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^i_k\Gamma^k_j - \Gamma(\Gamma^i_j).$$

We remark that in general, for any connection on $J^1\pi$, the horizontal lift of $\boldsymbol{T}$ is a SODE on $J^1\pi$, but it is to be noticed that the connection defined by this SODE is not the original one. This is the case only when the connection is a SODE-connection as above. Notice also that not every connection is the SODE-connection for some SODE.

Proposition 1 *A linear connection on $\pi_{10}^*(TE)$ is the linear connection defined by* SODE *if and only if it is torsionless, i.e.*

$$D_{X^H}Y - D_{Y^H}X = [X,Y]$$

for every pair of basic vectorfields $X, Y \in \mathfrak{X}(E)$.

The torsionless condition can be equivalently written in the form $[X^H, Y^V] - [Y^H, X^V] = [X,Y]^V$ for every pair of basic vectorfields X, Y.

In the local base $\{\Gamma, H_i, V_i\}$ of vectorfields in $J^1\pi$, where $V_i = \partial/\partial v^i$, and the local base $\{\boldsymbol{T}, \partial/\partial x^i\}$ of vectofields along π_{10} the linear connection is determined by

$$D_{V_i}\left(\frac{\partial}{\partial x^j}\right) = 0, \qquad D_{V_i}\boldsymbol{T} = \frac{\partial}{\partial x^i},$$

$$D_{H_i}\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial \Gamma^k_i}{\partial v^j}\frac{\partial}{\partial x^k}, \qquad D_{H_i}\boldsymbol{T} = 0,$$

$$D_\Gamma\left(\frac{\partial}{\partial x^j}\right) = \Gamma^k_j\frac{\partial}{\partial x^k}, \qquad D_\Gamma\boldsymbol{T} = 0.$$

In particular, from this expressions it is clear that D restricts to a connection on $\pi_{10}^*(\mathrm{Ver}(\pi))$: if Y is vertical over $\mathbb{R}$, then the expression of the covariant derivative simplifies to

$$D_Z X = \kappa([P_H Z, Y^V]) + j([P_V Z, Y^H]).$$

In what follows, to simplify as much as possible, we will consider the restriction of the connection to $\pi_{10}^*(\mathrm{Ver}(\pi))$. Therefore, we will consider only vertical vectorfields along π_{10}.

4 Parallel transport

The parallel transport map associated to the linear connection defined above was interpreted in the case of the autonomous formalism by Crampin in [6], and extended to the non-autonomous case in [10]. In this section this interpretation is presented (in a slightly different way).

Let X be a vector field on E and $X^H \in \mathfrak{X}(J^1\pi)$ its horizontal lift with respect to the given connection. Denote by φ_t the flow of X and by ϕ_t the flow of X^H. When the connection is linear, then ϕ_t is a linear map in the tangent bundle TE, which is but the parallel transport map along the integral curves of X. When the connection is nonlinear, we can linearize the flow obtaining a linear bundle map $\Psi_t\colon \pi_{10}^*(\mathrm{Ver}(\pi)) \to \pi_{10}^*(\mathrm{Ver}(\pi))$ over ϕ_t as follows. We take $p \in J^1\pi$ and $v \in \mathrm{Ver}(\pi)$, over the same point $m \in E$, so that $(p, v) \in \pi_{10}^*(\mathrm{Ver}(\pi))$. Then we consider the linearization $z = \frac{d}{ds}\phi_t(p+sv)\Big|_{s=0}$. This is a vector at the point $\phi_t(p)$ which is vertical, since X^H is projectable and thus $\pi_{10} \circ \phi_t = \varphi_t \circ \pi_{10}$. Therefore, there exists a vector $P_t(p, v) \in \mathrm{Ver}(\pi)_{\varphi_s(m)}$ whose vertical lifting to the point $\phi_t(p)$ is the above vector z, i.e. $z = \xi^V(\phi_t(p), P_t(p, v))$. Then we have the map $\Psi_t(p, v) = (\phi_t(p), P_t(p, v))$, which is the parallel transport map along the flow of X^H (and hence along horizontal curves).

In other words, if we consider the inverse $\kappa\colon \mathrm{Ver}(\pi_{10}) \to \pi_{10}^*(\mathrm{Ver}(\pi))$ of the vertical isomorphism $\xi^V\colon \pi_{10}^*(\mathrm{Ver}(\pi)) \to \mathrm{Ver}(\pi_{10}) \subset T(J^1\pi)$, then the map Ψ_t is defined by

$$\Psi_t = \kappa \circ T\phi_t \circ \xi^V.$$

Therefore, it follows that for every vectorfield $Y \in \mathfrak{X}(E)$ vertical over $\mathbb{R}$, we have

$$D_{X^H}Y = \frac{d}{dt}(\Psi_t \circ Y \circ \phi_t)\Big|_{t=0},$$

which is equivalent to the relation $D_{X^H}Y = \kappa([X^H, Y^V])$.

For parallel transport along vertical curves we can prescribe a complete parallelism rule as follows. Let $\gamma\colon [a, b] \to J^1\pi$ be a curve in the fibre $\pi_{10}^{-1}(m)$, i.e. $\pi_{10}(\gamma(t)) = m$, and let $p_i = \gamma(a)$ and $p_f = \gamma(b)$ the endpoints of the curve. We take an element of our bundle $z \in \pi_{10}^*(\mathrm{Ver}(\pi))$ at the initial point p_i, that is $z = (p_i, v)$ for some $v \in \mathrm{Ver}(\pi)_m$. Then the parallel transport of z from p_i to p_f along the curve γ is $P^\gamma_{p_i,p_f}(p_i, v) = (p_f, v)$. It is clear that parallel transport is independent of the curve γ that joins the point p_i and p_f as long as this curve is vertical, and it is in this sense that we speak about complete parallelism.

5 Parallel distributions are integrable

The relation between parallel transport and Lie transport that we have seen in the last section, suggests that there must be some relation between the properties for a distribution of being parallel and being invariant under Lie transport by the flows in the distribution, which is known to be equivalent to the integrability of the distribution. In the case we are considering (the case of the SODE-connection) with the help of the torsion-free condition we can see that parallel distributions along π_{10} are integrable. The precise meaning of this terminology is as follows.

By a distribution along π_{10} we mean a subbundle $\mathcal{D}$ of $\pi_{10}^*(TE)$. A distribution along π_{10} is said to be basic if there exists a distribution $\mathcal{E}$ on E such that $\mathcal{D} = \mathcal{E} \circ \pi_{10}$, that is, $\mathcal{D}_v = \mathcal{E}_{\pi_{10}(v)}$ for every $v \in J^1\pi$. A distribution $\mathcal{D}$ along π_{10} is said to be integrable if it is basic $\mathcal{D} = \mathcal{E} \circ \pi_{10}$ and the distribution $\mathcal{E}$ is integrable.

On the other hand we will say that a distribution $\mathcal{D}$ along π_{10} is parallel if it is invariant under parallel transport $P^\gamma \mathcal{D} \subset \mathcal{D}$ for every curve γ in $J^1\pi$. It is easy to see that $\mathcal{D}$ is parallel if and only if it is invariant by covariant differentiation, i.e. $D_W \mathcal{D} \subset \mathcal{D}$ for all $W \in \mathfrak{X}(J^1\pi)$.

As we mentioned before, we only consider the subbundle $\pi_{10}^*(\mathrm{Ver}(\pi)) \subset \pi_{10}^*(TE)$, and therefore in the rest of the paper we consider only $\mathbb{R}$-vertical distributions.

Theorem 2 *If a distribution along π_{10} is parallel then it is integrable.*

Proof If $\mathcal{D}$ is invariant by parallel transport over vertical curves then it is a basic distribution. Indeed, parallel transport along vertical curves is $(p_i, z) \mapsto (p_f, z)$, and therefore, a distribution $\mathcal{D}$ is parallel along vertical curves iff $\mathcal{D}_p$ depends only on the point $m = \pi_{10}(p)$. Defining $\mathcal{E}_m = \mathcal{D}_{0_m}$ then we have that $\mathcal{D}_p = \mathcal{E}_m$.

Moreover, since the connection is torsionless we have that $D_{X^H}Y - D_{Y^H}X = [X, Y]$ for all $X, Y \in \mathfrak{X}(E)$. Therefore if X, Y are vectorfields in the distribution $\mathcal{E}$ we have that $D_{X^H}Y$ and $D_{X^H}Y$ are in $\mathcal{E}$, and hence $[X, Y]$ is also in $\mathcal{E}$. Thus $\mathcal{E}$ is involutive and therefore integrable. $\square$

It is to be noticed that from the properties of the linear connection D, one can see that it is enough to impose the invariance condition $D_Z\mathcal{D} \subset \mathcal{D}$ for π_{10}-vertical vector fields Z and for $Z = \Gamma$, being the invariance under D_Z for Z a horizontal vectorfield a consequence of those.

It would be nice to understand in more detail the implications of the torsionfree condition in what respect to parallel transport along horizontal curves.

6 Submersive sodes

In this section we will use the results in the last section in order to characterize those systems of second-order differential equations that can be decoupled. The following is an adaptation to the non-autonomous case of the definition of submersive SODE given in [7] for the time independent case.

Definition 3 *A* SODE $\Gamma \in \mathfrak{X}(J^1\pi)$ *is submersive if there exists a bundle* $\bar{\pi}\colon \bar{E} \to \mathbb{R}$*, a submersion* $\varphi\colon E \to \bar{E}$ *over the identity in* $\mathbb{R}$*, and a* SODE $\bar{\Gamma}$ *on* $J^1\bar{\pi}$ *such that* Γ *and* $\bar{\Gamma}$ *are* $J^1\varphi$*-related, i.e.* $T(J^1\varphi) \circ \Gamma = \bar{\Gamma} \circ J^1\varphi$*.*

We say that a SODE Γ *is locally submersive at a point* $m \in E$ *if there is an open neighbourhood* $U \subset E$ *of* m *fibred over the real line* $\pi_U\colon U \to \mathbb{R}$*, such that the restriction of* Γ *to* $J^1\pi_U$ *is submersive.*

We will only consider the local problem, i.e. the word 'submersive' must be understood as 'locally submersive'.

We can take coordinates adapted to the submersion φ, i.e. (t, x^i) on $\bar{E}$ and (t, x^i, x^A) on E, with $i = 1, \ldots, k$, $A = k+1, \ldots, n$, such that the coordinate expression of φ is $\varphi(t, x^i, x^A) = (t, x^i)$. Then the SODE Γ with forces $f^i(t, x^j, x^B, v^j, v^B)$ and $f^A(t, x^j, x^B, v^j, v^B)$ is submersive iff the coefficients f^i depends only of the coordinates (t, x^i, v^i) and does not depend on (x^B, v^B). It follows that the differential equations for the integral curves can be written as

$$\ddot{x}^i = f^i(t, x^j, v^j) \qquad \ddot{x}^A = f^A(t, x^j, x^A, v^j, v^A),$$

i.e. the evolution of the x^i coordinates is independent of the evolution of the coordinates x^A, v^A.

Theorem 4 *A* SODE Γ *is submersive if and only if there exists a distribution* $\mathcal{D}$ *along* π_{10} *which is parallel and* Φ*-invariant, that is*

- $D_W\mathcal{D} \subset \mathcal{D}$ *for all* $W \in \mathfrak{X}(J^1\pi)$*, and*
- $\Phi(\mathcal{D}) \subset \mathcal{D}$*.*

Proof ($\Rightarrow$) Since $\mathcal{D}$ is parallel we have that it is involutive. Let (t, x^i, x^A), $i = 1, \ldots, d, A = d+1, \ldots, n$ a system of local coordinates, such that $\mathcal{D} = \operatorname{span}\{\partial/\partial x^A | A = d+1, \ldots, n\}$. Moreover, since $D_\Gamma \mathcal{D} \subset \mathcal{D}$ we have that $D_\Gamma \frac{\partial}{\partial x^A} = \Gamma^i_A \frac{\partial}{\partial x^i} + \Gamma^B_A \frac{\partial}{\partial x^B} \in \mathcal{D}$, from where it follows that $\Gamma^i_A = 0$. Thus f^i does not depend on the coordinates v^A, i.e. $f^i = f^i(t, x^j, x^A, v^j)$.

Moreover, $\mathcal{D}$ is also Φ-invariant, so that $\Phi(\frac{\partial}{\partial x^A}) = \Phi^i_A \frac{\partial}{\partial x^i} + \Phi^B_A \frac{\partial}{\partial x^B} \in \mathcal{D}$, form where we have $\Phi^i_A = 0$. From the local expression of Φ and taking into account that $\Gamma^i_A = 0$ we get

$$\Phi^i_A = -\frac{\partial f^i}{\partial x^A} - \Gamma^i_j \Gamma^j_A - \Gamma^i_B \Gamma^B_A - \Gamma(\Gamma^i_A) = -\frac{\partial f^i}{\partial x^A} = 0.$$

Thus f^i does not depend on x^A, v^A, i.e. $f^i = f^i(t, x^j, v^j)$. We conclude that the first d equations decouple from the others.

($\Leftarrow$) If in some coordinate system (t, x^i, x^A) the SODE is submersive, then $f^i = f^i(t, x^j, v^j)$ from where we have that $\frac{\partial f^i}{\partial v^A} = 0$ and $\frac{\partial f^i}{\partial x^A} = 0$. It follows that in this coordinates $\Gamma^i_A = 0$ and $\Phi^i_A = 0$, from where it is clear that the distribution $\mathcal{D} = \langle \frac{\partial}{\partial x^A} \rangle$ is parallel and Φ-invariant. □

Definition 5 *We say that a* SODE $\Gamma \in \mathfrak{X}(J^1\pi)$ *decouples if there exist two bundles* $\pi_i \colon E_i \to \mathbb{R}$, $i = 1, 2$, *a diffeomorphism* $\varphi \colon E \to E_1 \times_{\mathbb{R}} E_2$ *over the identity in* $\mathbb{R}$ *and two* SODE*s* Γ_i *on* $J^1\pi_i$, $i = 1, 2$ *such that* Γ *and* (Γ_1, Γ_2) *are* $J^1\varphi$*-related.*

We say that the SODE Γ *locally decouples at a point* $m \in E$ *if there is an open neighbourhood* $U \subset E$ *of* m *fibred over the real line* $\pi_U \colon U \to \mathbb{R}$, *such that the restriction of* Γ *to* $J^1\pi_U$ *decouples.*

It is clear that Γ decouples if and only if it is submersive with respect to two complementary subbundles E_1 and E_2, where by complementary we mean that $E_1 \times_{\mathbb{R}} E_2$ is (diffeomorphic to) E. Therefore, in adapted coordinates (t, x^i, x^A), we have that the differential equations for the integral curves of Γ are the union of two separate subsystem of second-order differential equations:

$$\ddot{x}^i = f^i(t, x^j, v^j) \qquad \ddot{x}^A = f^A(t, x^A, v^A),$$

¿From the above observation and the results in this section we immediately get the following result, where as above the word ‘decouples’ must be understood as ‘locally decouples’.

Theorem 6 *Let* $\mathcal{D}_1, \mathcal{D}_2$ *be two complementary distributions, in the sense that,* $\pi^*_{10}(\mathrm{Ver}(\pi)) = \mathcal{D}_1 \oplus \mathcal{D}_2$, *with dimension* d_1 *and* d_2, *respectively (and* $d_1 + d_2 = n$*). If both distributions are parallel and* Φ*-invariant, then the second-order differential equation decouples in two subsystems of dimension* d_1 *and* d_2*.*

And iterating the above process

Theorem 7 *Let* $\mathcal{D}_1, \ldots, \mathcal{D}_r$ *be complementary distributions* $\pi^*_{10}(\mathrm{Ver}(\pi)) = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r$, *of dimension* $d_1, \ldots, d_r$, *respectively, which are parallel and* Φ*-invariant. Then* Γ *decouples in* r *subsystems of dimension* $d_1, \ldots, d_r$*.*

Note that since every distribution $\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_i$ is parallel (being a sum of parallel distributions) then it is integrable, and therefore it is possible to find coordinates simultaneously adapted to all the distributions.

7 Complete decoupling

Taking into account the results in the above section, it is easy to characterize those systems which completely decouples in n scalar second-order equations, as we did in [1]. In the case of a SODE such that Φ is diagonalizable with n different eigenvalues we have the following simple result, where the bracket of (1,1) tensor fields is just the commutator bracket $[A, B] = A \circ B - B \circ A$.

Theorem 8 *Let Γ be a* SODE *such that Φ is diagonalizable with n different eigenvalues. Then Γ completely decouples if and only if $[D_W\Phi, \Phi] = 0$.*

Proof If we denote by $\mathcal{D}_i$ the i-th eigendistribution of Φ, we obviously have that they are Φ-invariant, and since Φ is diagonalizable we have that they are complementary. Moreover, the condition given above implies that if Z is an eigenvector with eigenvalue λ, then

$$[D_W\Phi, \Phi](Z) = (\lambda - \Phi)^2 D_W Z = 0.$$

Therefore $D_W Z$ is in the same eigendistribution, and it follows that the eigendistributions are parallel.

Conversely, if the system completely decouples, then Φ is a diagonal sum of the Φ of each individual SODE, and the off-diagonal connection coefficients Γ^i_j vanish. Therefore, $[D_W\Phi, \Phi]$ is a diagonal sum of 1-dimensional commutators, and hence vanishes. $\square$

It is to be noticed that in the above theorem we have really proved that if Φ is diagonalizable then the system decouples into subsystems accordingly to the eigendistributions, whether these are one dimensional or not. In the case of degenerated eigenvalues one has to impose additional conditions, as it was done in [1].

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The Legendre transformation in mechanics: a gauge–theoretical approach

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Abstract

A gauge-invariant formulation of the Legendre transformation in mechanics, extending to arbitrary non-autonomous systems the symplectic approach of W.M. Tulczyjew is presented.

1 Introduction

In recent papers [8,11,10] a new geometrical framework for analytical mechanics automatically embodying the gauge invariance of the theory under arbitrary transformations $L \to L + \frac{df}{dt}$ of the Lagrangian has been proposed. The construction relies on the introduction of a principal fiber bundle $\pi : P \to \mathcal{V}_{n+1}$ over the configuration space–time $\mathcal{V}_{n+1}$, with structural group $(\mathbb{R}, +)$, referred to as the bundle of *affine scalars*.

A formulation of the Legendre transformation, extending to the newer context the symplectic approach originally developed by Tulczyjew in time-independent mechanics [14–18] was subsequently worked out in [9].

In this paper, the results obtained so far are briefly reviewed, mainly in connection with the gauge–theoretical aspects underlying the construction of the Legendre transformation.

The foundations of the method are dealt with in § 2. These include a revisitation of the Lagrangian and Hamiltonian bundles, as well as a motivation for their introduction.

In the subsequent analysis, in § 3, a diffeomorphism between three higher jet extensions of the Lagrangian and Hamiltonian bundles is established. The argument, extending to the newer context the so called *Tulczyjew triple*

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$T^*(T^*(M)) \simeq T(T^*(M)) \simeq T^*(T(M))$, provides a convenient geometrical setup for the study of the Legendre transformation. The dynamical implications of the scheme are discussed.

2 Geometrical preliminaries

2.1 The Lagrangian bundles

For later use, we present here a brief review of the gauge–invariant formulation of Lagrangian mechanics developed in [8], with special emphasis on the basic definitions and concepts.

To every a mechanical system subject to (smooth) positional constrains we associate a double fibration $P \xrightarrow{\pi} \mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, where:

i) $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ is the configuration space-time of the system;

ii) $P \xrightarrow{\pi} \mathcal{V}_{n+1}$ is a principal fiber bundle, with structural group $(\mathbb{R}, +)$, called the bundle of *affine scalars* over $\mathcal{V}_{n+1}$.

The action of $(\mathbb{R}, +)$ on P results into a 1-parameter group of diffeomorphisms $\psi_\xi : P \to P$, conventionally expressed through the additive notation

$$\psi_\xi(\nu) := \nu + \xi \qquad \forall\, \xi \in \mathbb{R}\,,\ \nu \in P \tag{1}$$

The bundle P is diffeomorphic — in a non canonical way — to the cartesian product $\mathcal{V}_{n+1} \times \mathbb{R}$. Every function $u : P \to \mathbb{R}$ satisfying

$$u(\nu + \xi) = u(\nu) + \xi$$

is called a (global) trivialization of P. If u, u' is any pair of trivializations, the difference $u - u'$ is (the pull-back of) a function over $\mathcal{V}_{n+1}$.

The assignment of a trivialization u allows to lift every local coordinate system $t, q^1, \ldots, q^n$ over $\mathcal{V}_{n+1}$ to a corresponding "fibered" coordinate system $t, q^1, \ldots, q^n, u$ over P. The group of fibered coordinate transformations has the form

$$\bar{t} = t + c\,, \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n)\,, \qquad \bar{u} = u + f(t, q^1, \ldots, q^n)$$

In particular, in any fibered coordinate system, the generator of the group action (1), usually referred to as the *fundamental vector field* of P, coincides with the field $\frac{\partial}{\partial u}$.

The (pull-back of the) absolute time function determines a fibration $P \xrightarrow{t} \mathbb{R}$. The associated first-jet space will be indicated by $j_1(P, \mathbb{R}) \xrightarrow{\pi} P$. As usual, we shall refer $j_1(P,\mathbb{R})$ to local jet-coordinates $t, q^i, u, \dot q^i, \dot u$, with transformation laws

$$\bar t = t + c\ , \qquad \bar q^{\,i} = \bar q^{\,i}(t,q)\ , \qquad \bar u = u + f(t,q) \tag{2a}$$

$$\dot{\bar q}^{i} = \frac{\partial \bar q^{i}}{\partial q^k}\,\dot q^k + \frac{\partial \bar q^{i}}{\partial t}\ , \qquad \dot{\bar u} = \dot u + \frac{\partial f}{\partial q^k}\,\dot q^k + \frac{\partial f}{\partial t} := \dot u + \dot f \tag{2b}$$

The manifold $j_1(P,\mathbb{R})$ is naturally embedded into the tangent space $T(P)$. In local coordinates, this results into the identification

$$z \in j_1(P,\mathbb{R}) \quad \Leftrightarrow \quad z = \left[\frac{\partial}{\partial t} + \dot q^i(z)\,\frac{\partial}{\partial q^i} + \dot u(z)\,\frac{\partial}{\partial u}\right]_{\pi(z)} \tag{3}$$

The geometrical properties of $j_1(P,\mathbb{R})$ include, in the first place, all attributes coming from the jet-bundle structure (contact bundle, fundamental tensor, fiber differential, etc.). These will be regarded as known [3,12,6,7,4,13]. For the notation, terminology, etc. the reader is referred to [11] and references therein.
A local basis for the contact bundle $C(j_1(P,\mathbb{R}))$ is provided by the 1-forms

$$\omega^0 := du - \dot u\, dt\,, \qquad \omega^k := dq^k - \dot q^k\, dt\,, \qquad k = 1,\ldots,n \tag{4}$$

In terms of these, the fundamental tensor J of $j_1(P,\mathbb{R})$ and the fiber differential of an arbitrary function g over $j_1(P,\mathbb{R})$ are respectively expressed in components as

$$J = \frac{\partial}{\partial \dot u}\otimes\omega^0 + \frac{\partial}{\partial \dot q^k}\otimes\omega^k\ , \qquad d_v\, g = \frac{\partial g}{\partial \dot u}\,\omega^0 + \frac{\partial g}{\partial \dot q^k}\,\omega^k \tag{5}$$

In addition to the jet attributes, the space $j_1(P,\mathbb{R})$ carries two distinguished actions of the group $(\mathbb{R},+)$, both arising from the principal bundle structure of P, and related in a straightforward way to the identification (3).
The first one is simply the push-forward of the action (1), restricted to the submanifold $j_1(P,\mathbb{R}) \subset T(P)$. In jet coordinates, a comparison with eq. (3) provides the local representation

$$\psi_{\xi *}(z) = \left[\frac{\partial}{\partial t} + \dot q^i(z)\,\frac{\partial}{\partial q^i} + \dot u(z)\,\frac{\partial}{\partial u}\right]_{\pi(z)+\xi} \tag{6a}$$

expressed symbolically as

$$\psi_{\xi *} : (t, q^i, u, \dot q^i, \dot u) \ \longrightarrow\ (t, q^i, u + \xi, \dot q^i, \dot u) \tag{6b}$$

The quotient of $j_1(P,\mathbb{R})$ by this action is a $(2n+2)$–dimensional manifold, henceforth denoted by $\mathcal{L}(\mathcal{V}_{n+1})$. As shown in [8], the quotient map makes

$j_1(P,\mathbb{R})$ into a principal fiber bundle over $\mathcal{L}(\mathcal{V}_{n+1})$, with structural group $(\mathbb{R},+)$. By eq. (6b) it is also clear that $\mathcal{L}(\mathcal{V}_{n+1})$ is an affine bundle over $\mathcal{V}_{n+1}$, with coordinates $t, q^i, \dot{q}^i, \dot{u}$.

The second action of $(\mathbb{R},+)$ on $j_1(P,\mathbb{R})$ comes from the invariant character of the field $\frac{\partial}{\partial u}$, and is given by the addition

$$\phi_\xi(z) := z + \xi \left(\frac{\partial}{\partial u}\right)_{\pi(z)} = \left[\frac{\partial}{\partial t} + \dot{q}^i(z)\,\frac{\partial}{\partial q^i} + (\dot{u}(z)+\xi)\,\frac{\partial}{\partial u}\right]_{\pi(z)} \tag{7a}$$

or, more synthetically

$$\phi_\xi : (t, q^i, u, \dot{q}^i, \dot{u}) \;\longrightarrow\; (t, q^i, u, \dot{q}^i, \dot{u}+\xi) \tag{7b}$$

The quotient of $j_1(P,\mathbb{R})$ by this action is again a $(2n+2)$–dimensional manifold, henceforth denoted by $\mathcal{L}^c(\mathcal{V}_{n+1})$. More specifically, in view of eq. (7b), $\mathcal{L}^c(\mathcal{V}_{n+1})$ is a fiber bundle over P (and therefore also on $\mathcal{V}_{n+1}$), with coordinates $t, q^i, u, \dot{q}^i$. The quotient map makes $j_1(P,\mathbb{R}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$ into a principal fiber bundle, with structural group $(\mathbb{R},+)$ and group action (7a).

The final step in the definition of the Lagrangian bundles relies on the observation that the group actions (6a), (7a) *commute*. Each of them may therefore be used to induce a group action on the quotient space generated by the other. As illustrated in [8], this makes both $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ into principal fiber bundles over a common "double quotient" space, canonically diffeomorphic to the velocity space $j_1(\mathcal{V}_{n+1})$.

The situation is summarized into the commutative diagram

$$\begin{tikzcd} j_1(P,\mathbb{R}) \arrow[r] \arrow[d] & \mathcal{L}^c(\mathcal{V}_{n+1}) \arrow[d] \\ \mathcal{L}(\mathcal{V}_{n+1}) \arrow[r] & j_1(\mathcal{V}_{n+1}) \end{tikzcd} \tag{8}$$

in which all arrows denote principal fibrations, with structural group isomorphic to $(\mathbb{R},+)$, and group actions obtained in a straightforward way from eqs. (6b), (7b).
The principal fiber bundles $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ are respectively called the *Lagrangian* and the *co-Lagrangian bundle* over $j_1(\mathcal{V}_{n+1})$.

The geometrical setup arising from diagram (8) provides the natural environment for a gauge-invariant formulation of Lagrangian mechanics. As pointed out in [8], this is achieved by giving up the traditional approach, based on the interpretation of the Lagrangian $L(t, q^i, \dot{q}^i)$ as the representation of a (gauge-dependent) *scalar field* over $j_1(\mathcal{V}_{n+1})$, and introducing instead the concept of *Lagrangian section*, meant as a section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ of the Lagrangian bundle. For each choice of the trivialization u of P, the description of l takes the local form

$$\dot{u} = L(t, q^i, \dot{q}^i) \tag{9}$$

i.e. it does indeed rely on the assignment of a function $L(t, q^i, \dot{q}^i)$ on $j_1(\mathcal{V}_{n+1})$. However, according to eq. (2b), as soon as the trivialization is changed into $\bar{u} = u + f$, the representation (9) undergoes the transformation law

$$\dot{\bar{u}} = \dot{u} + \dot{f} = L(t, q^i, \dot{q}^i) + \dot{f} := L'(t, q^i, \dot{q}^i)$$

involving a different, gauge equivalent "Lagrangian".

Referring to [8] for further comments, we conclude this Subsection with a review of the algorithm assigning to each Lagrangian section l a corresponding Poincaré–Cartan 1-form on $j_1(\mathcal{V}_{n+1})$. To this end, starting with l, we consider in turn:

- the trivialization ψ_l of the bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ induced by l, described locally by the function $\dot{u} - L(t, q^i, \dot{q}^i)$@;
- the pull-back of ψ_l on $j_1(P, \mathbb{R})$, denoted by $\hat{\psi}_l$, and described locally by the function $\hat{\psi}_l(t, q^i, u, \dot{q}^i, \dot{u}) = \dot{u} - L(t, q^i, \dot{q}^i)$.

It is then an easy matter to verify the validity of the following assertions:

i) $\hat{\psi}_l$ is a trivialization of the bundle $j_1(P, \mathbb{R}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$; as such, it determines a section $\hat{l} : \mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$, expressed locally as $\dot{u} = L(t, q^i, \dot{q}^i)$. The sections l and $\hat{l}$, together, give rise to a principal bundle homomorphism, summarized into the commutative diagram

$$\begin{array}{ccc} \mathcal{L}^c(\mathcal{V}_{n+1}) & \xrightarrow{\hat{l}} & j_1(P, \mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{l} & \mathcal{L}(\mathcal{V}_{n+1}) \end{array} \tag{10}$$

ii) the fiber differential of $\hat{\psi}_l$, expressed locally as

$$d_v \hat{\psi}_l = d_v \left[\dot{u} - L(t, q^i, \dot{q}^i) \right] = \omega^0 - \frac{\partial L}{\partial \dot{q}^k}\, \omega^k \tag{11}$$

determines a *connection* on the principal bundle $j_1(P, \mathbb{R}) \to \mathcal{L}(\mathcal{V}_{n+1})$.

In view of i) and ii), the pull–back of $d_v \hat{\psi}_l$ through the diagram (10) defines a connection on $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$, described locally by the 1-form

$$\hat{l}^*(d_v \hat{\psi}_l) = du - L\, dt - \frac{\partial L}{\partial \dot{q}^k}\, \omega^k \tag{12}$$

The difference $du - \hat{l}^*(d_v \hat{\psi}_l)$ is then (the pull-back of) a 1-form ϑ_l over $j_1(\mathcal{V}_{n+1})$, called the *Poincaré–Cartan 1-form* of l, expressed in coordinates as

$$\vartheta_l = L\,dt + \frac{\partial L}{\partial \dot{q}^k}\,\omega^k \tag{13}$$

The behaviour of ϑ_l under an arbitrary change of trivialization is obtained by comparing the representations

$$l^*(d_v\,\hat{\psi}_l) = du - \pi^*(\vartheta_l) = d\bar{u} - \pi^*(\bar{\vartheta}_l)$$

π denoting the projection $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$. From this, setting $\bar{u} = u + f$, we get the transformation law

$$\pi^*(\bar{\vartheta}_l - \vartheta_l) = d(\bar{u} - u) = \pi^*(df) \quad \Rightarrow \quad \bar{\vartheta}_l = \vartheta_l + df \tag{14}$$

The exterior differential $\Omega_l := d\vartheta_l$, known as the *Poincaré–Cartan 2-form* of l, is therefore a gauge-invariant object over $j_1(\mathcal{V}_{n+1})$, identical, up to a sign, to the *curvature* of the connection (12).

By means of the correspondence $l \to \Omega_l$ one recovers the whole content of classical Lagrangian mechanics. The argument is standard, and will be regarded as known. For further information, see e.g. [8] and references therein.

2.2 The Hamiltonian bundles

Paralleling the discussion in § 2.1, let us now consider the fibration $P \to \mathcal{V}_{n+1}$, as well as the associated first jet space $j_1(P, \mathcal{V}_{n+1}) \xrightarrow{\pi} P$. Every fibered coordinate system t, q^i, u on P induces local coordinates t, q^i, u, p_0, p_i on $j_1(P, \mathcal{V}_{n+1})$, with transformation laws

$$\bar{t} = t + c\ , \qquad \bar{q}^{\,i} = \bar{q}^{\,i}(t, q)\ , \qquad \bar{u} = u + f(t, q) \tag{15a}$$

$$\bar{p}_0 = p_0 + \frac{\partial f}{\partial t} + \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial t}\ , \qquad \bar{p}_i = \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial \bar{q}^{\,i}} \tag{15b}$$

Eqs. (15a, b) ensure the invariance of the contact 1-form

$$\Theta = du - p_0\,dt - p_i\,dq^i \tag{16}$$

henceforth referred to as the *Liouville 1-form* of $j_1(P, \mathcal{V}_{n+1})$.

The manifold $j_1(P, \mathcal{V}_{n+1})$ is naturally embedded into the cotangent space $T^*(P)$. In local coordinates, this results into the identification

$$\eta = \Big[\, du - p_0(\eta)\,dt - p_i(\eta)\,dq^i\,\Big]_{\pi(\eta)} \qquad \forall\,\eta \in j_1(P, \mathcal{V}_{n+1}) \tag{17}$$

On the basis of eq. (17), one can easily establish two distinguished actions of the group $(\mathbb{R}, +)$ on $j_1(P, \mathcal{V}_{n+1})$, respectively denoted by $\psi_{\xi*} : j_1(P, \mathcal{V}_{n+1}) \to$

$j_1(P, \mathcal{V}_{n+1})$ and $\phi_\xi : j_1(P, \mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$, and expressed locally as

$$\psi_{\xi *}(\eta) \;:=\; (\psi_{-\xi})_*^{\;*}(\eta) \;=\; \Big[du - p_0(\eta)\,dt - p_i(\eta)\,dq^i\Big]_{\pi(\eta)+\xi} \tag{18a}$$

$$\phi_\xi(\eta) \;:=\; \eta \;-\; \xi\,(dt)_{\pi(\eta)} \;=\; \Big[du - (p_0(\eta) + \xi)\,dt - p_i(\eta)\,dq^i\Big]_{\pi(\eta)} \tag{18b}$$

$\forall\, \xi \in \mathbb{R}\,,\ \eta \in j_1(P, \mathcal{V}_{n+1})$.

The first one, written symbolically as

$$\psi_{\xi *} \;:\; (t, q^i, u, p_0, p_i) \;\longrightarrow\; (t, q^i, u + \xi, p_0, p_i)$$

is essentially the pull-back of the (inverse of) the action (1). Let $\mathcal{H}(\mathcal{V}_{n+1})$ denote the quotient of $j_1(P, \mathcal{V}_{n+1})$ by this action. The following properties are entirely straightforward [8]:

- $\mathcal{H}(\mathcal{V}_{n+1})$ is an affine bundle over $\mathcal{V}_{n+1}$, with coordinates t, q^i, p_0, p_i, modelled on the cotangent bundle $T^*(\mathcal{V}_{n+1})$;
- the quotient map makes $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ into a principal fiber bundle, with structural group $(\mathbb{R}, +)$ and fundamental vector field $\frac{\partial}{\partial u}$;
- the contact 1-form (16) determines a connection on $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$, henceforth called the *Liouville connection*. The curvature of the latter, described, up to a sign, by the exterior 2-form

$$\Omega := -d\Theta = dp_0 \wedge dt + dp_i \wedge dq^i \tag{19}$$

  endows the base manifold $\mathcal{H}(\mathcal{V}_{n+1})$ with a canonical *symplectic structure*.

The second action of $(\mathbb{R}, +)$ on $j_1(P, \mathcal{V}_{n+1})$, described by eq. (18b), and summarized into the symbolic relation

$$\phi_\xi \;:\; (t, q^i, u, p_0, p_i) \;\longrightarrow\; (t, q^i, u, p_0 + \xi, p_i)$$

comes from the invariant character of the 1-form dt. The quotient of $j_1(P, \mathcal{V}_{n+1})$ by this action will be denoted by $\mathcal{H}^c(\mathcal{V}_{n+1})$. Once again, one has the properties:

- $\mathcal{H}^c(\mathcal{V}_{n+1})$ is a fiber bundle over $\mathcal{V}_{n+1}$, with coordinates t, q^i, u, p_i;
- the action (18b) makes $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}^c(\mathcal{V}_{n+1})$ into a principal fiber bundle, with structural group $(\mathbb{R}, +)$ and fundamental vector field $\frac{\partial}{\partial p_0}$.

Exactly as in the Lagrangian case, an important aspect of the previous construction is the fact that the group actions (18a, b) commute. Each of them may therefore be used to induce an action on the space of orbits associated with the other. As illustrated in [8], this makes both $\mathcal{H}(\mathcal{V}_{n+1})$ and $\mathcal{H}^c(\mathcal{V}_{n+1})$ into principal fiber bundles over a common "double quotient" $\Pi(\mathcal{V}_{n+1})$, identified with the *phase space* of the system. The situation is summarized into the commutative diagram

$$
\begin{array}{ccc}
j_1(P, \mathcal{V}_{n+1}) & \longrightarrow & \mathcal{H}^c(\mathcal{V}_{n+1}) \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{H}(\mathcal{V}_{n+1}) & \longrightarrow & \Pi(\mathcal{V}_{n+1})
\end{array}
\tag{20}
$$

in which all arrows denote principal fibrations, with structural group isomorphic to $(\mathbb{R}, +)$, and group actions arising in a straightforward way from eqs. (18a, b).

The principal bundles $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$, $\mathcal{H}^c(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$ are respectively called the *Hamiltonian* and the *co-Hamiltonian* bundle over $\Pi(\mathcal{V}_{n+1})$. Every section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ is called a *Hamiltonian section*.
Consistently with the traditional notation, for each choice of the trivialization u of P, we shall express h locally as

$$p_0 = -H(t, q^1, \dots, q^n, p_1, \dots, p_n) \tag{21}$$

The function at the right-hand-side of eq. (21) will be called the *Hamiltonian*.

The previous arguments provide a convenient geometrical setting for the Hamiltonian formulation of classical mechanics.
Referring to [10] for the necessary details, we focus once again on the algorithm assigning to every Hamiltonian section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ a corresponding *Poincaré–Cartan 1-form* ϑ_h over $\Pi(\mathcal{V}_{n+1})$. To this end we lift h to a section $\hat{h} : \mathcal{H}^c(\mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$, described locally by the same equation (21).
The sections h and $\hat{h}$, together, give rise to a principal bundle homomorphism

$$
\begin{array}{ccc}
\mathcal{H}^c(\mathcal{V}_{n+1}) & \xrightarrow{\hat{h}} & j_1(P, \mathcal{V}_{n+1}) \\
\Big\downarrow & & \Big\downarrow \\
\Pi(\mathcal{V}_{n+1}) & \xrightarrow{h} & \mathcal{H}(\mathcal{V}_{n+1})
\end{array}
\tag{22}
$$

By means of the latter, the Liouville connection of $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ may be pulled back to a connection over $\mathcal{H}^c(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$, with connection 1-form

$$\hat{h}^*(\Theta) = du + H(t, q^i, p_i)\, dt - p_i\, dq^i \tag{23}$$

The difference $du - \hat{h}^*(\Theta)$ is then (the pull-back of) a 1-form ϑ_h over $\Pi(\mathcal{V}_{n+1})$, expressed in coordinates as

$$\vartheta_h = -H\, dt + p_i\, dq^i \tag{24}$$

and called the *Poincaré–Cartan 1-form* induced by the section h.

Exactly as in § 2.1, the behaviour of ϑ_h under an arbitrary change of trivialization $u \to u + f$ is easily recognized to be

$$\bar{\vartheta}_h = \vartheta_h + df \tag{25}$$

The exterior differential $\Omega_h = d\vartheta_h$, known as the *Poincaré–Cartan 2-form* of h, is therefore a gauge-invariant object over $\Pi(\mathcal{V}_{n+1})$, identical, up to a sign, to the curvature of the connection (23).

In terms of Ω_h, the dynamical flow of the system is completely characterized as the unique vector field Z over $\Pi(\mathcal{V}_{n+1})$ satisfying the conditions

$$\langle Z, dt \rangle = 1 \, , \qquad Z \,\lrcorner\, \Omega_h = 0 \tag{26}$$

In local coordinates, a straightforward comparison with eq. (24) provides the representation

$$Z = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \tag{27}$$

The integral curves of Z are therefore solutions of the *Hamilton equations*

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \qquad ; \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \tag{28}$$

3 The Legendre transformation

3.1 Higher jet spaces

The identifications (3), (17) provide a natural pairing between the fibers of the first jet spaces $j_1(P, \mathbb{R}) \xrightarrow{\pi} P$ and $j_1(P, \mathcal{V}_{n+1}) \xrightarrow{\pi} P$, expressed in coordinates as

$$\langle z, \eta \rangle = \left\langle \left[\frac{\partial}{\partial t} + \dot{q}^i(z) \frac{\partial}{\partial q^i} + \dot{u}(z) \, \frac{\partial}{\partial u} \right]_{\pi(z)} , \left[du - p_0(\eta) \, dt - p_i(\eta) \, dq^i \right]_{\pi(\eta)} \right\rangle \tag{29}$$

($z \in j_1(P, \mathbb{R})$, $\eta \in j_1(P, \mathcal{V}_{n+1})$, $\pi(z) = \pi(\eta)$). In view of eqs. (6a), (18a), the correspondence (29) satisfies the invariance property

$$\Big\langle \psi_{\xi *}(z) \, , \, \psi_{\xi *}(\eta) \Big\rangle = \Big\langle \psi_{\xi *}(z) \, , \, (\psi_{-\xi})_*^{\;*} \, (\eta) \Big\rangle = \Big\langle z, \eta \Big\rangle \qquad \forall \, \xi \in \mathbb{R} \tag{30}$$

thereby giving rise to an analogous pairing between the fibers of the bundles $\mathcal{L}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ and $\mathcal{H}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$, or, what is the same, to bi–affine map of the fibered product $\mathcal{L}(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} \mathcal{H}(\mathcal{V}_{n+1})$ onto $\mathbb{R}$, expressed in coordinates as

$$\zeta, \sigma \;\longrightarrow\; F(\zeta, \sigma) \,:=\, \dot{u}(\zeta) - p_0(\sigma) - p_i(\sigma) \, \dot{q}^i(\zeta) \tag{31}$$

Let $\mathcal{S}$ denote the submanifold of $\mathcal{L}(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} \mathcal{H}(\mathcal{V}_{n+1})$ described by the equation

$$\mathcal{S} = \Big\{(\zeta,\sigma) \in \mathcal{L}(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} \mathcal{H}(\mathcal{V}_{n+1})\,, \;\; F(\zeta,\,\sigma) = 0\Big\} \tag{32}$$

A straightforward argument, based on eq. (31), shows that $\mathcal{S}$ is fibered on both $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{H}(\mathcal{V}_{n+1})$. The former circumstance is made explicit by referring $\mathcal{S}$ to local coordinates $t, q^i, \dot q^i, \dot u, p_i$, with the p_i's regarded as fiber coordinates. The second circumstance is similarly accounted for by referring $\mathcal{S}$ to coordinates $t, q^i, p_0, p_i, \dot q^i$, related to the previous ones by the transformation

$$\dot u \;=\; p_0 \,+\, p_i\,\dot q^i \tag{33}$$

and with the $\dot q^i$'s playing the role of fiber coordinates.

A useful characterization of the manifold $\mathcal{S}$ comes from the fact that all spaces under consideration have the nature of *affine bundles* over the configuration space–time $\mathcal{V}_{n+1}$. From this, using a suffix $(\cdot\cdot)_{|x}$ to denote the fiber of the manifold $(\cdot\cdot)$ at the point $x \in \mathcal{V}_{n+1}$, one can easily draw the following conclusions:

i) both maps $\mathcal{L}(\mathcal{V}_{n+1})_{|x} \to j_1(\mathcal{V}_{n+1})_{|x}$, $\mathcal{H}(\mathcal{V}_{n+1})_{|x} \to \Pi(\mathcal{V}_{n+1})_{|x}$ are affine surjections as well as principal fibrations, with structural group $(\mathbb{R},+)$;

ii) for each $\sigma \in \mathcal{H}(\mathcal{V}_{n+1})_{|x}$, the *annihilator* $[\,\sigma\,]^0$, defined as the totality of points $\zeta \in \mathcal{L}(\mathcal{V}_{n+1})_{|x}$ satisfying $F(\zeta,\,\sigma) = 0$, is an affine section of the bundle $\mathcal{L}(\mathcal{V}_{n+1})_{|x} \to j_1(\mathcal{V}_{n+1})_{|x}$, described in coordinates as

$$\dot u = p_0(\sigma) + p_i(\sigma)\,\dot q^i \tag{34a}$$

Conversely, *every* affine section $\varphi : j_1(\mathcal{V}_{n+1})_{|x} \to \mathcal{L}(\mathcal{V}_{n+1})_{|x}$ may be expressed in the form (34a) for precisely one choice of the element $\sigma \in \mathcal{H}(\mathcal{V}_{n+1})_{|x}$. We have therefore a canonical identification of $\mathcal{H}(\mathcal{V}_{n+1})_{|x}$ with the space of affine sections of the bundle $\mathcal{L}(\mathcal{V}_{n+1})_{|x} \to j_1(\mathcal{V}_{n+1})_{|x}$;

iii) in a perfectly symmetric way, for each $\zeta \in \mathcal{L}(\mathcal{V}_{n+1})_{|x}$, the annihilator $[\,\zeta\,]^0$ is an affine section of $\mathcal{H}(\mathcal{V}_{n+1})_{|x} \to \Pi(\mathcal{V}_{n+1})$, described in coordinates as

$$p_0 = \dot u(\zeta) - \dot q^i(\zeta)\,p_i \tag{34b}$$

Once again, the correspondence $\zeta \to [\,\zeta\,]^0$ allows to identify $\mathcal{L}(\mathcal{V}_{n+1})_{|x}$ with the space of affine sections of the bundle $\mathcal{H}(\mathcal{V}_{n+1})_{|x} \to \Pi(\mathcal{V}_{n+1})_{|x}$.

Denoting by $j_1(\mathcal{L}(\mathcal{V}_{n+1})_{|x}\,, j_1(\mathcal{V}_{n+1})_{|x})$ and $j_1(\mathcal{H}(\mathcal{V}_{n+1})_{|x}\,, \Pi(\mathcal{V}_{n+1})_{|x})$ the first jet spaces associated with the principal fibrations described in i), the previous results are summarized into the following

Theorem 3.1 *Both spaces $j_1(\mathcal{L}(\mathcal{V}_{n+1})_{|x}, j_1(\mathcal{V}_{n+1})_{|x})$, $j_1(\mathcal{H}(\mathcal{V}_{n+1})_{|x}, \Pi(\mathcal{V}_{n+1})_{|x})$ are canonically diffeomorphic to the fiber $\mathcal{S}_{|x}$ of the manifold $\mathcal{S}$ at x.*

Proof Given any pair $(\zeta, \sigma) \in \mathcal{L}(\mathcal{V}_{n+1})_{|x} \times \mathcal{H}(\mathcal{V}_{n+1})_{|x}$, the condition for the point ζ to belong to the section $[\,\sigma\,]^0$, or, reciprocally, for the point σ to belong to the section $[\,\zeta\,]^0$ are both expressed by the same equation $F(\zeta, \sigma) = 0$.
From this, given any $(\zeta, \sigma) \in \mathcal{S}_{|x}$, by the very definition of first jet space we conclude that the pair $(\zeta, [\,\sigma\,]^0)$ is an element of $j_1(\mathcal{L}(\mathcal{V}_{n+1})_{|x}, j_1(\mathcal{V}_{n+1})_{|x})$, namely a point ζ in $\mathcal{L}(\mathcal{V}_{n+1})_{|x}$ and a hyperplane through ζ, while the pair $(\sigma, [\,\zeta\,]^0)$ is similarly an element of $j_1(\mathcal{H}(\mathcal{V}_{n+1})_{|x}, \Pi(\mathcal{V}_{n+1})_{|x})$, namely a point $\sigma \in \mathcal{H}(\mathcal{V}_{n+1})_{|x}$ and a hyperplane through σ.
A straightforward argument, left to the reader, shows that both correspondences $\mathcal{S}_{|x} \to j_1(\mathcal{L}(\mathcal{V}_{n+1})_{|x}, j_1(\mathcal{V}_{n+1})_{|x})$ and $\mathcal{S}_{|x} \to j_1(\mathcal{H}(\mathcal{V}_{n+1})_{|x}, \Pi(\mathcal{V}_{n+1})_{|x})$ obtained in this way are affine diffeomorphisms. $\square$

By varying x, Theorem 3.1 provides the identification [9]

$$\mathcal{S} = \bigcup_{x \in \mathcal{V}_{n+1}} j_1(\mathcal{L}(\mathcal{V}_{n+1})_{|x}, j_1(\mathcal{V}_{n+1})_{|x}) = \bigcup_{x \in \mathcal{V}_{n+1}} j_1(\mathcal{H}(\mathcal{V}_{n+1})_{|x}, \Pi(\mathcal{V}_{n+1})_{|x}) \tag{35}$$

A significant enhancement of eq. (35) is obtained by considering the first jet spaces $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ and $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$ respectively associated with the bundles $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ and $H \to \Pi(\mathcal{V}_{n+1})$.
These are naturally fibered over the spaces $\bigcup_x j_1(\mathcal{L}(\mathcal{V}_{n+1})_{|x}, j_1(\mathcal{V}_{n+1})_{|x})$ and $\bigcup_x j_1(\mathcal{H}(\mathcal{V}_{n+1})_{|x}, \Pi(\mathcal{V}_{n+1})_{|x})$, the fiber projections having the nature of *quotient maps* associated with suitable equivalence relations. More specifically:

- every section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, restricted to a fiber $j_1(\mathcal{V}_{n+1})_{|x}$, determines a section $\hat{l} : j_1(\mathcal{V}_{n+1})_{|x} \to \mathcal{L}(\mathcal{V}_{n+1})_{|x}$. When two sections l, l' have a first order contact at a point $z \in j_1(\mathcal{V}_{n+1})_{|x}$, their restrictions $\hat{l}, \hat{l}'$ also do. Conversely, a necessary and sufficient condition for two restrictions $\hat{l}, \hat{l}'$ to have a first order contact at z is that the original sections satisfy

  $$l(z) = l'(z)\,, \qquad \Big[d_v\Big(l - l'\Big)\Big]_{|z} = 0$$

  As an affine bundle over $j_1(\mathcal{V}_{n+1})$, the manifold $\bigcup_x j_1(\mathcal{L}(\mathcal{V}_{n+1})_{|x}, j_1(\mathcal{V}_{n+1})_{|x})$ is therefore the quotient of $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ by the vector bundle formed by the totality of *semibasic* 1–forms on $j_1(\mathcal{V}_{n+1})$;
- a similar argument characterizes $j_1(\mathcal{H}(\mathcal{V}_{n+1})_{|x}, \Pi(\mathcal{V}_{n+1})_{|x})$ as the quotient of $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$ by the bundle of semibasic 1–forms on $\Pi(\mathcal{V}_{n+1})$.

Collecting all stated results we conclude that both spaces $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ and $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$ are fibered over $\mathcal{S}$. This fact will be explicitly accounted for by adopting common base coordinates (indifferently $t, q^i, \dot{q}^i, \dot{u}, p_i$ or $t, q^i, \dot{q}^i, p_0, p_i$) on both manifolds, and completing them with fiber coordinates w_0, w_i on $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ and v_0, v_i on $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$.

With this choice, the Liouville 1-forms of the bundles $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ and $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$, respectively denoted by $\Theta_{\mathcal{L}}$ and $\Theta_{\mathcal{H}}$, are expressed in coordinates as

$$\Theta_{\mathcal{L}} = d\dot u - w_0\,dt - w_i\,dq^i - p_i\,d\dot q^i \tag{36a}$$

$$\Theta_{\mathcal{H}} = d\dot u - v_0\,dt - v_i\,dq^i - p_i\,d\dot q^i = dp_0 - v_0\,dt - v_i\,dq^i + \dot q^i\,dp_i \tag{36b}$$

the last equality depending on the transformation law (33).

On this basis, we can state

Proposition 3.1 *There exists a unique affine isomorphism*

$$\begin{tikzcd} j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1})) \arrow[r, "\psi"] \arrow[d] & j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1})) \arrow[d] \\ \mathcal{S} \arrow[r, equal] & \mathcal{S} \end{tikzcd}$$

fibered over $\mathcal{S}$ and satisfying $\psi^*(\Theta_{\mathcal{H}}) = \Theta_{\mathcal{L}}$.

Proof In coordinates, the assignment of ψ relies on the choice of $n+1$ affine functions $\psi^*(v_0)$, $\psi^*(v_i)$. Together with eq. (36b), these provide the evaluation

$$\psi^*(\Theta_{\mathcal{H}}) = d\dot u - \psi^*(v_0)\,dt - \psi^*(v_i)\,dq^i - p_i\,d\dot q^i$$

The condition $\psi^*(\Theta_{\mathcal{H}}) = \Theta_{\mathcal{L}}$ is therefore equivalent to the requirement

$$\psi^*(v_0) = w_0\,, \qquad \psi^*(v_i) = w_i$$

This establishes at one time the existence and the uniqueness of ψ. □

Proposition 3.1 provides a canonical identification between the first-jet spaces $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ and $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$ [9]. This result is further enhanced making use of the fibration $\mathcal{H}(\mathcal{V}_{n+1}) \to \mathbb{R}$ determined by the composition $\mathcal{H}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1} \to \mathbb{R}$. Denoting by $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$ the associated first jet space, and recalling that the manifold $\mathcal{H}(\mathcal{V}_{n+1})$ is canonically endowed with the symplectic structure (19), we have in fact the following

Theorem 3.2 *The manifolds $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$ and $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$ are canonically diffeomorphic.*

Proof By definition, both manifolds may be viewed as affine subbundles, respectively of the tangent space $T(\mathcal{H}(\mathcal{V}_{n+1}))$ and of the cotangent space $T^*(\mathcal{H}(\mathcal{V}_{n+1}))$, according to the identifications

$$j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R}) = \left\{ X \;\middle|\; X \in T(\mathcal{H}(\mathcal{V}_{n+1}))\,,\ \langle X\,, dt\rangle = 1 \right\} \tag{37a}$$

$$j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1})) = \left\{ \omega \;\middle|\; \omega \in T^*(\mathcal{H}(\mathcal{V}_{n+1}))\,,\ \left\langle \frac{\partial}{\partial p_0}\,, \omega \right\rangle = 1 \right\} \tag{37b}$$

The conclusion then follows from the identity

$$\Big\langle \frac{\partial}{\partial p_0}\,,\, -X \,\lrcorner\, \Omega \Big\rangle = \Big\langle -X \wedge \frac{\partial}{\partial p_0} \;\Big|\; dp_0 \wedge dt + dp_i \wedge dq^i \Big\rangle = \big\langle X\,,\, dt \big\rangle$$

showing that the correspondence $X \to -X \,\lrcorner\, \Omega$ maps the subbundle (37a) injectively onto the subbundle (37b). $\square$

In view of the previous results, all spaces $j_1(\mathcal{H}(\mathcal{V}_{n+1}),\mathbb{R})$, $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ and $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$ are mutually diffeomorphic, and may be identified [9]. For definiteness, and without any loss in generality, we choose to regard all of them as different copies of the manifold $j_1(\mathcal{H}(\mathcal{V}_{n+1}),\mathbb{R})$. Depending on the context, we shall refer the latter to ordinary jet coordinates $t, q^i, p_0, p_i, \dot q^i, \dot p_0, \dot p_i$, or to coordinates $t, q^i, \dot u, p_i, \dot q^i, \dot p_0, \dot p_i$, related to the previous ones by the transformation (see eq. (33))

$$\dot u - p_0 - p_i\,\dot q^i = 0 \tag{38}$$

With the stated choice, the Liouville 1-forms (36a, b) collapse into a single geometrical object, henceforth denoted by Θ, expressed in coordinates as

$$\Theta := d\dot u - \dot p_0\,dt - \dot p_i\,dq^i - p_i\,d\dot q^i = dp_0 - \dot p_0\,dt - \dot p_i\,dq^i + \dot q^i\,dp_i \tag{39}$$

The previous expression helps determining the coordinate representation of the first jet extension of an arbitrary section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ to a section $j_1(l) : j_1(\mathcal{V}_{n+1}) \to j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$, (respectively, the first jet extension of a section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ to a section $j_1(h) : \Pi(\mathcal{V}_{n+1}) \to j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$), the significant requirement being in any case the *vanishing* of the pull-back of the Liouville 1-form (39) under the extended map.

As a final remark we observe that, by an argument similar to the one discussed in § 2, the manifold $j_1(\mathcal{H}(\mathcal{V}_{n+1}),\mathbb{R})$ is naturally endowed with two distinguished actions of the group $(\mathbb{R},+)$, both arising from the principal bundle structure of $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$. Analogous conclusions hold for the spaces $j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$, $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$, the group actions being respectively inherited from the principal bundle structures of $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ and of $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$.
A straightforward argument, left to the reader, shows that, under the identification $j_1(\mathcal{H}(\mathcal{V}_{n+1}),\mathbb{R}) \simeq j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1})) \simeq j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$, all these actions collapse into a single pair of actions, respectively generated by the vector field $\frac{\partial}{\partial p_0} = \frac{\partial}{\partial \dot u}$ and $\frac{\partial}{\partial \dot p_0}$.

The quotient of $j_1(\mathcal{H}(\mathcal{V}_{n+1}),\mathbb{R})$ under the action of $\frac{\partial}{\partial p_0}$ will be denoted by $\mathcal{B}$, and will be referred to coordinates $t, q^i, \dot q^i, p_i, \dot p_0, \dot p_i$. Exactly as in § 2 we have the properties:

- the quotient map makes $j_1(\mathcal{H}(\mathcal{V}_{n+1}),\mathbb{R}) \to \mathcal{B}$ into a principal fiber bundle;

- the 1-form (39) endows $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R}) \to \mathcal{B}$ with a canonical connection;
- the curvature 2-form of Θ, defined up to a sign by

$$\Upsilon := -d\,\Theta = d\dot{p}_0 \wedge dt + d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i \tag{40}$$

  endows $\mathcal{B}$ with a canonical *symplectic structure*.

3.2 Legendre maps

The arguments of § 3.1, summarized into the commutative diagram

$$\begin{tikzcd}
j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1})) \arrow[r, equal] \arrow[d] & j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R}) \arrow[r, equal] \arrow[d] & j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1})) \arrow[d] \\
\mathcal{L}(\mathcal{V}_{n+1}) \arrow[d] & \mathcal{B} \arrow[dl] \arrow[dr] & \mathcal{H}(\mathcal{V}_{n+1}) \arrow[d] \\
j_1(\mathcal{V}_{n+1}) & & \Pi(\mathcal{V}_{n+1})
\end{tikzcd} \tag{41}$$

provide the necessary tools for a revisitation of the Legendre transformation in time dependent analytical mechanics [9].

The line of approach, extending the one originally exploited by Tulczyjew [14,17], may be traced as follows: given any section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, consider the jet extension $j_1(l) : j_1(\mathcal{V}_{n+1}) \to j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$. Composing the latter with the (significant) vertical arrows of diagram (41) gives rise to correspondences $\Lambda_l : j_1(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$, $\lambda_l : j_1(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$ and $\kappa_l : j_1(\mathcal{V}_{n+1}) \to \mathcal{B}$.

In coordinates, expressing l as $\dot{u} = L(t, q^i, \dot{q}^i)$, and recalling eqs. (38), as well as the characterization of $j_1(l)$ in terms of the Liouville 1-form (39), we get the representations

$$\Lambda_l : \qquad p_i = \frac{\partial L}{\partial \dot{q}^i} \ ; \qquad p_0 = L - \dot{q}^i\,\frac{\partial L}{\partial \dot{q}^i} \tag{42a}$$

$$\lambda_l : \qquad p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{42b}$$

$$\kappa_l : \qquad p_i = \frac{\partial L}{\partial \dot{q}^i} \ ; \qquad \dot{p}_0 = \frac{\partial L}{\partial t} \ ; \quad \dot{p}_i = \frac{\partial L}{\partial q^i} \tag{42c}$$

In view of eqs. (40), (42c), the map κ_l satisfies the identity

$$\begin{aligned} {\kappa_l}^*(\Upsilon) = {\kappa_l}^* \left(d\dot{p}_0 \wedge dt + d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i \right) = \\ = d\left(\frac{\partial L}{\partial t}\, dt + \frac{\partial L}{\partial q^i}\, dq^i + \frac{\partial L}{\partial \dot{q}^i}\, d\dot{q}^i \right) \equiv 0 \end{aligned} \tag{43}$$

showing that the image space $\kappa_l\,(j_1(\mathcal{V}_{n+1}))$ is a *Lagrangian submanifold* of $\mathcal{B}$.

A perfectly symmetric construction holds starting with a Hamiltonian section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$.
Once again the jet extension $j_1(h) : \Pi(\mathcal{V}_{n+1}) \to j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$, composed with the significant vertical arrows of diagram (41), generates maps $\Lambda_h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, $\lambda_h : \Pi(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ and $\kappa_h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{B}$, expressed in coordinates as

$$\Lambda_h \; : \qquad \dot q^i = \frac{\partial H}{\partial p_i} \;\; ; \qquad \dot u = -H + \frac{\partial H}{\partial p_i} p_i \tag{44a}$$

$$\lambda_h \; : \qquad \dot q^i = \frac{\partial H}{\partial p_i} \tag{44b}$$

$$\kappa_h \; : \qquad \dot q^i = \frac{\partial H}{\partial p_i} \;\; ; \qquad \dot p_0 = -\frac{\partial H}{\partial t} \;\; ; \quad \dot p_i = -\frac{\partial H}{\partial q^i} \tag{44c}$$

$H(t, q^i, p_i)$ denoting the Hamiltonian function involved in the local representation of h. Exactly as above, eqs. (40), (44c) provide the identity

$$\begin{aligned} \kappa_h^*(\Upsilon) = \kappa_h^* \left(d\dot p_0 \wedge dt + d\dot p_i \wedge dq^i + dp_i \wedge d\dot q^i \right) = \\ = d\left(-\frac{\partial H}{\partial t}\, dt - \frac{\partial H}{\partial q^i}\, dq^i - \frac{\partial H}{\partial p_i}\, dp_i \right) \equiv 0 \end{aligned} \tag{45}$$

showing that the image space $\kappa_h(\mathcal{H}(\mathcal{V}_{n+1}))$ is a Lagrangian submanifold of $\mathcal{B}$.

A special instance of the previous construction occurs when the map λ_l associated with the Lagrangian section l is a *diffeomorphism*, i.e. when the image space $\Lambda_l\,(j_1(\mathcal{V}_{n+1}))$ projects injectively onto $\Pi(\mathcal{V}_{n+1})$. Under the stated assumption, the correspondence $h := \Lambda_l \cdot \lambda_l^{-1} : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ is a *section* of the bundle $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$, described in coordinates as

$$p_0 = L\left(t, q^i, \dot q^i\right) - p_i\, \dot q^i \;:= -H\left(t, q^i, p_i\right) \tag{46}$$

with the variables $\dot q^i$ defined implicitly in terms of the p_i's through eqs. (42b).
From eqs. (42b), (46), by elementary computations, we get the identities

$$\frac{\partial H}{\partial p_i} \;=\; \frac{\partial L}{\partial \dot q^j}\, \frac{\partial \dot q^j}{\partial p_i} \;+\; \dot q^i \;-\; p_j\, \frac{\partial \dot q^j}{\partial p_i} \;=\; \dot q^i \;; \qquad L = -H \;+\; p_i\, \frac{\partial H}{\partial p_i} \tag{47}$$

Comparison with eqs. (44a, b) provides the identifications

$$\lambda_h = \lambda_l^{-1} \;; \qquad\qquad l = \Lambda_h \cdot \lambda_l$$

pointing out the perfectly symmetric role played by the sections l and h.

Every diffeomorphism $j_1(\mathcal{V}_{n+1}) \longleftrightarrow \Pi(\mathcal{V}_{n+1})$ determined by a Lagrangian section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, or by a Hamiltonian one $h : \Pi(\mathcal{V}_{n+1}) \to$

$\mathcal{H}(\mathcal{V}_{n+1})$ through the algorithm described above is called a *Legendre transformation*.

3.3 Dynamics

As a final topic, we discuss the Lagrangian and Hamiltonian formulation of dynamics within the geometrical framework developed so far [9]. The analysis will provide a gauge–invariant extension to non-autonomous systems of the classical results of Tulczyjew [14–18].

Let $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ denote a Lagrangian section, expressed in coordinates as $\dot{u} = L(t, q^i, \dot{q}^i)$. On account of the discussion following eq. (39), the first jet extension $j_1(l) : j_1(\mathcal{V}_{n+1}) \to j_1(\mathcal{L}(\mathcal{V}_{n+1}), j_1(\mathcal{V}_{n+1}))$ is then described by the equations

$$\dot{u} = L(t, q^i, \dot{q}^i) \ , \qquad \dot{p}_0 = \frac{\partial L}{\partial t} \ , \qquad \dot{p}_i = \frac{\partial L}{\partial q^i} \ , \qquad p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{48}$$

The map $j_1(l)$ carries a complete information on dynamics. Indeed, according to diagram (41), the image space $\mathcal{E} := j_1(l)(j_1(\mathcal{V}_{n+1}))$ may be viewed as a submanifold of $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$. Switching to coordinates $t, q^i, p_0, p_i, \dot{q}^i, \dot{p}_0, \dot{p}_i$ through eq. (38), let us accordingly rephrase eqs. (48) in the equivalent form

$$p_0 = L(t, q^i, \dot{q}^i) - \frac{\partial L}{\partial \dot{q}^i}\, \dot{q}^i \ , \quad \dot{p}_0 = \frac{\partial L}{\partial t} \ , \quad p_i = \frac{\partial L}{\partial \dot{q}^i} \ , \quad \dot{p}_i = \frac{\partial L}{\partial q^i} \tag{49}$$

By the very definition of $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$, eqs. (49) provide a system of ordinary differential equations, not in normal form, for the determination of the family of sections $\gamma : \Re \to \mathcal{H}(\mathcal{V}_{n+1})$ $(\Leftrightarrow \gamma(t) \equiv (t, q^i(t), p_0(t), p_i(t)))$ whose jet extension $\dot{\gamma} := j_1(\gamma)$ satisfies $\dot{\gamma}(t) \in \mathcal{E} \ \forall\, t \in \mathbb{R}$. In the resulting context, the last pair of relations (49) reproduce the content of Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \qquad\qquad i = 1, \ldots, n$$

while the first pair describes the evolution of the Hamiltonian $H := -L + \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i$.

Precisely the same state of affairs occurs if one considers a Hamiltonian section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$, expressed in coordinates as $p_0 = -H(t, q^i, p_i)$. In view of eq. (39), the first jet extension $j_1(h) : \Pi(\mathcal{V}_{n+1}) \to j_1(\mathcal{H}(\mathcal{V}_{n+1}), \Pi(\mathcal{V}_{n+1}))$ is now described by the system

$$p_0 = -H(t, q^i, p_i) \ , \quad \dot{p}_0 = -\frac{\partial H}{\partial t} \ , \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \ , \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \tag{50}$$

Once again, according to diagram (41), the image $\mathcal{E} := j_1(h)(\Pi(\mathcal{V}_{n+1}))$ may be viewed as a $(2n+1)$–dimensional submanifold of $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$. Eqs. (50)

play then the role of a system of ordinary differential equations (now in normal form) characterizing the totality of sections $\gamma : \mathbb{R} \to \mathcal{H}(\mathcal{V}_{n+1})$ whose jet extension satisfies $\dot\gamma(t) \in \mathcal{E} \ \forall\, t$. More specifically, the last pair of eqs. (50) reproduces the content of Hamilton's equations, while the first pair determines the evolution of the Hamiltonian.

A significant implication of the previous discussion is the fact that, in the environment $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$, the Lagrangian and Hamiltonian approaches to mechanics are nothing but different representations of the same $(2n+1)$–dimensional submanifold, described indifferently as $\mathcal{E} = j_1(l)(j_1(\mathcal{V}_{n+1}))$ or $\mathcal{E} = j_1(h)(\Pi(\mathcal{V}_{n+1}))$. This aspect is further enhanced by observing that, by construction, the embedding $\mathcal{E} \xrightarrow{i} j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$ satisfies the identity

$$i^*(\Theta) = 0 \tag{51}$$

The hypersurface $\mathcal{E}$ is therefore *horizontal* with respect to the canonical connection of $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R}) \xrightarrow{\pi} \mathcal{B}$.
On the other hand, a straightforward argument shows that *every* horizontal submanifold $i : \mathfrak{S} \to j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$ has dimension $\le 2n+1$ [1].
Regular dynamical systems may therefore be viewed as *horizontal submanifolds of maximal dimension* in $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$, projecting injectively onto both $j_1(\mathcal{V}_{n+1})$ and $\Pi(\mathcal{V}_{n+1})$.

The previous arguments extend to the newer context the results originally established by Tulczyjew in the autonomous case [14–18] (in this connection see also [1,2,5,19]). The analogies are easily understood by observing that the projection $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R}) \xrightarrow{\pi} \mathcal{B}$ sets up a 1-1 correspondence between horizontal *slicings* of maximal dimension in $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R})$ and Lagrangian submanifolds in $\mathcal{B}$. The details are straightforward, and are left to the reader. In coordinates, the previous assertions have their analytical counterpart in eqs. (48), (50).

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[1] Indeed, by eqs. (39), (51), the projection $j_1(\mathcal{H}(\mathcal{V}_{n+1}), \mathbb{R}) \xrightarrow{\pi} \mathcal{B}$ is locally injective on $\mathfrak{S}$. At the same time, eq. (51) implies $i^*(d\Theta) = 0$. From this, taking the non singular character of $d\Theta$ into account, we get the condition $\dim \mathfrak{S} = \dim \pi(\mathfrak{S}) \le \frac{1}{2} \dim \mathcal{B} = 2n+1$.

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A note on integrability and closure conditions in the inverse problem in the calculus of variations.

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Abstract

The inverse problem in the calculus of variations for a given set of second order ordinary differential equations consists of deciding whether their solutions are those of Euler–Lagrange equations and exhibiting the non-uniqueness of the resulting Lagrangians when they occur.

The necessary and sufficient conditions for the existence of equivalent Euler–Lagrange equations are called the Helmholtz conditions and this paper discusses the integrability of these conditions in an elementary way which nonetheless gives a new overview of their structure.

Mike Crampin

This paper is dedicated to Mike Crampin on the occasion of his 60^{th} birthday. It has been a pleasure to know and work with Mike over 20 years and to acknowledge his influence and assistance over that period. I wish him many more happy years of influential mathematics.

1 The inverse problem in the calculus of variations

The inverse problem in the calculus of variations involves deciding whether the solutions of a given system of second-order ordinary differential equations

$$\ddot{x}^a = F^a(t, x^b \dot{x}^b), \quad a, b = 1, \ldots, n$$

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are the solutions of a set of Euler-Lagrange equations

$$\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b}\ddot{x}^b + \frac{\partial^2 L}{\partial x^b \partial \dot{x}^a}\dot{x}^b + \frac{\partial^2 L}{\partial t \partial \dot{x}^a} = \frac{\partial L}{\partial x^a}$$

for some Lagrangian function $L(t, x^b, \dot{x}^b)$. The problem dates to the end of the 19^{th} century and it still has deep importance for mathematics and mathematical physics.

Because the Euler-Lagrange equations are not generally in normal form, the problem is to find a so-called multiplier matrix $g_{ab}(t, x^c, \dot{x}^c)$ such that

$$g_{ab}(\ddot{x}^b - F^b) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial \dot{x}^a}.$$

The most commonly used set of necessary and sufficient conditions for the existence of the g_{ab} are the so–called *Helmholtz conditions* due to Douglas [11] and put in the following form by Sarlet [22]:

$$g_{ab} = g_{ba}, \quad \Gamma(g_{ab}) = g_{ac}\Gamma^c_b + g_{bc}\Gamma^c_a, \quad g_{ac}\Phi^c_b = g_{bc}\Phi^c_a, \quad \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b},$$

where

$$\Gamma^a_b := -\frac{1}{2}\frac{\partial F^a}{\partial \dot{x}^b}, \quad \Phi^a_b := -\frac{\partial F^a}{\partial x^b} - \Gamma^c_b\Gamma^a_c - \Gamma(\Gamma^a_b),$$

and where

$$\Gamma := \frac{\partial}{\partial t} + u^a\frac{\partial}{\partial x^a} + F^a\frac{\partial}{\partial u^a}.$$

This inverse problem has spawned significant advances in the theory of second order ordinary differential equations [6,15,16,20,19] and tangent bundle geometry [1,17,18,26] and the most recent work on the inverse problem [1,2,9,10,13,14,23–25] uses some very sophisticated and often purpose-built differential geometry. The purpose here is to step back a pace or two and to explore the integrability conditions on the Helmholtz conditions using simple exterior calculus. While it is true that the detailed analysis of the inverse problem for arbitrary n and even for $n = 2$ and 3 requires complicated mathematical machinery, an overview is often obscured and I hope to show that some useful observations about specific integrability conditions can be obtained by looking at them in their entirety as closure conditions on a particular class of two forms.

The next section contains that part (dating from the 1970's and 80's) of the geometric formulation of second order ordinary differential equations that we will need. There then follows a standard presentation of the inverse problem which integrates the Cartan form approach to Lagrangian dynamics and the foregoing theory of second order O.D.E.'s. This part contains an examination of the sources of the algebraic and differential parts of the Helmholtz conditions.

The last section is a statement and exploration of the entire set of closure conditions on the two–form of the Helmholtz conditions. These closure conditions are at most second order differential conditions on the multiplier g_{ab} and I show that in fact only one subset of them are at most second order. I also give some results on lower order consequences of these conditions. These results account for some features of the inverse problem which have arisen in more complex analyses. I have attempted to make the presentation self-contained but the paper is by its nature brief and the reader might find the book chapter [21] a helpful starting place for a more detailed study. Mike Crampin and Felix Pirani's book, *Applicable Differential Geometry* [7] is still a good reference for the underlying differential geometry of differential equations.

2 Geometric formulation

Mike Crampin was one of the pioneers of the geometric formulation of both second order O.D.E.'s and Lagrangian dynamics [3–5,8] and most of the material presented in this section bears his stamp although I have been influenced by Jon Aldridge's presentation of the Helmholtz conditions as closure conditions on a certain set of two forms [1].

2.1 2^{nd} order o.d.e's

Suppose that M is some differentiable manifold with generic local co-ordinates (x^a). The *evolution space* is defined as $E := \mathbb{R} \times TM$, with projection onto the first factor being denoted by $t : E \to \mathbb{R}$ and bundle projection $\pi : E \to \mathbb{R} \times M$. E has adapted co-ordinates (t, x^a, u^a) associated with t and (x^a).

A system of second order differential equations with local expression

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b), \;\; a, b, = 1, \ldots, n$$

is associated with a smooth vector field Γ on E given in the same co-ordinates by

$$\Gamma := \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a}.$$

Γ is called a *second order differential equation field* or SODE. It can be thought of as the total derivative operator associated with the differential equations. The integral curves of Γ are just the parametrised and lifted solution curves of the differential equations.

The evolution space E is equipped with the *vertical endomorphism* S, defined locally by $S := V_a \otimes \theta^a$ (see [8] for an intrinsic characterisation). S combines the *contact structure* and *vertical sub–bundle*, $V(E)$, of E, θ^a being the local contact forms $\theta^a := dx^a - u^a dt$ and $V_a := \frac{\partial}{\partial u^a}$ forming a basis for vector fields tangent to the fibres of $\pi : E \to \mathbb{R} \times M$ (the vertical sub–bundle).

It is natural to study the deformation of S produced by the flow of Γ, $\mathcal{L}_\Gamma S$. The eigenspaces of this $(1,1)$ tensor field produce a direct sum decomposition of each tangent space of E. It is shown in [8] that $\mathcal{L}_\Gamma S$ (acting on vectors) has eigenvalues $0, +1$ and -1. The eigenspace at a point of E corresponding to the eigenvalue 0 is spanned by Γ, while the eigenspace corresponding to $+1$ is the *vertical subspace* of the tangent space. The remaining eigenspace (of dimension n) is called the *horizontal subspace*. Unlike the vertical subspaces these eigenspaces are not integrable; their failure to be so is due to the curvature of this nonlinear connection (induced by Γ) which has components

$$\Gamma^a_b := -\frac{1}{2}\frac{\partial F^b}{\partial u^a}.$$

The most useful basis for the horizontal eigenspaces has elements with local expression

$$H_a = \frac{\partial}{\partial x^a} - \Gamma^b_a \frac{\partial}{\partial u^b}$$

so that a local basis of vector fields for the direct sum decomposition of the tangent spaces of E is $\{\Gamma, H_a, V_a\}$ with corresponding dual basis $\{dt, \theta^a, \psi^a\}$ where

$$\psi^a = du^a - F^a dt + \Gamma^a_b \theta^b.$$

The components of the curvature manifest themselves in the expression for the commutators of the horizontal fields:

$$[H_a, H_b] = R^d_{ab} V_d$$

where the curvature of the connection is defined by

$$R^d_{ab} := \frac{1}{2}\left(\frac{\partial^2 F^d}{\partial x^a \partial u^b} - \frac{\partial^2 F^d}{\partial x^b \partial u^a} + \frac{1}{2}\left(\frac{\partial F^c}{\partial u^a}\frac{\partial^2 F^d}{\partial u^c \partial u^b} - \frac{\partial F^c}{\partial u^b}\frac{\partial^2 F^d}{\partial u^c \partial u^a}\right)\right).$$

It will be useful to have some other commutators:

$$[H_a, V_b] = -\frac{1}{2}\left(\frac{\partial^2 F^c}{\partial u^a \partial u^b}\right)V_c = V_b(\Gamma^c_a)V_c = V_a(\Gamma^c_b)V_c = [H_b, V_a],$$

$$[\Gamma, H_a] = \Gamma^b_a H_b + \Phi^b_a V_b, \qquad [\Gamma, V_a] = -H_a + \Gamma^b_a V_b,$$

and, of course, $[V_a, V_b] = 0$.

2.2 The Helmholtz conditions

Now we turn to the geometric formulation of the ordinary problem in the calculus of variations. The extremals of the variational problem with regular Lagrangian $L \in C^\infty(E)$ are the base integral curves (on M) of the Euler-Lagrange equation, now represented by a SODE Γ, called the Euler-Lagrange field. The primary object of the Cartan-Hamilton formulation is the Cartan one-form, θ_L, of the Lagrangian L:

$$\theta_L := Ldt + dL \circ S.$$

In co-ordinates,

$$\theta_L = Ldt + \frac{\partial L}{\partial u^a}\theta^a.$$

The key result is (see Goldschmidt and Sternberg [12] and Sternberg [27]):

Proposition 1 *If L is a regular Lagrangian (so that the matrix whose entries are $\dfrac{\partial^2 L}{\partial u^a \partial u^b}$ is everywhere nonsingular), then there is a unique vector field Γ on E such that*

$$\Gamma \lrcorner\, d\theta_L = 0 \quad \textit{and} \quad dt(\Gamma) = 1.$$

This vector field is a SODE, *and the equations satisfied by its integral curves are the Euler-Lagrange equations for L.*

The coefficients F^a in the expression for the Euler-Lagrange field Γ are determined by the equation

$$\frac{\partial^2 L}{\partial u^a \partial u^b} F^b = \frac{\partial L}{\partial x^a} - \frac{\partial^2 L}{\partial u^a \partial x^b} u^b - \frac{\partial^2 L}{\partial u^a \partial t}.$$

If we use the basis $\{dt, \theta^a, \psi^a\}$ we obtain a particularly simple expression for the Cartan 2-form $d\theta_L$:

$$d\theta_L = \frac{\partial^2 L}{\partial u^a \partial u^b} \psi^a \wedge \theta^b$$

and this points the way to obtaining the Helmholtz conditions in a geometric form.

Notice first of all that $\dfrac{\partial^2 L}{\partial u^a \partial u^b}$ must satisfy all the conditions on the multiplier matrix g_{ab}, and secondly that, in the basis $\{dt, \theta^a, \psi^a\}$, $d\theta_L$ is completely determined by $\dfrac{\partial^2 L}{\partial u^a \partial u^b}$. These two facts indicate that we should look for a closed 2–form of maximal rank amongst 2-forms in $Sp\{\theta^a \wedge \psi^b\}$. The following theorem from [8] gives a transparent geometric version of the Helmholtz conditions.

Theorem 2 *Given a* SODE Γ, *the necessary and sufficient conditions for there to be Lagrangian for which Γ is the Euler–Lagrange field is that there should exist a 2–form Ω such that*

$$\begin{aligned}
&\Omega(V_1, V_2) = 0, \quad \forall\ V_1, V_2 \in V(E)\\
&\Gamma \lrcorner\, \Omega = 0\\
&d\Omega = 0\\
&\Omega \quad \textit{is of maximal rank.}
\end{aligned}$$

(The second and third conditions imply that $\mathcal{L}_\Gamma \Omega = 0$ and the fourth condition means that Ω has a one-dimensional kernel which the second condition shows is spanned by Γ.)

Necessity is obvious and the proof of sufficiency is a simple matter of starting with an arbitrary two-form $\Omega \in Sp\{\theta^a \wedge \theta^b, \psi^a \wedge \theta^b\}$ and showing that

$$\Omega = g_{ab}\psi^a \wedge \theta^b$$

where g_{ab} satisfies Douglas's Helmholtz conditions.

The simplest way to see how the Helmholtz conditions arise from Theorem 2 is to put $\Omega := g_{ab}\psi^a \wedge \theta^b$ and compute $d\Omega$:

$$\begin{aligned}
d\Omega = {} & (\Gamma(g_{ab}) - g_{cb}\Gamma^c_a - g_{ac}\Gamma^c_b) dt \wedge \psi^a \wedge \theta^b \\
& + (H_d(g_{ab}) - g_{cb}V_a(\Gamma^c_d))\psi^a \wedge \theta^b \wedge \theta^d \\
& + V_c(g_{ab})\psi^c \wedge \psi^a \wedge \theta^b \\
& + g_{ab}\psi^a \wedge \psi^b \wedge dt \\
& + g_{ca}\Phi^c_b \theta^a \wedge \theta^b \wedge dt \\
& + g_{ca} H_b(\Gamma^c_d)\theta^a \wedge \theta^b \wedge \theta^d.
\end{aligned}$$

The four Helmholtz conditions are

$$\begin{aligned}
&d\Omega(\Gamma, V_a, V_b) = 0, \qquad && d\Omega(\Gamma, V_a, H_b) = 0, \\
&d\Omega(\Gamma, H_a, H_b) = 0, \qquad && d\Omega(H_a, V_b, V_c) = 0.
\end{aligned}$$

The remaining conditions arising from $d\Omega = 0$, namely

$$d\Omega(H_a, H_b, V_c) = 0 \qquad \text{and} \qquad d\Omega(H_a, H_b, H_c) = 0,$$

can be shown to be derivable from the first four (notice that this last condition is void in dimension 2).

At this point it is worthwhile identifying the separate sources of the algebraic and differential conditions. Recall the identity

$$\begin{aligned}
d\Omega(X, Y, Z) = {} & X(\Omega(Y, Z)) + Y(\Omega(Z, X)) + Z(\Omega(X, Y)) \\
& - \Omega([X, Y], Z) - \Omega([Y, Z], X) - \Omega([Z, X], Y)
\end{aligned} \tag{1}$$

for an arbitrary 2-form Ω and vector fields X, Y, Z. The first 3 terms involve the derivatives of Ω and the last 3 do not. In our case using

$$\{X, Y, Z\} := \{\Gamma, H_a, H_b\}, \{\Gamma, V_a, V_b\}, \{H_a, H_b, H_c\},$$

in turn makes the corresponding conditions, $d\Omega(X, Y, Z) = 0$ purely algebraic (in g_{ab}). Any other choices produce (first order) differential conditions (except $d\Omega(V_a, V_b, V_c) = 0$ which is identically satisfied).

3 Closure conditions

We see from the foregoing we can treat the inverse problem as the one of finding closed two forms, Ω, in the submodule $\mathrm{Sp}\{\psi^a \wedge \theta^b\}$. This problem has been treated using exterior differential systems theory [2,1], but our task here is to obtain an overview at a simpler level, so we write this closure problem on an appropriate subset U of E as

$$d\Omega(X, Y, Z) = 0, \quad \forall\ X, Y, Z \in \mathfrak{X}(U), \tag{2}$$

the necessary and sufficient conditions for (2) are

$$d(d\Omega(X, Y, Z)) = 0 \quad \text{on} \quad U$$

$$\Leftrightarrow W(d\Omega(X, Y, Z)) = 0 \quad \forall\ X, Y, Z, W \in \mathfrak{X}(U) \tag{3}$$

Notice firstly that we do not have to know that $d\Omega(X, Y, Z) = 0$ at a point in U for (3) to be necessary and sufficient for (2) because if (as a result of (3)), for fixed $X, Y, Z,\ d\Omega(X, Y, Z) = k \in \mathbb{R}$ on U and $d\Omega(fX, Y, Z) = m \in \mathbb{R}$ for some non-constant $f \in \bigwedge^0(U)$ then $k = m = 0$.

However, we would like to deal with the closure conditions in terms of our $\{\Gamma, H_a, V_b\}$ basis by considering

$$W(d\Omega(X, Y, Z)) = 0 \quad \text{for all basis elements} \quad X, Y, Z, W.$$

Because

$$\begin{aligned} &d\Omega(X, Y, Z) = 0, \quad \forall\ X, Y, Z \in \mathfrak{X}(U) \\ \Longleftrightarrow\ &d\Omega(X, Y, Z) = 0, \quad \forall\ X, Y, Z \quad \text{in a basis for} \quad \mathfrak{X}(U), \end{aligned}$$

and because $W(f) = 0\ \forall\, W \in \mathfrak{X}(U) \iff W(f) = 0\ \forall\, W$ in a basis for $\mathfrak{X}(U)$, in place of (3) we only need to consider

$$W(d\Omega(X, Y, Z)) = 0 \quad \text{for basis elements} \quad W, X, Y, Z, \tag{4}$$

if we know that $d\Omega(X, Y, Z) = 0$ at some point in U.

Conditions (3) and (4) are, on the face of it, second order in g and it is of interest to identify lower order consequences of the conditions (4) in particular.

Proposition 3 *Suppose that $\Omega \in \bigwedge^2(E)$. Then $W(d\Omega(X, Y, Z)) = 0$ for all $W, X, Y, Z \in \mathfrak{X}(U)$ if and only if*

$$X(d\Omega(W, Y, Z)) + Y(d\Omega(W, Z, X)) + Z(d\Omega(W, X, Y)) = 0$$

for all $W, X, Y, Z \in \mathfrak{X}(U)$.

Proof The forward direction is straightforward since $W(d\Omega(X,Y,Z)) = 0$ is assumed true for all $W,X,Y,Z \in \mathfrak{X}(U)$. To see the reverse direction, assume for all $W,X,Y,Z \in \mathfrak{X}(U)$ that $X(d\Omega(W,Y,Z)) + Y(d\Omega(W,Z,X)) + Z(d\Omega(W,X,Y)) = 0$ and generate three similar equalities by interchange W with X,Y,Z respectively. This gives

$$\begin{aligned} W(d\Omega(X,Y,Z)) + Y(d\Omega(X,Z,W)) + Z(d\Omega(X,W,Y)) &= 0, \\ X(d\Omega(Y,W,Z)) + W(d\Omega(Y,Z,X)) + Z(d\Omega(Y,X,W)) &= 0, \\ X(d\Omega(Z,Y,W)) + Y(d\Omega(Z,W,X)) + W(d\Omega(Z,X,Y)) &= 0. \end{aligned}$$

Adding twice the original equality to the sum of the three above gives the result. □

Remark: The statement of proposition 3 continues to hold if W,X,Y,Z are taken to be basis elements.

The next lemma gives a neat first order consequence of the closure conditions (3).

Lemma 4 *If $W(d\Omega(X,Y,Z)) = 0$ for all $W,X,Y,Z \in \mathfrak{X}(U)$ then*

$$\begin{aligned} &- d\Omega(W,[X,Y],Z) - d\Omega(W,[Y,Z],X) - d\Omega(W,[Z,X],Y) \\ &+ d\Omega([W,X],Y,Z) + d\Omega(X,[W,Y],Z) + d\Omega(X,Y,[W,Z]) = 0 \end{aligned} \tag{5}$$

Proof Using the three form identity (1) twice and the Leibniz rule we get the identity:

$$\begin{aligned} W(d\Omega(X,Y,Z)) = {} & X(d\Omega(W,Y,Z)) + Y(d\Omega(W,Z,X)) + Z(d\Omega(W,X,Y)) \\ & - d\Omega(W,[X,Y],Z) - d\Omega(W,[Y,Z],X) - d\Omega(W,[Z,X],Y) \\ & + d\Omega([W,X],Y,Z) + d\Omega(X,[W,Y],Z) + d\Omega(X,Y,[W,Z]) \end{aligned} \tag{6}$$

and the result follows immediately from the assumption of the lemma. □

Remark: This lemma also holds if W,X,Y,Z are taken to be basis elements.

The next theorem will allow us to identify the redundant second order conditions amongst (4).

Theorem 5 *For all basis elements $W,X,Y,Z \in \mathfrak{X}(U)$,*

$$X(d\Omega(W,Y,Z)) = W(d\Omega(X,Z,Y)) \tag{7}$$

and

$$\begin{aligned} &-d\Omega(W,[X,Y],Z) - d\Omega(W,[Y,Z],X) - d\Omega(W,[Z,X],Y) \\ &+d\Omega([W,X],Y,Z) + d\Omega(X,[W,Y],Z) + d\Omega(X,Y,[W,Z]) = 0 \end{aligned} \tag{8}$$

if and only if $W(d\Omega(X,Y,Z)) = 0$ *for all basis elements* $W, X, Y, Z \in \mathfrak{X}(U)$.

Proof Forward direction: using the identity (6) we see that

$$
\begin{aligned}
&-d\Omega(W,[X,Y],Z) - d\Omega(W,[Y,Z],X) - d\Omega(W,[Z,X],Y) \\
&+d\Omega([W,X],Y,Z) + d\Omega(X,[W,Y],Z) + d\Omega(X,Y,[W,Z]) = 0 \\
\Longleftrightarrow\ & W(d\Omega(X,Y,Z)) = X(d\Omega(W,Y,Z)) + Y(d\Omega(W,Z,X)) + Z(d\Omega(W,X,Y)).
\end{aligned}
$$

Using $X(d\Omega(W,Y,Z)) = W(d\Omega(X,Z,Y)), Y(d\Omega(W,Z,X)) = W(d\Omega(Y,X,Z))$ and $Z(d\Omega(W,X,Y)) = W(d\Omega(Z,Y,X)$ on the right hand side of this expression gives the result.
Backward direction: this follows directly from lemma 4 and the trivial statement that

$$
(\forall\ W,X,Y,Z)\ W(d\Omega(X,Y,Z)) = 0 \implies X(d\Omega(W,Y,Z)) = W(d\Omega(X,Z,Y))
$$

□

Remark: Conditions (8) appear because W, X, Y, Z are basis elements: if (7) holds for all $W, X, Y, Z \in \mathfrak{X}(U)$ then (3) holds immediately (for example, replace X by fX in (7)).

Now we will deduce some properties of the basis expressions $W(d\Omega(X,Y,Z))$ and the conditions (4) as a result of the equivalence demonstrated in theorem 5 of the joint conditions (8), (7) to (4).

We order the conditions (4) as

$$
W(d\Omega(\Gamma,H_a,H_b)) = 0,\ W(d\Omega(\Gamma,V_a,V_b)) = 0,\ W(d\Omega(H_a,H_b,H_c)) = 0, \tag{9}
$$
$$
W(d\Omega(V_a,V_b,V_c)) = 0,\ W(d\Omega(W,Y,Z)) = 0, \tag{10}
$$

$$
H_a(d\Omega(V_c,V_c,H_b)) = 0,\ V_c(d\Omega(H_a,H_b,V_d)) = 0, \tag{11}
$$

$$
\Gamma(d\Omega(V_c,H_a,H_b)) = 0,\ \Gamma(d\Omega(H_c,V_a,V_b)) = 0, \tag{12}
$$
$$
H_a(d\Omega(V_c,\Gamma,H_b)) = 0,\ V_a(d\Omega(H_c,\Gamma,V_b)) = 0, \tag{13}
$$
$$
H_c(d\Omega(V_d,H_a,H_b)) = 0,\ V_a(d\Omega(H_d,V_b,V_c)) = 0. \tag{14}
$$

Because $d\Omega(\Gamma,H_a,H_b),\ d\Omega(\Gamma,V_a,V_b),\ d\Omega(H_a,H_b,H_c)$ are algebraic in g_{ab}, conditions (9) are first order and the first of conditions (10) is identically true because $d\Omega(V_a,V_b,V_c)) = 0$. The remaining conditions are potentially second order.

Suppose now that (7) holds for all basis elements W, X, Y, Z ((8) does not necessarily hold so that we are not assuming (4)). The second of conditions (10) are now identically zero because conditions (7) impose symmetry in Y, Z on

$W(d\Omega(W,Y,Z))$. Further, if (7) holds then conditions (9) and (10) imply conditions (12), (13) and (14). The contrapositive of this statement is useful: if (9) and (10) do not imply conditions (12), (13) and (14) then (7) does not hold and there is no solution of the Helmholtz conditions. In this way the second order conditions (11) are distinguished amongst the second order closure conditions (4) as not necessarily being a consequence of the lower order conditions (9) and (10) in order for solutions to exist.

We now examine condition

$$H_a(d\Omega(V_c,H_b,V_d)=V_c(d\Omega(H_a,V_d,H_b)$$

of theorem 5, the "A conditions" of [24,25], along with the corresponding joint condition (8).

The expanded internal structure of the generic condition (5) is informative: using the three form identity (1) and the Jacobi identity on (5) gives

$$\begin{aligned}
&- W(\Omega([X,Y],Z)) - W(\Omega([Y,Z],X)) - W(\Omega([Z,X],Y))\\
&+ Z(\Omega([X,Y],W)) + Z(\Omega([W,X],Y)) + Z(\Omega([Y,W],X))\\
&+ X(\Omega([Y,Z],W)) + X(\Omega([W,Y],Z)) + X(\Omega([Z,W],Y))\\
&+ Y(\Omega([Z,X],W)) + Y(\Omega([X,W],Z)) + Y(\Omega([W,Z],X))\\
&- [X,Y](\Omega(Z,W)) - [Y,Z](\Omega(X,W)) - [Z,X](\Omega(Y,W))\\
&+ [W,X](\Omega(Y,Z)) + [W,Y](\Omega(Z,X)) + [W,Z](\Omega(X,Y)) = 0.
\end{aligned} \tag{15}$$

Putting $W:=H_a,\ X:=V_c,\ Y:=V_d$ and $Z:=H_b$ in (15) gives a case of (8)

$$\begin{aligned}
&V_c(\Omega([V_d,H_b],H_a)) + V_c(\Omega([H_a,V_d],H_b))\\
&+ V_d(\Omega([H_b,V_c],H_a)) + V_d(\Omega([V_c,H_a],H_b))\\
&- [V_d,H_b](\Omega(V_c,H_a)) - [H_b,V_c](\Omega(V_d,H_a))\\
&+ [H_a,V_c](\Omega(V_d,H_b)) + [H_a,V_d](\Omega(H_b,V_c)) = 0,
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&V_d(\Gamma^e_b)\,(V_e(g_{ca})-V_c(g_{ea})) + V_d(\Gamma^e_a)\,(V_c(g_{eb})-V_e(g_{cb}))\\
&+ V_c(\Gamma^e_b)\,(V_d(g_{ea})-V_e(g_{da})) + V_a(\Gamma^e_c)\,(V_e(g_{db})-V_d(g_{eb}))=0.
\end{aligned}$$

The identity (6) shows that the above expression is equivalent to the necessary condition

$$\begin{aligned}
H_a(d\Omega(V_c,V_d,H_b)) &= V_c(d\Omega(H_a,V_d,H_b))+V_d(d\Omega(H_a,H_b,V_c))\\
&\quad + H_b(d\Omega(H_a,V_c,V_d)).
\end{aligned}$$

In a similar manner to the proof of theorem 5, this condition taken with the "A conditions" $H_a(d\Omega(V_c,H_b,V_d)=V_c(d\Omega(H_a,V_d,H_b)$ is sufficient to establish $H_a(d\Omega(V_c,V_d,H_b)=0$ and $V_c(d\Omega(H_a,H_b,V_d)=0$.

In this note we have shown that the A conditions appear naturally as members of the class (7) which, along with the first order conditions (8), are equivalent to the standard closure conditions (4). Moreover, the corresponding conditions (11) are distinguished amongst the other potentially second order conditions in (4) by not being necessary consequences of the lower order closure conditions (9) and (10) in order for solutions to exist.

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Lagrangian equations and affine Lie algebroids

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1 Introduction

Lagrangian equations on Lie algebroids are the leitmotiv for this text, but large parts of it are excursions into general features such as the concept of an affine Lie algebroid and, even more generally, generalised connections on an affine bundle and affineness of such connections. As such, it is a review of recent work which was carried out jointly with Eduardo Martínez and Tom Mestdag [7,10,11] and which constitutes also part of the PhD work to be submitted by Tom Mestdag in 2003. I am grateful to these co-authors for letting me use the results of our joint efforts for this occasion.

My contribution to the Colloquium in Ghent (November 2002) was the presentation of an overview of the activities at the *Workshop on differential geometric methods in theoretical mechanics*, since its creation in 1986. The reason for that is the fact that Mike Crampin was to a large extent the initiator of this workshop and that it proved to be a very successful organisation over the years. I therefore chose to let my presentation at the 17th edition of this workshop in Levico Terme, Italy (September 2002), be the core of my contribution to this special volume.

2 Lagrangian equations on a Lie algebroid

Let us first have a look at the analytical format of Lagrangian equations on a Lie algebroid. The by now familiar analytical expression of such equations read:

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha(x)\, y^\alpha \\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) &= \rho^i_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}\,, \qquad L \in C^\infty(V). \end{aligned} \tag{1}$$

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The underlying geometrical structure is that the coordinates y^α are the fibre coordinates of a vector bundle $\pi : V \to M$, x^i being the coordinates on the base M; the functions ρ^i_α represent the so-called anchor map, which is a vector bundle map from V into TM; the $C^\gamma_{\alpha\beta}$ are the structure functions coming from a bracket defined on sections of π, and there are some compatibility conditions to be satisfied, roughly coming from a compatibility between the bracket on $Sec(\pi)$ and the Lie bracket of vector fields on M. Among these, we mention

$$\rho^i_\alpha \frac{\partial \rho^j_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^j_\alpha}{\partial x^i} = \rho^j_\gamma C^\gamma_{\alpha\beta}\,. \tag{2}$$

For more details, see for example [6,13].

What I would like to indicate here already is that, if the main interest would be to model equations of type (1)(2), there is room for generalisation. For example, if one tries to derive such kind of equations from a (formal) calculus of variations approach, there is no need to assume that the bracket on $Sec(\pi)$ satisfies a Jacobi identity.

My own involvement in the subject (always in collaboration with Eduardo and Tom) started from the question: "What would be a time-dependent generalisation of such systems?" The claim is that such a generalisation will give rise to equations of the following type:

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha(t,x) y^\alpha + \rho^i_0(t,x) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) &= \rho^i_\alpha \frac{\partial L}{\partial x^i} + (C^\gamma_{\beta\alpha} y^\beta + C^\gamma_{0\alpha}) \frac{\partial L}{\partial y^\gamma}\,, \end{aligned} \tag{3}$$

where this time the $\rho^i_\alpha, \rho^i_0, C^\gamma_{\alpha\beta}, C^\gamma_{0\alpha}$ are functions of t and x satisfying,

$$\rho^i_\alpha \frac{\partial \rho^j_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^j_\alpha}{\partial x^i} = \rho^j_\gamma C^\gamma_{\alpha\beta} \tag{4}$$

$$\frac{\partial \rho^j_\beta}{\partial t} + \rho^i_0 \frac{\partial \rho^j_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^j_0}{\partial x^i} = \rho^j_\alpha C^\alpha_{0\beta}. \tag{5}$$

Notice that one sees a certain affineness entering the equations here and of course, the extra time coordinate makes that there is a zero component of the structure functions and a corresponding extra compatibility condition. The usual framework for time-dependent mechanics in general and time-dependent Lagrangian mechanics in particular, is the first jet bundle J^1M of a manifold M fibred over $\mathbb{R}$ (cf. [2]). Therefore, a natural extension of the Lie algebroid generalisation is to consider an anchor map with values in J^1M rather than in TM and whose domain may then just as well be an affine bundle $E \to M$ rather than a vector bundle. If one does that, the result is a theory which we described in [11] and is centred around the following diagram.

$$
\begin{tikzcd}
 & J^1E \arrow[dd] \arrow[dr] & & \\
J^1_\rho E \arrow[ur, "\rho^1"] \arrow[rr, "\pi_1"'] \arrow[dd, "\pi_2"] & & E \arrow[dd, "\pi"'] \arrow[dr] & \\
 & J^1M \arrow[dr] & & \mathbb{R} \\
E \arrow[ur, "\rho"] \arrow[rr, "\pi"'] & & M \arrow[ur, "\tau"'] &
\end{tikzcd}
$$

Without going into any detail now, let me briefly point out the main ingredients and features of this diagram. The bottom part is the scheme of an affine Lie algebroid. The bundle $E \to M$ appears on the right again, with its own first jet bundle J^1E. $J^1_\rho E$ is in fact the pullback bundle of J^1E under ρ. An important point is, however, as discussed first by Mackenzie [5] and fully exploited for standard Lie algebroids in [6], that one should look at the total space of this pullback bundle as being fibred over E via $\tau_E \circ \rho^1$ (with less emphasis on the usual projections of a pullback bundle, here called π_2 and ρ^1). If one does so, one discovers that there is a kind of complete lift from the Lie algebroid structure at the bottom to one at the top, and the Lagrange equations shown above should be regarded as coming from special sections of the *prolonged bundle* $J^1_\rho E \to E$.

However, it is possible to work in a more general framework. This will in particular be fruitful for exploring the affine nature of a Lie algebroid in all generality, and for some of the aspects of the digression I want to make now, one does not even need the full structure of an algebroid.

3 Playing with diagrams to understand generalised connections

Consider the following general scheme as depicted on the diagram below: $\tau : V \to M$ is a vector bundle; $\rho : V \to TM$ is an "anchor map", to be understood here as a vector bundle morphism about which no further structure is assumed at the moment; $\mu : P \to M$ is an arbitrary fibre bundle.

Definition: *A ρ-connection on the bundle μ is a linear bundle map $h : \mu^*V \to TP$, such that the following diagram commutes: $\rho \circ p_V = T\mu \circ h$.*

The best source for a general study of ρ-connections is [1].

$$\begin{tikzcd}
 & TP \arrow[dd] \arrow[dr, "\tau_P"] & \\
\mu^*V \arrow[ur, "h"] \arrow[rr, "p"'] \arrow[dd, "p_V"'] & & P \arrow[dd, "\mu"] \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}$$

The form of the preceding picture no doubt is reminiscent of the first one. For comparison, therefore, let us discuss the prolongation idea in some more detail in this more general context. In the next picture, the right part is the same as in the preceding one, but rather than pulling V back along μ, we pull TP back along ρ. The total space

$$T^\rho P = \{(v, X_p) \in V \times TP \mid \rho(v) = T\mu(X_p)\}$$

is not called $\rho^* TP$, however, because the fibration we are primarily interested in is not $\rho^1 : \rho^* TP \to TP$ or $\mu^2 : \rho^* TP \to V$, but $\mu^1 = \tau_P \circ \rho^1$. The bundle $\mu^1 : T^\rho P \to P$ is called the *ρ-prolongation* of $\mu : P \to M$.

$$\begin{tikzcd}
 & TP \arrow[dd] \arrow[dr, "\tau_P"] & \\
T^\rho P \arrow[ur, "\rho^1"] \arrow[rr, "\mu^1"'] \arrow[dd, "\mu^2"'] & & P \arrow[dd, "\mu"] \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}$$

The ρ-prolongation in many respects has the features of a tangent bundle. For example, there is a vertical subbundle $\mathcal{V}^\rho P := \ker \mu^2 = \{(0, Q)\} \subset T^\rho P$, and there are also deeper similarities, upon which we will not dwell here, however.

The sort of overall structure of this diagram is the same as in the previous one, in the sense that, by the very construction of a pullback bundle, there is a commuting diagram around the anchor map here as well. In fact, this is the reason why I prefer to keep representing points of $T^\rho P$ as a couple of elements, one from V and the other one a tangent vector to P at some point p, whereby this base point in the fibration over P thus is not given a separate entry in the notation.

In view of the similarity in structure, it is tempting to put the last diagram on top of the previous one, which would require pushing the two competing spaces apart. This, in fact, can easily be done because $T^\rho P$ is naturally fibred over μ^*V. The result is the following overall diagram.

$$
\begin{tikzcd}
 & TP \arrow[ddr, "\tau_P"] \arrow[dddd, "T\mu"] & \\
T^\rho P \arrow[ur, "\rho^1"] \arrow[drr, "\mu^1"] \arrow[dd, "j"'] & & \\
 & & P \arrow[ddd, "\mu"] \\
\mu^*V \arrow[uuur, "h"] \arrow[urr, "p"] \arrow[dd, "p_V"'] & & \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}
$$

Having brought the fibration j into the picture and thinking of the injection of the vertical subbundle $\mathcal{V}^\rho P$ into $T^\rho P$, we are facing a short exact sequence

$$0 \to \mathcal{V}^\rho P \to T^\rho P \xrightarrow{j} \mu^*V \to 0.$$

This suggests a way of defining a possibly different kind of ρ-connection on μ, namely as a splitting of this sequence or, in other words, as a horizontal lift operation ^{H} from μ^*V (or sections of it) to $T^\rho P$. We then have a direct sum decomposition

$$T^\rho P = \mathcal{H}^\rho P \oplus \mathcal{V}^\rho P,$$

with corresponding horizontal and vertical projectors P_H and P_V, as in the usual theory of non-linear connections on a tangent bundle. The point is that these two different looking notions of generalised connection are completely equivalent [10], and we have $\rho^1 \circ {}^H = h$.

It is of some interest, however, to point out that the second view on ρ-connections has some advantages over the first. To begin with, there is no ambiguity in the decomposition of sections of μ^1 into horizontal and vertical ones, as opposed to attempts to use the map h for defining horizontality in TP, which then creates a number of complications [1]. Also the concept of *connection map* (see e.g. [12]) may be somewhat more transparent in the second point of view. In the first approach, we immediately spot from our overall diagram, more particularly from the two commuting diagrams over ρ, that $\rho^1 - h \circ j$ yields a vertical vector on P. In the particular case that P is a vector bundle, this can be identified with an element of P itself, yielding a map $K : T^\rho P \to P$. In the second approach, K is essentially P_V. Note: the connection map is a very useful instrument to define an associated covariant derivative operator when the ρ-connection is linear.

4 Affineness of a ρ-connection

Let us make a further digression now and replace for a start the arbitrary bundle $\mu : P \to M$ by an *affine* bundle $\pi : E \to M$, modelled on the vector bundle $\overline{\pi} : \overline{E} \to M$, say. Put $E^\dagger_m := \mathrm{Aff}(E_m, \mathbb{R})$, the set of affine functions on E_m, let $E^\dagger = \bigcup_{m\in M} E^\dagger_m$ denote the 'extended dual' of E, which is a vector bundle over M, and consider the bidual $\tilde{\pi} : \tilde{E} := (E^\dagger)^* \to M$, which is a vector bundle containing E and $\overline{E}$ via canonical injections: $\iota(E)$ and $\boldsymbol{\iota}(\overline{E})$.

Now I take two of the overall diagrams, one with the affine $E \to M$ in the position of the general bundle $P \to M$, and the other with P replaced by $\tilde{E} \to M$.

$$\begin{tikzcd}
 & TE \arrow[dddd, "\pi^1"] \arrow[ddr, "\tau_E"] & \\
T^\rho E \arrow[ur, "\rho^1_E"] \arrow[drr] \arrow[dd, "j"'] & & \\
 & & E \arrow[ddd, "\pi"] \\
\pi^* V \arrow[uuur, "h"] \arrow[urr, "p"] \arrow[dd, "p_V"'] & & \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T\tilde{E} \arrow[dddd, "\tilde{\pi}^1"] \arrow[ddr, "\tau_{\tilde{E}}"] & \\
T^\rho \tilde{E} \arrow[ur, "\rho^1_{\tilde{E}}"] \arrow[drr] \arrow[dd, "\tilde{j}"'] & & \\
 & & \tilde{E} \arrow[ddd, "\tilde{\pi}"] \\
\tilde{\pi}^* V \arrow[uuur, "\tilde{h}"] \arrow[urr, "\tilde{p}"] \arrow[dd, "\tilde{p}_V"'] & & \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}$$

Again, a good definition of affineness of a ρ-connection h becomes quite apparent by inspection of these diagrams: it is essentially the commutation of the diagram which links h to $\tilde{h}$ via canonical injections.

Definition: *A ρ-connection h on $\pi : E \to M$ is affine, if there exists a linear ρ-connection $\tilde{h} : \tilde{\pi}^* V \to T\tilde{E}$ on $\tilde{\pi} : \tilde{E} \to M$ such that $\tilde{h} \circ \iota = T\iota \circ h$, as maps from $\pi^* V$ into $T\tilde{E}$.*

It is about time to illustrate these notions by looking at coordinate expressions now.

With x^i, y^α coordinates on $\pi : E \to M$ and $(e_0; \{\mathbf{e}_\alpha\})$ a local frame for $Sec(\pi)$, denote the induced basis for $Sec(\pi^\dagger)$ by (e^0, e^α), meaning that $\forall\, a \in Sec(\pi),\ a(x) = e_0(x) + a^\alpha(x)\boldsymbol{e}_\alpha(x)$ say, we have

$$e^0(a)(x) = 1,\ \forall x, \qquad e^\alpha(a)(x) = a^\alpha(x).$$

Observe hereby that e^0 is actually globally defined! Let then (e_0, e_α) denote the dual basis for $Sec(\tilde{\pi})$, so that $\iota(e_0) = e_0$ and $\boldsymbol{\iota}(\boldsymbol{e}_\alpha) = e_\alpha$. Finally we write

$(x^i, y^A) = (x^i, y^0, y^\alpha)$ for the induced coordinates on $\tilde{E}$, and (x^i, v^a) for the coordinate representation of a point $v \in V$.

The anchor map $\rho : V \to TM$ is of the form $(x^i, v^a) \mapsto \rho^i_a(x) v^a \frac{\partial}{\partial x^i}$; the map $h : \pi^*V \to TE$ in general will look as follows:

$$h(x^i, y^\alpha, v^a) = (x^i, y^\alpha, \rho^i_a(x)v^a, -\Gamma^\alpha_a(x,y)v^a), \tag{6}$$

the minus sign before the connection coefficients being a matter of convention. Now, affineness of the ρ-connection on π means that the connection coefficients are of the form:

$$\Gamma^\alpha_a(x,y) = \Gamma^\alpha_{a0}(x) + \Gamma^\alpha_{a\beta}(x)y^\beta. \tag{7}$$

In the equivalent representation ${}^H : \pi^*V \to T^\rho E$ of the ρ-connection, this becomes:

$$(x^i, y^\alpha, v^a) \overset{H}{\mapsto} \left((x^i, v^a), v^a\left(\rho^i_a \frac{\partial}{\partial x^i} - \Gamma^\alpha_a \frac{\partial}{\partial y^\alpha}\right)\right). \tag{8}$$

Now that we are bringing the prolonged bundle into the picture, if $\boldsymbol{v}_a$ denotes a local basis of sections of $\tau : V \to M$, then a standard local basis of sections of $\pi^1 : T^\rho E \to E$ is given by

$$\mathcal{X}_a(e) = \left(\boldsymbol{v}_a(x),\, \rho^i_a(x)\frac{\partial}{\partial x^i}\bigg|_e\right), \qquad \mathcal{V}_\alpha(e) = \left(0,\, \frac{\partial}{\partial y^\alpha}\bigg|_e\right). \tag{9}$$

Notice that there is a canonical vertical lift ${}^V : \pi^*\overline{E} \to T^\rho E$, which is such that the $\mathcal{V}_\alpha$ are roughly the vertical lifts of the basis vectors for $\overline{E}$, more precisely: $\mathcal{V}_\alpha(e) = \big(e, \mathbf{e}_\alpha(\pi(e))\big)^V$. Whenever there is a given a ρ-connection, it will be more suitable to do coordinate calculations on the prolonged bundle with respect to an adapted local basis, which consists of horizontal and vertical sections. A basis for the horizontal sections is given by:

$$\mathcal{H}_a = P_H(\mathcal{X}_a) = \mathcal{X}_a - \Gamma^\alpha_a(x,y)\mathcal{V}_\alpha. \tag{10}$$

Just a few words now, to finish this section, about covariant derivatives in this context. If the ρ-connection is *affine*, there is an associated covariant derivative operator $\nabla : Sec(\tau) \times Sec(\pi) \to Sec(\overline{\pi})$, which in coordinates will look as follows. For $\zeta = \zeta^a(x)\boldsymbol{v}_a \in Sec(\tau)$ and $\sigma = e_0 + \sigma^\alpha(x)\mathbf{e}_\alpha \in Sec(\pi)$:

$$\nabla_\zeta \sigma = \left(\frac{\partial \sigma^\alpha}{\partial x^i}\rho^i_a(x) + \Gamma^\alpha_{a0}(x) + \Gamma^\alpha_{a\beta}(x)\sigma^\beta(x)\right)\zeta^a(x)\,\mathbf{e}_\alpha. \tag{11}$$

In fact, the affine ρ-connection can be completely characterised by such a ∇, having the intrinsic properties: for all $f \in C^\infty(M)$,

$$\nabla_{f\zeta}\sigma = f\,\nabla_\zeta\sigma \tag{12}$$

$$\nabla_\zeta(\sigma + f\boldsymbol{\eta}) = \nabla_\zeta\sigma + f\,\overline{\nabla}_\zeta\boldsymbol{\eta} + \rho(\zeta)(f)\,\boldsymbol{\eta}, \tag{13}$$

where $\overline{\nabla}$ is the covariant derivative associated to the linear ρ-connection $\overline{h}$ on $\overline{\pi}$, obtained for example by restricting $\tilde{h}$ to $\overline{\pi}^*V$.

5 Back to algebroids

Now that we know what affineness of a ρ-connection means, I want to put more structure in the anchor map again and define affineness of a Lie algebroid in all generality (see [7,3]). With $\pi : E \to M$ still being an affine bundle, and putting the concept of connections aside for the moment, the picture of interest now arises from taking as vector bundle $V \to M$ the dual of the extended dual of E, i.e. $\tilde{\pi} : \tilde{E} \to M$.

$$\begin{tikzcd}
 & TE \arrow[dr, "\tau_E"] \arrow[dd] & \\
T^{\tilde{\rho}}E \arrow[ur, "\tilde{\rho}^1"] \arrow[rr, "\pi^1"'] \arrow[dd, "\pi^2"'] & & E \arrow[dd, "\pi"] \\
 & TM \arrow[dr, "\tau_M"] & \\
\tilde{E} \arrow[ur, "\tilde{\rho}"] \arrow[rr, "\tilde{\pi}"'] & & M
\end{tikzcd}$$

The plan is to define a Lie algebroid structure on π and explain its relation to an algebroid structure on $\tilde{\pi}$. In fact, since $\tilde{E}$ contains E, I want to start from an anchor map ρ on E and explain how this extends to an anchor $\tilde{\rho}$ on $\tilde{E}$.

Definition: *A Lie algebroid on an affine bundle $\pi : E \to M$ (modelled on $\overline{\pi} : \overline{E} \to M$), consists of:*

(i) a Lie algebra structure on $Sec(\overline{\pi})$ (over $\mathbb{R}$), with associated bracket $[\ ,\ ]$;
(ii) an action by derivations of $Sec(\pi)$ on $Sec(\overline{\pi})$ (over $\mathbb{R}$)

$$D_\zeta(\lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2) = \lambda_1 D_\zeta\boldsymbol{\sigma}_1 + \lambda_2 D_\zeta\boldsymbol{\sigma}_2 \in Sec(\overline{\pi}), \quad \lambda_i \in \mathbb{R}$$

$$D_\zeta[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] = [D_\zeta\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] + [\boldsymbol{\sigma}_1, D_\zeta\boldsymbol{\sigma}_2],$$

compatible with the bracket on $Sec(\overline{\pi})$, in the sense that

$$D_{\zeta+\boldsymbol{\sigma}}\boldsymbol{\eta} = D_\zeta\boldsymbol{\eta} + [\boldsymbol{\sigma}, \boldsymbol{\eta}];$$

[(i) and (ii) define an affine Lie algebra structure]
(iii) an affine anchor map $\rho : E \to TM$, such that

$$D_\zeta(f\boldsymbol{\sigma}) = f\, D_\zeta\boldsymbol{\sigma} + \rho(\zeta)(f)\,\boldsymbol{\sigma}, \qquad f \in C^\infty(M).$$

It is often convenient to write $D_\zeta\boldsymbol{\sigma}$ as a bracket $[\zeta, \boldsymbol{\sigma}]$ also, and in fact this makes even more sense since it is easy to extend the affine Lie algebroid to a vector Lie algebroid on $\tilde{\pi} : \tilde{E} \to M$ as follows. For $\zeta = f\,\iota(\zeta_0) + \boldsymbol{\iota}(\boldsymbol{\eta}) \in Sec(\tilde{\pi})$, where $\zeta_0 \in Sec(\pi)$ is an arbitrary reference section, define the anchor map $\tilde{\rho}$ as

$$\tilde{\rho}(\zeta) = f\,\rho(\zeta_0) + \boldsymbol{\rho}(\boldsymbol{\eta}), \qquad \boldsymbol{\rho}: \text{ linear part of } \rho, \tag{14}$$

and the bracket of two such ζ_i by

$$[\zeta_1, \zeta_2] = \Big(\tilde{\rho}(\zeta_1)(f_2) - \tilde{\rho}(\zeta_2)(f_1)\Big)\iota(\zeta_0) + \boldsymbol{\iota}\Big([\boldsymbol{\eta}_1, \boldsymbol{\eta}_2] + f_1 D_{\zeta_0}\boldsymbol{\eta}_2 - f_2 D_{\zeta_0}\boldsymbol{\eta}_1\Big). \tag{15}$$

It can be shown that these definitions do not depend on the choice of ζ_0.

The following result was proved in [7]: there is a one to one correspondence between Lie algebroids on the affine bundle $\pi : E \to M$ and Lie algebroids on the bidual $\tilde{\pi} : \tilde{E} \to M$ which have the property that the bracket of two elements belonging to E, belongs to the vector bundle $\overline{E}$ on which E is modelled:

$$[\iota(\sigma_1), \iota(\sigma_2)] \subset \operatorname{Im} \boldsymbol{\iota}.$$

In coordinates, if $(e_0; \{\mathbf{e}_\alpha\})$ is a local basis for $Sec(\pi)$ and we consider the induced basis $(e_A) = (e_0, e_\alpha)$ for $Sec(\tilde{\pi})$ as before, the brackets of an affine Lie algebroid structure are of the form

$$[e_0, e_0] = 0, \quad [e_o, e_\alpha] = C^\gamma_{0\alpha} e_\gamma, \quad [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma, \tag{16}$$

and for the anchor and its extension, we have

$$\rho(e_0 + y^\alpha \mathbf{e}_\alpha) = (\rho^i_0 + \rho^i_\alpha y^\alpha)\frac{\partial}{\partial x^i} \tag{17}$$

$$\tilde{\rho}(y^0 e_0 + y^\alpha e_\alpha) = (\rho^i_0 y^0 + \rho^i_\alpha y^\alpha)\frac{\partial}{\partial x^i} = \rho^i_A y^A \frac{\partial}{\partial x^i}. \tag{18}$$

For completeness, this is the way the compatibility property $[\tilde{\rho}(e_A), \tilde{\rho}(e_\alpha)] = \tilde{\rho}([e_A, e_\alpha])$ looks like in coordinates:

$$\rho^i_A \frac{\partial \rho^j_\alpha}{\partial x^i} - \rho^i_\alpha \frac{\partial \rho^j_A}{\partial x^i} = C^\gamma_{A\alpha}\rho^j_\gamma, \tag{19}$$

and the Jacobi identity reads

$$\sum_{A,B,\gamma}\left(\rho^i_A \frac{\partial C^\mu_{B\gamma}}{\partial x^i} + C^\mu_{A\nu} C^\nu_{B\gamma}\right) = 0. \tag{20}$$

The extension of the affine Lie algebroid to its vector counterpart on $\tilde{E}$ often simplifies matters when it comes to defining further concepts and operations. Let us look at the concept of differential forms on an affine algebroid to illustrate this point. The problem is of course that one roughly wants to think of a skew-symmetric multilinear map, but sections of π cannot be multiplied by functions. A definition of a k-form without recourse to $\tilde{E}$ overcomes this difficulty as follows.

Definition: *A k-form on $Sec(\pi)$, $\omega \in \bigwedge^k(\pi^\dagger)$, is a map $\omega : Sec\,\pi \times \cdots \times Sec\,\pi \to C^\infty(M)$ for which there exist maps $\omega_0, \boldsymbol{\omega}$, where*

$$\omega_0 : Sec\,\pi \times Sec\,\overline{\pi} \times \cdots \times Sec\,\overline{\pi} \to C^\infty(M)$$

is skew-symmetric and linear in its $k-1$ vector arguments, and $\boldsymbol{\omega}$ is a (standard) k-form on $Sec\,\overline{\pi}$, such that

$$\omega_0(\zeta + \boldsymbol{\sigma}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{k-1}) = \omega_0(\zeta, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{k-1}) + \boldsymbol{\omega}(\boldsymbol{\sigma}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{k-1}), \tag{21}$$

and for any reference section ζ_0:

$$\omega(\zeta_1, \ldots, \zeta_k) = \sum_{i=1}^{k} (-1)^{i-1} \omega_0(\zeta_0, \boldsymbol{\zeta}_1, \ldots, \hat{\boldsymbol{\zeta}}_i, \ldots, \boldsymbol{\zeta}_k) + \boldsymbol{\omega}(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k)\,. \tag{22}$$

This construction is of some interest in its own right, but life becomes easier if one observes that a k-form on $Sec(\pi)$ is just the pullback of a form on $Sec(\tilde{\pi})$ under the canonical injection: $\omega = \iota^*(\tilde{\omega})$ say. Once this is clear, one can for example immediately define the exterior derivative of forms on $Sec(\pi)$ by: $d\omega = \iota^*(d\tilde{\omega})$.

In coordinates, a k-form on $Sec(\pi)$ is of the form

$$\omega = \frac{1}{(k-1)!}\, \omega_{0\mu_1\cdots\mu_{k-1}}\, e^0 \wedge e^{\mu_1} \wedge \cdots \wedge e^{\mu_{k-1}} + \frac{1}{k!}\, \omega_{\mu_1\cdots\mu_k}\, e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}, \tag{23}$$

with coefficients in $C^\infty(M)$, skew-symmetric in all indices. The exterior derivative of forms is determined by: $df = \rho^i_A \frac{\partial f}{\partial x^i}\, e^A$, for $f \in C^\infty(M)$, and

$$de^0 = 0, \qquad de^\alpha = -C^\alpha_{0\beta} e^0 \wedge e^\beta - \tfrac{1}{2} C^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma. \tag{24}$$

The following observations are more important now. Going back to our prolongation picture of the beginning of this section, let us move upwards in the diagram. As we know [6], there is an inherited (vector) Lie algebroid structure on the prolonged bundle $\pi^1 : T^{\tilde{\rho}}E \to E$. The point is that this is again one of the type which gives rise to (or comes from) an affine Lie algebroid. Indeed, the space

$$\mathcal{J}^\rho E = \{\, (e, X_e) \in E \times TE \mid \rho(e) = T\pi(X_e) \,\}\,,$$

is the affine bundle of which the bidual is $T^{\tilde{\rho}}E$. With respect to the local frame of sections $(\mathcal{X}_A, \mathcal{V}_\alpha)$ of $Sec(\pi^1)$, which was discussed in the more general context of the preceding section, the Lie algebroid brackets of the prolonged bundle are given by

$$[\mathcal{X}_A, \mathcal{X}_B] = C^\alpha_{AB} \mathcal{X}_\alpha, \quad [\mathcal{X}_A, \mathcal{V}_\alpha] = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0. \tag{25}$$

The corresponding exterior derivative is determined by

$$dx^i = \rho^i_A \mathcal{X}^A, \qquad dy^\alpha = \mathcal{V}^\alpha \tag{26}$$

$$d\mathcal{X}^\alpha = -\tfrac{1}{2} C^\alpha_{AB} \mathcal{X}^A \wedge \mathcal{X}^B, \quad d\mathcal{X}^0 = 0, \quad d\mathcal{V}^\alpha = 0. \tag{27}$$

6 And now Lagrangian equations again

I need two more concepts now before I can return to my starting point, Lagrangian equations, but now on general affine bundles, without the underlying motivation of time-dependent mechanics, i.e. without the assumption of a further fibration $M \to \mathbb{R}$. The first one is a vertical endomorphism, the second is a notion of "second-order differential equation field" on the prolonged bundle.

There is a *canonical map* $\vartheta : \pi^*\tilde{E} \to \pi^*\overline{E} \subset \pi^*\tilde{E}$, which is defined as follows. For a given $a \in E$, any $z \in \tilde{E}$ is of the form $z = \lambda(z)\iota(a) + \boldsymbol{\iota}(\boldsymbol{v})$, with $\lambda(z) \in \mathbb{R}$. As a result, we can define

$$\vartheta(a, z) = (a, z - \lambda(z)\iota(a)), \tag{28}$$

leading to an operator which extends to sections of the corresponding bundles and has coordinate representation: $\vartheta = (e^\alpha - y^\alpha e^0) \otimes e_\alpha$.

It was already mentioned that there is a vertical lift from $\pi^*\overline{E}$ to $T^{\tilde{\rho}}E$ (with $V = \tilde{E}$ here). With the aid of ϑ, it can now be extended to

$$^{V} : \pi^*\tilde{E} \to T^{\tilde{\rho}}E, \quad ^{V} : (a, z) \mapsto \left(0_{\pi(a)}, \vartheta(a, z)^{V}\right).$$

Applied to sections, we have: if $\zeta = \zeta^0 e_0 + \zeta^\alpha e_\alpha \in Sec(\tilde{\pi})$, then $\zeta^{V} = (\zeta^\alpha - y^\alpha \zeta^0)\mathcal{V}_\alpha \in Sec(\pi^1)$. In turn this leads, just as in the standard theory of first-jet bundles, to the *vertical endomorphism* $S : Sec(\pi^1) \to Sec(\pi^1)$, given in coordinates by

$$S = (\mathcal{X}_\alpha - y^\alpha \mathcal{X}_0) \otimes \mathcal{V}_\alpha. \tag{29}$$

As for the second ingredient, we actually talk about pseudo-Sodes here, because the resulting differential equations will not strictly be second-order ordinary differential equations. Now that we have S at our disposal, and remembering that the section $\mathcal{X}^0$ of the extended dual is actually globally defined, a simple way of defining pseudo-Sodes goes as follows.

Definition: *A pseudo-*Sode *on the affine $\pi : E \to M$ is a section Γ of $\pi^1 : T^{\tilde{\rho}}E \to E$ such that*

$$S(\Gamma) = 0, \qquad \langle \Gamma, \mathcal{X}^0 \rangle = 1.$$

Locally, Γ is of the form

$$\Gamma = \mathcal{X}_0 + y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha, \tag{30}$$

and the vector field $\tilde{\rho}^1(\Gamma)$ determines the differential equations

$$\dot{x}^i = \rho_0^i(x) + \rho_\alpha^i(x)y^\alpha, \qquad \dot{y}^\alpha = f^\alpha(x,y). \tag{31}$$

There are a number of equivalent ways for defining pseudo-SODEs, one of which is that its integral curves, by which we mean of course the integral curves of the corresponding vector field on E, all are *admissible curves* in the following sense: for $\gamma : \mathbb{R} \to E$, with projection $\gamma_M = \pi \circ \gamma : \mathbb{R} \to M$, we have $\rho \circ \gamma = \dot{\gamma}_M$. Note further that an admissible curve γ can be lifted to a curve $t \overset{\gamma^c}{\mapsto} (\gamma, \dot{\gamma})$, which belongs to $\mathcal{J}^\rho E$ for all t and by construction is such that $\tilde{\rho}^1 \circ \gamma^c = \dot{\gamma}$, hence is admissible for the prolonged algebroid.

Contact forms on $Sec(\pi^1)$ are 1-forms vanishing on all pseudo-SODEs. Locally, they are spanned by

$$\theta^\alpha = \mathcal{X}^\alpha - y^\alpha \mathcal{X}^0. \tag{32}$$

There also is a complete lift from $Sec(\tilde{\pi})$ to $Sec(\pi^1)$, determined by requiring that contact forms be preserved.

So now, to close the circle for this review of recent work, let me describe in two words how Lagrangian systems on an affine Lie algebroid can be defined, and how $\tilde{\rho}$-connections, both non-linear and linear ones, naturally make their appearance in dealing with pseudo-SODEs.

For $L \in C^\infty(E)$, define the Poincaré-Cartan type 1-form $\theta_L = S^*(dL) + L\,\mathcal{X}^0$ and the 2-form $\Omega_L = d\theta_L$. A pseudo-SODE Γ is of Lagrangian type if

$$i_\Gamma \Omega_L = 0.$$

The corresponding differential equations are of the form

$$\dot{x}^i = \rho_\alpha^i y^\alpha + \rho_0^i, \tag{33}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho_\alpha^i \frac{\partial L}{\partial x^i} + C_\alpha^\gamma \frac{\partial L}{\partial y^\gamma}\,, \tag{34}$$

where $C_\alpha^\gamma = C_{0\alpha}^\gamma + C_{\beta\alpha}^\gamma y^\beta$.

Now for any given pseudo-SODE Γ, the operator

$$P_H = \tfrac{1}{2}(I - d_\Gamma S + \mathcal{X}^0 \otimes \Gamma) \tag{35}$$

defines a (non-linear) $\tilde{\rho}$-connection on π, with connection coefficients (see [8])

$$\Gamma_\beta^\alpha = -\tfrac{1}{2}\left(\frac{\partial f^\alpha}{\partial y^\beta} + C_\beta^\alpha\right), \quad \Gamma_0^\alpha = -f^\alpha - y^\beta \Gamma_\beta^\alpha. \tag{36}$$

There further is an associated "linearisation", a Berwald-type connection, which is a linear $\tilde{\rho}^1$-connection on $\pi^*\tilde{E} \to E$, corresponding to an *affine* $\tilde{\rho}^1$-connection on $\pi^*E \to E$. The latter statement actually refers to work which is still under construction [9].

Finally, here are a couple of closing observations which are worth mentioning. Recall that, starting from $(e_0; \{\mathbf{e}_\alpha\})$, a local basis of sections of the affine bundle $E \to M$, and constructing an induced basis (e^0, e^α) for $Sec(\pi^\dagger)$, one encounters, somewhat surprisingly, the globally defined section e^0: $e^0_m(a_m) = 1, \forall a_m \in E_m$. Interestingly, additional properties of e^0 characterise the aspects of affineness we have been discussing.

First of all, a Lie algebroid on the vector bundle $\tilde{E} \to M$ restricts to an affine Lie algebroid on $E \to M$ if and only if $de^0 = 0$. Furthermore, in the special case that M is fibred over $\mathbb{R}$ and $\rho(E) \subset J^1M$, we have $e^0 = dt$. Note in passing that, in the theory of Lie bi-algebroids developed in [4], a central role is played by a 1-cocycle; the link with affine algebroids is explained in [3].

Secondly, a linear ρ-connection on $\tilde{\pi}$ is associated to an affine ρ-connection on π if and only if e^0 is parallel.

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"S"— the vertical endomorphism

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1 Introduction

Mike Crampin was my Ph.D. supervisor, and during the course of my studies he gave me a paper [2] describing some of the properties of the vertical endomorphism S on a tangent bundle. Now Mike was not the first person to use this operator; but he was, perhaps, the person who suggested calling it S rather than J (or, occasionally, v) as had been used previously in the literature. Over the years S, or variants or generalizations of it, have appeared in various papers that Mike and his colleagues have written, and it does indeed seem to be a very useful tool.

This present paper, based on a talk given at a colloquium to celebrate Mike's sixtieth birthday, describes some of the different manifestations of S and their uses. It does not attempt to be comprehensive, and contains rather few references: it tries, instead, to tell a story showing the importance of these constructions and how they are linked together.

Throughout this paper, differentiable manifolds are assumed to be of class C^∞, finite-dimensional, paracompact, Hausdorff and connected (even when not all of these properties are strictly necessary for the definitions to make sense or for the conclusions to be valid).

2 The original vertical endomorphism S

The original vertical endomorphism, as a 1-covariant, 1-contravariant tensor field on a tangent bundle TM, traces its origin to a fundamental property of real vector spaces and their tangent vectors. This property, applied pointwise to each fibre of a vector bundle, gives an operator which (when the vector

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bundle is the tangent bundle of some manifold) may be represented as a tensor field.

2.1 The vector space property

Let V be a finite-dimensional real vector space. At any point $v \in V$, there is a natural isomorphism between V and the tangent space T_vV. We construct the isomorphism by regarding elements of T_vV as equivalence classes of curves $\gamma\colon \mathbf{R} \to V$ satisfying $\gamma(0) = v$, where $\gamma_1 \sim \gamma_2$ if, for every function $f\colon V \to \mathbf{R}$,

$$\frac{d}{dt}(f \circ \gamma_1)\bigg|_0 = \frac{d}{dt}(f \circ \gamma_2)\bigg|_0 ;$$

then each equivalence class contains a unique curve of the form $\gamma_w(t) = v + tw$ for some $w \in V$. The isomorphism $V \to T_vV$ is given by $w \mapsto [\gamma_w]$, and is called the *vertical lift*.

By duality, there is a natural isomorphism between the cotangent space T_v^*V and V^*. But we can also describe the dual isomorphism directly, and this dual description is perhaps more natural than the direct one. We start with the algebra $C^\infty(V)$ of functions on V: this has an ideal $\mathfrak{n}_v$ containing the functions vanishing at v, and the cotangent space T_v^*V may be defined as the quotient $\mathfrak{n}_v/\mathfrak{n}_v^2$. This construction gives a natural vector space structure to T_v^*V (indeed it gives an algebra structure, but the multiplication is trivial), whereas obtaining the linear structure on the tangent space T_vV requires more work. Now each element of $\mathfrak{n}_v/\mathfrak{n}_v^2$ contains a unique representative that is an affine function η, and hence is of the form

$$\eta(w) = \lambda(w) - \lambda(v)$$

for some linear function $\lambda \in V^*$; the dual isomorphism $T_v^*V \to V^*$ is given by $[\eta] \mapsto \lambda$. We might call this the *vertical collapse*.

2.2 From vector spaces to vector bundles

We now apply this idea to the fibres of a vector bundle $\pi\colon E \to M$. At each point $v \in E$ the vertical tangent space $V_v\pi$ is a subspace of T_vE; then

$$V_v\pi \cong T_vE_{\pi(v)} \cong E_{\pi(v)},$$

where $T_vE_{\pi(v)}$ is the tangent space to the fibre $E_{\pi(v)}$. Similarly, at each point $v \in E$ the cotangent space T_v^*E projects to the "vertical cotangent space" $V_v^*\pi$ (containing equivalence classes of cotangent vectors). Then

$$V_v^*\pi \cong T_v^*E_{\pi(v)} \cong E_{\pi(v)}^*,$$

where $T^*_v E_{\pi(v)}$ is the cotangent space to the fibre $E_{\pi(v)}$, and $E^*_{\pi(v)}$ is the fibre of the dual bundle. The maps

$$E_{\pi(v)} \to T_v E, \qquad T^*_v E \to E^*_{\pi(v)}$$

are, respectively, a linear inclusion and a linear projection, and these give rise to morphisms

$$\pi^*(E) \to TE, \qquad T^*E \to \pi^*(E^*)$$

of vector bundles over E.

2.3 The tangent bundle

If we specialize the construction defined above to the case where the vector bundle $\pi\colon E \to M$ is the tangent bundle $\tau_M\colon TM \to M$, we can compose the push-forward map

$$TTM \to \tau^*_M(TM), \qquad \xi \mapsto (\tau_{TM}(\xi), T\tau_M(\xi))$$

with our vertical lift morphism to give a vector bundle morphism $TTM \to TTM$; similarly we can compose the vertical collapse morphism with the pull-back map

$$\tau^*_M(T^*M) \to T^*TM, \qquad (v, \eta) \mapsto T^*_v \tau_M(\eta)$$

to give a vector bundle morphism $T^*TM \to T^*TM$. These two vector bundle morphisms are dual to each other, and they give rise to a section S of the tensor bundle

$$T^*TM \otimes TTM \to TM$$

called the *almost tangent structure* or the *vertical endomorphism* of the tangent manifold TM.

To see a representation of this tensor field in coordinates, choose a local chart on M with coordinate functions (q^i); then in the induced fibred chart on TM with coordinate functions $(q^i, \dot{q}^i)$ we have

$$S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}.$$

3 Some properties of S

The tensor field S has some characteristic properties. First, as a map $TTM \to TTM$ its image equals its kernel (so that, in particular, $S^2 = 0$). Secondly, S is integrable in the sense that its Nijenhuis tensor vanishes: for any two vector fields X, Y on TM

$$[X \lrcorner S, Y \lrcorner S] - [X \lrcorner S, Y] \lrcorner S - [X, Y \lrcorner S] \lrcorner S = 0.$$

These properties are characteristic of S in that the existence on a manifold of a tensor with these properties may be used (with some other conditions) to show that the manifold is the total space of a tangent bundle.

One of the most important uses of S arises in the context of dynamical systems and second-order differential equations, in particular those arising from variational principles.

3.1 Lagrangian mechanics

Let $L\colon TM \to \mathbf{R}$ be a Lagrangian function. We may use S to construct a semi-basic 1-form $\vartheta_L = S \lrcorner\, dL$, known as the *Cartan form* of L; as a section of the cotangent bundle, ϑ_L is a map $TM \to T^*TM$. Extremals of the Lagrangian are also extremals of the Cartan form.

We may also construct the Legendre transformation $\mathcal{F}L\colon TM \to T^*M$ by taking the "fibre derivative" of L, and if this map has maximal rank then extremals of the Cartan form are also extremals of the Lagrangian. In these circumstances we say that the Lagrangian is *regular*.

The Cartan form and the Legendre transformation are related by the pull-back map

$$\vartheta_L(x) = \tau_M^*(\mathcal{F}L(x)),$$

given in coordinates by

$$\vartheta = \frac{\partial L}{\partial \dot{q}^i} dq^i, \qquad p_i \circ \mathcal{F}L = \frac{\partial L}{\partial \dot{q}^i}$$

Of course, taking the fibre derivative involves taking the unique affine function in an equivalence class of functions, and so is the "vertical collapse" operator used in the dual definition of S.

If L is positively homogeneous (a Finsler function) then the Cartan form is called the Hilbert form and is used extensively in the study of Finsler geometry.

3.2 The Euler-Lagrange equations

The Cartan form, constructed using S, may be used to give the Euler-Lagrange equations of a variational problem. We let Δ be the dilation field of the vector bundle $TM \to M$, and let $E = \Delta(L) - L$ be the energy of the Lagrangian L. If L is regular then $d\vartheta_L$ is a symplectic form; when this is the case the unique vector field Γ_L satisfying

$$\Gamma_L \lrcorner\, d\vartheta_L = -dE$$

is second-order (that is, $\Gamma_L \lrcorner S = \Delta$) and the curves in its flow are the solutions of the Euler-Lagrange equations. Even if L is not regular, the 1-form

$$\varepsilon_L = (\tau_M^{2,1})^*(dL) - d_{\mathbf{T}}\vartheta_L$$

defined on the second-order tangent bundle T^2M vanishes on a submanifold $\mathcal{E} \subset T^2M$ representing the Euler-Lagrange equations in invariant form. (Here, the map $\tau_M^{2,1}: T^2M \to TM$ is the canonical projection.)

If L is regular, the submanifold $\mathcal{E}$ is $2m$-dimensional, and is the image of the section γ_L of $T^2M \to TM$ corresponding to the second-order vector field Γ_L.

3.3 S and α

Another approach to Lagrangian mechanics, used in particular by W.M. Tulczyjew, involves a diffeomorphism

$$\alpha: TT^*M \to T^*TM \qquad (q^i, \dot{q}^i; p_i, \dot{p}_i) \circ \alpha = (q^i, \dot{p}_i; \dot{q}^i, p_i).$$

Given a Lagrangian L, we may use the following procedure. First, the image of dL (regarded as a map $TM \to T^*TM$) is a submanifold $\mathcal{D} \subset T^*TM$, and so we may consider $\alpha^{-1}(\mathcal{D}) \subset TT^*M$. Taking the tangent to this submanifold gives $T\alpha^{-1}(\mathcal{D}) \subset TTT^*M$ the *prolongation* of $\alpha^{-1}(\mathcal{D})$, and the intersection of this tangent submanifold with $T^2T^*M \subset TTT^*M$ gives the *holonomic prolongation*. The image of the holonomic prolongation $\alpha^{-1}(\mathcal{D}) \cap T^2T^*M$ under the projection $T^2\tau_M^*: T^2T^*M \to T^2M$ is then the Euler-Lagrange manifold $\mathcal{E}$.

The diffeomorphism α is related to the vertical collapse operator $\xi \mapsto \xi_{\mathrm{v}}$, because if $\xi \in T^*TM$ then

$$\tau_{T^*M}(\alpha^{-1}(\xi)) = \xi_{\mathrm{v}}.$$

The submanifold $\alpha^{-1}(\mathcal{D}) \subset TT^*M$ contains information about the Lagrangian similar to that encoded in the Cartan form ϑ_L, and the holonomic prolongation processes that information in the same way as the total time derivative operator $d_{\mathbf{T}}$ operating on ϑ_L.

3.4 The SODE *connection*

Let Γ be a general second-order vector field on TM: that is, one not necessarily arising from a variational problem. Any such SODE gives rise to a connection on the bundle $TTM \to TM$ in the following way. First, the tensor field $\mathcal{L}_\Gamma S$ satisfies $(\mathcal{L}_\Gamma S)^2 = I$; using this, we may check that the tensor fields $P = \frac{1}{2}(I - \mathcal{L}_\Gamma S)$ and $Q = \frac{1}{2}(I + \mathcal{L}_\Gamma S)$ are the horizontal and vertical projectors of such a connection. In coordinates, if

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \dot{q}^i}$$

then

$$\begin{aligned} P &= dq^i \otimes \left(\frac{\partial}{\partial q^i} + \tfrac{1}{2} \frac{\partial f^j}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{q}^j} \right) \\ Q &= \left(d\dot{q}^j - \tfrac{1}{2} \frac{\partial f^j}{\partial \dot{q}^i} dq^i \right) \otimes \frac{\partial}{\partial \dot{q}^j}. \end{aligned}$$

Properties of this connection are important in trying to determine whether Γ arises from a Lagrangian.

In a later section, we shall see how to obtain this connection by using a "second-order" version of S.

4 Generalizations of the vertical endomorphism

We can generalize S in a number of different ways, to give operators defined on manifolds different from on ordinary tangent bundle. Four direct extensions give operators that:

(1) act on higher-order tangent bundles;
(2) act on Lie algebroids;
(3) act on jet bundles, involving time explicitly;
(4) act on frame bundles, with more independent variables.

And we can combine some of these generalizations, to:

- $(1+2)$ higher-order algebroids;
- $(1+3)$ higher-order jet bundles (over $\mathbf{R}$);
- $(3+4)$ jet bundles with more independent variables;
- $(1+4)$ higher-order frame bundles;
- $(1+3+4)$ higher-order jet bundles with more independent variables, in a limited way.

The following subsections describe each of these generalizations in turn.

4.1 A generalization to higher-order tangent bundles

In order to generalize S to higher-order tangent bundles, we first construct a vertical lift operator from TT^kM to $TT^{k+1}M$. Given a point $p \in T^{k+1}M$ and a vector $w \in TT^kM$ such that

$$\tau_M^{k+1,k}(p) = \tau_{T^kM}(w),$$

choose a map $\chi\colon \mathbf{R}^2 \to M$ such that

$$\frac{\partial^{k+1}}{\partial s^{k+1}}\,\chi(s,t)\bigg|_{(0,0)} = p, \qquad \frac{\partial^{k+1}}{\partial s^k \partial t}\,\chi(s,t)\bigg|_{(0,0)} = w.$$

Now define the vector $u \in T_p T^{k+1}M$ by

$$u = \frac{\partial^{k+2}}{\partial s^{k+1}\partial t}\,\chi(s,st)\bigg|_{(0,0)};$$

then u is the vertical lift of w to p (and is independent of the choice of map χ satisfying the required condition). If $k=0$ then the vertical lift defined in this way is the same as the one defined earlier. Indeed, the idea of representing a vector on a tangent bundle by a function of two real variables will turn out to be useful for generalizations of the verical lift to several other contexts. [3]

Given this, compose the push-forward map

$$TT^{k+1}M \to (\tau_M^{k+1,k})^*(TT^kM), \qquad v \mapsto (\tau_{T^{k+1}M}(v), T\tau_M^{k+1,k}(v))$$

with the vertical lift to get an endomorphism of $T_p^{k+1}TM$, and let S be the representation of this endomorphism as section of the tensor bundle

$$T^*T^{k+1}M \otimes TT^{k+1}M \to T^{k+1}M.$$

In coordinates $(q^i_{(0)}, \ldots, q^i_{(k+1)})$ on $T^{k+1}M$,

$$S = \sum_{r=0}^{k}(r+1)dq^i_{(r)} \otimes \frac{\partial}{\partial q^i_{(r+1)}}.$$

The infinite tangent bundle $T^\infty M$ may be defined as the inverse limit of the sequence of finite-order tangent bundles. It may be shown that any cotangent vector on $T^\infty M$ is of *finite order*, in that it is the pull-back of a cotangent vector on some finite-order tangent bundle T^kM. We can use this to define S on $T^\infty M$ as the linear map $T^*T^\infty M \to T^*T^\infty M$ given by

$$S((\tau_M^{\infty,k})^*\eta) = (\tau_M^{\infty,k})^*(S(\eta)).$$

4.1.1 An alternative description

The canonical involution $J_1\colon TTM \to TTM$ has a fixed point set $\iota_{1,1}(T^2M)$ (where $\iota_{1,1}$ is the canonical inclusion), and this generalizes to a diffeomorphism

$$J_k\colon T^kTM \to TT^kM$$

which maps $\iota_{k,1}(T^{k+1}M)$ onto $\iota_{1,k}(T^{k+1}M)$. We can use this to give an alternative method of describing the higher-order vertical endomrphisms.

Given a 1-form η on TT^kM, consider the sum

$$\iota^*_{1,k}(S_{TT^kM} \lrcorner\, \eta) + \iota^*_{k,1}(S_{T^kTM} \lrcorner\, J^*_k\eta);$$

we may check, using the coordinate formula for S, that this equals

$$S_{T^{k+1}M} \lrcorner\, \iota^*_{1,k}\eta.$$

So if we have a form on $T^{k+1}M$ we can spread it out arbitrarily to a tubular neighbourhood, and then use the S tensors on T^kTM and on TT^kM to give a well-defined result corresponding to the S tensor on $T^{k+1}M$.

4.1.2 S and $d_{\mathbf{T}}$; the higher-order Cartan form

We may check, again using the coordinate formula, that the S tensors are related to the total time derivative operators $d_{\mathbf{T}}$ in their action on forms by the rule

$$S \circ d_{\mathbf{T}} - d_{\mathbf{T}} \circ S = (\tau_M^{k+1,k})^*.$$

(On the infinite tangent bundle $T^\infty M$ this simplifies to

$$S \circ d_{\mathbf{T}} - d_{\mathbf{T}} \circ S = \mathrm{id}$$

so that S is a homotopy operator for $d_{\mathbf{T}}$. As we shall see later, a generalization of this property is useful when considering the local exactness of the variational bicomplex.)

If $L \colon T^kM \to \mathbf{R}$ is a Lagrangian then its Cartan form is defined on $T^{2k-1}M$ and is given by

$$\vartheta_L = \sum_{r=0}^{k-1} \frac{(-1)^r}{r!} d^r_{\mathbf{T}} S^{r+1} dL;$$

this form is semi-basic over $T^{k-1}M$. The 1-form

$$\varepsilon_L = (\tau_M^{2k,k})^*(dL) - d_{\mathbf{T}}\vartheta_L$$

vanishes on a submanfold of $T^{2k}M$ that is an invariant representation of the Euler-Lagrange equations for L. If $d\vartheta_L$ is symplectic then this submanifold is $2km$-dimensional, and is the image of a $2k$-th order vector field on $T^{2k-1}M$.

4.1.3 The alternative SODE *connection*

If Γ is a second-order vector field on TM, its SODE conection may be constructed in an alternative way. Let γ be the section of $T^2M \to TM$ corresponding to Γ, and let C be the image of γ as a submanifold of T^2M. At each point $p \in C$, take the annihilator $(T_pC)^\circ$ of T_pC in T_pT^2M, and then act on this annihilator with the second-order S tensor to give $S((T_pC)^\circ)$. Pulling this back by γ to TM then gives $\gamma^*S((T_pC)^\circ)$. The resulting subspaces of cotangent vectors on TM are just the vertical cotangent spaces of the SODE connection. [11]

This construction generalizes immediately to the SODE of a $(k+1)$-th order vector field on T^kM. The method of using $\mathcal{L}_\Gamma S$ for the construction also generalizes, but involves a slightly more complicated formula.

4.2 A generalization to Lie algebroids

If $\tau\colon A \to M$ is a Lie algebroid with anchor map $\rho\colon A \to TM$ then a prolonged algebroid may be constructed on the fibre product $A \times_{TM} TA$ (where $T\tau\colon TA \to TM$) with projection

$$\tau_1\colon A \times_{TM} TA \to A, \qquad \tau_1(a, v) = \tau_A(v)$$

and with anchor

$$\rho_1\colon A \times_{TM} TA \to TA, \qquad \rho_1(a, v) = v.$$

We say that an element of $A \times_{TM} TA$ is vertical if $a = 0$ (so that $T\tau(v) = 0$ and hence $v \in V\tau$). The vertical lift is then the map

$$A \times_M A \to A \times_{TM} TA, \qquad (a, b) \mapsto (0, b^{\mathrm{v}}_a)$$

and the vector bundle endomorphism S on $A \times_{TM} TA$ may be defined to be the projection $(a, v) \mapsto (\tau_E(v), a)$ followed by the vertical lift. For the canonical Lie algebroid $\tau_M\colon TM \to M$, this construction just gives the usual vertical endomorphism. [8]

4.3 A generalization to jet bundles, involving time explicitly

Take a fibration $\pi\colon E \to \mathbf{R}$; we may define a version of S on the jet bundle $J^1\pi$. An instructive way of doing this is to use the existing vertical endomorphism on the tangent bundle TE.

To do this, we use the fact that $J^1\pi$ is a closed codimension 1 submanifold of TE, given by $\dot{t} = 1$. Now S (on TE) does not restrict to $J^1\pi$, but we can modify it so that it does. To do this, we use the total time derivative operator $d_{\mathbf{T}}$, regarded as a vector field along the projection $\pi_{1,0}\colon J^1\pi \to E$. Then for each vector field X on $J^1\pi$ there is a unique multiple f of $d_{\mathbf{T}}$ such that $\pi_{1,0} \circ X + f d_{\mathbf{T}}$ gives a vertical lift to TE tangent to $J^1\pi$. This operation then defines the tensor field S on $J^1\pi$.

In coordinates $(t, q^i, \dot{q}^i)$ on $J^1\pi$

$$S = \theta^i \otimes \frac{\partial}{\partial \dot{q}^i}$$

where $\theta^i = dq^i - \dot{q}^i\, dt$ are the contact forms.

4.3.1 Time-dependent Lagrangian mechanics

As well as being a closed submanifold of TE, we may also consider the jet bundle $J^1\pi$ as being an open submanifold of the projective tangent bundle

$$PTE = (TE - \{0\})/(\alpha \neq 0), \qquad \rho\colon (TE - \{0\}) \to PTE.$$

We may use this to define the Cartan form for a time-dependent Lagrangian system.

A Lagrangian 1-form $\lambda = L\, dt$ (where $L\colon J^1\pi \to \mathbf{R}$) defines a homogeneous Lagrangian function $\widehat{L}$ on an open submanifold of TE by $\widehat{L} = \langle d_{\mathbf{T}}, \rho^*\lambda \rangle$. In coordinates,

$$\widehat{L}(t, q^i, \dot{t}, \dot{q}^i) = \dot{t} L(t, q^i, \dot{q}^i/\dot{t}).$$

The Cartan form $\vartheta_{\widehat{L}}$ is projectable to $J^1\pi \subset PTE$, and in coordinates is

$$\vartheta_\lambda = \vartheta_{\widehat{L}} = L\, dt + \frac{\partial L}{\partial \dot{q}^i}\theta^i.$$

We may also write the Cartan form directly using the version of S defined on $J^1\pi$, with the formula

$$\vartheta_\lambda = L\, dt + S \lrcorner\, dL.$$

4.3.2 Non-holonomic mechanics

Given a submanifold C of $J^1\pi$ fibred over E (a *constraint submanifold*), a non-holonomic mechanical system may be described by a SODE given at points of C and tangent to C. The "force" constraining the system to remain on C is given by a 1-form that is a section of a bundle constructed in the following way. [9]

At each point $p \in C$, take the annihilator $(T_pC)^\circ$ of T_pC in $T_pJ^1\pi$; then act on this annihilator with S to give $S((T_pC)^\circ)$. This bundle is called the *Chetaev bundle*. If C is an affine sub-bundle of $J^1\pi$ then the Chetaev bundle is projectable from C to E. If, in addition, the constraint submanifold is defined by some auxiliary fibration $E \to E_0 \to \mathbf{R}$ then the projected Chetaev bundle defines a connection on the fibration $E \to E_0$.

This construction of a connection in the context of non-holonomic mechanics is remarkably similar to that of the SODE connection (in its alternative formulation).

4.4 A generalization to frame bundles, with more independent variables

Now let $\dim E = N$, and let $\mathcal{F}_{(m)}E$ be the bundle of m-frames on E; we may regard this is a sub-bundle of the vector bundle $\bigoplus^m TE$. So if we take two frames $\xi, \zeta \in \mathcal{F}_{(m)}E$ projecting to the same point of E, we may consider $\xi, \zeta \in \bigoplus^m TE$ and take the vertical lift of ζ to a vertical vector $\zeta^{\mathrm{v}} \in T_\xi \bigoplus^m TE$. As $\mathcal{F}_{(m)}E \subset FE$. In coordinates (u^A, u^A_i) on $\mathcal{F}_{(m)}E$,

$$\left(\zeta^A_1 \frac{\partial}{\partial u^A}, \ldots, \zeta^A_m \frac{\partial}{\partial u^A}\right) \mapsto \zeta^A_i \frac{\partial}{\partial u^A_i}.$$

The i-th component of this map, preceded by the projection $\mathcal{F}_{(m)}E \to E$, may be regarded as a globally-defined tensor field

$$S^i = du^A \otimes \frac{\partial}{\partial u^A_i}$$

and the family $(S^1, \ldots, S^m)$ may be regarded as a generalization of the vertical endomorphism to $\mathcal{F}_{(m)}E$. [4]

4.4.1 Lagrangian field theories

In the context of frame bundles, a Lagrangian is a map $L\colon \mathcal{F}_{(m)}E \to \mathbf{R}$, and it is homogeneous if $L(A\xi) = (\det A)L(\xi)$ for any $A \in \mathrm{GL}(m)$; furthermore, it is positively homogeneous if this condition holds for $A \in \mathrm{GL}^+(m)$.

Let $\vartheta^i_L = S^i \lrcorner\, dL$: we call these the *Hilbert forms* for L. The 1-form

$$\varepsilon_L = dL - \mathbf{d}_i \vartheta^i_L$$

(where $\mathbf{d}_i = u^A_i \partial/\partial u^A$ is the i-th total derivative) is defined on the second-order frame bundle, and vanishes on a submanifold described by the Euler-Lagrange equations for L, generalizing the similar equation for single-integral problems.

As well as the family $(\vartheta^1_L, \ldots, \vartheta^m_L)$ of Hilbert 1-forms, there are also two m-forms closely associated with the variational problem defined by L. If L is non-vanishing then the m-form

$$\Theta_C = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \vartheta^i$$

has the same extremals as L. On the other hand, the m-form

$$\Theta_F = S^1 \lrcorner\, d(S^2 \lrcorner\, d \ldots (S^m \lrcorner\, dL) \ldots)$$

is closed precisely when L is null. [6] In coordinates

$$\Theta_C = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \frac{\partial L}{\partial u_i^A} du^A$$

and

$$\Theta_F = \frac{\partial^m L}{\partial u_1^{A_1} \cdots \partial u_m^{A_m}} du^{A_1} \wedge \ldots \wedge du^{A_m}.$$

4.5 A generalization to higher-order algebroids

If the Lie algebroid A is integrable (so that it is the algebroid $A(G)$ of a Lie groupoid G) then the prolonged algebroid $A \times_{TM} TA$ is the algebroid $A(PG)$ of a prolonged groupoid

$$PG = V\alpha \times_G V\beta$$

where α, β are the source and target maps of G. This construction can be generalized to give a k-th order prolonged groupoid

$$P^k G = V^k \alpha \times_G V^k \beta,$$

and also a k-th order "generalized algebroid"

$$A^k G = V^k \alpha \Big|_M ,$$

so that $A^k G$ is the identity submanifold of $P^k G$ (and hence the base manifold of the Lie algebroid $A(P^k G) \to A^k G$).

There is then a vertical lift operation

$$A(P^k G) \times_{A^k G} A^{k+1} G \to A(P^{k+1})G$$

defined by using a generalization of the $\chi(s, st)$ formula for higher-order tangent bundles, and correspondingly an operator S on sections of $A(P^{k+1}G) \to A^{k+1}G$. [12]

4.6 A generalization to higher-order jet bundles (over $\mathbf{R}$*)*

This works in the same way as the first-order generalization, from TE to $J^1\pi$ (where $\pi \colon E \to \mathbf{R}$). We consider $J^k\pi$ as a submanifold of $T^k E$, defined by the equations $t_{(1)} = 1$, $t_{(r)} = 0$ for $2 \leq r \leq k$.

If X is a vector field on $T^k E$ then $X \lrcorner S$ is not tangent to $J^k\pi$, but there is a unique multiple f of the total time derivative $d_{\mathbf{T}}$ such that $\pi_{k,k-1} \circ X + f d_{\mathbf{T}}$ gives a vertical lift to $T^k E$ tangent to $J^k\pi$. This operation defines the tensor field S on $J^k\pi$.

In coordinates $(t, q^i_{(r)})$ on $J^1\pi$

$$S = \sum_{r=0}^{k-1} (r+1)\theta^i_{(r)} \otimes \frac{\partial}{\partial q^i_{(r+1)}}$$

where $\theta^i_{(r)}$ are the contact forms $dq^i_{(r)} - q^i_{(r+1)}dt$.

4.7 A generalization to jet bundles with more independent variables

In the context of jet bundles, the move from one independent variable to several turns out to be more complicated than in the case of tangent (and frame) bundles. The reason for this is connected with the need for invariance under reparametrization: if $\pi\colon E \to M$ is a fibration with $\dim M = m$ and $\dim E = n$ giving the jet bundle $J^1\pi$ then there are no canonical basis directions at points of M.

We can, however, construct a directional vertical lift with an extended set of ingredients, involving a choice of direction in M. So take a point $j^1_p\phi \in J^1\pi$, a vector $w \in T_{\phi(p)}E$ and a cotangent vector $\eta \in T^*_pM$. Given these items, let $f \in C^\infty M$ satisfy $df_p = \eta$, and let $\chi\colon M \times \mathbf{R} \to E$ be such that for each t, $q \mapsto \chi(q,t)$ is a section of π, and such that

$$j^1_p(q \mapsto \chi(q,0)) = j^1_p\phi, \qquad \left.\frac{\partial\chi}{\partial t}\right|_{(p,0)} = w.$$

If we define $v \in T_{j^1_p\phi}J^1\pi$ by

$$v = \frac{\partial}{\partial t} j^1(q \mapsto \chi(q, f(q)t))\Big|_{(p,0)}$$

then u is vertical over E. In coordinates $(x^i, u^\alpha, u^\alpha_i)$ on $J^1\pi$, if

$$w = w^i\frac{\partial}{\partial x^i} + w^\alpha\frac{\partial}{\partial u^\alpha}, \qquad \eta = \eta_j dx^j$$

then

$$v = \eta_j(w^\alpha - u^\alpha_i w^i)\frac{\partial}{\partial u^\alpha_j}.$$

Once again, this construction is a generalization of the one described above for higher-order tangent bundles. [10]

4.7.1 The S operators on $J^1\pi$

The directional vertical lift may be used to define several different types of S operator on $J^1\pi$. Most straightforwardly, choosing a 1-form η on M gives a $(1,1)$ tensor field S^η by taking the appropriate pointwise vertical lifts; in coordinates

$$S^\eta = \eta_i \, \theta^\alpha \otimes \frac{\partial}{\partial u^\alpha_i}$$

where θ^α are the contact forms $du^\alpha - u^\alpha_j dx^j$. We can also construct a type (2,1) tensor field S "along π_1" giving the tensors S^η by tensor contraction; in coordinates

$$S = \theta^\alpha \otimes \frac{\partial}{\partial u^\alpha_i} \otimes \frac{\partial}{\partial x^i}.$$

And finally, if M is oriented with volume form ω then contraction of S with ω gives a vector-valued m-form S_ω; in coordinates

$$S_\omega = (\theta^\alpha \wedge \omega_i) \otimes \frac{\partial}{\partial u^\alpha_i}$$

where $\omega_i = \partial/\partial x^i \lrcorner\, \omega$.

The third of these S operators may be used to construct a *truncated Cartan form* for a Lagrangian $\lambda = L\omega$ defined on $J^1\pi$. This m-form is $\Theta = L\omega + S_\omega \lrcorner\, dL$; in coordinates where $\omega = d^m x$

$$\Theta = L\omega + \frac{\partial L}{\partial u^\alpha_i} \theta^\alpha \lrcorner\, \omega_i$$

(using the notation $\omega_i = \partial/\partial x^i \lrcorner\, \omega$).

4.7.2 Other Cartan forms for multiple integral problems on jet bundles

As for single integral problems, the lagrangian form $\lambda = L\omega$ on $J^1\pi$ defines a homogeneous Lagrangian function $\widehat{L}$ on (an open subset of) the frame bundle $\mathcal{F}_{(m)}E$ by

$$\widehat{L} = \langle d_{\mathbf{T}}, \rho^* \lambda \rangle$$

where $d_{\mathbf{T}}$ is the tautological m-frame along $\mathcal{F}_{(m)}E \to E$ given by the identity map $\mathcal{F}_{(m)}E \to \mathcal{F}_{(m)}E$; the components of $d_{\mathbf{T}}$ are the total derivative operators $\mathbf{d}_i$. But now, unlike in the single-integral case, there are two candidates for an m-form that could project to a "Cartan form", and they are both different from the truncated form described above. We find that the m-forms Θ_C and Θ_F defined by $\widehat{L}$ both project to the Grassmannian bundle $\mathcal{G}_{(m)}E = \mathcal{F}_{(m)}E/\mathrm{GL}(m)$, and we consider $J^1\pi$ as an open submanifold of $\mathcal{G}_{(m)}E$.

The projection of Θ_C is the *Carathéodory form* [1] of L, in coordinates

$$\frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \left(L \, dx^i + \frac{\partial L}{\partial u^\alpha_i} \theta^\alpha \right).$$

The projection of Θ_F is the *fundamental Lepage equivalent* [7] of L, in coordinates

$$\sum_{r=0}^{\min\{m,n\}} \frac{1}{(r!)^2} \frac{\partial^r L}{\partial u^{\alpha_1}_{i_1} \cdots \partial u^{\alpha_r}_{i_r}} \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r} \wedge \omega_{i_1 \cdots i_r}$$

where $\omega_{i_1 \cdots i_s} = \partial/\partial x^{i_s} \lrcorner\, \omega_{i_1 \cdots i_{s-1}}$.

4.8 A generalization to higher-order frame bundles

The vertical lift operator may be generalized from first-order frame bundles to higher-order frame bundles in a fairly straightforward way: this uses the representation of a frame on a frame bundle by a function of two m-vector variables. So let $\mathcal{F}^k_{(m)}E$ be the bundle of k-th order m-frames in E: it is the open sub-bundle of the bundle $T^k_{(m)}E$ containing the non-singular m-dimensional k-velocities, with coordinates (u^A_I) where I is a multi-index.

We can construct a vertical lift from $\mathcal{F}_{(m)}\mathcal{F}^{k-1}_{(m)}E$ to $T\mathcal{F}^k_{(m)}E$ in the following way. Given $\xi \in \mathcal{F}^k_{(m)}E$ and $\zeta \in \mathcal{F}_{(m)}\mathcal{F}^{k-1}_{(m)}E$ such that

$$\tau^{k,k-1}_{(m)E}(\xi) = \tau_{\mathcal{F}^{k-1}_{(m)}E}(\zeta),$$

choose a map $\chi\colon \mathbf{R}^m \times \mathbf{R}^m \to E$ such that

$$j^k_0(x \mapsto \chi(x,0)) = \xi, \qquad j^1_0(y \mapsto (x \mapsto j^{k-1}_0(\chi(x,y)))) = \zeta.$$

Define $u \in T_v\mathcal{F}^k_{(m)}E$ by

$$u = \frac{d}{dt}(j^k_0(x \mapsto \chi(x,tx)))\bigg|_0 ;$$

then u is a vertical vector on $\mathcal{F}^k_{(m)}E$ and is the vertical lift of ζ to ξ. If $k=1$ then this is the same as the vertical lift defined previously. The i-th component of this map, preceded by a projection to $\mathcal{F}^{k-1}_{(m)}E$, is a globally-defined tensor field [5]

$$S^i = \sum_{|I|=0}^{k-1} (I_i + 1) du^A_I \otimes \frac{\partial}{\partial u^A_{I+1_i}}.$$

4.9 A (limited) generalization to higher-order jet bundles with more independent variables

Although the generalization of the vertical lift from first-order to higher-order frame bundles was straightforward, a similar generalization from first-order to higher-order jet bundles (over an m-dimensional base) turns out to be more difficult.

We may start in the same way, by letting $\pi\colon E \to M$ be a fibration, and taking a point $j^k_p\phi \in J^k\pi$ and a vector $w \in T_{j^{k-1}_p\phi}J^{k-1}\pi$. However it is no longer adequate to define a direction on M by taking a cotangent vector; instead we need to take a closed 1-form η on M. With these ingredients, let $f \in C^\infty M$ satisfy $df = \eta$ in a neighbourhood of p, and let $\chi\colon M \times \mathbf{R} \to E$ be such that for each t, $q \mapsto \chi(q,t)$ is a section of π, and such that

$$j^k_p(q \mapsto \chi(q,0)) = j^k_p\phi, \qquad \frac{\partial}{\partial t}(j^{k-1}(q \mapsto \chi(q,t)))\bigg|_{(p,0)} = w.$$

If we define $v \in T_{j^k_p\phi}J^k\pi$ by

$$v = \frac{\partial}{\partial t}\, j^k(q \mapsto \chi(q, f(q)t))\bigg|_{(p,0)};$$

then u is vertical over $J^{k-1}\pi$. In coordinates (x^i, u^α_I) on $J^k\pi$, if

$$w = w^i\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k-1} w^\alpha_I \frac{\partial}{\partial u^\alpha_I}, \qquad \eta = \eta_j dx^j$$

then

$$v = \sum_{|I+J|=0}^{k-1} \frac{(I+J+1_j)!}{I!(J+1_j)!}(w^\alpha_K - u^\alpha_{K+1_i}w^i)\frac{\partial^J\eta_j}{\partial x^J}\frac{\partial}{\partial u^\alpha_{I+J+1_j}}.$$

So for each closed 1-form η on M we get a tensor field

$$S^\eta = \sum_{|I+J|=0}^{k-1} \frac{(I+J+1_j)!}{I!(J+1_j)!}\frac{\partial^J\eta_j}{\partial x^J}\,\theta^\alpha_K \otimes \frac{\partial}{\partial u^\alpha_{I+J+1_j}}$$

depending on the derivatives of the components of η.

4.9.1 Exactness of d_h in the variational bicomplex

Versions of the S operator have appeared earlier in formlæ for Cartan forms, and where the Lagrangian had order greater than one these operators were combined with total derivative operators. Indeed, the relationship between S and total differentiation is very close, as the homotopy formula on $T^\infty M$ demonstrates.

A similar relationship holds for the case of several independent variables, though here there is much greater complexity. An important construction in this context is that of the *variational bicomplex*. Here, we consider a fibration $\pi\colon E \to M$ and let Φ^r_s denote the module of $(r+s)$-forms on $J^\infty\pi$ containing r horizontal components (semi-basic over$M$) and s contact components. The modules Ξ_s are quotient modules. The horizontal differential $d_h\colon \Phi^r_s \to \Phi^{r+1}_s$

has coordinate form

$$d_h f = \frac{df}{dx^i} dx^i$$

and is an invariant object incorporating the total derivative operators on jet bundles. In the context of finite-order jet bundles it would map forms on $J^k\pi$ to forms on $J^{k+1}\pi$, but when considering $J^\infty\pi$ it is a mapping between forms on the same manifold. On the other hand, the vertical differential $d_v = d - d_h$ has many of the properties of the ordinary exterior derivative on the fibres of π, and in particular the map $d_v \colon \Phi_0^m \to \Phi_1^m$ passes to the quotient to give a map $\delta \colon \Xi_0 \to \Xi_1$ which is the Euler-Lagrange operator. (See the Appendix for a diagram of the variational bicomplex on $J^\infty\pi$.)

The local exactness of d_h may be shown using the tensors S^η where the closed 1-forms η are the coordinate differentials dx^i; we write S^i for S^{dx^i} and $S^I = (S^1)^{I_1} \ldots (S^m)^{I_m}$. [13] In this particular coordinate system, S^i is represented as

$$S^i = \sum_J (J_i + 1)\, \theta^\alpha_J \otimes \frac{\partial}{\partial u^\alpha_{J+1_i}}.$$

The homotopy operator for d_h acting on Φ^r_s $(s > 0)$ is then

$$H^r_s(\theta) = \frac{1}{s} \left(\sum_{i_1 < \ldots < i_{m-r}} \omega_{i_1 \cdots i_{m-r} i} \wedge F_i(\theta^{i_1 \cdots i_{m-r}}) \right),$$

where

$$\theta = \sum_{i_1 < \ldots < i_{m-r}} \omega_{i_1 \cdots i_{m-r}} \wedge \theta^{i_1 \cdots i_{m-r}},$$

$$F_i(\theta) = - \sum_{I \in \mathcal{M}_i} (-1)^{|I|} \frac{d^{|I|-1}}{dx^{I-1_i}} (S^I \lrcorner\, \theta)$$

and

$$\mathcal{M}_i = \{ I \in \mathbf{N}^m : I(i) > 0 \text{ but } I(j) = 0 \text{ for } j > i \}.$$

5 Summary

The various applications of S described above demonstrate its importance in the study of problems in the calculus of variations, and in the application of those studies to nechanics and field theories. Although the *definition* of S commonly involves its action on vectors fields, or on frames, its *use* tends to arise through the transpose, as an operator on forms, or on coframes. And in this sense, as the homotopy formulæ suggest, its role is really that of a jet integration operator.

And this, perhaps, is why S is quite a good name for this operator. For we use $\int$ as a symbol for integration, and that is supposed to be an elongated letter "S", representing summation. It is surely, therefore, appropriate to use the letter S itself for the standard jet integration operator.

Appendix: The variational bicomplex

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & & \\
 & \mathbf{R} \arrow[d] & \mathbf{R} \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigwedge^0 M \arrow[r, "\pi_\infty^*"] \arrow[d, "d"'] & \Phi_0^0 \arrow[r, "d_v"] \arrow[d, "d_h"'] & \Phi_1^0 \arrow[d, "-d_h"'] & \cdots & \Phi_s^0 \arrow[r, "d_v"] \arrow[d, "(-1)^s d_h"'] & \Phi_{s+1}^0 \arrow[d, "(-1)^{s+1} d_h"'] & \cdots \\
0 \arrow[r] & \bigwedge^1 M \arrow[r, "\pi_\infty^*"] & \Phi_0^1 \arrow[r, "d_v"] & \Phi_1^1 & \cdots & \Phi_s^1 \arrow[r, "d_v"] & \Phi_{s+1}^1 & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 \arrow[r] & \bigwedge^{m-1} M \arrow[r, "\pi_\infty^*"] \arrow[d, "(-1)^{m-1} d"'] & \Phi_0^{m-1} \arrow[r, "d_v"] \arrow[d, "(-1)^{m-1} d_h"'] & \Phi_1^{m-1} \arrow[d, "(-1)^m d_h"'] & \cdots & \Phi_s^{m-1} \arrow[r, "d_v"] \arrow[d, "(-1)^{m+s-1} d_h"'] & \Phi_{s+1}^{m-1} \arrow[d, "(-1)^{m+s} d_h"'] & \cdots \\
0 \arrow[r] & \bigwedge^m M \arrow[r, "\pi_\infty^*"] \arrow[d] & \Phi_0^m \arrow[r, "d_v"] \arrow[d, "p_0"'] & \Phi_1^m \arrow[d, "p_1"'] & \cdots & \Phi_s^m \arrow[r, "d_v"] \arrow[d, "p_s"'] & \Phi_{s+1}^m \arrow[d, "p_{s+1}"'] & \cdots \\
 & 0 \arrow[r] & \Xi_0 \arrow[r, "\delta"] \arrow[d] & \Xi_1 \arrow[d] & \cdots & \Xi_s \arrow[r, "\delta"] \arrow[d] & \Xi_{s+1} \arrow[d] & \cdots \\
 & & 0 & 0 & & 0 & 0 &
\end{tikzcd}
$$

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A retrospective on the inverse problem of Lagrangian mechanics

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Abstract

I begin with some personal reminiscences of the last twenty five years. Then in the main body of the article I review some of the recent developments in the inverse problem of Lagrangian mechanics and discuss the inverse problem for the canonical Lie group connection. I outline an algorithm for solving the Lie group problem. Finally I give some some specific Lie group examples and associated Lagrangians, solutions to the inverse problem.

1 Reminiscences and Introduction

It seems to me to be appropriate in an article such as this one to offer a perspective that is more personal and perhaps more polemical than one is accustomed to see in mathematical research papers. However, in such a platform, which is a celebration of Mike's career there is much to record both about Mike's work and the workshop that has drawn so many people together during the last eighteen years. I first met Mike Crampin in the Spring of 1978 and began work on my doctorate in October of that year. At that first meeting he warned me that I might be putting myself on the path to becoming "unemployable" but I was determined to press on. The first topic that Mike suggested for me to work on was the role played by Killing tensors in classical mechanics. It proved to be a fruitful area and there have been a number of exciting discoveries, principally due to Mike in the last few years, concerning Killing tensors as they relate to Hamilton-Jacobi theory. In fact that circle of ideas is still very much"work in progress" but I will not be touching upon it in this article. After less than two years of working under Mike and suffering somewhat from the loneliness of the long distance doctoral student, I was in

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1980 on the lookout for opportunities to explore other vistas. Such an opportunity was occasioned to a large extent by the circumstance of Mike becoming Dean of the Faculty of Mathematics at the Open University and the realization that his time for meeting with me, with which hitherto he had been most generous, would be severely curtailed. In the event I went off to North America to search for my fortune and am still searching, without realizing that I was making a momentous decision after which life would never be the same again. After spending a felicitous year in North Carolina, I was still nowhere near being able to present a doctoral dissertation so the logical step was to remain enrolled as a graduate student there. I was in no hurry to return to the UK because the prospects of permanent employment in a university seemed remote in the extreme; on the other hand life in North Carolina was proving to be most agreeable. It was after all only 15 years after the reforms of the 1960's and the south of the United States was already being transformed. At least there was no longer the problem of isolation: before I went to the US I had not the least idea of "Graduate School", a concept which really did not exist in the UK. On the other hand now, horror of horrors, I was going to have to take courses and examinations. At this point I must also acknowledge my debt to my other mentor Robert Gardner, who most unfortunately is no longer with us. He passed away in 1998. I owe my passage through Graduate School in large measure to him. In the body of this article his influence is evident, through his dissemination of the work of Elie Cartan.

Even though I was now a graduate student in Chapel Hill I was able to maintain strong ties to Mike and finally in 1984, after having had two papers accepted for publication, I suggested that I had enough material for a doctoral dissertation. Mike was enthusiastic about the idea and I spent most of that summer writing it up. Mike having been released previously from his administrative duties was able to devote much of that summer to me and I particularly value the painstaking line by line criticsm he made of my dissertation. I think that such devotion to detail should be the model for any doctoral supervisor and of course it is characteristic of all Mike's work. Concerning attention to detail, one of the members of my doctoral committee was a certain Willy Sarlet, whom I had the pleasure of meeting for the first time at my viva. In the following year I finished my doctorate in Chapel Hill. Let me say that my reason for so doing was not because I was suffering from a case of meglomania. To my surprise I had been awarded, as it was in those days, an SRC fellowship to support my studies at the Open University. I was cognisant of the fact that it would reflect very poorly on Mike and of course myself if the fellowship never came to fruition. On the other hand since employment prospects in the UK were still as dismal as ever and of course I was equally indebted to the faculty in Chapel Hill, the only sensible course seemed to be to write up both dissertations.

Part of my OU dissertation involved some theory about jet bundles in mechanics and again I am indebted to Robbie Gardner for introducing me to that subject. In fact Gardner was one of the leading proponents of the theory of jet bundles and now, many years later its utility is self-evident. At issue was how to generalize from autonomous to time-dependent systems in mechanics. In 1981 Mike had written a very elegant paper in which he showed how to recast Douglas' fundamental differential system [8] in the inverse problem of Lagrangian mechanics in invariant terminology, using the machinery of tangent bundle geometry [3]. Then again in 1983 he wrote a lengthy article in which he went into the subject of tangent bundle geometry much more extensively and again looked at the same inverse problem [4]. He also credited the afore-mentioned Willy Sarlet [22] with a "...somewhat unexpected and ingenious achievement", referring to the additional algebraic conditions that Willy had been able to generate in the inverse problem and that have been the focus of so much attention since then. As a result of conversations with one Geoffrey Eamon Prince and some rather naive calculations of myself Mike crafted a paper [7] which extended his 1981 analysis to the non-autonomous case. However, perhaps [7] lacks some of the elegance because it uses local coordinates; on the other hand as Jesse Douglas said in the context of another aspect of the inverse problem " ...it is in the nature of things". As happens perhaps more nowadays than at that time, because of the spectacular advance of the internet and the use of email, I was not to meet my coauthor Geoffrey Prince for another two years until the all important year of 1986.

After moving on from Chapel Hill I spent the academic year of 1985-6 in the northern climes of The University of Waterloo, Ontario, Canada and then by some strange hazard obtained an advanced SERC fellowship to be held at the University of Edinburgh in Scotland for the years 1986-8. Mike indicated to me that since I was coming back to the UK I might be interested in going to a small get-together to be held at the the University of Gent in Belgium, that was being organized by that same ubiquitous Willy Sarlet. When I flew into Brussels I was aware of wild scenes of cheering and enthusiasm which, much to my disappointment I learned had nothing whatever to do with our meeting. In fact the period of the meeting coincided with the conclusion of the 1986 football World Cup and the cheering fans were waiting the arrival of the Belgium team which had exceeded all expectations by finishing in fourth place. I was seriously beginning to doubt the usefulness of any such academic meeting. After all I had been forced to miss the final of the World Cup so as to arrive at the meeting on time. Could anything justify my missing the gyrations of Maradona et al?

When I was a graduate student in Chapel Hill there was a seminar entitled "Fluid Dynamics on the Village Green" which always struck me as being a little strange. In the first place in most mathematics departments in the US there is a fierce separation between pure mathematics and physics or engineering, which is not unlike the constitutionally mandated division between church and state. Why then would a pure mathematics department be holding seminars on an applied topic at 5:00 pm on Fridays? And would Americans know what a "Village Green" was even if they saw one? It took me some time to realize that the research, such as it was, was being conducted on the viscosity of certain alcoholic beverages and their propensity to percolate down the human throat. Furthermore, the "Village Green" in question actually alluded to a bar where there were areas of grass to sit on and quaintly entitled "He's not here", which for a long time believe it or not could well have referred to me. As for the meeting in Gent one could well have been forgiven for believing at first glance that its primary purpose too was to conduct research into fluids. However, such an assessment would be very wide of the mark because it was at that meeting that was formed the nucleus of the group that has held a workshop on Differential Geometry and Theoretical Mechanics every year since. It was also at that first workshop that Willy posed his famous "so what" question about life, the universe and everything and for which the bureaucrats who decide about giving out grants are still waiting for an answer. The investigation into the behaviour of fluids has continued to be an important, not to say, indispensable feature of subsequent meetings, mostly notably at a certain famous seminar at Gent in 1992.

"...and now for something completely different", as the famous saying goes. The rest of the article is supposed to be a little more serious. I wanted to take this opportunity to put some of the work that has been done on the inverse problem into context and try to devine some future directions of research. I am concerned primarily with developments of the last ten years. Many of the ideas on Lie groups will be treated in more detail elsewhere. As regards notation the summation convention on repeated indices applies throughout the text. In Section 5 we use (q, x, y, z, w) as local coordinates on $\mathbb{R}^5$ to describe our connections. In order to avoid having an excessive number of dots, the corresponding derivative or velocity variables will be denoted by (p, u, v, s, t). The method that we follow amounts to solving the Helmholtz conditions, which consist of solving in turn some algebraic, ordinary differential and finally some partial differential equations. When we solve the algebraic conditions (equations (3) and (4) below) we generally denote the parameters of the solution by lower case Greek letters. The arbitrary functions that enter from integrating the ODE conditions (equations (7) below) are generally denoted by capital Roman letters and are first integrals of the geodesics. We also acknowledge the indispensable role that MAPLE played in carrying out and checking many of our calculations.

2 The inverse problem

The inverse problem of Lagrangian dynamics consists of finding necessary and sufficient conditions for a system of second order ordinary differential equations to be the Euler-Lagrange equations of a regular Lagrangian function and in case they are, to describe all possible such Lagrangians. Work on the problem had begun even at the end of the nineteenth century but by far the most important contribution was the 1941 article of Douglas [8]. However, Douglas' analysis of the two degrees of freedom case turned out to be so involved that work on the problem was effectively stalled for more than thirty years. Only with the rise of global differential geometric methods was progress possible. Three important contributions were the papers of Crampin et al [3,4,7] Henneaux and Shepley [14] and Sarlet [22]. In [8] the fundamental differantial system of Douglas was recast into coordinate-free language. It is a very elegant paper which illustrates the advantages of the invariant formalism and one to which I returned many times for formulae and inspiration. In [14] the Kepler problem in dimension three was studied and a wide class of non-standard Lagrangians was obtained. Lastly in [22] it was shown how the Helmholtz conditions could be manipulated so as to derive some hidden, purely algebraic conditions. An excellent and comprehensive analysis of the state of the art in 1990, which owes much to the workshops that had already been held, is given in the article by Morandi et al [19]. However, it is clear from [19], that further progress would require the development of yet new techniques and methods.

In the 1990's investigations advanced on three fronts. In [1] Anderson and Thompson presented an algorithm for solving the inverse problem in a concrete situation, which consists of formulating the Helmholtz conditions as an exterior differential system. We shall outline this method for solving the inverse problem, but we suppress the EDS aspects and deal directly with the closure conditions as a PDE system. We consider a system of second order ODE of the form

$$\ddot{x}^i = f^i(x^j, \dot{x}^j). \tag{1}$$

In fact, we shall denote $\dot{x}^i$ by u^i. The first step in the method is to construct the $n \times n$ matrix of functions Φ defined by

$$\Phi^i_j = \frac{1}{2}\,\frac{d}{dt}\left(\frac{\partial f^i}{\partial u^j}\right) - \frac{\partial f^i}{\partial x^j} - \frac{1}{4}\,\frac{\partial f^i}{\partial u^k}\,\frac{\partial f^k}{\partial u^j}. \tag{2}$$

Actually the Φ^i_j are in a certain sense the components of a tensor field known as the Jacobi endomorphism field [5]. One now finds the algebraic solution for g of the equation

$$g\Phi = (g\Phi)^t \tag{3}$$

which expresses the self-adjointness of Φ relative to g. The symmetric matrix g will represent the Hessian with respect to the u^i variables of a putative Lagrangian L. Since there is just a single matrix Φ, one can always find non-degenerate solutions to (3), whatever the algebraic normal of Φ may be. In fact, (3) imposes at most $\frac{n(n-1)}{2}$ conditions on the $\frac{n(n+1)}{2}$ components of g.

In the general theory there is a hierarchy $\overset{n}{\Phi}$ of matrices defined recursively by

$$\overset{n+1}{\Phi} = \frac{d}{dt}\left(\overset{n}{\Phi}\right) - \frac{1}{2}\left[\frac{\partial f}{\partial u}, \overset{n}{\Phi}\right] \tag{4}$$

and the multiplier g is such that each $\overset{n}{\Phi}$ is self-adjoint relative to g. There is also a second hierarchy of algebraic conditions that must be satisfied by g. Define functions Ψ^i_{jk} by

$$\Psi^i_{jk} = \frac{1}{3}\left(\frac{\partial \Phi^i_j}{\partial u^k} - \frac{\partial \Phi^i_k}{\partial u^j}\right). \tag{5}$$

Then the Ψ^i_{jk} are the principal components of the curvature of the linear connection associated to the ODE system (1) (see [5] for further details). The first set of conditions in the hierarchy is given by

$$g_{mi}\Psi^m_{jk} + g_{mk}\Psi^m_{ij} + g_{mj}\Psi^m_{ki} = 0, \tag{6}$$

and the higher order Ψ^i_{jk}'s are obtained from Ψ^i_{jk} much in the same way that the $\overset{n}{\Phi}$'s are obtained from the Φ. As I mentioned in Section 1 Willy Sarlet was the first to consider these extra algebraic conditions in [22].

According to the general theory we now assume that we have a basis of solutions to the double hierarchy of algebraic conditions. If we cannot find a nonsingular solution then we can be sure at this stage that no regular Lagrangian exists for the problem under consideration. Using our basis of solutions we can think of each basis element as giving a "Cartan two-form" for (1). The problem is that such a two-form need not be closed. One of the auxiliary conditions that must be satisfied by g if the corresponding two-form is to be closed is

$$\frac{dg_{ij}}{dt} + \frac{1}{2}\frac{\partial f^k}{\partial u^i} g_{kj} + \frac{1}{2}\frac{\partial f^k}{\partial u^j} g_{ki} = 0. \tag{7}$$

Now (7) is a system of ODE's and it is possible, in principle, to scale basis elements which are solutions to (3) by first integrals of (1) so as to satisfy (7). Being able to carry out the preceding step in practice depends on having explicit first integrals of (1) available. After we have obtained a basis of solutions for (3), each of which satisfies (7), the final step is to impose the so-called closure conditions

$$\frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^j} = 0. \tag{8}$$

This step is accomplished by looking for linear combinations of the basis elements over the ring of first integrals for (1) so that (8) is satisfied. Then (3) and (7) still hold and the resulting closed two-forms, if indeed they exist, will be Cartan two-forms, albeit possibly degenerate. We remark that (3), (7) and (8) together with the symmetry and non-degeneracy of g constitute the Helmholtz conditions for the inverse problem for (1). The double hierarchy of algebraic conditions are actually integrability conditions of zeroth order that can be deduced from (3),(7) and (8) so that there is no loss of generality in assuming that they are satisfied from the outset. See [1,23] for more details. One of the curious aspects of the inverse problem is that, in practice, these auxiliary algebraic conditions rarely do seem to materially decrease the difficulty of solving the Helmholtz conditions.

The reader should observe that the Helmholtz conditions and its modification in [1] are concerned with finding all possible Hessians of Lagrangians for a given second order ODE system. It is unrealistic to hope to give all possible Lagrangians in closed form, even for a free particle system. To pass from the Hessian to the Lagrangian requires two integrations and Douglas [8] explains how the linear and zeroth order terms may added so as to obtain a bona fide Lagrangian; in particular, the fact that appropriate terms can be found is a consequence of the Helmholtz conditions and the only ambiguity in passing from Hessian to Lagrangian is the trivial one of scaling by a constant and adding a total time derivative. The drawback, if such it is, of this approach is that it does not make evident all the integrability conditions that arise from (1). It is not even clear how to obtain the extra conditions obtained by Muzsnay and Grifone [11,12].

The second approach to the inverse problem is due to Muzsnay and Grifone who completely by-pass the Helmholtz conditions. They work directly with the Euler-Lagrange operator and employ the techniques of Spencer cohomology [11,12]. They obtain yet more purely algebraic conditions but their results are difficult to reconcile with the Helmholtz conditions approach, all the more so because of the dearth of examples. In their approach, the first obstruction in prolonging solutions from order to two to order three is expressed by the condition that the horizontal distribution should be Lagrangian for the required Cartan two-form. The Euler-Lagrange operator is now augmented by this second order condition. In the Helmholtz formulation, this condition is automatically satisfised as too, of course, is the same condition for the vertical distribution. The condition that permits one to prolong a solution to the augmented system from order two to order three is precisely expressed by (3) and (7) above. Two major themes of Muzsnay and Grifone are whether the variational multiplier is diagonal or not and whether the second order system ("spray" in their terminology) is "typical" or "atypical", the distinction between the two being that in the former case, the spray is an eigenvector of Φ whereas in the latter it is not. The reason for the term "typical" is that geodesic sprays of linear connections begin to this class, though paradoxically, it is obviously a non-generic condition. It is much easier to test whether the symbols of the prolongation of "typical" systems are involutive, whereas in the atypical case, more prolongations and messy arguments are required.

The third direction in the inverse problem was initiated by Martinez, Sarlet and Crampin [5,23]. They and others developed a powerful calclulus associated to any second order ODE system. The most interesting feature of this calculus is that all objects are considered to be defined "along the tangent bundle projection" and a remarkable efficiency in computation is thereby obtained. They used this calculus to solve a number of problems related to the inverse problem, such as determining necessary and sufficient conditions for a second order ODE system to decouple into scalar equations under a point transformation [5,17,24]. Let me record here some small related results. The key construct in the decoupling problem is the H-tensor: it is in fact a concomitant associated to any field of endomrphisms. Now in [17] the situation is more complicated because all the tensorial objects are defined along the tangent bundle projection. As such there are both vertical and horizontal H-tensors. In fact the H-tensor in [17] is introduced by means of an auxiliary tensor C that is defined and actually depends on a connection, which, in the context of [5,17,24], is provided by the vertical and horiziontal covariant derivatives that are associated to the system of second order ODE in question. However, the H-tensor can be defined directly, and so is independent of any such connection. Indeed we have:

Proposition 2.1 *Let U be a field of endomorphisms defined on an n-manifold M. Then H defined by the formula*

$$H_U(X,\cdot) = [L_{U^2X}U, U] - 2U \circ [L_{UX}U, U] + U^2 \circ [L_X U, U] \tag{9}$$

is a tensor of type (1,2), where X is an arbitrary vector field on M and the bracket denotes the algebraic commutator of endomorphisms and not the Nijenhuis bracket.

Proof The proof is a routine calculation using the formula

$$L_X fU = fL_X U + U(X) \otimes df - X \otimes U(df), \tag{10}$$

where f, X and U denote a function, vector field and field of endomorphisms, respectively. □

Recent investigations of myself and my student Chamath Hettiarchchi suggest that more remains to be discovered about H although in practice it turns out to be a quite complicated object. Another computation leads to:

Proposition 2.2 *Let U and H be defined as above. Then*

$$H_U(X,\cdot) = U \circ [N_U(\cdot, X), U] + [U, N_U(\cdot, UX)] \tag{11}$$

where N_U denotes the Nijenhuis tensor of U. □

The local expression for H is given by

$$H^l_{mn} = U^l_j N^j_{kn} U^k_m - U^l_j U^k_n N^j_{km} + U^l_j N^j_{mk} U^k_n - N^j_{nk} U^k_m U^l_j. \tag{12}$$

Turning now to the two degrees of freedom case, there are now three comprehensive analyses. First of all there is Douglas' version [8]. Douglas claimed to have investigated every possible eventuality. Since Douglas was a Fields medal winner and not prone to making errors one is inclined to believe him. The difficult cases arise from having to take many prolongations in the application of Riquier's theory of differential systems. Unfortunately the calculations are so involved, with so many "dots", that it is not clear that all the cases considered by Douglas can indeed occur; and one does not have much incentive for investing a lot of time in very complicated cases, which after the fact, may prove to be non-existent. Douglas does supply lots of examples, however, it is not clear whether the "missing " examples are oversights on his part or simply that he chose not to write them down. For example, he offers no systems which belong to case IIa3, though he does claim that they are quite extensive. After a considerable effort Ian Anderson and I did find a class of examples of that type. Another example that we were unable to find was for case IIIa where the system is not variational. That case depends on the closure of a certain one-form but in all the systems with which I am familiar, the one-form is closed and there is a Lagrangian. After expending a lot of effort in the early nineteen nineties trying to construct many of these examples, I reached the conclusion that, although one may be able to establish the existence of certain types of Lagrangians by using Riquier, Cartan-Kahler or Spencer theory and although some of these cases may in a certain sense be "generic" within a subclass, it may not be possible to write down a concrete exmple in terms of elementary functions. See also page 124 in Douglas, where he gives an "example" belonging to case IIb. At that point one is bound to ask whether the whole enterprise is any longer worthwhile. I have never been attracted to the kind of research that produces six mathematical theorems in search of a single example.

The second comprehensive analysis of the two degrees of freedom case is by Muzsnay and Grifone [12]. They too claim to cover every eventuality. The really difficult case to analyze is in Douglas' classification case II. Muzsnay and Grifone devote 57 pages to the subcase of II in which the spray is atypical. Again for this author, as in the version of Douglas, the remoteness from concrete examples is extremely disturbing. To take one case in point Muzsnay and Grifone discuss in the case where the multiplier is diagonal and distinguish various subcases as "reducible", "semi-reducible" and "irreducible", which they say, " ...is very close to but not exactly the same as Douglas' classification of separable, semi-separable and non-separable sprays." However, they fail to tell us precisely what the differences are and the interested reader is faced with working through pages of calculations. To be fair, perhaps this author is asking for something that is impossible, namely, a comprehensive collection of examples that illustrate all the principal cases.

The third account of the two degrees of freedom case is by Sarlet, Thompson and Prince [25]. This analysis uses the tangent bundle-projection calculus and while it does not claim to be exhaustive, it does provide an effective means of obtaining the principal cases.

3 The Lie group problem

One aspect of the inverse problem which until recently was little explored, is the very special case of the geodesic equations of the canonical symmetric connection, that we shall denote by ∇, belonging to any Lie group G. The canonical connection ∇ was introduced by Cartan and Schouten in [2]. In fact ∇ is defined on left invariant vector fields X and Y by

$$\nabla_X Y = \frac{1}{2}\,[X, Y] \tag{13}$$

and then extended to arbitrary vector fields by making ∇ tensorial in the X argument and satisfy the Leibnitz rule in the Y argument. Following the conventions of [13] a left invariant vector field X is denoted by $\tilde{X}$, that is, $\tilde{X}(g) = L_{g*}X$. Likewise the right invariant vector field induced by X is denoted by $\tilde{X}^{R(g)}$ so that $\tilde{X}^{R(g)} = (R_g)_* X$. It follows that

$$\tilde{X}^{R(g)} = (Ad(g^{-1})X)^{\sim},$$

where Ad denotes the adjoint representation. If Y is a second tangent vector then

$$\begin{aligned}\nabla_{\tilde{X}^{R(g)}} \tilde{Y}^{R(g)} &= \nabla_{(Ad(g^{-1})X)^{\sim}} (Ad(g^{-1})Y)^{\sim} \\ &= 1/2\ [(Ad(g^{-1})X)^{\sim},\ (Ad(g^{-1})Y)^{\sim}] \\ &= 1/2\ [\tilde{X}^{R(g)},\ \tilde{Y}^{R(g)}].\end{aligned}$$

Thus in (1) X and Y could equally well denote right invariant rather than left invariant vector fields.

It can be shown that ∇ is symmetric, bi-invariant and that the curvature tensor on left or right invariant vector fields is given by

$$R(X, Y)Z = \frac{1}{4}\,[Z, [X, Y]]. \tag{14}$$

Furthermore, G is a symmetric space in the sense that R is a parallel tensor field. Indeed suppose that W, X, Y and Z are left-invariant vector fields. Then from (1) and (2) we have that

$$\begin{aligned}
4\nabla_W R(X,Y)Z &= 1/2[W,[Z,[X,Y]]] - 4R(\nabla_W X,Y)Z \\
&\quad -4R(X,\nabla_W Y)Z - 4R(X,Y)\nabla_W Z \\
&= 1/2[W,[Z,[X,Y]]] - [Z,[\nabla_W X,Y]] \\
&\quad -[Z,[X,\nabla_W Y]] - [\nabla_W Z,[X,Y]] \\
&= 1/2[W,[Z,[X,Y]]] - 1/2[Z,[[W,X],Y]] \\
&\quad -1/2[Z,[X,[W,Y]]] - 1/2[[W,Z],[X,Y]] \\
&= 1/2([Z,[W,[X,Y]]] - [Z,[[W,X],Y]] - [Z,[X,[W,Y]]]) \\
&= 0.
\end{aligned}$$

It follows from (2) that ∇ is flat if and only if the Lie algebra g of G is nilpotent of order two. The geodesics of ∇ are translates of one parameter subgroups of G. The Ricci tensor R_{ij} of ∇ is symmetric and bi-invariant. In fact, if $\{E_i\}$ is a basis of left invariant vector fields then

$$[E_i, E_j] = C^k_{ij} E_k \tag{15}$$

where C^k_{ij} are the structure constants and relative to this basis the Ricci tensor R_{ij} is given by

$$R_{ij} = \frac{1}{4}\, C^l_{jm} C^m_{il} \tag{16}$$

from which the symmetry of R_{ij} becomes apparent. Indeed, R_{ij} is obtained by translating to the left or right one quarter of the Killing form. Since R^i_{jkl} is a parallel tensor field and R_{ij} is symmetric, it follows that Ricci gives rise to a quadratic Lagrangian which may, however, not be regular. In fact it is a natural question to ask whether or not there is a metric of some signature for which ∇ is the Levi-Civita connection. If G is semi-simple then the Killing form provides a bi-invariant metric whose Levi-Civita connection is ∇. For this reason, I will usually assume that G is not semi-simple. Now it is known that if L is the standard quadratic Lagrangian, any smooth function of it will also be a Lagrangian, subject only to regularity and I conjecture that it is the only ambiguity in the description of the Lagrangian. I shall also assume that G is indecomposable in the sense that the Lie algebra g of G is not a direct sum of lower dimensional algebras. It should be noted however, that, generally, in solving the inverse problem, it is not sufficient to restrict to indecomposable algebras. Certainly, if a decomposable algebra is a sum of algebras, each of which possesses a variational connection, then the sum will certainly be variational. However, the most general Lagrangian for the sum need not be just the sum of individual Lagrangians. Furthermore, even if each component is not variational, it is not clear that the same is true for the sum, though I am unaware of a counterexample. The smallest dimension in which such a phenomenon could occur is five.

I have investigated the situation for Lie groups of dimension two and three in [29]. It was found in [29] that in all these cases the geodesics were the Euler -Lagrange equations of a suitable Lagrangian defined on an open subset of the tangent bundle TG. In [10] the inverse problem for the canonical connection in the case of Lie groups of dimension four was studied and some rather comprehensive results were obtained. We began our investigations at the Lie algebra level and worked from the list of four-dimensional Lie algebras given in [20]. The Lagrangians are constructed by implementing the algorithm described in detail in [1]. The fact that the procedure can be carried out can be traced ultimately to the fact that every left and right-invariant one form on G gives rise to a first integral for the geodesics of ∇. However, we have found it convenient in several cases to modify the usual procedure, for example, by simplifying the system of geodesics before implementing the algorithm. In other cases it is profitable to introduce complex coordinates and regard two of the geodesic equations as the real and imaginary parts of a single complex equation. It was found that there are several classes of group whose geodesic equations are not the Euler-Lagrange equations of any regular Lagrangian and this primary existence test can be performed as a pure Lie algebra calculation. For the remaining groups, classes of Lagrangian were obtained depending on arbitrary functions and in several cases a complete description of all possible Hessians was obtained. The case of dimension four is probably the last case in which it is feasible to obtain comprehensive results. In forthcoming investigations we hope to be able to study canonical connections in higher dimensions and for nilpotent algebras and construct the symmetry groups as well as considering the special case where the connection is of Levi-Civita type. We shall also consider the Helmholtz conditions as a differential system using right-invariant coordinates, as opposed to the straightforward approach adopted here, and investigate various integrability conditions that arise. There are two cases in dimension four where the connection is the Levi-Civita connection of a bi-invariant metric, but for these cases, the closure conditions are the most involved and at the moment a complete understanding is lacking. Notice that if a canonical connection is of Levi-Civita type the associated metric is not necessarily bi-invariant. In dimension five there are two flat nilpotent connections for which the corresponding metrics are not bi-invariant. In fact I would make the following conjecture:

Conjecture: *If a Lagrangian associated to an invariant connection is bi-invariant, then there is another equivalent Lagrangian which is the quadratic Lagrangian of a bi-invariant metric. Furthermore, the only ambiguity in the Lagrangian arises from scaling by constants and the addition of symmetrized products of parallel one-forms.*

At the moment I do have not a method for proving this conjecture, but I know of no counterexamples. On the other hand one has to decide exactly what one means by "bi-invariant Lagrangian", because such a Lagrangian is defined on TG and not on G. If the conjecture is in the appropriate sense true, then it says that the Lie group project is only of limited interest.

Since our starting point is the Lie algebra g of a Lie group and since we are assuming that G is not semi-simple, it is of interest to ask how the ideals of g are related to ∇. To this end we shall quote the following result [16].

Proposition 3.1 *Let ∇ denote a symmetric connection on a smooth manifold M. Necessary and sufficient conditions that there exist a submersion from M to a quotient space Q such that ∇ is projectable to Q are that there exists an integrable distribution D on M that satisfies:*

(i) $\nabla_X Y$ belongs to D whenever Y belongs to D and X is arbitrary;

(ii) $R(Z, X)Y$ belongs to D whenever Z belongs to D and X and Y are arbitrary, where R denotes the curvature of ∇. □

Clearly in the case of the canonical connection on G the two conditions of the last Proposition coalesce into just one. Furthermore if h is an ideal in g it gives rise to an integrable distribution on G for which this single condition holds. Thus we have:

Proposition 3.2 *Every ideal h of g gives rise to a quotient space Q consisting of the leaf space of the integrable distribution determined by h and ∇ on G projects to Q.* □

The center of g is of course an ideal and it has the property that any element of it gives rise to a parallel vector field on G. A very interesting situation occurs where g possesses two ideals h_1 and h_2 such that $h_1 \cap h_2$ is zero. Denote the corresponding distributions on G by D_1 and D_2, respectively. Since we are always assuming that g is indecomposable, g cannot be the direct sum of h_1 and h_2 and hence $D_1 \cap D_2$ is non-zero. In fact $D_1 \cap D_2$ is the integrable distribution on G that corresponds to the ideal $h_1 + h_2$ of g and simliarly $D_1 + D_2$ corresponds to the ideal $h_1 \cap h_2$. In the most favourable of circumstances, one is able to construct all Lagrangians from a knowledge of Lagrangians on two quotient spaces [10].

We turn our attention next to properties of the geodesic flow Γ of ∇. We note first of all that since ∇ is bi-invariant any left-invariant vector field will be a Killing vector field or affine collineation of ∇. Indeed if X and Y are also left-invariant one finds that the Lie derivative of ∇ along Z is given by

$$\begin{aligned}
(L_Z\nabla)_X Y &= [Z, \nabla_X Y] - \nabla_{[Z,X]} Y - \nabla_X [Z, Y] \\
&= \tfrac{1}{2}([Z, [X.Y]] + [[X, Z], Y]) + [X, [Y, Z]]) \\
&= 0
\end{aligned}$$

because of the Jacobi identity. The same argument applies equally to right-invariant vector fields. Again if Z is a left or right-invariant vector field it follows that on TG the fields Γ and Z^C commute where Z^C is the complete lift of Z to TG. A very interesting consequence of the latter remark is that whenever L is a Lagrangian that engenders Γ as its Euler-Lagrange vector field, the function Z^CL is another, possibly degenerate Lagrangian. See [21] for a further discussion of this point.

The case of dimension five will be the subject of the doctoral dissertation of Igor Strugar [26] and will continue the investigations in [29,10]. The primary concern is to ascertain whether or not a particular connection is derivable from a Lagrangian function and, if so, to give at least one such Lagrangian. In the future I would like to be able look at the inverse problem for the six-dimensional nilpotent Lie algebras of which there are 24 classes. At that point the the low dimensional classification of Winternitz et al will have been exhausted, though Ian Anderson is in the process of completing a new description of all six-dimensional algebras of which there are 99 classes.

4 Lie group algorithm

In this section let us explain next how the general theory of Section 3 simplifies for the case of the geodesic equations associated to a canonical connection. In the case of a symmetric linear connection the matrix Φ is of the form

$$\Phi^i_j = R^i_{kjl} u^k u^l \tag{17}$$

where R^i_{kjl} are the components of the curvature R of the connection relative to a coordinate system (x^i). The higher order Φ-tensors in this case just correspond to covariant derivatives of the curvature so that, for example,

$$\overset{1}{\Phi}{}^i_j = R^i_{kjl;m} u^k u^l u^m. \tag{18}$$

Since R is parallel all the higher order Φ-tensors vanish. Similarly for the case of a linear connection, one finds that

$$\Psi^i_{jk} = R^i_{ljk} u^l \tag{19}$$

and again the higher order Ψ's correspond to covariant derivatives of R. Thus, for example,

$$\overset{1}{\Psi}{}^i_{jk} = R^i_{ljk;m} u^l u^m. \tag{20}$$

Again since R is parallel the higher order Ψ-tensors vanish. The condition coming from Φ is

$$\left(g_{mi} R^i_{pjq} - g_{ji} R^i_{pmq}\right) u^p u^q = 0, \tag{21}$$

while the condition coming from Ψ is

$$(g_{mi}R^i_{pjq} + g_{qi}R^i_{pmj} + g_{ji}R^i_{pqm})u^p = 0. \tag{22}$$

If we contract u^q into (22) we find from (21) that

$$g_{qi}R^i_{pmj}u^p u^q = 0. \tag{23}$$

Thus, for the special case of a canonical connection, we can use (22) and (23) as the first and only algebraic conditions in the double hierarchy and the calculation can be done at the Lie algebra level without the need for having a group representation. In some cases we find that, even at that level, the matrix g_{ij} is forced to be singular. In such a case we can be sure that there will be no Lagrangian corresponding to the geodesic equations of any Lie group that has g as its Lie algebra. Suppose, however, that conditions (22) and (23) do not entail that g_{ij} should be singular. One is now faced with the problem of finding a Lie group G so that g is its Lie algebra. An answer of sorts is furnished by Ado's theorem [15], which asserts, in the first instance, that any finite dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ has a faithful finite -dimensional linear representation. The corresponding group can then be obtained, in principle, by exponentiation. If g has only a trivial center then the adjoint representation is faithful. If the center is non-trivial then there is no obvious representation available. Ado's theorem appears to offer no information on the order of the representating matrices.

It is very convenient, if not essential for our purposes, to work with linear representations of order n for algebras and groups of order n. Now it would seem to be impossible to obtain a classification of all finite-dimensional Lie algebras, let alone Lie groups. Nonetheless such classifications, or perhaps descriptions, are available in dimensions 2,3,4 and 5 [20]. The cases of dimensions 2 and 3 have been discussed in [29]. As for dimension 4, we have in every case been able to find a faithful linear representation by 4×4 matrices without recourse to Ado's theorem [10]. Thus

Theorem 4.1 *Every Lie algebra in dimensions two, three and four has a faithful representation by matrices of order two, three and four, respectively.*

Let us assume now that we have a Lie algebra g, that conditions (22) and (23) do not entail that the matrix g_{ij} is singular and that we have a matrix representation and that we are able to determine a corresponding Lie group G by exponentiation. On G we construct the right invariant Maurer-Cartan one-form and then by dualizing, we obtain a basis for the right invariant vector fields. We obtain thereby a representation of g by vector fields. From Section 4 we see that no further algebraic conditions can arise and we proceed to formulate conditions (7). We solve these conditions with the help of first integrals of the geodesic equations of the canonical connection ∇ on G. Finally, we formulate and solve to the extent that is possible, the closure PDE conditions (8).

We state next without proof several results about first integrals. The existence of these integrals in the Lie group context is the ultimate reason that conditions (7) and (8) can be implemented in practice.

Proposition 4.1 *Any left or right invariant one-form on G gives rise to a linear first integral on TG.* □

Proposition 4.2 *Consider the following conditions for a one-form α on G:*

(i) *α is right-invariant and closed;*
(ii) *α is left-invariant and closed;*
(iii) *α is bi-invariant;*
(iv) *α is parallel.*

Then we have the following implications: (iii) implies (i); (iii) implies (ii); each of (i),(ii) or (iii) implies (iv). □

Proposition 4.3 *Suppose that a basis for a Lie algebra g of a Lie group G consists of*

$$X_i = \frac{\partial}{\partial x^i}, \qquad W = \frac{\partial}{\partial w} + a^k_j x^j \frac{\partial}{\partial x^k}, \tag{24}$$

where a^k_j is a constant $n \times n$ matrix. Then the geodesic equations for the canonical connection on G are given by

$$\ddot{x}^i = a^i_j \dot{x}^j \dot{w}, \qquad \ddot{w} = 0. \tag{25}$$

This last proposition pertains to the class of Lie algebras that have a codimension one abelian nilradical. Such algebras are characterized by a single "ad" matrix.

5 Examples

In keeping with my usual philosophy I will exhibit several Lie group examples in this final section. The numbering of the Lie algebras comes from Winternitz' 1976 classification [20].

Example 1: The Euclidean group of the plane

This example is particularly nice and the entire algorithm can be carried through completely. The corresponding Lie algebra is denoted by $A_{3,6}$ in [20] and has basis e_1, e_2, e_3 with non-zero brackets, $[e_1, e_3] = -e_2$ and $[e_2, e_3] = e_1$. As in [29] the geodesics are given by

$$\dot{u} = tv, \quad \dot{v} = -tu, \quad \dot{t} = 0, \tag{26}$$

where u, v and t denote $\dot{x}, \dot{y}$ and $\dot{w}$, respectively. The connection form θ is given by

$$-2\theta = \begin{bmatrix} 0 & -dw & -dy \\ dw & 0 & dx \\ 0 & 0 & 0 \end{bmatrix}$$

and the curvature two-form is given by

$$4\Omega = \begin{bmatrix} 0 & 0 & dxdw \\ 0 & 0 & dydw \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence we see that the curvature tensor has essentially only the following non-zero components

$$4R^1_{313} = 1, \qquad 4R^2_{323} = 1. \tag{27}$$

Conditions (22) and (23) entail that g_{ij} satisfies the conditions

$$g_{1q}u^q = g_{2q}u^q = 0$$

and the solution of the ODE conditions (7) imply that g_{ij} is given by

$$g_{ij} = M \begin{bmatrix} t^2u & t^2v & -t(u^2+v^2) \\ t^2v & -t^2u & 0 \\ -t(u^2+v^2) & 0 & u(u^2+v^2) \end{bmatrix} + P \begin{bmatrix} t^2 & 0 & -tu \\ 0 & t^2 & -tv \\ -tu & -tv & u^2+v^2 \end{bmatrix}$$
$$+ N \begin{bmatrix} -t^2v & t^2u & 0 \\ t^2u & t^2v & -t(u^2+v^2) \\ 0 & -t(u^2+v^2) & v(u^2+v^2) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}.$$

The closure conditions (8) turn out to be

$$uM_u + vM_v + tM_t + 4M = 0 \tag{28}$$

$$uN_u + vN_v + tN_t + 4N = 0 \tag{29}$$

$$uP_u + vP_v + tP_t + 3P = 0 \tag{30}$$

$$F_u = 0 \tag{31}$$

$$F_v = 0 \tag{32}$$

$$P_v + 2N + tN_t + uM_v - vM_u = 0 \tag{33}$$

$$P_u + 2M + tM_t - uN_v + vN_u = 0. \tag{34}$$

The closure conditions can be solved by introducing the following first integrals: $\alpha = \frac{u}{t} - y$, $\beta = \frac{v}{t} + x$, $\gamma = \frac{cos(w)u - sin(w)v}{t}$, $\delta = \frac{cos(w)v + sin(w)u}{t}$. Thus

$$F = F(t) \tag{35}$$
$$M = \frac{m(\alpha,\beta)}{t^4} \tag{36}$$
$$N = \frac{n(\alpha,\beta)}{t^4} \tag{37}$$
$$P = \frac{(2n + \delta m_\gamma - \gamma m_\delta)\beta + (2m - \delta n_\gamma + \gamma n_\delta)\alpha}{t^3} \tag{38}$$

where m, n and F are arbitrary smooth functions.

A very simple Lagrangian in this case is given by

$$L = \frac{(u^2 + v^2)}{t} + xv - yu + t^2. \tag{39}$$

Example 2: A4,4

The Lie algebra relations are:

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \quad [e_3, e_4] = e_2 + e_3. \tag{40}$$

On exponentiating the matrices E_1, E_2, E_3, E_4 one finds that a typical element S of the Lie group associated to the Lie algebra is given by

$$S = \begin{bmatrix} e^w & e^w w & \frac{e^w w^2}{2} & x \\ 0 & e^w & e^w w & y \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The one-forms $dw, dx - (x+y)dw,\ dy - (y+z)dw,\ dz - zdw$ comprise a right-invariant coframe. The corresponding right-invariant frame of vector fields is given by

$$\begin{aligned} W &= \frac{\partial}{\partial w} + (x+y)\frac{\partial}{\partial x} + (y+z)\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \\ X &= \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}. \end{aligned} \tag{41}$$

The corresponding system of geodesic equations is given by

$$\dot{t} = 0, \quad \dot{u} = tu, \quad \dot{v} = t(u+v), \quad \dot{s} = t(v+s) \tag{42}$$

where s, t, u and v denote $\dot{z}, \dot{w}, \dot{x}$ and $\dot{y}$, respectively. For later use we define the following first integrals of the geodesics:

$$\alpha = \frac{e^{-x}t}{u}, \quad \beta = \frac{t - wu}{u}, \quad \gamma = \frac{e^{-x}(s - xt)}{u}, \quad \delta = \frac{s}{u} - (z + w),$$

$$\zeta = \frac{v}{u} - (y + z), \qquad \eta = \frac{e^{-x}(v - xs + \frac{x^2 t}{2})}{u}.$$

The connection form θ is given by

$$-2\theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ dy + dz & dx & dx & 0 \\ dz + dw & 0 & dx & dx \\ dw & 0 & 0 & dx \end{bmatrix}$$

and the curvature two-form is given by

$$4\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ dx(dy + 2dz + dw) & 0 & 0 & 0 \\ dx(dz + 2dw) & 0 & 0 & 0 \\ dxdw & 0 & 0 & 0 \end{bmatrix}.$$

We see that the curvature tensor has essentially only the following non-zero components

$$4R^2_{112} = 1, \; 2R^2_{113} = 1, \; 4R^3_{113} = 1, \; 4R^2_{114} = 1, \; 2R^3_{114} = 1, \; 4R^4_{114} = 1. \quad (43)$$

Conditions (22) and (23) entail that g_{ij} satisfies the following condition:

$$g_{ij} = \begin{bmatrix} \lambda & t^3\mu & t^2\sigma & t\rho \\ t^3\mu & 0 & 0 & -t^2\mu \\ \sigma t^2 & 0 & -\mu t^2 & -\sigma tu + \mu stu \\ t\rho & -t^2\mu & -\sigma tu + \mu stu & \mu(tuv - s^2u) + \sigma su - \rho u \end{bmatrix}.$$

The ODE conditions (7) are given by

$$\dot{\lambda} + (s + v)t^3\mu + (s + t)t^2\sigma + t^2\rho = 0 \quad (44)$$

$$\dot{\mu}t + 2\mu\dot{t} + ut\mu = 0 \quad (45)$$

$$t\dot{\rho} + \rho\dot{t} = 0 \quad (46)$$

$$t\dot{\sigma} + 2\dot{t}\sigma = 0. \quad (47)$$

The solution to these ODE conditions is given by

$$g_{ij} = M \begin{bmatrix} -\frac{v}{u} & 1 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \\ 0 & 0 & -\frac{u}{t} & -\frac{su}{t^2} \\ 0 & -\frac{u}{t} & -\frac{su}{t^2} & \frac{tuv-s^2u}{t^3} \end{bmatrix} + S \begin{bmatrix} -\frac{s}{u} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \\ 0 & 0 & -\frac{u}{t} & \frac{su}{t^2} \end{bmatrix}$$
$$+ \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + R \begin{bmatrix} -\frac{t}{u} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \end{bmatrix}$$

where L, M, R, S are first integrals.

After some rearrangement, the closure conditions turn out to be equivalent to:

$$L_s = L_t = L_v = 0 \tag{48}$$
$$uM_u + tM_t + M = 0 \tag{49}$$
$$sS_s + tS_t + uS_u + S = 0 \tag{50}$$
$$sR_s + tR_t + uR_u + vR_v + R = 0 \tag{51}$$
$$S_s = M_t, M_t = R_v, S_t = R_s. \tag{52}$$

Using the first integrals introduced earlier we can write the solutions for L, M, R, S as

$$L = L(u) \tag{53}$$
$$uM = a(\alpha, \beta) \tag{54}$$
$$uS = b(\alpha, \beta) \tag{55}$$
$$uR = c((\alpha, \beta, \gamma, \delta, \eta, \zeta)). \tag{56}$$

There are three closure conditions that remain to be satisfied. However, they already imply that

$$R_{vv} = 0, R_{sv} = 0, R_{sss} = 0.$$

It follows that we may write

$$uS = A(\alpha, \beta)\gamma + B(\alpha, \beta)\delta \tag{57}$$

and

$$\begin{aligned} uR = {} & H(\alpha, \beta)\eta + J(\alpha, \beta)\zeta + C(\alpha, \beta)(\delta)^2 \\ & + D(\alpha, \beta)\delta + E(\alpha, \beta)(\gamma)^2 + F(\alpha, \beta)\gamma + G(\alpha, \beta). \end{aligned} \tag{58}$$

In (57) and (58) $A, B, ..., J$ are arbitrary smooth functions of their arguments. If we substitute (57) and (58) into (52) these functions may be identified via the following equations:

$$uS = a_\alpha \gamma + a_\beta \delta \tag{59}$$

and

$$uR = a_\alpha \eta + a_\beta \zeta + a_{\beta,\beta}\frac{\delta^2}{2} + a_{\alpha,\alpha}\frac{\gamma^2}{2} + G(\alpha, \beta). \tag{60}$$

Thus the Hessian is parametrized by the functions a, G and L. A concrete Lagrangian is given by

$$L = (2s + u - v)\ln(\frac{u}{t}) - 2ws - wv + \frac{v^2}{u} + t^2. \tag{61}$$

Example 3: $A_{4,9}\,(b = 0)$

The system of geodesic equations is given by

$$\dot{t} = 0,\ \dot{u} = 0,\ \dot{v} = tv,\ \dot{s} = uv + st, \tag{62}$$

where the same notation as in the previous example is used. The curvature tensor has essentially only the following non-zero components

$$4R^4_{123} = 1, \quad 4R^3_{113} = 1, \quad 4R^4_{213} = 1, \quad 4R^4_{114} = 1. \tag{63}$$

Conditions (22) and (23) entail that g_{ij} satisfies the following conditions:

$$u^q g_{q3} = u^q g_{q4} = g_{24} = g_{44} = g_{14} - g_{23} = 0. \tag{64}$$

The algebraic solution for g may be written as

$$g = \begin{bmatrix} \lambda & \mu & \sigma & -\frac{v\tau}{t} \\ \mu & \rho & -\frac{v\tau}{t} & 0 \\ \sigma & -\frac{v\tau}{t} & \frac{(uv-st)\tau}{tv} & \tau \\ -\frac{v\tau}{t} & 0 & \tau & 0 \end{bmatrix}.$$

The ODE conditions (7) are given by

$$\dot{\lambda} + v\sigma - \frac{sv\tau}{t} = 0, \quad \dot{\rho} = 0, \quad \dot{\mu} - \frac{v^2\tau}{t} = 0, \quad \dot{\sigma} = 0, \dot{\tau} + t\tau = 0. \tag{65}$$

When (65) are integrated we find that a new solution for g is given by

$$g = P\begin{bmatrix} \frac{s}{t^2}-\frac{2uv}{t^3} & \frac{v}{t^2} & \frac{u}{t^2} & -\frac{1}{t} \\ \frac{v}{t^2} & 0 & -\frac{1}{t} & 0 \\ \frac{u}{t^2} & -\frac{1}{t} & -\frac{s}{v^2} & \frac{1}{v} \\ -\frac{1}{t} & 0 & \frac{1}{v} & 0 \end{bmatrix} + Ne^{-w}\begin{bmatrix} \frac{2v^2}{t^3} & 0 & -\frac{2v}{t^2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2v}{t^2} & 0 & \frac{2}{t} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} K & L & 0 & 0 \\ L & M & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

At this point we cannot make a definitive decision about the existence of a Lagrangian and we are thus compelled to explore the closure conditions (8). After a long calculation we find that they are equivalent to the following conditions:

$$\begin{aligned} &K = K(t,u) \\ &L = L(t,u) \\ &M = M(t,u) \\ &M_t - L_u = 0 \\ &K_u - L_t = 0 \end{aligned} \tag{66}$$

and

$$\begin{aligned} &tN_t + uN_u + vN_v + sN_s = 0 \\ &N_s - P_v = 0 \\ &P_t - N_u = 0 \\ &vP_v + tP_t = 0 \\ &P_u = P_s = 0. \end{aligned} \tag{67}$$

Thus the conditions involving K, L and M decouple from the ones for N and P and we analyze the latter two functions in the following way. We note first of all that the following seven functions constitute a maximally functionally independent set of linear first integrals: $t, u, e^{-w}v, xu - wu, e^{-w}(s - vx), yt - v, s - yu - zt$. For P we note that it is annihilated by $\frac{\partial}{\partial u}, \frac{\partial}{\partial s}, t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$ and the geodesic vector field Γ where

$$\Gamma = t\frac{\partial}{\partial w} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + s\frac{\partial}{\partial z} + vt\frac{\partial}{\partial v} + (uv + st)\frac{\partial}{\partial s}. \tag{68}$$

A short calculation shows that these differential operators span a six-dimensional integrable distribution and that the general solution for P is given by

$$P = P(\alpha, \beta) \tag{69}$$

where

$$\alpha = y - \frac{v}{t}, \beta = e^{-w}\frac{v}{t}. \tag{70}$$

As for N, at the outset it is subject to three conditions. Two of them are embodied in (67) and the third arises from the fact that N is a first integral and so is annihilated by the geodesic vector field Γ. In addition to Γ and $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial s}$ define the following two differential operators:

$$\Delta = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + s\frac{\partial}{\partial s} \tag{71}$$

$$T = t\frac{\partial}{\partial u} + v\frac{\partial}{\partial s}. \tag{72}$$

We find now that the operators Δ, Γ, T and $[T, \Gamma]$ close as a four-dimensional integrable module. Hence the general solution for N involves four arbitrary functions. Finally there remain the two conditions relating N and P in (67) and when they are imposed one finds the following general solution:

$$N = (z + \frac{(uv - st)}{t^2})\frac{\partial P}{\partial \alpha} + \frac{e^{-w}(uv - st + xtv)}{t^2}\frac{\partial P}{\partial \beta} + R(\alpha, \beta) \tag{73}$$

where R is an arbitrary function. Together (69) and (73) furnish a complete solution for the Hessian of the Lagrangian that we are seeking. A particular Lagrangian is given by

$$L = s\ln(\frac{v}{t}) - \frac{uv}{t} + tu + zt - xv. \tag{74}$$

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