Lagrangian equations and affine Lie algebroids

Willy Sarlet

Department of Mathematical Physics and Astronomy
Ghent University, Krijgslaan 281, B-9000 Ghent, Belgium
email: willy.sarlet@rug.ac.be

1 Introduction

Lagrangian equations on Lie algebroids are the leitmotiv for this text, but large parts of it are excursions into general features such as the concept of an affine Lie algebroid and, even more generally, generalised connections on an affine bundle and affineness of such connections. As such, it is a review of recent work which was carried out jointly with Eduardo Martínez and Tom Mestdag [11,7,10] and which constitutes also part of the PhD work to be submitted by Tom Mestdag in 2003. I am grateful to these co-authors for letting me use the results of our joint efforts for this occasion.

My contribution to the Colloquium in Ghent (November 2002) was the presentation of an overview of the activities at the *Workshop on differential geometric methods in theoretical mechanics*, since its creation in 1986. The reason for that is the fact that Mike Crampin was to a large extent the initiator of this workshop and that it proved to be a very successful organisation over the years. I therefore chose to let my presentation at the 17th edition of this workshop in Levico Terme, Italy (September 2002), be the core of my contribution to this special volume.

2 Lagrangian equations on a Lie algebroid

Let us first have a look at the analytical format of Lagrangian equations on a Lie algebroid. The by now familiar analytical expression of such equations read:

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha(x)\, y^\alpha \\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) &= \rho^i_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}, \qquad L \in C^\infty(V). \end{aligned} \tag{1}$$

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The underlying geometrical structure is that the coordinates y^α are the fibre coordinates of a vector bundle $\pi : V \to M$, x^i being the coordinates on the base M; the functions ρ^i_α represent the so-called anchor map, which is a vector bundle map from V into TM; the $C^\gamma_{\alpha\beta}$ are the structure functions coming from a bracket defined on sections of π, and there are some compatibility conditions to be satisfied, roughly coming from a compatibility between the bracket on $Sec(\pi)$ and the Lie bracket of vector fields on M. Among these, we mention

$$\rho^i_\alpha \frac{\partial \rho^j_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^j_\alpha}{\partial x^i} = \rho^j_\gamma C^\gamma_{\alpha\beta} \,. \tag{2}$$

For more details, see for example [13,6].

What I would like to indicate here already is that, if the main interest would be to model equations of type (1)(2), there is room for generalisation. For example, if one tries to derive such kind of equations from a (formal) calculus of variations approach, there is no need to assume that the bracket on $Sec(\pi)$ satisfies a Jacobi identity.

My own involvement in the subject (always in collaboration with Eduardo and Tom) started from the question: "What would be a time-dependent generalisation of such systems?" The claim is that such a generalisation will give rise to equations of the following type:

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha(t,x) y^\alpha + \rho^i_0(t,x) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) &= \rho^i_\alpha \frac{\partial L}{\partial x^i} + (C^\gamma_{\beta\alpha} y^\beta + C^\gamma_{0\alpha}) \frac{\partial L}{\partial y^\gamma} \,, \end{aligned} \tag{3}$$

where this time the $\rho^i_\alpha, \rho^i_0, C^\gamma_{\alpha\beta}, C^\gamma_{0\alpha}$ are functions of t and x satisfying,

$$\rho^i_\alpha \frac{\partial \rho^j_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^j_\alpha}{\partial x^i} = \rho^j_\gamma C^\gamma_{\alpha\beta} \tag{4}$$

$$\frac{\partial \rho^j_\beta}{\partial t} + \rho^i_0 \frac{\partial \rho^j_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^j_0}{\partial x^i} = \rho^j_\alpha C^\alpha_{0\beta}. \tag{5}$$

Notice that one sees a certain affineness entering the equations here and of course, the extra time coordinate makes that there is a zero component of the structure functions and a corresponding extra compatibility condition. The usual framework for time-dependent mechanics in general and time-dependent Lagrangian mechanics in particular, is the first jet bundle J^1M of a manifold M fibred over $\mathbb{R}$ (cf. [2]). Therefore, a natural extension of the Lie algebroid generalisation is to consider an anchor map with values in J^1M rather than in TM and whose domain may then just as well be an affine bundle $E \to M$ rather than a vector bundle. If one does that, the result is a theory which we described in [11] and is centred around the following diagram.

$$
\begin{tikzcd}
 & J^1E \arrow[dd] \arrow[dr] & & \\
J^1_\rho E \arrow[ur, "\rho^1"] \arrow[rr, "\pi_1"'] \arrow[dd, "\pi_2"] & & E \arrow[dd, "\pi"'] \arrow[dr] & \\
 & J^1M \arrow[dr] & & \mathbb{R} \\
E \arrow[ur, "\rho"] \arrow[rr, "\pi"'] & & M \arrow[ur, "\tau"'] &
\end{tikzcd}
$$

Without going into any detail now, let me briefly point out the main ingredients and features of this diagram. The bottom part is the scheme of an affine Lie algebroid. The bundle $E \to M$ appears on the right again, with its own first jet bundle J^1E. $J^1_\rho E$ is in fact the pullback bundle of J^1E under ρ. An important point is, however, as discussed first by Mackenzie [5] and fully exploited for standard Lie algebroids in [6], that one should look at the total space of this pullback bundle as being fibred over E via $\tau_E \circ \rho^1$ (with less emphasis on the usual projections of a pullback bundle, here called π_2 and ρ^1). If one does so, one discovers that there is a kind of complete lift from the Lie algebroid structure at the bottom to one at the top, and the Lagrange equations shown above should be regarded as coming from special sections of the *prolonged bundle* $J^1_\rho E \to E$.

However, it is possible to work in a more general framework. This will in particular be fruitful for exploring the affine nature of a Lie algebroid in all generality, and for some of the aspects of the digression I want to make now, one does not even need the full structure of an algebroid.

3 Playing with diagrams to understand generalised connections

Consider the following general scheme as depicted on the diagram below: $\tau : V \to M$ is a vector bundle; $\rho : V \to TM$ is an "anchor map", to be understood here as a vector bundle morphism about which no further structure is assumed at the moment; $\mu : P \to M$ is an arbitrary fibre bundle.

Definition: *A ρ-connection on the bundle μ is a linear bundle map $h : \mu^*V \to TP$, such that the following diagram commutes: $\rho \circ p_V = T\mu \circ h$.*

The best source for a general study of ρ-connections is [1].

$$
\begin{tikzcd}
 & TP \arrow[dd] \arrow[dr, "\tau_P"] & \\
\mu^*V \arrow[ur, "h"] \arrow[rr, "p"'] \arrow[dd, "p_V"'] & & P \arrow[dd, "\mu"] \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}
$$

The form of the preceding picture no doubt is reminiscent of the first one. For comparison, therefore, let us discuss the prolongation idea in some more detail in this more general context. In the next picture, the right part is the same as in the preceding one, but rather than pulling V back along μ, we pull TP back along ρ. The total space

$$T^\rho P = \{(v, X_p) \in V \times TP \mid \rho(v) = T\mu(X_p)\}$$

is not called ρ^*TP, however, because the fibration we are primarily interested in is not $\rho^1 : \rho^*TP \to TP$ or $\mu^2 : \rho^*TP \to V$, but $\mu^1 = \tau_P \circ \rho^1$. The bundle $\mu^1 : T^\rho P \to P$ is called the *ρ-prolongation* of $\mu : P \to M$.

$$
\begin{tikzcd}
 & TP \arrow[dd] \arrow[dr, "\tau_P"] & \\
T^\rho P \arrow[ur, "\rho^1"] \arrow[rr, "\mu^1"'] \arrow[dd, "\mu^2"'] & & P \arrow[dd, "\mu"] \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}
$$

The ρ-prolongation in many respects has the features of a tangent bundle. For example, there is a vertical subbundle $\mathcal{V}^\rho P := \ker \mu^2 = \{(0, Q)\} \subset T^\rho P$, and there are also deeper similarities, upon which we will not dwell here, however.

The sort of overall structure of this diagram is the same as in the previous one, in the sense that, by the very construction of a pullback bundle, there is a commuting diagram around the anchor map here as well. In fact, this is the reason why I prefer to keep representing points of $T^\rho P$ as a couple of elements, one from V and the other one a tangent vector to P at some point p, whereby this base point in the fibration over P thus is not given a separate entry in the notation.

In view of the similarity in structure, it is tempting to put the last diagram on top of the previous one, which would require pushing the two competing spaces apart. This, in fact, can easily be done because $T^\rho P$ is naturally fibred over $\mu^* V$. The result is the following overall diagram.

$$\begin{tikzcd}
 & TP \arrow[dddd, "T\mu"] \arrow[ddr, "\tau_P"] & \\
T^\rho P \arrow[ur, "\rho^1"] \arrow[drr, "\mu^1"] \arrow[dd, "j"'] & & \\
 & & P \arrow[ddd, "\mu"] \\
\mu^* V \arrow[uuur, "h"] \arrow[urr, "p"] \arrow[dd, "p_V"'] & & \\
 & TM \arrow[dr, "\tau_M"] & \\
V \arrow[ur, "\rho"] \arrow[rr, "\tau"'] & & M
\end{tikzcd}$$

Having brought the fibration j into the picture and thinking of the injection of the vertical subbundle $\mathcal{V}^\rho P$ into $T^\rho P$, we are facing a short exact sequence

$$0 \to \mathcal{V}^\rho P \to T^\rho P \xrightarrow{j} \mu^* V \to 0.$$

This suggests a way of defining a possibly different kind of ρ-connection on μ, namely as a splitting of this sequence or, in other words, as a horizontal lift operation ^{H} from $\mu^* V$ (or sections of it) to $T^\rho P$. We then have a direct sum decomposition

$$T^\rho P = \mathcal{H}^\rho P \oplus \mathcal{V}^\rho P,$$

with corresponding horizontal and vertical projectors P_H and P_V, as in the usual theory of non-linear connections on a tangent bundle. The point is that these two different looking notions of generalised connection are completely equivalent [10], and we have $\rho^1 \circ {}^H = h$.

It is of some interest, however, to point out that the second view on ρ-connections has some advantages over the first. To begin with, there is no ambiguity in the decomposition of sections of μ^1 into horizontal and vertical ones, as opposed to attempts to use the map h for defining horizontality in TP, which then creates a number of complications [1]. Also the concept of *connection map* (see e.g. [12]) may be somewhat more transparent in the second point of view. In the first approach, we immediately spot from our overall diagram, more particularly from the two commuting diagrams over ρ, that

$\rho^1 - h \circ j$ yields a vertical vector on P. In the particular case that P is a vector bundle, this can be identified with an element of P itself, yielding a map $K : T^\rho P \to P$. In the second approach, K is essentially P_V. Note: the connection map is a very useful instrument to define an associated covariant derivative operator when the ρ-connection is linear.

4 Affineness of a ρ-connection

Let us make a further digression now and replace for a start the arbitrary bundle $\mu : P \to M$ by an *affine* bundle $\pi : E \to M$, modelled on the vector bundle $\overline{\pi} : \overline{E} \to M$, say. Put $E_m^\dagger := \mathrm{Aff}(E_m, \mathbb{R})$, the set of affine functions on E_m, let $E^\dagger = \bigcup_{m\in M} E_m^\dagger$ denote the 'extended dual' of E, which is a vector bundle over M, and consider the bidual $\tilde{\pi} : \tilde{E} := (E^\dagger)^* \to M$, which is a vector bundle containing E and $\overline{E}$ via canonical injections: $\iota(E)$ and $\boldsymbol{\iota}(\overline{E})$.

Now I take two of the overall diagrams, one with the affine $E \to M$ in the position of the general bundle $P \to M$, and the other with P replaced by $\tilde{E} \to M$.

$$\begin{tikzcd}
 & & TE \arrow[ddd, "\pi^1"] \arrow[dr, "\tau_E"] & \\
T^\rho E \arrow[urr, "\rho^1_E"] \arrow[rrr] \arrow[d, "j"'] & & & E \arrow[ddd, "\pi"] \\
\pi^*V \arrow[uurr, "h"] \arrow[urrr, "p"] \arrow[dd, "p_V"'] & & & \\
 & & TM \arrow[dr, "\tau_M"] & \\
V \arrow[urr, "\rho"] \arrow[rrr, "\tau"'] & & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & T\tilde{E} \arrow[ddd, "\tilde{\pi}^1"] \arrow[dr, "\tau_{\tilde{E}}"] & \\
T^\rho \tilde{E} \arrow[urr, "\rho^1_{\tilde{E}}"] \arrow[rrr] \arrow[d, "\tilde{j}"'] & & & \tilde{E} \arrow[ddd, "\tilde{\pi}"] \\
\tilde{\pi}^*V \arrow[uurr, "\tilde{h}"] \arrow[urrr, "\tilde{p}"] \arrow[dd, "\tilde{p}_V"'] & & & \\
 & & TM \arrow[dr, "\tau_M"] & \\
V \arrow[urr, "\rho"] \arrow[rrr, "\tau"'] & & & M
\end{tikzcd}$$

Again, a good definition of affineness of a ρ-connection h becomes quite apparent by inspection of these diagrams: it is essentially the commutation of the diagram which links h to $\tilde{h}$ via canonical injections.

Definition: *A ρ-connection h on $\pi : E \to M$ is affine, if there exists a linear ρ-connection $\tilde{h} : \tilde{\pi}^*V \to T\tilde{E}$ on $\tilde{\pi} : \tilde{E} \to M$ such that $\tilde{h} \circ \iota = T\iota \circ h$, as maps from π^*V into $T\tilde{E}$.*

It is about time to illustrate these notions by looking at coordinate expressions now.

With x^i, y^α coordinates on $\pi : E \to M$ and $(e_0; \{\mathbf{e}_\alpha\})$ a local frame for $Sec(\pi)$, denote the induced basis for $Sec(\pi^\dagger)$ by (e^0, e^α), meaning that $\forall\, a \in$

$Sec(\pi),\ a(x) = e_0(x) + a^\alpha(x)\boldsymbol{e}_\alpha(x)$ say, we have

$$e^0(a)(x) = 1,\ \forall x, \qquad e^\alpha(a)(x) = a^\alpha(x).$$

Observe hereby that e^0 is actually globally defined! Let then (e_0, e_α) denote the dual basis for $Sec(\tilde{\pi})$, so that $\iota(e_0) = e_0$ and $\boldsymbol{\iota}(\boldsymbol{e}_\alpha) = e_\alpha$. Finally we write $(x^i, y^A) = (x^i, y^0, y^\alpha)$ for the induced coordinates on $\tilde{E}$, and (x^i, v^a) for the coordinate representation of a point $v \in V$.

The anchor map $\rho : V \to TM$ is of the form $(x^i, v^a) \mapsto \rho^i_a(x) v^a \frac{\partial}{\partial x^i}$; the map $h : \pi^* V \to TE$ in general will look as follows:

$$h(x^i, y^\alpha, v^a) = (x^i, y^\alpha, \rho^i_a(x) v^a, -\Gamma^\alpha_a(x, y) v^a), \tag{6}$$

the minus sign before the connection coefficients being a matter of convention. Now, affineness of the ρ-connection on π means that the connection coefficients are of the form:

$$\Gamma^\alpha_a(x, y) = \Gamma^\alpha_{a0}(x) + \Gamma^\alpha_{a\beta}(x) y^\beta. \tag{7}$$

In the equivalent representation $\ ^H : \pi^* V \to T^\rho E$ of the ρ-connection, this becomes:

$$(x^i, y^\alpha, v^a) \ \overset{H}{\mapsto}\ \left((x^i, v^a), v^a \left(\rho^i_a \frac{\partial}{\partial x^i} - \Gamma^\alpha_a \frac{\partial}{\partial y^\alpha}\right)\right). \tag{8}$$

Now that we are bringing the prolonged bundle into the picture, if $\boldsymbol{v}_a$ denotes a local basis of sections of $\tau : V \to M$, then a standard local basis of sections of $\pi^1 : T^\rho E \to E$ is given by

$$\mathcal{X}_a(e) = \left(\boldsymbol{v}_a(x), \rho^i_a(x) \frac{\partial}{\partial x^i}\bigg|_e\right), \qquad \mathcal{V}_\alpha(e) = \left(0, \frac{\partial}{\partial y^\alpha}\bigg|_e\right). \tag{9}$$

Notice that there is a canonical vertical lift $\ ^V : \pi^* \overline{E} \to T^\rho E$, which is such that the $\mathcal{V}_\alpha$ are roughly the vertical lifts of the basis vectors for $\overline{E}$, more precisely: $\mathcal{V}_\alpha(e) = \Big(e, \mathbf{e}_\alpha(\pi(e))\Big)^V$. Whenever there is a given a ρ-connection, it will be more suitable to do coordinate calculations on the prolonged bundle with respect to an adapted local basis, which consists of horizontal and vertical sections. A basis for the horizontal sections is given by:

$$\mathcal{H}_a = P_H(\mathcal{X}_a) = \mathcal{X}_a - \Gamma^\alpha_a(x, y) \mathcal{V}_\alpha. \tag{10}$$

Just a few words now, to finish this section, about covariant derivatives in this context. If the ρ-connection is *affine*, there is an associated covariant derivative operator $\nabla : Sec(\tau) \times Sec(\pi) \to Sec(\overline{\pi})$, which in coordinates will look as follows. For $\zeta = \zeta^a(x) \boldsymbol{v}_a \in Sec(\tau)$ and $\sigma = e_0 + \sigma^\alpha(x) \mathbf{e}_\alpha \in Sec(\pi)$:

$$\nabla_\zeta \sigma = \left(\frac{\partial \sigma^\alpha}{\partial x^i} \rho^i_a(x) + \Gamma^\alpha_{a0}(x) + \Gamma^\alpha_{a\beta}(x) \sigma^\beta(x)\right) \zeta^a(x)\, \mathbf{e}_\alpha. \tag{11}$$

In fact, the affine ρ-connection can be completely characterised by such a ∇, having the intrinsic properties: for all $f \in C^\infty(M)$,

$$\nabla_{f\zeta}\sigma = f\,\nabla_\zeta\sigma \tag{12}$$
$$\nabla_\zeta(\sigma + f\boldsymbol{\eta}) = \nabla_\zeta\sigma + f\,\overline{\nabla}_\zeta\boldsymbol{\eta} + \rho(\zeta)(f)\,\boldsymbol{\eta}, \tag{13}$$

where $\overline{\nabla}$ is the covariant derivative associated to the linear ρ-connection $\overline{h}$ on $\overline{\pi}$, obtained for example by restricting $\tilde{h}$ to $\overline{\pi}^*V$.

5 Back to algebroids

Now that we know what affineness of a ρ-connection means, I want to put more structure in the anchor map again and define affineness of a Lie algebroid in all generality (see [7,3]). With $\pi : E \to M$ still being an affine bundle, and putting the concept of connections aside for the moment, the picture of interest now arises from taking as vector bundle $V \to M$ the dual of the extended dual of E, i.e. $\tilde{\pi} : \tilde{E} \to M$.

$$\begin{tikzcd}
& TE \arrow[dd, "\pi^1"] \arrow[dr, "\tau_E"] & \\
T^{\tilde\rho}E \arrow[ur, "\tilde\rho^1"] \arrow[rr] \arrow[dd, "\pi^2"'] & & E \arrow[dd, "\pi"] \\
& TM \arrow[dr, "\tau_M"] & \\
\tilde{E} \arrow[ur, "\tilde\rho"] \arrow[rr, "\tilde\pi"'] & & M
\end{tikzcd}$$

The plan is to define a Lie algebroid structure on π and explain its relation to an algebroid structure on $\tilde{\pi}$. In fact, since $\tilde{E}$ contains E, I want to start from an anchor map ρ on E and explain how this extends to an anchor $\tilde{\rho}$ on $\tilde{E}$.

Definition: *A Lie algebroid on an affine bundle $\pi : E \to M$ (modelled on $\overline{\pi} : \overline{E} \to M$), consists of:*

(i) a Lie algebra structure on $Sec(\overline{\pi})$ (over $\mathbb{R}$), with associated bracket $[\ ,\ ]$;
(ii) an action by derivations of $Sec(\pi)$ on $Sec(\overline{\pi})$ (over $\mathbb{R}$)

$$D_\zeta(\lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2) = \lambda_1 D_\zeta\boldsymbol{\sigma}_1 + \lambda_2 D_\zeta\boldsymbol{\sigma}_2 \in Sec(\overline{\pi}), \quad \lambda_i \in \mathbb{R}$$

$$D_\zeta[\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2]=[D_\zeta\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2]+[\boldsymbol{\sigma}_1,D_\zeta\boldsymbol{\sigma}_2],$$

compatible with the bracket on $Sec(\overline{\pi})$*, in the sense that*

$$D_{\zeta+\boldsymbol{\sigma}}\boldsymbol{\eta}=D_\zeta\boldsymbol{\eta}+[\boldsymbol{\sigma},\boldsymbol{\eta}];$$

[(i) and (ii) define an affine Lie algebra structure]

(iii) an affine anchor map $\rho: E\to TM$*, such that*

$$D_\zeta(f\boldsymbol{\sigma})=f\,D_\zeta\boldsymbol{\sigma}+\rho(\zeta)(f)\,\boldsymbol{\sigma},\qquad f\in C^\infty(M).$$

It is often convenient to write $D_\zeta\boldsymbol{\sigma}$ as a bracket $[\zeta,\boldsymbol{\sigma}]$ also, and in fact this makes even more sense since it is easy to extend the affine Lie algebroid to a vector Lie algebroid on $\tilde{\pi}:\tilde{E}\to M$ as follows. For $\zeta=f\,\iota(\zeta_0)+\boldsymbol{\iota}(\boldsymbol{\eta})\in Sec(\tilde{\pi})$, where $\zeta_0\in Sec(\pi)$ is an arbitrary reference section, define the anchor map $\tilde{\rho}$ as

$$\tilde{\rho}(\zeta)=f\,\rho(\zeta_0)+\boldsymbol{\rho}(\boldsymbol{\eta}),\qquad \boldsymbol{\rho}:\ \text{linear part of }\rho, \tag{14}$$

and the bracket of two such ζ_i by

$$[\zeta_1,\zeta_2]=\Big(\tilde{\rho}(\zeta_1)(f_2)-\tilde{\rho}(\zeta_2)(f_1)\Big)\iota(\zeta_0)+\boldsymbol{\iota}\Big([\boldsymbol{\eta}_1,\boldsymbol{\eta}_2]+f_1D_{\zeta_0}\boldsymbol{\eta}_2-f_2D_{\zeta_0}\boldsymbol{\eta}_1\Big). \tag{15}$$

It can be shown that these definitions do not depend on the choice of ζ_0.

The following result was proved in [7]: there is a one to one correspondence between Lie algebroids on the affine bundle $\pi: E\to M$ and Lie algebroids on the bidual $\tilde{\pi}:\tilde{E}\to M$ which have the property that the bracket of two elements belonging to E, belongs to the vector bundle $\overline{E}$ on which E is modelled:

$$[\iota(\sigma_1),\iota(\sigma_2)]\subset \operatorname{Im}\boldsymbol{\iota}.$$

In coordinates, if $(e_0;\{\mathbf{e}_\alpha\})$ is a local basis for $Sec(\pi)$ and we consider the induced basis $(e_A)=(e_0,e_\alpha)$ for $Sec(\tilde{\pi})$ as before, the brackets of an affine Lie algebroid structure are of the form

$$[e_0,e_0]=0,\quad [e_o,e_\alpha]=C^\gamma_{0\alpha}e_\gamma,\quad [e_\alpha,e_\beta]=C^\gamma_{\alpha\beta}e_\gamma, \tag{16}$$

and for the anchor and its extension, we have

$$\rho(e_0+y^\alpha\mathbf{e}_\alpha)=(\rho^i_0+\rho^i_\alpha y^\alpha)\frac{\partial}{\partial x^i} \tag{17}$$

$$\tilde{\rho}(y^0e_0+y^\alpha e_\alpha)=(\rho^i_0y^0+\rho^i_\alpha y^\alpha)\frac{\partial}{\partial x^i}=\rho^i_Ay^A\frac{\partial}{\partial x^i}. \tag{18}$$

For completeness, this is the way the compatibility property $[\tilde{\rho}(e_A),\tilde{\rho}(e_\alpha)]=\tilde{\rho}([e_A,e_\alpha])$ looks like in coordinates:

$$\rho^i_A\frac{\partial\rho^j_\alpha}{\partial x^i}-\rho^i_\alpha\frac{\partial\rho^j_A}{\partial x^i}=C^\gamma_{A\alpha}\rho^j_\gamma, \tag{19}$$

and the Jacobi identity reads

$$\sum_{A,B,\gamma}\left(\rho^i_A\frac{\partial C^\mu_{B\gamma}}{\partial x^i}+C^\mu_{A\nu}C^\nu_{B\gamma}\right)=0. \tag{20}$$

The extension of the affine Lie algebroid to its vector counterpart on $\tilde{E}$ often simplifies matters when it comes to defining further concepts and operations. Let us look at the concept of differential forms on an affine algebroid to illustrate this point. The problem is of course that one roughly wants to think of a skew-symmetric multilinear map, but sections of π cannot be multiplied by functions. A definition of a k-form without recourse to $\tilde{E}$ overcomes this difficulty as follows.

Definition: *A k-form on $Sec(\pi)$, $\omega\in\bigwedge^k(\pi^\dagger)$, is a map $\omega: Sec\,\pi\times\cdots\times Sec\,\pi\to C^\infty(M)$ for which there exist maps $\omega_0,\boldsymbol{\omega}$, where*

$$\omega_0: Sec\,\pi\times Sec\,\overline{\pi}\times\cdots\times Sec\,\overline{\pi}\to C^\infty(M)$$

is skew-symmetric and linear in its $k-1$ vector arguments, and $\boldsymbol{\omega}$ is a (standard) k-form on $Sec\,\overline{\pi}$, such that

$$\omega_0(\zeta+\boldsymbol{\sigma},\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{k-1})=\omega_0(\zeta,\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{k-1})+\boldsymbol{\omega}(\boldsymbol{\sigma},\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{k-1}), \tag{21}$$

and for any reference section ζ_0:

$$\omega(\zeta_1,\ldots,\zeta_k)=\sum_{i=1}^k(-1)^{i-1}\omega_0(\zeta_0,\boldsymbol{\zeta}_1,\ldots,\hat{\boldsymbol{\zeta}}_i,\ldots,\boldsymbol{\zeta}_k)+\boldsymbol{\omega}(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_k)\,. \tag{22}$$

This construction is of some interest in its own right, but life becomes easier if one observes that a k-form on $Sec(\pi)$ is just the pullback of a form on $Sec(\tilde{\pi})$ under the canonical injection: $\omega=\iota^*(\tilde{\omega})$ say. Once this is clear, one can for example immediately define the exterior derivative of forms on $Sec(\pi)$ by: $d\omega=\iota^*(d\tilde{\omega})$.

In coordinates, a k-form on $Sec(\pi)$ is of the form

$$\omega=\frac{1}{(k-1)!}\,\omega_{0\mu_1\cdots\mu_{k-1}}\,e^0\wedge e^{\mu_1}\wedge\cdots\wedge e^{\mu_{k-1}}+\frac{1}{k!}\,\omega_{\mu_1\cdots\mu_k}\,e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}, \tag{23}$$

with coefficients in $C^\infty(M)$, skew-symmetric in all indices. The exterior derivative of forms is determined by: $df=\rho^i_A\frac{\partial f}{\partial x^i}\,e^A$, for $f\in C^\infty(M)$, and

$$de^0=0,\qquad de^\alpha=-C^\alpha_{0\beta}e^0\wedge e^\beta-\tfrac{1}{2}C^\alpha_{\beta\gamma}e^\beta\wedge e^\gamma. \tag{24}$$

The following observations are more important now. Going back to our prolongation picture of the beginning of this section, let us move upwards in the diagram. As we know [6], there is an inherited (vector) Lie algebroid structure on the prolonged bundle $\pi^1 : T^{\tilde\rho}E \to E$. The point is that this is again one of the type which gives rise to (or comes from) an affine Lie algebroid. Indeed, the space

$$\mathcal{J}^\rho E = \{\,(e, X_e) \in E \times TE \mid \rho(e) = T\pi(X_e)\,\},$$

is the affine bundle of which the bidual is $T^{\tilde\rho}E$. With respect to the local frame of sections $(\mathcal{X}_A, \mathcal{V}_\alpha)$ of $Sec(\pi^1)$, which was discussed in the more general context of the preceding section, the Lie algebroid brackets of the prolonged bundle are given by

$$[\mathcal{X}_A, \mathcal{X}_B] = C^\alpha_{AB}\mathcal{X}_\alpha, \quad [\mathcal{X}_A, \mathcal{V}_\alpha] = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0. \tag{25}$$

The corresponding exterior derivative is determined by

$$dx^i = \rho^i_A \mathcal{X}^A, \qquad dy^\alpha = \mathcal{V}^\alpha \tag{26}$$

$$d\mathcal{X}^\alpha = -\tfrac{1}{2}C^\alpha_{AB}\mathcal{X}^A \wedge \mathcal{X}^B, \quad d\mathcal{X}^0 = 0, \quad d\mathcal{V}^\alpha = 0. \tag{27}$$

6 And now Lagrangian equations again

I need two more concepts now before I can return to my starting point, Lagrangian equations, but now on general affine bundles, without the underlying motivation of time-dependent mechanics, i.e. without the assumption of a further fibration $M \to \mathbb{R}$. The first one is a vertical endomorphism, the second is a notion of "second-order differential equation field" on the prolonged bundle.

There is a *canonical map* $\vartheta : \pi^*\tilde{E} \to \pi^*\overline{E} \subset \pi^*\tilde{E}$, which is defined as follows. For a given $a \in E$, any $z \in \tilde{E}$ is of the form $z = \lambda(z)\iota(a) + \boldsymbol{\iota}(\boldsymbol{v})$, with $\lambda(z) \in \mathbb{R}$. As a result, we can define

$$\vartheta(a, z) = (a, z - \lambda(z)\iota(a)), \tag{28}$$

leading to an operator which extends to sections of the corresponding bundles and has coordinate representation: $\vartheta = (e^\alpha - y^\alpha e^0) \otimes e_\alpha$.

It was already mentioned that there is a vertical lift from $\pi^*\overline{E}$ to $T^{\tilde\rho}E$ (with $V = \tilde{E}$ here). With the aid of ϑ, it can now be extended to

$${}^V : \pi^*\tilde{E} \to T^{\tilde\rho}E, \quad {}^V : (a, z) \mapsto \left(0_{\pi(a)}, \vartheta(a, z)^V\right).$$

Applied to sections, we have: if $\zeta = \zeta^0 e_0 + \zeta^\alpha e_\alpha \in Sec(\tilde\pi)$, then $\zeta^V = (\zeta^\alpha - y^\alpha\zeta^0)\mathcal{V}_\alpha \in Sec(\pi^1)$. In turn this leads, just as in the standard theory of first-jet bundles, to the *vertical endomorphism* $S : Sec(\pi^1) \to Sec(\pi^1)$, given in coordinates by

$$S = (\mathcal{X}_\alpha - y^\alpha \mathcal{X}_0) \otimes \mathcal{V}_\alpha. \tag{29}$$

As for the second ingredient, we actually talk about pseudo-SODEs here, because the resulting differential equations will not strictly be second-order ordinary differential equations. Now that we have S at our disposal, and remembering that the section $\mathcal{X}^0$ of the extended dual is actually globally defined, a simple way of defining pseudo-SODEs goes as follows.

Definition: *A pseudo-*SODE *on the affine* $\pi : E \to M$ *is a section* Γ *of* $\pi^1 : T^{\tilde{\rho}}E \to E$ *such that*

$$S(\Gamma) = 0, \qquad \langle \Gamma, \mathcal{X}^0 \rangle = 1.$$

Locally, Γ is of the form

$$\Gamma = \mathcal{X}_0 + y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha, \tag{30}$$

and the vector field $\tilde{\rho}^1(\Gamma)$ determines the differential equations

$$\dot{x}^i = \rho_0^i(x) + \rho_\alpha^i(x) y^\alpha, \qquad \dot{y}^\alpha = f^\alpha(x, y). \tag{31}$$

There are a number of equivalent ways for defining pseudo-SODEs, one of which is that its integral curves, by which we mean of course the integral curves of the corresponding vector field on E, all are *admissible curves* in the following sense: for $\gamma : \mathbb{R} \to E$, with projection $\gamma_M = \pi \circ \gamma : \mathbb{R} \to M$, we have $\rho \circ \gamma = \dot{\gamma}_M$. Note further that an admissible curve γ can be lifted to a curve $t \overset{\gamma^c}{\mapsto} (\gamma, \dot{\gamma})$, which belongs to $\mathcal{J}^\rho E$ for all t and by construction is such that $\tilde{\rho}^1 \circ \gamma^c = \dot{\gamma}$, hence is admissible for the prolonged algebroid.

Contact forms on $Sec(\pi^1)$ are 1-forms vanishing on all pseudo-SODEs. Locally, they are spanned by

$$\theta^\alpha = \mathcal{X}^\alpha - y^\alpha \mathcal{X}^0. \tag{32}$$

There also is a complete lift from $Sec(\tilde{\pi})$ to $Sec(\pi^1)$, determined by requiring that contact forms be preserved.

So now, to close the circle for this review of recent work, let me describe in two words how Lagrangian systems on an affine Lie algebroid can be defined, and how $\tilde{\rho}$-connections, both non-linear and linear ones, naturally make their appearance in dealing with pseudo-SODEs.

For $L \in C^\infty(E)$, define the Poincaré-Cartan type 1-form $\theta_L = S^*(dL) + L\,\mathcal{X}^0$ and the 2-form $\Omega_L = d\theta_L$. A pseudo-SODE Γ is of Lagrangian type if

$$i_\Gamma \Omega_L = 0.$$

The corresponding differential equations are of the form

$$\dot{x}^i = \rho^i_\alpha y^\alpha + \rho^i_0, \tag{33}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \frac{\partial L}{\partial x^i} + C^\gamma_\alpha \frac{\partial L}{\partial y^\gamma}\,, \tag{34}$$

where $C^\gamma_\alpha = C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} y^\beta$.

Now for any given pseudo-SODE Γ, the operator

$$P_H = \tfrac{1}{2}(I - d_\Gamma S + \mathcal{X}^0 \otimes \Gamma) \tag{35}$$

defines a (non-linear) $\tilde{\rho}$-connection on π, with connection coefficients (see [8])

$$\Gamma^\alpha_\beta = -\tfrac{1}{2}\left(\frac{\partial f^\alpha}{\partial y^\beta} + C^\alpha_\beta\right), \quad \Gamma^\alpha_0 = -f^\alpha - y^\beta \Gamma^\alpha_\beta. \tag{36}$$

There further is an associated "linearisation", a Berwald-type connection, which is a linear $\tilde{\rho}^1$-connection on $\pi^*\tilde{E} \to E$, corresponding to an *affine* $\tilde{\rho}^1$-connection on $\pi^*E \to E$. The latter statement actually refers to work which is still under construction [9].

Finally, here are a couple of closing observations which are worth mentioning. Recall that, starting from $(e_0; \{\mathbf{e}_\alpha\})$, a local basis of sections of the affine bundle $E \to M$, and constructing an induced basis (e^0, e^α) for $Sec(\pi^\dagger)$, one encounters, somewhat surprisingly, the globally defined section e^0: $e^0_m(a_m) = 1, \forall a_m \in E_m$. Interestingly, additional properties of e^0 characterise the aspects of affineness we have been discussing.

First of all, a Lie algebroid on the vector bundle $\tilde{E} \to M$ restricts to an affine Lie algebroid on $E \to M$ if and only if $de^0 = 0$. Furthermore, in the special case that M is fibred over $\mathbb{R}$ and $\rho(E) \subset J^1M$, we have $e^0 = dt$. Note in passing that, in the theory of Lie bi-algebroids developed in [4], a central role is played by a 1-cocycle; the link with affine algebroids is explained in [3].

Secondly, a linear ρ-connection on $\tilde{\pi}$ is associated to an affine ρ-connection on π if and only if e^0 is parallel.

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