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Contact symmetries of the Helmholtz form

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Abstract

In this paper, vector fields which are symmetries of the contact ideal are studied. It is shown that contact symmetries of the Helmholtz form transform a dynamical form to a dynamical form which is variational (i.e. comes as the Euler–Lagrange form from a Lagrangian). The case of dynamical forms representing first-order classes in the variational sequence is analysed in detail, which means, by the variational sequence theory, that systems of ordinary differential equations of order $\leqslant 3$ are concerned.

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1. Introduction

One of the results of the variational sequence theory, related to the inverse problem of the calculus of variations, states that a dynamical form ε, representing a system of ordinary or partial differential equations, is locally variational if and only if the Helmholtz form $H(\varepsilon)$ vanishes. Invariance properties of classes in the variational sequence then suggest a new idea, namely that there should exist a close correspondence between the notions of variationality of a differential form and invariance of its exterior derivative. The aim of this paper is to study a relationship between the Lie derivatives of ε and $H(\varepsilon)$. We prove that invariance of the Helmholtz form $H(\varepsilon)$ with respect to a vector field Z, preserving contact forms, is equivalent with local variationality of the Lie derivative $\partial_Z \varepsilon$ of ε by Z, meaning that the dynamical form $\partial_Z \varepsilon$ is the Euler–Lagrange form of a Lagrangian. This result is then analysed in detail for the case of dynamical forms on $J^3 Y$ which are known to represent classes in the $(n+1)$st column of the first-order variational sequence.

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First, in Section 2, we present a survey of basic concepts of the theory of global higher-order variational functionals in fibred spaces as developed in Goldschmidt and Sternberg [5], Krupka [6–8], and Trautman [21], and of the variational sequence theory due to Krupka [9,10]. Then we introduce *contact symmetries* as vector fields, preserving contact differential forms (Garcia [4]), and we prove a key result that *the Lie derivative of a dynamical form ε by a contact symmetry Z is variational if and only if Z leaves invariant the Helmholtz form of ε*. We call vector fields Z, transforming a non-variational dynamical form ε to a variational one $\partial_Z\varepsilon$, *variational vector fields*.

The idea to transform a non-variational form to a variational one by means of a contact symmetry was announced in the conference paper [12]. In that paper we studied this problem for second-order dynamical forms with components affine in the highest derivatives, representing second-order ordinary differential equations. It turned out that this problem can have interesting applications, namely to a system of equations which is not variational and even does not possess any variational multiplier (Douglas [3]) one can find a variational system related by a contact transformation. In this paper we continue studies in this direction. The aim is to investigate contact transformations related with objects in the *first-order variational sequence in mechanics*. From the side of dynamical forms (and differential equations) this means that our results concern systems of *first-order* ODE, of *second-order* ODE affine in the second derivatives, and a class of *third-order* ODE, namely those defined by dynamical forms $\varepsilon = \varepsilon_\sigma dq^\sigma \wedge dt$, of the following structure

$$\varepsilon_\sigma = A_\sigma + \left(C_{\sigma\nu} - 2\frac{dD_{\sigma\nu}}{dt}\right)\ddot{q}^\nu - \frac{dE_\sigma}{dt} - 2D_{\sigma\nu}\dddot{q}^\nu, \tag{1.1}$$

where $A_\sigma, C_{\sigma\nu}, D_{\sigma\nu}$ and E_σ are functions of $t, q^\tau, \dot{q}^\tau$. In Section 3 we derive all important formulas for main objects appearing in the first three columns of the first-order variational sequence: Lagrangians, dynamical forms and Helmholtz forms. In proofs we present explicit computations in order to provide techniques and useful tricks of computation of classes and representatives. Then we find an explicit characterisation of contact symmetries, and of transformed classes in the variational sequence. Finally, we derive a general form of equations for variational contact symmetries.

In Section 4, illustrative examples are presented. We discuss briefly the case of second-order ODE's, and bring concrete examples of contact symmetries of Helmholtz forms, including Douglas' equations mentioned above.

2. Variational sequences

Throughout this section, Y is a fibred manifold with base X and projection π. We denote $n = \dim X$, $n+m = \dim Y$. J^rY, $r > 0$, is the r-jet prolongation of Y, and $\pi^{r,s} : J^rY \to J^sY$, $\pi^r : J^rY \to X$ are the canonical jet projections. The points of J^rY are r-jets $J^r_x\gamma$ of sections γ of Y at $x \in X$; the r-jet prolongation of γ is the mapping $x \to J^r\gamma(x) = J^r_x\gamma$. A vector ϑ at $y \in Y$ is π-*vertical*, if $T_y\pi \cdot \vartheta = 0$; a differential form ρ on Y is π-*horizontal*, if it vanishes whenever one of its arguments is a π-vertical vector. Any fibred chart (V, χ), $\chi = (x^i, y^\sigma)$, on Y induces the associated charts (U, ω), $\omega = (x^i)$, on X, and (V^r, χ^r), $\chi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, on J^rY, where $U = \pi(V)$, and $V^r = (\pi^{r,0})^{-1}(V)$, $1 \leqslant i, j_1, j_2, \ldots, j_r \leqslant n$, $1 \leqslant \sigma \leqslant m$.

2.1. Differential forms on a fibred manifold

For any open set $W \subset Y$ we denote by Ω^rW the exterior algebra on $W^r = (\pi^{r,0})^{-1}(W)$. Ω^r_0W and Ω^r_kW are the ring of smooth functions and the Ω^r_0W-module of smooth k-forms on W^r, respectively. We also use some submodules, the submodule of π^r-horizontal k-forms $\Omega^r_{k,X}W \subset \Omega^r_kW$, and the submodule of $\pi^{r,0}$-horizontal k-forms $\Omega^r_{k,Y}W \subset \Omega^r_kW$. We have a morphism of exterior algebras

$$h : \Omega^r_kW \to \Omega^{r+1}_{k,X}W \tag{2.1}$$

defined by

$$hf = f\pi^{r+1,r}, \quad h\,dx^i = dx^i, \quad h\,dy^\sigma_{j_1 j_2 \ldots j_l} = y^\sigma_{j_1 j_2 \ldots j_l p}\,dx^p, \tag{2.2}$$

where $f : V^r \to \mathbb{R}$ is a function; obviously, $J^r\gamma^*\rho = J^{r+1}\gamma^*h\rho$ for every section γ of Y. We call h the π-*horizontalisation*. We say that a form $\rho \in \Omega^r_kW$ is *contact*, if $h\rho = 0$. For any fibred chart (V, χ), $\chi = (x^i, y^\sigma)$, the 1-forms

$$\omega^\sigma_{j_1 j_2 \dots j_l} = dy^\sigma_{j_1 j_2 \dots j_l} - y^\sigma_{j_1 j_2 \dots j_l p}\, dx^p, \tag{2.3}$$

where $0 \leqslant l \leqslant r-1$, are examples of contact forms. The system of forms

$$dx^i, \quad \omega^\sigma, \quad \dots, \quad \omega^\sigma_{j_1 j_2 \dots j_{r-1}}, \quad dy^\sigma_{j_1 j_2 \dots j_r} \tag{2.4}$$

is a basis of linear forms on V^r. By the *contact ideal* on W we mean the ideal $\Theta^r W$ in the exterior algebra $\Omega^r W$ locally generated by the forms $\omega^\sigma_{j_1 j_2 \dots j_l}$, $d\omega^\sigma_{j_1 j_2 \dots j_l}$, where $0 \leqslant l \leqslant r-1$. Since

$$d\omega^\sigma_{j_1 j_2 \dots j_l} = -\omega^\sigma_{j_1 j_2 \dots j_l s} \wedge dx^s, \tag{2.5}$$

the contact ideal is also generated by the forms

$$\omega^\sigma, \quad \omega^\sigma_{j_1}, \quad \omega^\sigma_{j_1 j_2}, \quad \dots, \quad \omega^\sigma_{j_1 j_2 \dots j_{r-1}}, \quad d\omega^\sigma_{j_1 j_2 \dots j_{r-1}}. \tag{2.6}$$

A form $\rho \in \Omega^r_k W$ has a unique decomposition

$$(\pi^{r+1,r})^* \rho = h\rho + p_1\rho + p_2\rho + \dots + p_k\rho, \tag{2.7}$$

in which $p_i\rho$ contains, in any fibred chart, exactly i exterior factors $\omega^\sigma_{j_1 j_2 \dots j_l}$; transformation properties of the forms (2.3) guarantee invariance of the decomposition (2.7). In (2.7), $p_i\rho$ is called the *i-contact component* of ρ. If $k \geqslant n+1$, then we define $\rho \in \Omega^r_k W$ to be *strongly contact*, if $p_{k-n}\rho = 0$.

By a *π-projectable* vector field we mean a vector field ϑ on Y such that there exists a vector field φ on X satisfying $T\pi \cdot \vartheta = \varphi \circ \pi$. We denote by $J^r\vartheta$ the r-jet prolongation of ϑ.

We shall need the behaviour of the projections $h, p_1, p_2, \dots, p_k$ under the Lie derivatives $\partial_{J^r\vartheta}$. Since for any π-projectable vector field ϑ the operator $\partial_{J^r\vartheta}$ preserves contact forms, and

$$(\pi^{r+1,r})^* \partial_{J^r\vartheta}\rho = \partial_{J^{r+1}\vartheta} (\pi^{r+1,r})^* \rho, \tag{2.8}$$

we have

$$h\partial_{J^r\vartheta}\rho = \partial_{J^{r+1}\vartheta} h\rho, \tag{2.9}$$

and for all $i = 1, 2, \dots, k$,

$$p_i \partial_{J^r\vartheta}\rho = \partial_{J^{r+1}\vartheta} p_i\rho. \tag{2.10}$$

2.2. *Lagrangians, variational functionals*

By a *Lagrangian* (of order r) for Y we mean an element λ of the module $\Omega^r_{n,X} W$, where W is an open subset of Y. In a fibred chart (V, χ), $\chi = (x^i, y^\sigma)$,

$$\lambda = \mathcal{L}\omega_0, \tag{2.11}$$

where

$$\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n. \tag{2.12}$$

The component $\mathcal{L} : V^r \to \mathbb{R}$ is the Lagrange function. Let Ω be a piece of X, i.e., a compact, n-dimensional submanifold with boundary $\partial\Omega$, and let $\Gamma_\Omega Y$ be the set of sections of Y, defined on Ω. λ gives rise to the *variational functional*

$$\Gamma_\Omega Y \ni \gamma \to \lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda \in \mathbb{R}. \tag{2.13}$$

Let $U \subset X$ be an open set, let $\gamma : U \to Y$ be a section. Let ϑ be a π-projectable vector field on an open set $W \subset Y$ such that $\gamma(U) \subset W$. If α_t is the flow of ϑ, and $\alpha_{(0)t}$ is its π-projection, then since $\pi\alpha_t = \alpha_{(0)t}\pi$ for all t, $\gamma_t = \alpha_t \gamma \alpha^{-1}_{(0)t}$ is a 1-parameter family of sections of Y, depending smoothly on the parameter t. Sometimes γ_t is called the *variation*, or the *deformation* of γ, induced by ϑ. We get a real-valued function on a neighbourhood $(-\varepsilon,\varepsilon)$ of the origin $0\in\mathbb{R}$,

$$(-\varepsilon,\varepsilon)\ni t\to\lambda_{\alpha_{(0)t}(\Omega)}(\alpha_t\gamma\alpha_{(0)t}^{-1})=\int_{\alpha_{(0)t}(\Omega)}J^r(\alpha_t\gamma\alpha_{(0)t}^{-1})^*\lambda\in\mathbb{R}. \tag{2.14}$$

Differentiating this function at $t=0$ we obtain

$$(\partial_{J^r\vartheta}\lambda)_\Omega(\gamma)=\int_\Omega J^r\gamma^*\partial_{J^r\vartheta}\lambda. \tag{2.15}$$

The number (2.15) is the *variation of the variational functional* λ_Ω at γ, induced by the vector field ϑ. This formula shows, in particular, that the function

$$\Gamma_\Omega Y\ni\gamma\to(\partial_{J^r\vartheta}\lambda)_\Omega(\gamma)\in\mathbb{R} \tag{2.16}$$

is the variational functional (over Ω) associated with a new Lagrangian $\partial_{J^r\vartheta}\lambda$. We call this function the (*first*) *variational derivative*, or the (*first*) *variation* of λ_Ω by ϑ. Formula (2.15) can be used in a standard way to define extremals, and higher order variational derivatives of λ_Ω.

2.3. The Euler–Lagrange mapping

Now we shall analyse the structure of the variational derivatives by means of invariant differential-geometric operations. We know that if η is a differential form on a manifold X, and φ is a vector field on X, then the Lie derivative $\partial_\varphi\eta$ decomposes into two terms,

$$\partial_\varphi\eta=i_\varphi d\eta+di_\varphi\eta; \tag{2.17}$$

in this formula, i_φ denotes the contraction of a form by φ. We wish to apply this decomposition to formula (2.15).

Let $W\subset Y$ be an open set, and let $\lambda\in\Omega^r_{n,X}W$ be a Lagrangian. Formally, we define a form $\rho\in\Omega^s_nW$ to be a *Lepage equivalent* of λ, if

(a) $h\rho=\lambda$ (up to a canonical jet projection), and
(b) the form $p_1\,d\rho$ is $\pi^{s+1,0}$-horizontal.

If $\rho\in\Omega^s_nW$ is a Lepage equivalent of λ, then condition (a) implies that

$$\int_\Omega J^s\gamma^*\rho=\int_\Omega J^{s+1}\gamma^*h\rho=\int_\Omega J^r\gamma^*\lambda. \tag{2.18}$$

This means, in particular, that ρ defines the same variational functional as λ.

We have noticed in Section 2.1 that for any π-projectable vector field ϑ on W, the Lie derivative operator with respect to $J^r\vartheta$ commutes with the horizontalisation h. Then

$$\partial_{J^r\vartheta}\lambda=\partial_{J^r\vartheta}h\rho=h\partial_{J^s\vartheta}\rho, \tag{2.19}$$

so we have for any section γ of Y with values in W,

$$J^r\gamma^*\partial_{J^r\vartheta}\lambda=J^s\gamma^*i_{J^s\vartheta}\,d\rho+dJ^s\gamma^*i_{J^s\vartheta}\rho. \tag{2.20}$$

This is the *first variation formula* for the Lagrangian λ. Equivalently, formula (2.20) can be expressed as

$$\partial_{J^r\vartheta}\lambda=i_{J^{s+1}\vartheta}p_1\,d\rho+h\,di_{J^s\vartheta}\rho. \tag{2.21}$$

Let $\lambda\in\Omega^r_{n,X}W$ be a Lagrangian, expressed in a fibred chart by (2.11). Denote

$$\omega_i=(-1)^{i-1}dx^1\wedge dx^2\wedge\cdots\wedge dx^{i-1}\wedge dx^{i+1}\wedge\cdots\wedge dx^n. \tag{2.22}$$

A form $\rho \in \Omega_n^s W$ is a Lepage equivalent of λ if and only if $(\pi^{s+1,s})^*\rho$ has an expression

$$(\pi^{s+1,s})^*\rho = \Theta_\lambda + d\eta + \mu, \tag{2.23}$$

where

$$\Theta_\lambda = \mathcal{L}\omega_0 + \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l d_{p_1} d_{p_2}\dots d_{p_l}\frac{\partial\mathcal{L}}{\partial y^\sigma_{j_1 j_2\dots j_k p_1 p_2\dots p_l i}}\right)\omega^\sigma_{j_1 j_2\dots j_k}\wedge\omega_i, \tag{2.24}$$

η is a contact $(n-1)$-form, and the order of contactness of μ is $\geqslant 2$. The n-form Θ_λ, defined by (2.24), is the *principal Lepage equivalent* of λ. Θ_λ is defined on the corresponding coordinate neighbourhood. Existence of *global* Lepage equivalents can be proved by partitions of unity. In general, Lepage equivalent of a Lagrangian is non-unique.

Applying the decomposition of $d\rho$ into contact components (formula (2.7)) to (2.23) we obtain a decomposition

$$(\pi^{s+1,s})^*d\rho = E(\lambda) + F, \tag{2.25}$$

enjoying the following properties:

(a) $E(\lambda)$ is the 1-contact component of $d\rho$, i.e., $E(\lambda) = p_1\, d\rho$; moreover, $E(\lambda)$ does not depend on the choice of the Lepage equivalent ρ of λ. In a fibred chart

$$E(\lambda) = E_\sigma(\mathcal{L})\omega^\sigma\wedge\omega_0, \tag{2.26}$$

where

$$E_\sigma(\mathcal{L}) = \sum_{l=0}^{r}(-1)^l d_{p_1} d_{p_2}\dots d_{p_l}\frac{\partial\mathcal{L}}{\partial y^\sigma_{p_1 p_2\dots p_l}}. \tag{2.27}$$

(b) $hF = 0$ and $p_1 F = 0$, i.e., the order of contactness of F is $\geqslant 2$.

The form $E(\lambda)$, called the *Euler–Lagrange form* associated with λ, can be regarded as an element of the module $\Omega^{2r}_{n+1,Y}W$; its components $E_\sigma(\mathcal{L})$ are the *Euler–Lagrange expressions*. The mapping

$$\Omega^r_{n,X}W \ni \lambda \to E(\lambda) \in \Omega^{2r}_{n+1,Y}W, \tag{2.28}$$

assigning to a Lagrangian its Euler–Lagrange form, is $\mathbb{R}$-linear, and is called the *Euler–Lagrange mapping*. The forms belonging to the kernel of the Euler–Lagrange mapping are called *variationally trivial*; elements of the image are called *variational forms*. A 1-contact form $\varepsilon \in \Omega^s_{n+1,Y}W$ is called a *dynamical form* (cf. [14]; Takens [20] calls these forms *source forms*). *The inverse problem of the calculus of variations for a dynamical form* ε consists in finding a Lagrangian λ such that $\varepsilon = E(\lambda)$.

2.4. Invariance transformations

Recall that a form η on a manifold M is said to be *invariant* with respect to a vector field φ on M, if $\partial_\varphi\eta = 0$. We now apply this definition in the context of higher order calculus of variations in fibred manifolds.

Let λ be a Lagrangian of order r for Y, and let ϑ be a π-projectable vector field. We say that λ is *invariant* with respect to ϑ if $\partial_{J^r\vartheta}\lambda = 0$. Analogously, we say that the Euler–Lagrange form $E(\lambda)$ is *invariant* with respect to ϑ if $\partial_{J^{2r}\vartheta}E(\lambda) = 0$. Since the Lie derivative $\partial_{J^s\vartheta}$ commutes with the mappings $h, p_1, p_2, \dots, p_k$ (Section 2.1), we have $\partial_{J^{s+1}\vartheta}\, p_1\, d\rho = p_1\, d\partial_{J^s\vartheta}\rho$ for any Lepage equivalent of λ. But $h\partial_{J^s\vartheta}\rho = \partial_{J^{s+1}\vartheta}h\rho$, so the form $\partial_{J^s\vartheta}\rho$ is a Lepage equivalent of $\partial_{J^{s+1}\vartheta}\lambda$, and

$$\partial_{J^{s+1}\vartheta}E(\lambda) = E(\partial_{J^s\vartheta}\lambda). \tag{2.29}$$

Thus, $E(\lambda)$ *is invariant with respect to* ϑ *if and only if the transformed Lagrangian* $\partial_{J^s\vartheta}\lambda$ *belongs to the kernel of the Euler–Lagrange mapping*.

2.5. The variational sequence

We now recall the main steps of the construction of an exact sequence of sheaves, the variational sequence, in which the Euler–Lagrange mapping E appears as a sequence morphism. By means of this sequence we obtain more information about the structure of the Euler–Lagrange mapping, and discover new objects, describing its local and global properties.

Let $\Omega^r_{0,c} = \{0\}$, and let $\Omega^r_{k,c}$ be the sheaf of contact k-forms, if $k \leqslant n$, or the sheaf of strongly contact k-forms, if $k > n$, on $J^r Y$. We set

$$\Theta^r_k = \Omega^r_{k,c} + d\Omega^r_{k-1,c}, \tag{2.30}$$

where $d\Omega^r_{k-1,c}$ is the image sheaf of $\Omega^r_{k-1,c}$ by the exterior derivative d. It can be shown that we get an exact sequence of soft sheaves

$$0 \to \Theta^r_1 \to \Theta^r_2 \to \Theta^r_3 \to \cdots, \tag{2.31}$$

where the morphisms are the exterior derivative, i.e., a subsequence of the De Rham sequence

$$0 \to \mathbb{R} \to \Omega^r_0 \to \Omega^r_1 \to \Omega^r_2 \to \Omega^r_3 \to \cdots. \tag{2.32}$$

The quotient sequence

$$0 \to \mathbb{R} \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3 \to \cdots \tag{2.33}$$

which is also exact, is called the r-th order *variational sequence* on Y. We denote the sequence (2.33) symbolically by

$$0 \to \mathbb{R} \to \mathcal{V}^r, \tag{2.34}$$

and the quotient mappings by

$$E_k : \Omega^r_k/\Theta^r_k \to \Omega^r_{k+1}/\Theta^r_{k+1}. \tag{2.35}$$

The class of a form ρ is denoted by $[\rho]$.

The variational sequence is an acyclic resolution of the constant sheaf $\mathbb{R}$ over Y. Let $\Gamma(Y, \mathcal{V}^r)$ denote the cochain complex of global sections of (2.33),

$$0 \to \Gamma(Y, \mathbb{R}) \to \Gamma(Y, \Omega^r_0) \to \Gamma(Y, \Omega^r_1/\Theta^r_1) \to \Gamma(Y, \Omega^r_2/\Theta^r_2) \to \cdots. \tag{2.36}$$

As a corollary to the abstract De Rham theorem we get the following identification of the cohomology groups $H^k(\Gamma(Y, \mathcal{V}^r))$ of this complex with the De Rham cohomology groups of the manifold Y:

$$H^k\big(\Gamma(Y, \mathcal{V}^r)\big) = H^k Y. \tag{2.37}$$

To understand the meaning of variational sequences for global higher order variational theory, first note that the quotient sheaves Ω^r_k/Θ^r_k are determined up to natural isomorphisms of Abelian groups. Thus, the classes in Ω^r_k/Θ^r_k admit various equivalent characterisations. A simple analysis shows that the sections of the quotient sheaf Ω^r_n/Θ^r_n can be identified, in a fibred chart, with some n-forms $\lambda = \mathcal{L}\omega_0$, i.e., with some *Lagrangians*. Elements of $\Omega^r_{n+1}/\Theta^r_{n+1}$ can be identified with some $(n+1)$-forms $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$, i.e., with *dynamical forms*. More precisely, we can prove that the sheaf Ω^r_n/Θ^r_n is isomorphic with a subsheaf of the sheaf of Lagrangians $\Omega^{r+1}_{n,X}$, and $\Omega^r_{n+1}/\Theta^r_{n+1}$ is isomorphic with a subsheaf of the sheaf of dynamical forms $\Omega^{2r+1}_{n+1,Y}$; the quotient mapping

$$E_n : \Omega^r_n/\Theta^r_n \to \Omega^r_{n+1}/\Theta^r_{n+1} \tag{2.38}$$

in this representation of the sheaves coincides with the *Euler–Lagrange mapping*.

We say that a dynamical form $\varepsilon \in \Omega^s_{n+1,Y} W$ is *associated* with a 2-form $\rho \in \Omega^r W$ if $\varepsilon = [\rho]$. Then we call the class $E_{n+1}(\varepsilon) = [d\rho]$ *the Helmholtz class* of ε. The mapping

$$E_{n+1} : \Omega^r_{n+1}/\Theta^r_{n+1} \to \Omega^r_{n+2}/\Theta^r_{n+2} \tag{2.39}$$

is called the *Helmholtz mapping*. When we do not want to stress the context of the variational sequence, we also write for the Helmholtz class of $\varepsilon = [\rho]$

$$H(\varepsilon) = E_{n+1}(\varepsilon). \tag{2.40}$$

Now it is clear what kind of results are described by the variational sequence:

(i) Assume that a Lagrangian $\lambda = [\rho]$ satisfies $E_n(\lambda) = 0$. Then by exactness of (2.33), there always exists a class $[\eta]$ such that $E_{n-1}([\eta]) = [\rho] = [d\eta]$. This means that, locally, ρ decomposes as the sum of a closed form and a contact form. Condition

$$E_n(\lambda) = 0 \tag{2.41}$$

is the *local variational triviality condition*. If in addition, $H^n Y = \{0\}$, (2.37) says that η may be chosen *globally* defined on $J^r Y$. The local variational triviality condition strongly determines the structure of Lagrangians whose Euler–Lagrange forms vanish identically.

(ii) Suppose that we have a dynamical form $\varepsilon = [\rho]$. In analogy with Lagrangians, we have the *local variationality condition*

$$E_{n+1}(\varepsilon) = 0, \tag{2.42}$$

stating that ε is locally variational if and only if the associated Helmholtz class vanishes. If ε satisfies the local variationality condition, then there exists a class $[\eta]$ such that $E_n([\eta]) = \varepsilon = [\rho] = [d\eta]$. Thus, locally, ρ can be expressed as the sum of a closed form and a strongly contact form. If in addition, $H^{n+1} Y = \{0\}$, (2.37) guarantees that η may be chosen *globally* defined on $J^r Y$. The local variationality condition strongly determines the structure of dynamical forms.

2.6. Variational vector fields

We say that a vector field Z on $J^r Y$ *preserves contact forms*, if for any contact form ρ on $J^r Y$, the Lie derivative $\partial_Z \rho$ is again a contact form; we also say that Z is a *contact symmetry*.

If Z is a contact symmetry, then for any two k-forms ρ_1, ρ_2 belonging to the same class in the variational sequence, the k-forms $\partial_Z \rho_1$, $\partial_Z \rho_2$ also belong to the same class. Thus, we can define the *Lie derivative of a class* $[\rho]$ to be the class

$$\partial_Z[\rho] = [\partial_Z \rho]. \tag{2.43}$$

For any π-projectable vector field ϑ on an open subset of Y, the r-jet prolongation $Z = J^r\vartheta$ is a contact symmetry. This property of the vector field $J^r\vartheta$ implies, among others, the commutativity of the Lie derivative $\partial_{J^r\vartheta}$ and the Euler–Lagrange mapping E (cf. (2.29)). One can easily show that an analogous property holds for any contact symmetry, and any morphism $E_k : \Omega^r_k/\Theta^r_k \to \Omega^r_{k+1}/\Theta^r_{k+1}$.

Theorem 1. *Let $W \subset Y$ be an open set, and let a vector field Z, defined on W^r, be a contact symmetry. Then for all k,*

$$\partial_Z E_k\big([\rho]\big) = E_k\big(\partial_Z[\rho]\big) = E_k\big([i_Z\, d\rho]\big). \tag{2.44}$$

Proof. Since the Lie derivative commutes with the exterior derivative, we have for any k-form ρ on $J^r Y$,

$$\partial_Z[d\rho] = [\partial_Z\, d\rho] = [d\partial_Z \rho] = [d i_Z\, d\rho]. \tag{2.45}$$

Writing this formula in terms of the morphism E_k, we get (2.44). □

Our main goal is to introduce the concept of a variational vector field for a given dynamical form. Let $\varepsilon \in \Omega^s_{n+1,Y} W$ be a dynamical form such that $\varepsilon = [\rho]$ for some $\rho \in \Omega^r_{n+1} W$. We say that a vector field Z on $W^s \subset J^s Y$ is a *variational vector field* for $\varepsilon = [\rho]$, if the Lie derivative $\partial_Z \varepsilon$ is a locally variational form.

The proof of the following theorem is based on a simple observation explaining the meaning of the identity (2.44) for the Helmholtz mapping E_{n+1}.

Theorem 2. *Let $W \subset Y$ be an open set, let ε be a dynamical form on $W^s \subset J^s Y$, and let Z be a vector field on W^s. Suppose that Z is a contact symmetry. Then the following two conditions are equivalent*:

(1) *Z is a variational vector field for ε, i.e.,*

$$E_{n+1}(\partial_Z \varepsilon) = 0. \tag{2.46}$$

(2) *Z leaves invariant the Helmholtz class, i.e.,*

$$\partial_Z E_{n+1}(\varepsilon) = 0. \tag{2.47}$$

Proof. We choose $\rho \in \Omega^r_{n+1} W$ such that $\varepsilon = [\rho]$ and then apply Theorem 1. We obtain

$$\partial_Z E_{n+1}(\varepsilon) = E_{n+1}(\partial_Z \varepsilon). \tag{2.48}$$

Theorem 2 is a direct consequence of this formula. □

3. Variationality and invariance: Fibred mechanics

In this section, Y is a fibred manifold with 1-dimensional base X and projection π, and $\dim Y = m + 1$. A fibred chart on Y is denoted by (V, χ), $\chi = (t, q^\sigma)$. The associated fibred chart on $J^r Y$ is denoted by (V^r, χ^r), $\chi^r = (t, q^\sigma, q^\sigma_{(1)}, q^\sigma_{(2)}, \ldots, q^\sigma_{(r)})$, for $r = 3$ we usually write $\chi^3 = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, \dddot{q}^\sigma)$. For any fibred chart (V, χ), $\chi = (t, q^\sigma)$, and any differentiable function $f : V^r \to \mathbb{R}$, we define the "cut formal derivative operator" $d'f/dt$ by

$$\frac{d'f}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\nu} q^\nu_1 + \frac{\partial f}{\partial q^\nu_1} q^\nu_2 + \cdots + \frac{\partial f}{\partial q^\nu_{r-1}} q^\nu_r. \tag{3.1}$$

The formal derivative df/dt is defined on V^{r+1}, and $d'f/dt$ is a function on V^r.

3.1. Euler–Lagrange mapping

Let us consider the variational sequence on $J^1 Y$,

$$0 \to \mathbb{R} \to \Omega^1_0 \to \Omega^1_1/\Theta^1_1 \to \Omega^1_2/\Theta^1_2 \to \Omega^1_3/\Theta^1_3 \to \cdots. \tag{3.2}$$

We now give explicit formulas for elements of this sequence belonging to the terms

- Ω^1_1/Θ^1_1 (Lagrange classes, or Lagrangians),
- Ω^1_2/Θ^1_2 (Euler–Lagrange classes, or dynamical forms), and
- Ω^1_3/Θ^1_3 (Helmholtz classes),

and for the corresponding quotient mappings.

Theorem 3.

(1) *If ρ is a 1-form on $J^1 Y$,*

$$\rho = A\,dt + B_\sigma \omega^\sigma + C_\sigma\, d\dot{q}^\sigma, \tag{3.3}$$

then

$$[\rho] = \mathcal{L}\, dt, \tag{3.4}$$

where

$$\mathcal{L} = A + C_\sigma \ddot{q}^\sigma. \tag{3.5}$$

(2) *If ρ is a 2-form on J^1Y,*

$$\rho = \omega^\sigma \wedge (A_\sigma\, dt + B_{\sigma\nu}\omega^\nu + C_{\sigma\nu}\, d\dot{q}^\nu) + D_{\sigma\nu}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu + E_\sigma\, d\dot{q}^\sigma \wedge dt, \tag{3.6}$$

then

$$[\rho] = \varepsilon_\sigma \omega^\sigma \wedge dt, \tag{3.7}$$

where

$$\varepsilon_\sigma = A_\sigma + \left(C_{\sigma\nu} - 2\frac{dD_{\sigma\nu}}{dt}\right)\ddot{q}^\nu - \frac{dE_\sigma}{dt} - 2D_{\sigma\nu}\dddot{q}^\nu. \tag{3.8}$$

(3) *If ρ is a 1-form, the class $[d\rho]$ is given by*

$$[d\rho] = \mathcal{E}_\sigma \omega^\sigma \wedge dt, \tag{3.9}$$

where

$$\mathcal{E}_\sigma = \frac{\partial A}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial A}{\partial \dot{q}^\sigma} + \left(\frac{\partial C_\nu}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial C_\nu}{\partial \dot{q}^\sigma}\right)\ddot{q}^\nu - \frac{\partial C_\nu}{\partial \dot{q}^\sigma}\dddot{q}^\nu + \frac{d^2 C_\sigma}{dt^2}. \tag{3.10}$$

The functions $\mathcal{E}_\sigma$ depend on the class $[\rho]$ only, and can be written in the form

$$\mathcal{E}_\sigma = \frac{\partial \mathcal{L}}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\sigma} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}^\sigma}. \tag{3.11}$$

Proof. (1) Formula

$$(\pi^{2,1})^*\rho = (A + C_\sigma \ddot{q}^\sigma)\, dt + B_\sigma \omega^\sigma + C_\sigma \dot{\omega}^\sigma \tag{3.12}$$

gives (3.5).

(2) We have

$$(\pi^{3,1})^*\rho = (A_\sigma + C_{\sigma\nu}\ddot{q}^\nu)\omega^\sigma \wedge dt + (2D_{\sigma\nu}\ddot{q}^\nu + E_\sigma)\dot{\omega}^\sigma \wedge dt$$
$$+ B_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + C_{\sigma\nu}\omega^\sigma \wedge \dot{\omega}^\nu + D_{\sigma\nu}\dot{\omega}^\sigma \wedge \dot{\omega}^\nu. \tag{3.13}$$

But since $d\omega^\sigma = -\dot{\omega}^\sigma \wedge dt$, we obtain

$$(2D_{\sigma\nu}\ddot{q}^\nu + E_\sigma)\dot{\omega}^\sigma \wedge dt = -\frac{d(2D_{\sigma\nu}\ddot{q}^\nu + E_\sigma)}{dt}\omega^\sigma \wedge dt$$
$$+ pd(2D_{\sigma\nu}\ddot{q}^\nu + E_\sigma) \wedge \omega^\sigma - d\big((2D_{\sigma\nu}\ddot{q}^\nu + E_\sigma)\omega^\sigma\big), \tag{3.14}$$

which implies (3.8).

(3) We describe the class $[d\rho]$ for a form ρ defined by (3.3). We obtain

$$d\rho = \tilde{A}_\sigma \omega^\sigma \wedge dt + \tilde{E}_\sigma\, d\dot{q}^\sigma \wedge dt + \tilde{B}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + \tilde{C}_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + \tilde{D}_{\sigma\nu}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu, \tag{3.15}$$

where

$$\tilde{A}_\sigma = \frac{\partial A}{\partial q^\sigma} - \frac{d' B_\sigma}{dt},$$
$$\tilde{B}_{\sigma\nu} = \frac{1}{2}\left(\frac{\partial B_\sigma}{\partial q^\nu} - \frac{\partial B_\nu}{\partial q^\sigma}\right),$$
$$\tilde{C}_{\sigma\nu} = -\frac{\partial B_\sigma}{\partial \dot{q}^\nu} + \frac{\partial C_\nu}{\partial q^\sigma},$$
$$\tilde{D}_{\sigma\nu} = -\frac{1}{2}\left(\frac{\partial C_\sigma}{\partial \dot{q}^\nu} - \frac{\partial C_\nu}{\partial \dot{q}^\sigma}\right),$$
$$\tilde{E}_\sigma = \frac{\partial A}{\partial \dot{q}^\sigma} - B_\sigma - \frac{d' C_\sigma}{dt}. \tag{3.16}$$

Then writing the class $[d\rho]$ as in (3.8) we get from (3.13) $[d\rho] = \mathcal{E}_\sigma \omega^\sigma \wedge dt$, where

$$\mathcal{E}_\sigma = \tilde{A}_\sigma + \left(\tilde{C}_{\sigma\nu} - 2\frac{d\tilde{D}_{\sigma\nu}}{dt}\right)\ddot{q}^\nu - \frac{d\tilde{E}_\sigma}{dt} - 2\tilde{D}_{\sigma\nu}\dddot{q}^\nu. \tag{3.17}$$

From (3.16) we get

$$\mathcal{E}_\sigma = \frac{\partial A}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial A}{\partial \dot{q}^\sigma} + \frac{\partial C_\nu}{\partial q^\sigma}\ddot{q}^\nu - \frac{d}{dt}\frac{\partial C_\nu}{\partial \dot{q}^\sigma}\ddot{q}^\nu - \frac{\partial C_\nu}{\partial \dot{q}^\sigma}\dddot{q}^\nu$$
$$+ \frac{d}{dt}\frac{\partial C_\sigma}{\partial \dot{q}^\nu}\ddot{q}^\nu + \frac{\partial C_\sigma}{\partial \dot{q}^\nu}\dddot{q}^\nu + \frac{d}{dt}\frac{d' C_\sigma}{dt}. \tag{3.18}$$

We now obtain (3.10) by substituting to (3.18) from

$$\frac{d^2 C_\sigma}{dt^2} = \frac{d}{dt}\frac{dC_\sigma}{dt} = \frac{d}{dt}\frac{d' C_\sigma}{dt} + \frac{d}{dt}\frac{\partial C_\sigma}{\partial \dot{q}^\nu}\ddot{q}^\nu + \frac{\partial C_\sigma}{\partial \dot{q}^\nu}\dddot{q}^\nu. \tag{3.19}$$

Finally, on the other hand, we have using (3.5)

$$\frac{\partial \mathcal{L}}{\partial \ddot{q}^\sigma} = C_\sigma, \qquad \frac{\partial \mathcal{L}}{\partial \dot{q}^\sigma} = \frac{\partial A}{\partial \dot{q}^\sigma} + \frac{\partial C_\tau}{\partial \dot{q}^\sigma}\ddot{q}^\tau, \qquad \frac{\partial \mathcal{L}}{\partial q^\sigma} = \frac{\partial A}{\partial q^\sigma} + \frac{\partial C_\tau}{\partial q^\sigma}\ddot{q}^\tau, \tag{3.20}$$

hence

$$\frac{\partial \mathcal{L}}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\sigma} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}^\sigma} = \frac{\partial A}{\partial q^\sigma} + \frac{\partial C_\tau}{\partial q^\sigma}\ddot{q}^\tau - \frac{d}{dt}\frac{\partial A}{\partial \dot{q}^\sigma} - \frac{d}{dt}\frac{\partial C_\tau}{\partial \dot{q}^\sigma}\ddot{q}^\tau - \frac{\partial C_\tau}{\partial \dot{q}^\sigma}\dddot{q}^\tau + \frac{d^2 C_\sigma}{dt^2}. \tag{3.21}$$

Comparing this expression with Theorem 3, we get formula (3.11). □

Theorem 3 gives us an explicit expression for the Euler–Lagrange mapping

$$E_1 : \Omega_1^1/\Theta_1^1 \to \Omega_2^1/\Theta_2^1. \tag{3.22}$$

3.2. The Helmholtz mapping

In the following explicit description of the Helmholtz class we use the notation of Theorem 3.

Theorem 4.

(1) *If μ is a 3-form on J^1Y, expressed by*

$$\mu = A_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge dt + B_{\sigma\nu}\, d\dot{q}^\sigma \wedge \omega^\nu \wedge dt + C_{\sigma\nu}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu \wedge dt$$
$$+ D_{\sigma\nu\tau}\omega^\sigma \wedge \omega^\nu \wedge \omega^\tau + E_{\sigma\nu\tau}\, d\dot{q}^\sigma \wedge \omega^\nu \wedge \omega^\tau + F_{\sigma\nu\tau}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu \wedge \omega^\tau$$
$$+ G_{\sigma\nu\tau}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu \wedge d\dot{q}^\tau, \tag{3.23}$$

then

$$[\mu] = (P_{\sigma\nu}\omega^\sigma + Q_{\sigma\nu}\dot{\omega}^\sigma + R_{\sigma\nu}\ddot{\omega}^\sigma) \wedge \omega^\nu \wedge dt, \tag{3.24}$$

where

$$P_{\sigma\nu} = A_{\sigma\nu} + E_{\tau\sigma\nu}\ddot{q}^\tau - \frac{1}{2}\frac{d}{dt}\left(\frac{1}{2}(B_{\sigma\nu} - B_{\nu\sigma}) - (F_{\sigma\tau\nu} - F_{\nu\tau\sigma})\ddot{q}^\tau - \frac{d(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}\right),$$
$$Q_{\sigma\nu} = \frac{1}{2}(B_{\sigma\nu} + B_{\nu\sigma}) - (F_{\sigma\tau\nu} + F_{\nu\tau\sigma})\ddot{q}^\tau,$$
$$R_{\sigma\nu} = -C_{\sigma\nu} - 3G_{\sigma\nu\tau}\ddot{q}^\tau. \tag{3.25}$$

(2) *If ρ is a 2-form on J^1Y, expressed as in Theorem 3, (3.6), then*

$$[d\rho] = (F_{\sigma\nu}\omega^\sigma + G_{\sigma\nu}\dot{\omega}^\sigma + H_{\sigma\nu}\ddot{\omega}^\sigma) \wedge \omega^\nu \wedge dt, \tag{3.26}$$

where

$$\begin{aligned}
F_{\sigma\nu} = {} & \frac{1}{2}\left(-\frac{\partial A_\sigma}{\partial q^\nu} + \frac{\partial A_\nu}{\partial q^\sigma}\right) - \frac{1}{4}\frac{d}{dt}\left(\frac{\partial A_\nu}{\partial \dot{q}^\sigma} - \frac{\partial A_\sigma}{\partial \dot{q}^\nu}\right) - \frac{1}{2}\left(\frac{\partial C_{\sigma\tau}}{\partial q^\nu} - \frac{\partial C_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau \\
& + \frac{1}{4}\frac{d}{dt}\left(-\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\ddot{q}^\tau + \frac{1}{4}\left(-\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\dddot{q}^\tau \\
& + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial D_{\sigma\tau}}{\partial q^\nu} - \frac{\partial D_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau + \frac{1}{2}\left(\frac{\partial D_{\sigma\tau}}{\partial q^\nu} - \frac{\partial D_{\nu\tau}}{\partial q^\sigma}\right)\dddot{q}^\tau + \frac{1}{2}\frac{d^3 D_{\sigma\nu}}{dt^3} \\
& - \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\ddot{q}^\tau - \frac{d}{dt}\left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\dddot{q}^\tau \\
& - \frac{1}{2}\left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)q^\tau_{(4)} - \frac{1}{4}\frac{d^2}{dt^2}\left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\right) + \frac{1}{4}\frac{d}{dt}\left(\frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma}\right), \\
G_{\sigma\nu} = {} & \frac{1}{2}\left(\frac{\partial A_\nu}{\partial \dot{q}^\sigma} + \frac{\partial A_\sigma}{\partial \dot{q}^\nu}\right) - \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial q^\nu} + \frac{\partial E_\nu}{\partial q^\sigma}\right) - \left(\frac{\partial D_{\sigma\tau}}{\partial q^\nu} + \frac{\partial D_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau \\
& - \frac{1}{2}\frac{d(C_{\sigma\nu} + C_{\nu\sigma})}{dt} + \frac{1}{2}\left(\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\ddot{q}^\tau, \\
H_{\sigma\nu} = {} & -\frac{1}{2}(C_{\sigma\nu} - C_{\nu\sigma}) + \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\right) - \frac{dD_{\sigma\nu}}{dt} + \left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\ddot{q}^\tau.
\end{aligned} \tag{3.27}$$

The functions $F_{\sigma\nu}$, $G_{\sigma\nu}$, $H_{\sigma\nu}$ can be expressed by means of ε_σ (cf. (3.7), (3.8)) as follows:

$$\begin{aligned}
F_{\sigma\nu} &= -\frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial q^\nu} - \frac{\partial \varepsilon_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma} - \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial \varepsilon_\sigma}{\partial \dddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \dddot{q}^\sigma}\right)\right)\right), \\
G_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} + \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\right), \\
H_{\sigma\nu} &= -\frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma} - \frac{3}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_\sigma}{\partial \dddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \dddot{q}^\sigma}\right)\right).
\end{aligned} \tag{3.28}$$

Proof. (1) To derive formulas (3.25) note that

$$\begin{aligned}
p_2\mu = {} & (A_{\sigma\nu} + E_{\tau\sigma\nu}\ddot{q}^\tau)\omega^\sigma \wedge \omega^\nu \wedge dt \\
& + (B_{\sigma\nu} - 2F_{\sigma\tau\nu}\ddot{q}^\tau)\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt + (C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)\dot{\omega}^\sigma \wedge \dot{\omega}^\nu \wedge dt.
\end{aligned} \tag{3.29}$$

But

$$d(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) = (\ddot{\omega}^\sigma \wedge \omega^\nu - \ddot{\omega}^\nu \wedge \omega^\sigma) \wedge dt + 2\dot{\omega}^\sigma \wedge \dot{\omega}^\nu \wedge dt, \tag{3.30}$$

so that

$$\begin{aligned}
& (C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)\dot{\omega}^\sigma \wedge \dot{\omega}^\nu \wedge dt \\
& \quad = -\frac{1}{2}\frac{d(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \wedge dt \\
& \qquad - \frac{1}{2}(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)(\ddot{\omega}^\sigma \wedge \omega^\nu - \ddot{\omega}^\nu \wedge \omega^\sigma) \wedge dt + \eta,
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
2\eta = {} & -pd(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau) \wedge (\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \\
& + d\big((C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma)\big).
\end{aligned} \tag{3.32}$$

Hence

$$\begin{aligned}
p_2\mu = {} & (A_{\sigma\nu} + E_{\tau\sigma\nu}\ddot{q}^\tau)\omega^\sigma \wedge \omega^\nu \wedge dt \\
& + \frac{1}{2}\big(B_{\sigma\nu} + B_{\nu\sigma} - 2(F_{\sigma\tau\nu} + F_{\nu\tau\sigma})\ddot{q}^\tau\big)\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt \\
& + \frac{1}{2}(B_{\sigma\nu} - 2F_{\sigma\tau\nu}\ddot{q}^\tau)(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \wedge dt \\
& - \frac{1}{2}\frac{d(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \wedge dt \\
& - \frac{1}{2}(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)(\ddot{\omega}^\sigma \wedge \omega^\nu - \ddot{\omega}^\nu \wedge \omega^\sigma) \wedge dt \\
& - \frac{1}{2}pd(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau) \wedge (\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \\
& + \frac{1}{2}d\big((C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma)\big).
\end{aligned} \tag{3.33}$$

From the formula

$$(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \wedge dt = d(\omega^\sigma \wedge \omega^\nu) \tag{3.34}$$

we have

$$\begin{aligned}
& \frac{1}{2}(B_{\sigma\nu} - 2F_{\sigma\tau\nu}\ddot{q}^\tau)(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \wedge dt \\
& \quad - \frac{1}{2}\frac{d(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}(\dot{\omega}^\sigma \wedge \omega^\nu - \dot{\omega}^\nu \wedge \omega^\sigma) \wedge dt \\
& = \frac{1}{2}d\left(\left(B_{\sigma\nu} - 2F_{\sigma\tau\nu}\ddot{q}^\tau - \frac{d(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}\right)\omega^\sigma \wedge \omega^\nu\right) \\
& \quad - \frac{1}{2}d\left(B_{\sigma\nu} - 2F_{\sigma\tau\nu}\ddot{q}^\tau - \frac{d(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}\right) \wedge \omega^\sigma \wedge \omega^\nu.
\end{aligned} \tag{3.35}$$

Substituting (3.35) to (3.33) we easily recognise that

$$\begin{aligned}
[\mu] = {} & \left(A_{\sigma\nu} + E_{\tau\sigma\nu}\ddot{q}^\tau - \frac{1}{2}\frac{d}{dt}\left(B_{\sigma\nu} - 2F_{\sigma\tau\nu}\ddot{q}^\tau - \frac{d(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}\right)\right)\omega^\sigma \wedge \omega^\nu \wedge dt \\
& + \frac{1}{2}\big(B_{\sigma\nu} + B_{\nu\sigma} - 2(F_{\sigma\tau\nu} + F_{\nu\tau\sigma})\ddot{q}^\tau\big)\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt \\
& - \frac{1}{2}(C_{\sigma\nu} + 3G_{\sigma\nu\tau}\ddot{q}^\tau)(\ddot{\omega}^\sigma \wedge \omega^\nu - \ddot{\omega}^\nu \wedge \omega^\sigma) \wedge dt.
\end{aligned} \tag{3.36}$$

(2) From the expression of ρ we obtain

$$\begin{aligned}
d\rho = {} & \tilde{A}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge dt + \tilde{B}_{\sigma\nu}\, d\dot{q}^\sigma \wedge \omega^\nu \wedge dt + \tilde{C}_{\sigma\nu}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu \wedge dt \\
& + \tilde{D}_{\sigma\nu\tau}\omega^\sigma \wedge \omega^\nu \wedge \omega^\tau + \tilde{E}_{\sigma\nu\tau}\, d\dot{q}^\sigma \wedge \omega^\nu \wedge \omega^\tau + \tilde{F}_{\sigma\nu\tau}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu \wedge \omega^\tau \\
& + \tilde{G}_{\sigma\nu\tau}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu \wedge d\dot{q}^\tau,
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
\tilde{A}_{\sigma\nu} &= \frac{1}{2}\left(-\frac{\partial A_\sigma}{\partial q^\nu} + \frac{\partial A_\nu}{\partial q^\sigma}\right) + \frac{d' B_{\sigma\nu}}{dt}, \\
\tilde{B}_{\sigma\nu} &= 2B_{\sigma\nu} + \frac{\partial A_\nu}{\partial \dot{q}^\sigma} - \frac{d' C_{\nu\sigma}}{dt} - \frac{\partial E_\sigma}{\partial q^\nu}, \\
\tilde{C}_{\sigma\nu} &= \frac{1}{2}\left(C_{\sigma\nu} - C_{\nu\sigma} - \frac{\partial E_\sigma}{\partial \dot{q}^\nu} + \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\right) + \frac{d' D_{\sigma\nu}}{dt},
\end{aligned}$$

$$\begin{aligned}
\tilde{D}_{\sigma\nu\tau} &= -\frac{1}{3}\left(\frac{\partial B_{\sigma\nu}}{\partial q^\tau} - \frac{\partial B_{\tau\nu}}{\partial q^\sigma} - \frac{\partial B_{\sigma\tau}}{\partial q^\nu}\right),\\
\tilde{E}_{\sigma\nu\tau} &= \frac{\partial B_{\nu\tau}}{\partial \dot{q}^\sigma} - \frac{1}{2}\left(\frac{\partial C_{\nu\sigma}}{\partial q^\tau} - \frac{\partial C_{\tau\sigma}}{\partial q^\nu}\right),\\
\tilde{F}_{\sigma\nu\tau} &= \frac{1}{2}\left(-\frac{\partial C_{\tau\nu}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\tau\sigma}}{\partial \dot{q}^\nu}\right) + \frac{\partial D_{\sigma\nu}}{\partial q^\tau},\\
\tilde{G}_{\sigma\nu\tau} &= \frac{1}{3}\left(\frac{\partial D_{\sigma\nu}}{\partial \dot{q}^\tau} - \frac{\partial D_{\tau\nu}}{\partial \dot{q}^\sigma} - \frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu}\right).
\end{aligned}\tag{3.38}$$

Now we can compute the class $[d\rho]$ from formula (3.25). We get formula (3.26), where

$$\begin{aligned}
F_{\sigma\nu} &= \tilde{A}_{\sigma\nu} + \tilde{E}_{\tau\sigma\nu}\ddot{q}^\tau - \frac{1}{2}\frac{d}{dt}\left(\frac{1}{2}(\tilde{B}_{\sigma\nu} - \tilde{B}_{\nu\sigma}) - (\tilde{F}_{\sigma\tau\nu} - \tilde{F}_{\nu\tau\sigma})\ddot{q}^\tau - \frac{d(\tilde{C}_{\sigma\nu} + 3\tilde{G}_{\sigma\nu\tau}\ddot{q}^\tau)}{dt}\right),\\
G_{\sigma\nu} &= \frac{1}{2}(\tilde{B}_{\sigma\nu} + \tilde{B}_{\nu\sigma}) - (\tilde{F}_{\sigma\tau\nu} + \tilde{F}_{\nu\tau\sigma})\ddot{q}^\tau,\\
H_{\sigma\nu} &= -\tilde{C}_{\sigma\nu} - 3\tilde{G}_{\sigma\nu\tau}\ddot{q}^\tau.
\end{aligned}\tag{3.39}$$

Formulas (3.27) follow directly from (3.38). Formulas (3.28) then are derived by a direct computation. □

Theorem 4 describes the Helmholtz mapping

$$E_2 : \Omega^1_2/\Theta^1_2 \to \Omega^1_3/\Theta^1_3,\tag{3.40}$$

assigning to a class $\varepsilon = [\rho]$ (3.7) a class $E_2(\varepsilon) = [d\rho]$, expressed by formula (3.28). In the well-known sense, equations

$$F_{\sigma\nu} = 0, \qquad G_{\sigma\nu} = 0, \qquad H_{\sigma\nu} = 0\tag{3.41}$$

express necessary and sufficient conditions for existence of (local) Lagrangians for ε (*Helmholtz conditions*).

Remark 1. For a general 3rd order dynamical form $\varepsilon = \varepsilon_\sigma\omega^\sigma \wedge dt$, which is not necessarily of the form $\varepsilon = [\rho]$ (cf. (3.8)), the Helmholtz class $E_2(\varepsilon)$ is given by

$$\begin{aligned}
E_2(\varepsilon) = &-\frac{1}{2}\left(\frac{\partial\varepsilon_\sigma}{\partial q^\nu} - \frac{\partial\varepsilon_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial\varepsilon_\sigma}{\partial \dot{q}^\nu} - \frac{\partial\varepsilon_\nu}{\partial \dot{q}^\sigma} - \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial\varepsilon_\sigma}{\partial \dddot{q}^\nu} - \frac{\partial\varepsilon_\nu}{\partial \dddot{q}^\sigma}\right)\right)\right)\omega^\sigma \wedge \omega^\nu \wedge dt\\
&+ \frac{1}{2}\left(\frac{\partial\varepsilon_\sigma}{\partial \dot{q}^\nu} + \frac{\partial\varepsilon_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial\varepsilon_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial\varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\right)\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt\\
&- \frac{1}{2}\left(\frac{\partial\varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial\varepsilon_\nu}{\partial \ddot{q}^\sigma} - \frac{3}{2}\frac{d}{dt}\left(\frac{\partial\varepsilon_\sigma}{\partial \dddot{q}^\nu} - \frac{\partial\varepsilon_\nu}{\partial \dddot{q}^\sigma}\right)\right)\ddot{\omega}^\sigma \wedge \omega^\nu \wedge dt\\
&+ \frac{1}{2}\left(\frac{\partial\varepsilon_\sigma}{\partial \dddot{q}^\nu} + \frac{\partial\varepsilon_\nu}{\partial \dddot{q}^\sigma}\right)\dddot{\omega}^\sigma \wedge \omega^\nu \wedge dt.
\end{aligned}\tag{3.42}$$

Expression (3.42) is in accordance with the Helmholtz class of a dynamical form on a general fibred manifold ($\dim X = n$) obtained in (Krupka [11]).

Remark 2. It is worth notice that *for computing the Euler–Lagrange and Helmholtz classes* $[\rho]$ *and* $[d\rho]$, *we can restrict ourselves to representatives* ρ *such that* $B_{\sigma\nu} = 0$ *and* $E_\sigma = 0$, i.e., to

$$\rho = \omega^\sigma \wedge (A_\sigma\, dt + C_{\sigma\nu}\, d\dot{q}^\nu) + D_{\sigma\nu} d\dot{q}^\sigma \wedge d\dot{q}^\nu.\tag{3.43}$$

Indeed, writing

$$E_\sigma\, d\dot{q}^\sigma \wedge dt = -d(E_\sigma\omega^\sigma) + dE_\sigma \wedge \omega^\sigma\tag{3.44}$$

in formula (3.6), we can see that

$$\rho = \omega^\sigma \wedge (\bar{A}_\sigma\, dt + \bar{C}_{\sigma\nu}\, d\dot{q}^\nu) + D_{\sigma\nu}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu + \mu, \tag{3.45}$$

with $\bar{A}_\sigma$ and $\bar{C}_{\sigma\nu}$ given by

$$\bar{A}_\sigma = A_\sigma - \frac{\partial E_\sigma}{\partial t} - \frac{\partial E_\sigma}{\partial q^\nu}\dot{q}^\nu, \qquad \bar{C}_{\sigma\nu} = C_{\sigma\nu} - \frac{\partial E_\sigma}{\partial \dot{q}^\nu}, \tag{3.46}$$

and $\mu \in \Theta_2^1 W$,

$$\mu = B_{\sigma\nu}\omega^\sigma \wedge \omega^\nu - d(E_\sigma \omega^\sigma). \tag{3.47}$$

Hence, the reduced form

$$\bar{\rho} = \omega^\sigma \wedge (\bar{A}_\sigma\, dt + \bar{C}_{\sigma\nu}\, d\dot{q}^\nu) + D_{\sigma\nu}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu \tag{3.48}$$

satisfies $[\rho] = [\bar{\rho}]$ and $[d\rho] = [d\bar{\rho}]$.

3.3. Contact symmetries

In this section we give a complete description of contact symmetries on W^3, where $W \subset Y$ is an open set. Recall that in a fibred chart the contact ideal $\Theta^3 W$ is generated by the 1-forms

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma\, dt, \qquad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma\, dt, \qquad \ddot{\omega}^\sigma = d\ddot{q}^\sigma - \dddot{q}^\sigma\, dt, \tag{3.49}$$

and the 2-forms

$$d\ddot{\omega}^\sigma = -d\dddot{q}^\sigma \wedge dt. \tag{3.50}$$

Let Z be a vector field on $J^3 Y$,

$$Z = \zeta_0 \frac{\partial}{\partial t} + \zeta^\tau \frac{\partial}{\partial q^\tau} + \dot{\zeta}^\tau \frac{\partial}{\partial \dot{q}^\tau} + \ddot{\zeta}^\tau \frac{\partial}{\partial \ddot{q}^\tau} + \dddot{\zeta}^\tau \frac{\partial}{\partial \dddot{q}^\tau}. \tag{3.51}$$

To derive equations for Z to be a contact symmetry, we require the forms $\partial_Z \omega^\sigma$, $\partial_Z \dot{\omega}^\sigma$, and $\partial_Z \ddot{\omega}^\sigma$ be contact. We have

$$\begin{aligned}
\partial_Z \omega^\sigma = {} & \left(\frac{d'\zeta^\sigma}{dt} - \dot{\zeta}^\sigma - \frac{d'\zeta_0}{dt}\dot{q}^\sigma\right) dt + \left(\frac{\partial \zeta^\sigma}{\partial q^\tau} - \frac{\partial \zeta_0}{\partial q^\tau}\dot{q}^\sigma\right)\omega^\tau \\
& + \left(\frac{\partial \zeta^\sigma}{\partial \dot{q}^\tau} - \frac{\partial \zeta_0}{\partial \dot{q}^\tau}\dot{q}^\sigma\right)\dot{\omega}^\tau + \left(\frac{\partial \zeta^\sigma}{\partial \ddot{q}^\tau} - \frac{\partial \zeta_0}{\partial \ddot{q}^\tau}\dot{q}^\sigma\right)\ddot{\omega}^\tau \\
& + \left(\frac{\partial \zeta^\sigma}{\partial \dddot{q}^\tau} - \frac{\partial \zeta_0}{\partial \dddot{q}^\tau}\dot{q}^\sigma\right) d\dddot{q}^\tau,
\end{aligned} \tag{3.52}$$

and analogous expressions are obtained for $\partial_Z \dot{\omega}^\sigma$ and $\partial_Z \ddot{\omega}^\sigma$. Thus, we have the following system of equations for the components of Z:

$$\begin{aligned}
& \dot{\zeta}^\sigma - \zeta_0 \ddot{q}^\sigma = \frac{d'(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{dt}, \qquad \frac{\partial(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{\partial \dddot{q}^\tau} = 0, \\
& \ddot{\zeta}^\sigma - \zeta_0 \dddot{q}^\sigma = \frac{d'(\dot{\zeta}^\sigma - \zeta_0 \ddot{q}^\sigma)}{dt}, \qquad \frac{\partial(\dot{\zeta}^\sigma - \zeta_0 \ddot{q}^\sigma)}{\partial \dddot{q}^\tau} = 0, \\
& \dddot{\zeta}^\sigma = \frac{d'(\ddot{\zeta}^\sigma - \zeta_0 \dddot{q}^\sigma)}{dt}, \qquad \frac{\partial(\ddot{\zeta}^\sigma - \zeta_0 \dddot{q}^\sigma)}{\partial \dddot{q}^\tau} + \delta^\sigma_\tau \zeta_0 = 0.
\end{aligned} \tag{3.53}$$

We can integrate Eqs. (3.53) by elementary methods. To this end, it is convenient to consider the cases $\dim Y = 2$, and $\dim Y \geqslant 3$ separately.

Lemma 1. *Let* $\dim Y = 2$*, and let* Z *be a vector field on* J^3Y*,*

$$Z = \zeta_0 \frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial q} + \dot{\zeta} \frac{\partial}{\partial \dot{q}} + \ddot{\zeta} \frac{\partial}{\partial \ddot{q}} + \dddot{\zeta} \frac{\partial}{\partial \dddot{q}}. \tag{3.54}$$

The following three conditions are equivalent:

(1) *Z is a contact symmetry.*
(2) *There exists a function* $f = f(t, q, \dot{q})$ *such that*

$$\zeta_0 = -\frac{\partial f}{\partial \dot{q}}, \qquad \zeta = f - \frac{\partial f}{\partial \dot{q}} \dot{q}, \qquad \dot{\zeta} = \frac{d' f}{dt}, \qquad \ddot{\zeta} = \frac{d'}{dt} \frac{df}{dt}, \qquad \dddot{\zeta} = \frac{d'}{dt} \frac{d^2 f}{dt^2}. \tag{3.55}$$

(3) *The components* ζ_0 *and* ζ *depend on* $t, q, \dot{q}$ *only, and satisfy*

$$\frac{\partial \zeta}{\partial \dot{q}} - \frac{\partial \zeta_0}{\partial \dot{q}} \dot{q} = 0, \tag{3.56}$$

and

$$\dot{\zeta} = \frac{d\zeta}{dt} - \dot{q} \frac{d\zeta_0}{dt}, \qquad \ddot{\zeta} = \frac{d\dot{\zeta}}{dt} - \ddot{q} \frac{d\zeta_0}{dt}, \qquad \dddot{\zeta} = \frac{d\ddot{\zeta}}{dt} - \dddot{q} \frac{d\zeta_0}{dt}. \tag{3.57}$$

If Z is a contact symmetry and f is a function such that conditions (3.55) are satisfied, we say that Z is *associated* with f.

Lemma 2. *Let* $\dim Y \geqslant 3$*. A vector field* Z (3.51) *is a contact symmetry if and only if* ζ_0 *and* ζ^σ *depend on* t, q^ν *only, and*

$$\begin{aligned} \dot{\zeta}^\sigma &= \frac{d\zeta^\sigma}{dt} - \frac{d\zeta_0}{dt} \dot{q}^\sigma, \\ \ddot{\zeta}^\sigma &= \frac{d^2\zeta^\sigma}{dt^2} - \frac{d^2\zeta_0}{dt^2} \dot{q}^\sigma - 2\frac{d\zeta_0}{dt} \ddot{q}^\sigma, \\ \dddot{\zeta}^\sigma &= \frac{d^3\zeta^\sigma}{dt^3} - \frac{d^3\zeta_0}{dt^3} \dot{q}^\sigma - 3\frac{d^2\zeta_0}{dt^2} \ddot{q}^\sigma - 3\frac{d\zeta_0}{dt} \dddot{q}^\sigma. \end{aligned} \tag{3.58}$$

It is immediately seen that the set of contact symmetries includes all vector fields, expressible as the 3-jet prolongations of π-projectable vector fields. Recall that if ϑ is a π-projectable vector field, expressed by

$$\vartheta = \vartheta_0 \frac{\partial}{\partial t} + \vartheta^\sigma \frac{\partial}{\partial q^\sigma} \tag{3.59}$$

then the 3-jet prolongation $J^3\vartheta$ is given by

$$J^3\vartheta = \vartheta_0 \frac{\partial}{\partial t} + \vartheta^\sigma \frac{\partial}{\partial q^\sigma} + \dot{\vartheta}^\tau \frac{\partial}{\partial \dot{q}^\tau} + \ddot{\vartheta}^\tau \frac{\partial}{\partial \ddot{q}^\tau} + \dddot{\vartheta}^\tau \frac{\partial}{\partial \dddot{q}^\tau}, \tag{3.60}$$

where

$$\dot{\vartheta}^\tau = \frac{d\vartheta^\tau}{dt} - \dot{q}^\tau \frac{d\vartheta_0}{dt}, \qquad \ddot{\vartheta}^\tau = \frac{d\dot{\vartheta}^\tau}{dt} - \ddot{q}^\tau \frac{d\vartheta_0}{dt}, \qquad \dddot{\vartheta}^\tau = \frac{d\ddot{\vartheta}^\tau}{dt} - \dddot{q}^\tau \frac{d\vartheta_0}{dt}; \tag{3.61}$$

π-projectability means that ϑ_0 depends on t only.

Lemma 3. *For any* π*-projectable vector field* ϑ*, defined on an open set* $W \subset Y$*, the* 3*-jet prolongation* $J^3\vartheta$ *is a contact symmetry.*

Proof. If ϑ is a π-projectable vector field on W expressed by (3.59) then $\vartheta_0 = \vartheta_0(t)$ and $\vartheta^\sigma = \vartheta^\sigma(t, q^\nu)$. Thus, our assertion follows from Lemmas 1, 2, and formula (3.61). □

Remark 3. Eq. (3.56) can be easily solved. We obtain

$$\zeta = \dot{q}^2 \int_0^1 \frac{\partial \zeta_0(t, q, s\dot{q})}{\partial \dot{q}} s\, ds + \xi, \tag{3.62}$$

where $\xi = \xi(t, q)$ is an arbitrary function.

3.4. Variational vector fields

Let $W \subset Y$ be an open set, and let ε be a dynamical form on $W^3 \subset J^3Y$. Suppose that $\varepsilon = [\rho]$ for some $\rho \in \Omega_2^1 W$. Recall that a vector field Z on W^3 is a variational vector field for ε, if the class $\partial_Z \varepsilon$, is a variational form. We know that a contact symmetry Z is variational for ε if and only if it leaves invariant the Helmholtz class $H(\varepsilon) = E_2(\varepsilon)$ (Theorem 2). In this section we derive the variationality equations for contact symmetries.

Lemma 4.

(1) *Let* $\dim Y = 2$, *and let* Z (3.54) *be a contact symmetry, associated with a function* f. *Then*

$$\begin{aligned}
\partial_Z \omega &= \frac{\partial f}{\partial q}\omega, \\
\partial_Z \dot{\omega} &= \frac{\partial}{\partial q}\frac{df}{dt}\omega + \frac{\partial}{\partial \dot{q}}\frac{df}{dt}\dot{\omega}, \\
\partial_Z \ddot{\omega} &= \frac{\partial}{\partial q}\frac{d^2 f}{dt^2}\omega + \frac{\partial}{\partial \dot{q}}\frac{d^2 f}{dt^2}\dot{\omega} + \frac{\partial}{\partial \ddot{q}}\frac{d^2 f}{dt^2}\ddot{\omega}.
\end{aligned} \tag{3.63}$$

(2) *Let* $\dim Y \geqslant 3$, *and let* Z (3.51) *be a contact symmetry. Then*

$$\begin{aligned}
\partial_Z \omega^\sigma &= P^\sigma_\tau \omega^\tau, \\
\partial_Z \dot{\omega}^\sigma &= Q^\sigma_\tau \omega^\tau + R^\sigma_\tau \dot{\omega}^\tau, \\
\partial_Z \ddot{\omega}^\sigma &= S^\sigma_\tau \omega^\tau + T^\sigma_\tau \dot{\omega}^\tau + U^\sigma_\tau \ddot{\omega}^\tau,
\end{aligned} \tag{3.64}$$

where

$$\begin{aligned}
P^\sigma_\tau &= \frac{\partial(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{\partial q^\tau}, & Q^\sigma_\tau &= \frac{\partial}{\partial q^\tau}\frac{d(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{dt}, \\
R^\sigma_\tau &= \frac{\partial}{\partial \dot{q}^\tau}\frac{d(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{dt}, & S^\sigma_\tau &= \frac{\partial}{\partial q^\tau}\frac{d^2(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{dt^2}, \\
T^\sigma_\tau &= \frac{\partial}{\partial \dot{q}^\tau}\frac{d^2(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{dt^2}, & U^\sigma_\tau &= \frac{\partial}{\partial \ddot{q}^\tau}\frac{d^2(\zeta^\sigma - \zeta_0 \dot{q}^\sigma)}{dt^2}.
\end{aligned} \tag{3.65}$$

Proof. All these formulas can be obtained by a direct computation. □

Remark 4. Formulas (3.56), (3.57) for contact symmetries on 2-dimensional fibred manifolds Y can be written in the same form as (3.58), because

$$f = \zeta - \zeta_0 \dot{q}. \tag{3.66}$$

Thus, we can use formulas (3.58) independently of the dimension of Y, having in mind, however, differences in the dependence of ζ_0 and ζ on the coordinates. The same arguments apply to formulas (3.63) and (3.64), (3.65) above.

Let us turn to study transformations of classes in the variational sequence. To this end, note that we may suppose without loss of generality that the representative ρ of the class ε is already in the reduced form (3.43). Indeed, if we write $\rho = \rho_0 + \mu$, where $\mu \in \Theta_2^1 W$, then $[\rho] = [\rho_0]$, and

$$[\partial_Z d\rho] = [\partial_Z d\rho_0] + [\partial_Z d\mu] = [\partial_Z d\rho_0] \tag{3.67}$$

for any contact symmetry Z. However, by Theorems 3 and 4, and Remark 2, the class $\varepsilon = [\rho]$ is given by $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt$, where

$$\varepsilon_\sigma = A_\sigma + \left(C_{\sigma\nu} - 2\frac{dD_{\sigma\nu}}{dt}\right)\ddot{q}^\nu - 2D_{\sigma\nu}\dddot{q}^\nu, \tag{3.68}$$

and the Helmholtz class $H(\varepsilon) = E_2(\varepsilon)$ is expressed by

$$H(\varepsilon) = [d\rho] = (F_{\sigma\nu}\omega^\sigma + G_{\sigma\nu}\dot{\omega}^\sigma + H_{\sigma\nu}\ddot{\omega}^\sigma) \wedge \omega^\nu \wedge dt, \tag{3.69}$$

where the components $F_{\sigma\nu}$, $G_{\sigma\nu}$, $H_{\sigma\nu}$ are given by

$$\begin{aligned}
F_{\sigma\nu} = {} & \frac{1}{2}\left(-\frac{\partial A_\sigma}{\partial q^\nu} + \frac{\partial A_\nu}{\partial q^\sigma}\right) - \frac{1}{4}\frac{d}{dt}\left(\frac{\partial A_\nu}{\partial \dot{q}^\sigma} - \frac{\partial A_\sigma}{\partial \dot{q}^\nu}\right) \\
& - \frac{1}{2}\left(\frac{\partial C_{\sigma\tau}}{\partial q^\nu} - \frac{\partial C_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau + \frac{1}{4}\frac{d}{dt}\left(-\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\ddot{q}^\tau \\
& + \frac{1}{4}\left(-\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\dddot{q}^\tau + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial D_{\sigma\tau}}{\partial q^\nu} - \frac{\partial D_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau \\
& + \frac{1}{2}\left(\frac{\partial D_{\sigma\tau}}{\partial q^\nu} - \frac{\partial D_{\nu\tau}}{\partial q^\sigma}\right)\dddot{q}^\tau + \frac{1}{2}\frac{d^3 D_{\sigma\nu}}{dt^3} - \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\ddot{q}^\tau \\
& - \frac{d}{dt}\left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\dddot{q}^\tau - \frac{1}{2}\left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)q^\tau_{(4)}, \\
G_{\sigma\nu} = {} & \frac{1}{2}\left(\frac{\partial A_\nu}{\partial \dot{q}^\sigma} + \frac{\partial A_\sigma}{\partial \dot{q}^\nu}\right) - \left(\frac{\partial D_{\sigma\tau}}{\partial q^\nu} + \frac{\partial D_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau \\
& - \frac{1}{2}\frac{d(C_{\sigma\nu} + C_{\nu\sigma})}{dt} + \frac{1}{2}\left(\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\ddot{q}^\tau, \\
H_{\sigma\nu} = {} & -\frac{1}{2}(C_{\sigma\nu} - C_{\nu\sigma}) - \frac{dD_{\sigma\nu}}{dt} + \left(\frac{\partial D_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\ddot{q}^\tau.
\end{aligned} \tag{3.70}$$

Lemma 5. *Let Z be a contact symmetry on J^3Y. If ρ is expressed by (3.43) then*

$$\partial_Z \rho = \omega^\sigma \wedge (A^Z_\sigma\, dt + B^Z_{\sigma\nu}\omega^\nu + C^Z_{\sigma\nu}\, d\dot{q}^\nu) + D^Z_{\sigma\nu}\, d\dot{q}^\sigma \wedge d\dot{q}^\nu + E^Z_\sigma\, d\dot{q}^\sigma \wedge dt, \tag{3.71}$$

where

$$\begin{aligned}
A^Z_\sigma &= \partial_Z A_\sigma + \frac{\partial(\zeta^\tau - \zeta_0\dot{q}^\tau)}{\partial q^\sigma}A_\tau + \frac{d'\zeta_0}{dt}A_\sigma + \frac{d'\dot{\zeta}^\nu}{dt}C_{\sigma\nu}, \\
B^Z_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial\zeta_0}{\partial q^\nu}A_\sigma - \frac{\partial\zeta_0}{\partial q^\sigma}A_\nu + \frac{\partial\dot{\zeta}^\tau}{\partial q^\nu}C_{\sigma\tau} - \frac{\partial\dot{\zeta}^\tau}{\partial q^\sigma}C_{\nu\tau}\right), \\
C^Z_{\sigma\nu} &= \partial_Z C_{\sigma\nu} + C_{\tau\nu}\frac{\partial(\zeta^\tau - \zeta_0\dot{q}^\tau)}{\partial q^\sigma} + \frac{\partial\zeta_0}{\partial\dot{q}^\nu}A_\sigma + \frac{\partial\dot{\zeta}^\tau}{\partial\dot{q}^\nu}C_{\sigma\tau} + 2D_{\tau\nu}\frac{\partial\dot{\zeta}^\tau}{\partial q^\sigma}, \\
D^Z_{\sigma\nu} &= \partial_Z D_{\sigma\nu} + D_{\tau\nu}\frac{\partial\dot{\zeta}^\tau}{\partial\dot{q}^\sigma} - D_{\tau\sigma}\frac{\partial\dot{\zeta}^\tau}{\partial\dot{q}^\nu}, \\
E^Z_\sigma &= -2D_{\nu\sigma}\frac{d'\dot{\zeta}^\nu}{dt}.
\end{aligned} \tag{3.72}$$

Proof. To consider the cases $\dim Y = 2$ and $\dim Y \geqslant 3$ together, we suppose for the following computations that the components ζ_0, ζ^σ of Z depend on $t, q^\tau, \dot{q}^\tau$,

$$\zeta_0 = \zeta_0(t, q^\tau, \dot{q}^\tau), \qquad \zeta^\sigma = \zeta^\sigma(t, q^\tau, \dot{q}^\tau) \tag{3.73}$$

(cf. Remark 4). Using Lemma 4 and the expressions

$$\begin{aligned} \partial_Z\, dt &= d\zeta_0 = \left(\frac{\partial \zeta_0}{\partial t} + \frac{\partial \zeta_0}{\partial q^\tau}\dot{q}^\tau\right) dt + \frac{\partial \zeta_0}{\partial q^\tau}\omega^\tau + \frac{\partial \zeta_0}{\partial \dot{q}^\tau}\, d\dot{q}^\tau, \\ \partial_Z\, d\dot{q}^\sigma &= d\dot{\zeta}^\sigma = \left(\frac{\partial \dot{\zeta}^\sigma}{\partial t} + \frac{\partial \dot{\zeta}^\sigma}{\partial q^\tau}\dot{q}^\tau\right) dt + \frac{\partial \dot{\zeta}^\sigma}{\partial q^\tau}\omega^\tau + \frac{\partial \dot{\zeta}^\sigma}{\partial \dot{q}^\tau}\, d\dot{q}^\tau, \end{aligned} \tag{3.74}$$

we obtain (3.72) by a straightforward computation. □

Remark 5. In several particular cases, expressions (3.72) can be further simplified. We mention the following three cases:

(1) $\dim Y = 2$; in this case $m = \dim Y - \dim X = 1$, hence, $D \equiv 0$, and consequently, $D^Z = 0$, $E^Z = 0$.
(2) $\dim Y \geqslant 3$; then according to Lemma 2, $\partial \zeta_0 / \partial \dot{q}^\nu = 0$.
(3) $Z = J^3\vartheta$, where ϑ is a π-vertical vector field; in this case, $\zeta_0 = 0$, and $\zeta^\sigma = \zeta^\sigma(t, q^\tau)$.

We now compute the Lie derivative $\partial_Z \varepsilon$ of the class ε. Using notations introduced in Remark 2 we have:

Lemma 6. *If ρ is given by (3.43) and $\varepsilon = [\rho]$ is expressed by (3.68), then*

$$\partial_Z \varepsilon = \varepsilon_\sigma^Z \omega^\sigma \wedge dt, \tag{3.75}$$

where

$$\begin{aligned} \varepsilon_\sigma^Z &= A_\sigma^Z + \left(C_{\sigma\nu}^Z - 2\frac{dD_{\sigma\nu}^Z}{dt}\right)\ddot{q}^\nu - \frac{dE_\sigma^Z}{dt} - 2D_{\sigma\nu}^Z \dddot{q}^\nu \\ &= \bar{A}_\sigma^Z + \left(\bar{C}_{\sigma\nu}^Z - 2\frac{dD_{\sigma\nu}^Z}{dt}\right)\ddot{q}^\nu - 2D_{\sigma\nu}^Z \dddot{q}^\nu \end{aligned} \tag{3.76}$$

with

$$\bar{A}_\sigma^Z = A_\sigma^Z - \frac{\partial E_\sigma^Z}{\partial t} - \frac{\partial E_\sigma^Z}{\partial q^\nu}\dot{q}^\nu, \qquad \bar{C}_{\sigma\nu}^Z = C_{\sigma\nu}^Z - \frac{\partial E_\sigma^Z}{\partial \dot{q}^\nu}. \tag{3.77}$$

Proof. We know that $\partial_Z \varepsilon$ is the class $[\partial_Z \rho]$, represented by the two-form $\partial_Z \rho$ given by expressions (3.71) and (3.72). To compute the class, we can use either Theorem 3, or proceed with the corresponding equivalent reduced form (cf. Remark 2). This gives us the desired formulas. □

Remark 6. Lemma 6 characterises the general structure of dynamical forms $\partial_Z \varepsilon$ for given ε (3.68). In particular, formulas (3.76), (3.72) describe the structure of the corresponding dynamical equations.

Finally, we need a formula for the class $\partial_Z H(\varepsilon) = E_2([\partial_Z \rho]) = [d\partial_Z \rho]$.

Lemma 7. *If ρ is given by (3.43) and $\varepsilon = [\rho]$ is expressed by (3.68) then*

$$\partial_Z H(\varepsilon) = (F_{\sigma\nu}^Z \omega^\sigma + G_{\sigma\nu}^Z \dot{\omega}^\sigma + H_{\sigma\nu}^Z \ddot{\omega}^\sigma) \wedge \omega^\nu \wedge dt, \tag{3.78}$$

where (*with the notation* (3.77) *and* (3.72))

$$F_{\sigma\nu}^Z = \frac{1}{2}\left(\frac{\partial \bar{A}_\nu^Z}{\partial q^\sigma} - \frac{\partial \bar{A}_\sigma^Z}{\partial q^\nu}\right) - \frac{1}{4}\frac{d}{dt}\left(\frac{\partial \bar{A}_\nu^Z}{\partial \dot{q}^\sigma} - \frac{\partial \bar{A}_\sigma^Z}{\partial \dot{q}^\nu}\right)$$

$$
\begin{aligned}
&- \frac{1}{2}\left(\frac{\partial \bar{C}^Z_{\sigma\tau}}{\partial q^\nu} - \frac{\partial \bar{C}^Z_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau + \frac{1}{4}\frac{d}{dt}\left(-\frac{\partial \bar{C}^Z_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial \bar{C}^Z_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\ddot{q}^\tau \\
&+ \frac{1}{4}\left(-\frac{\partial \bar{C}^Z_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial \bar{C}^Z_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\dddot{q}^\tau + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial D^Z_{\sigma\tau}}{\partial q^\nu} - \frac{\partial D^Z_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau \\
&+ \frac{1}{2}\left(\frac{\partial D^Z_{\sigma\tau}}{\partial q^\nu} - \frac{\partial D^Z_{\nu\tau}}{\partial q^\sigma}\right)\dddot{q}^\tau + \frac{1}{2}\frac{d^3 D^Z_{\sigma\nu}}{dt^3} - \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial D^Z_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D^Z_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\ddot{q}^\tau \\
&- \frac{d}{dt}\left(\frac{\partial D^Z_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D^Z_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\dddot{q}^\tau - \frac{1}{2}\left(\frac{\partial D^Z_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D^Z_{\nu\tau}}{\partial \dot{q}^\sigma}\right)q^\tau_{(4)}, \\
G^Z_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial \bar{A}^Z_\nu}{\partial \dot{q}^\sigma} + \frac{\partial \bar{A}^Z_\sigma}{\partial \dot{q}^\nu}\right) - \left(\frac{\partial D^Z_{\sigma\tau}}{\partial q^\nu} + \frac{\partial D^Z_{\nu\tau}}{\partial q^\sigma}\right)\ddot{q}^\tau \\
&- \frac{1}{2}\frac{d(\bar{C}^Z_{\sigma\nu} + \bar{C}^Z_{\nu\sigma})}{dt} + \frac{1}{2}\left(\frac{\partial \bar{C}^Z_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial \bar{C}^Z_{\sigma\tau}}{\partial \dot{q}^\nu}\right)\ddot{q}^\tau, \\
H^Z_{\sigma\nu} &= -\frac{1}{2}(\bar{C}^Z_{\sigma\nu} - \bar{C}^Z_{\nu\sigma}) - \frac{dD^Z_{\sigma\nu}}{dt} + \left(\frac{\partial D^Z_{\sigma\tau}}{\partial \dot{q}^\nu} - \frac{\partial D^Z_{\nu\tau}}{\partial \dot{q}^\sigma}\right)\ddot{q}^\tau.
\end{aligned}
\tag{3.79}
$$

Proof. The form $\partial_Z \rho$ is given by Lemma 5. Formulas in (3.79) then follow from Theorem 4 and Remark 2. □

We now state a general form of equations for variational contact symmetries.

Theorem 5. *A contact symmetry Z is variational for a dynamical form $\varepsilon = [\rho]$ if and only if*

$$F^Z_{\sigma\nu} = 0, \qquad G^Z_{\sigma\nu} = 0, \qquad H^Z_{\sigma\nu} = 0. \tag{3.80}$$

Proof. This is an immediate consequence of Lemma 7. □

4. Examples

We shall illustrate some of the above results on examples concerning second-order dynamical forms. For more details on geometric structures related with second and higher-order ordinary differential equations, and for different aspects of the inverse variational problem and its connection to closed differential forms, we refer e.g. to Crampin, Prince, and Thompson [2] Krupková [13–15], and Sarlet, Thompson and Prince [19].

4.1. Second-order differential equations

A system of *regular* second-order ordinary differential equations (SODE) for the fibred manifold Y is given by a semispray connection $\Gamma : J^1Y \to J^2Y$, or equivalently, by a semispray distribution of rank one on J^1Y. If (V, χ) is a fibred chart on Y then on the open set $V^1 = (\pi^{1,0})^{-1}V \subset J^1Y$ the distribution is spanned by the vector field (denoted for simplicity by the same symbol)

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} + F^\sigma \frac{\partial}{\partial \dot{q}^\sigma}, \tag{4.1}$$

respectively, annihilated by the 1-forms

$$\omega^\sigma, \quad \dot{\omega}^\sigma_\Gamma = d\dot{q}^\sigma - F^\sigma\, dt, \quad 1 \leqslant \sigma \leqslant m. \tag{4.2}$$

Indeed, integral sections of Γ are solutions of the system of equations

$$\ddot{q}^\sigma = F^\sigma, \quad 1 \leqslant \sigma \leqslant m. \tag{4.3}$$

We say that a semispray connection Γ is *variational*, or equivalently, that Eqs. (4.3) have a *variational multiplier*, $(B_{\sigma\nu})$, if there exists a system of functions $B_{\sigma\nu}$, $1 \leqslant \sigma, \nu \leqslant m$, on V^1 such that the matrix $(B_{\sigma\nu})$ is regular at each point of V^1, and the dynamical form $\varepsilon = B_{\sigma\nu}(\ddot{q}^\nu - F^\nu)\omega^\sigma \wedge dt$ is variational (recall that this means that there exists a Lagrangian λ such that $\varepsilon = E(\lambda)$).

We have on V^1 a dynamical form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt$,

$$\varepsilon_\sigma = \delta_{\sigma\nu}(\ddot{q}^\nu - F^\nu), \tag{4.4}$$

naturally associated to Γ. In notations of Section 3, (3.68),

$$A_\sigma = -\delta_{\sigma\nu}F^\nu, \qquad C_{\sigma\nu} = \delta_{\sigma\nu}, \qquad D_{\sigma\nu} = 0. \tag{4.5}$$

By Theorem 3 and Remark 2, $\varepsilon = [\rho]$, where

$$\rho = \omega^\sigma \wedge \delta_{\sigma\nu}(d\dot{q}^\nu - F^\nu\,dt) = \delta_{\sigma\nu}\omega^\sigma \wedge \dot{\omega}^\nu_\Gamma \tag{4.6}$$

(cf. (3.48)). The Helmholtz class of (4.4) becomes according to (3.69) and (3.70)

$$\begin{aligned}
H(\varepsilon) &= [d\rho] \\
&= -\frac{1}{2}\left(\frac{\partial A_\sigma}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\right)\right)\omega^\sigma \wedge \omega^\nu \wedge dt \\
&\quad + \frac{1}{2}\left(\frac{\partial A_\nu}{\partial \dot{q}^\sigma} + \frac{\partial A_\sigma}{\partial \dot{q}^\nu}\right)\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt \\
&= \frac{1}{2}\delta_{\sigma\tau}\left(\frac{\partial F^\tau}{\partial q^\nu} - \frac{\partial F^\nu}{\partial q^\tau} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial F^\tau}{\partial \dot{q}^\nu} - \frac{\partial F^\nu}{\partial \dot{q}^\tau}\right)\right)\omega^\sigma \wedge \omega^\nu \wedge dt \\
&\quad - \frac{1}{2}\delta_{\sigma\tau}\left(\frac{\partial F^\tau}{\partial \dot{q}^\nu} + \frac{\partial F^\nu}{\partial \dot{q}^\tau}\right)\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt,
\end{aligned} \tag{4.7}$$

since $H_{\sigma\nu} = 0$. Note that condition $H(\varepsilon) = 0$ gives the well-known *variationality conditions for a first-order force* $F = (F^\sigma)$ (Whittaker [22]).

Using Theorem 5 and the preceding lemmas we can write down conditions for a vector field Z on J^1Y to be a *variational vector field* for the dynamical form ε (4.4): From (3.72) we can see that $D^Z_{\sigma\nu} = 0$ and $E^Z_\sigma = 0$, hence $\bar{A}^Z_\sigma = A^Z_\sigma$ and $\bar{C}^Z_{\sigma\nu} = C^Z_{\sigma\nu}$ (cf. (3.77)). Now, we shall consider the cases $\dim Y = 2$ and $\dim Y > 2$ separately.

(1) $\dim Y = 2$ (a single differential equation $\ddot{q} = F$): We have

$$\begin{aligned}
A^Z &= -\partial_Z F - \left(\frac{\partial \zeta}{\partial q} + \frac{\partial \zeta_0}{\partial t}\right)F + \frac{d'}{dt}\left(\frac{d\zeta}{dt} - \dot{q}\frac{d\zeta_0}{dt}\right), \\
C^Z &= 2\frac{\partial(\zeta - \zeta_0\dot{q})}{\partial q} - \frac{\partial \zeta_0}{\partial \dot{q}}F - \frac{d'\zeta_0}{dt},
\end{aligned} \tag{4.8}$$

where we have used the relation

$$\frac{\partial \dot{\zeta}}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial \zeta}{\partial \dot{q}} + \frac{\partial \zeta}{\partial q} - \frac{d\zeta_0}{dt} - \dot{q}\frac{d}{dt}\frac{\partial \zeta_0}{\partial \dot{q}} - \frac{\partial \zeta_0}{\partial q}\dot{q} = \frac{\partial(\zeta - \zeta_0\dot{q})}{\partial q} - \frac{d'\zeta_0}{dt} \tag{4.9}$$

following from (3.57) and (3.56). Hence, *Z is variational if and only if*

$$\frac{\partial \zeta}{\partial \dot{q}} - \frac{\partial \zeta_0}{\partial \dot{q}}\dot{q} = 0, \qquad \dot{\zeta} = \frac{d\zeta}{dt} - \dot{q}\frac{d\zeta_0}{dt}, \tag{4.10}$$

and

$$\frac{\partial A^Z}{\partial \dot{q}} - \frac{d'C^Z}{dt} = 0. \tag{4.11}$$

The corresponding *variational dynamical form* then reads

$$\varepsilon' = \partial_Z \varepsilon = (A^Z + C^Z\ddot{q})\,dq \wedge dt, \tag{4.12}$$

i.e., the obtained *variational equation* takes the form

$$\left(2\frac{\partial(\zeta - \zeta_0\dot{q})}{\partial q} - \frac{\partial \zeta_0}{\partial \dot{q}}F - \frac{d'\zeta_0}{dt}\right)\ddot{q} = \partial_Z F + \left(\frac{\partial \zeta}{\partial q} + \frac{\partial \zeta_0}{\partial t}\right)F - \frac{d'}{dt}\left(\frac{d\zeta}{dt} - \dot{q}\frac{d\zeta_0}{dt}\right). \tag{4.13}$$

It is regular if $C^Z \neq 0$.

(2) $\dim Y \geqslant 3$: We have

$$A^Z_\sigma = -\partial_Z F^\sigma - \frac{\partial(\zeta^\tau - \zeta_0 \dot q^\tau)}{\partial q^\sigma} F^\tau - \frac{d\zeta_0}{dt} F^\sigma + \frac{d'}{dt}\left(\frac{d\zeta^\sigma}{dt} - \dot q^\sigma \frac{d\zeta_0}{dt}\right),$$
$$C^Z_{\sigma\nu} = \frac{\partial(\zeta^\nu - \zeta_0 \dot q^\nu)}{\partial q^\sigma} + \frac{\partial(\zeta^\sigma - \zeta_0 \dot q^\sigma)}{\partial q^\nu} - \frac{d\zeta_0}{dt}\delta_{\sigma\nu} = C^Z_{\nu\sigma}. \tag{4.14}$$

Note that

$$\frac{\partial C^Z_{\sigma\nu}}{\partial \dot q^\tau} = -\left(\frac{\partial \zeta_0}{\partial q^\sigma}\delta_{\nu\tau} + \frac{\partial \zeta_0}{\partial q^\nu}\delta_{\sigma\tau} + \frac{\partial \zeta_0}{\partial q^\tau}\delta_{\sigma\nu}\right) = \frac{\partial C^Z_{\tau\nu}}{\partial \dot q^\sigma}. \tag{4.15}$$

Substituting into (3.78), (3.79), we can see that

$$\begin{aligned}\partial_Z H(\varepsilon) &= H(\partial_Z \varepsilon) = [d\partial_Z \rho] \\ &= \frac{1}{2}\left(\frac{\partial A^Z_\nu}{\partial q^\sigma} - \frac{\partial A^Z_\sigma}{\partial q^\nu} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} - \frac{\partial A^Z_\sigma}{\partial \dot q^\nu}\right) - \left(\frac{\partial C^Z_{\sigma\tau}}{\partial q^\nu} - \frac{\partial C^Z_{\nu\tau}}{\partial q^\sigma}\right)\ddot q^\tau\right)\omega^\sigma \wedge \omega^\nu \wedge dt \\ &\quad + \left(\frac{1}{2}\left(\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} + \frac{\partial A^Z_\sigma}{\partial \dot q^\nu}\right) - \frac{d' C^Z_{\sigma\nu}}{dt}\right)\dot\omega^\sigma \wedge \omega^\nu \wedge dt.\end{aligned} \tag{4.16}$$

This means that Helmholtz conditions for $\partial_Z \varepsilon$ take a simplified form

$$\frac{\partial A^Z_\nu}{\partial q^\sigma} - \frac{\partial A^Z_\sigma}{\partial q^\nu} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} - \frac{\partial A^Z_\sigma}{\partial \dot q^\nu}\right) - \left(\frac{\partial C^Z_{\sigma\tau}}{\partial q^\nu} - \frac{\partial C^Z_{\nu\tau}}{\partial q^\sigma}\right)\ddot q^\tau = 0,$$
$$\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} + \frac{\partial A^Z_\sigma}{\partial \dot q^\nu} - 2\frac{d' C^Z_{\sigma\nu}}{dt} = 0, \tag{4.17}$$

or, equivalently,

$$\frac{\partial A^Z_\nu}{\partial q^\sigma} - \frac{\partial A^Z_\sigma}{\partial q^\nu} - \frac{1}{2}\frac{d'}{dt}\left(\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} - \frac{\partial A^Z_\sigma}{\partial \dot q^\nu}\right) = 0,$$
$$\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} + \frac{\partial A^Z_\sigma}{\partial \dot q^\nu} - 2\frac{d' C^Z_{\sigma\nu}}{dt} = 0,$$
$$\frac{\partial}{\partial \dot q^\tau}\left(\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} - \frac{\partial A^Z_\sigma}{\partial \dot q^\nu}\right) - \left(\frac{\partial C^Z_{\sigma\tau}}{\partial q^\nu} - \frac{\partial C^Z_{\nu\tau}}{\partial q^\sigma}\right) = 0. \tag{4.18}$$

The last condition above is, however, a consequence of the preceding one, and (4.15). Summarising, we get that *a contact symmetry Z is variational if and only if*

$$\frac{\partial A^Z_\nu}{\partial q^\sigma} - \frac{\partial A^Z_\sigma}{\partial q^\nu} - \frac{1}{2}\frac{d'}{dt}\left(\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} - \frac{\partial A^Z_\sigma}{\partial \dot q^\nu}\right) = 0,$$
$$\frac{\partial A^Z_\nu}{\partial \dot q^\sigma} + \frac{\partial A^Z_\sigma}{\partial \dot q^\nu} - 2\frac{d' C^Z_{\sigma\nu}}{dt} = 0. \tag{4.19}$$

The corresponding *variational equations* then take the form

$$A^Z_\sigma + C^Z_{\sigma\nu}\ddot q^\nu = 0, \tag{4.20}$$

and they are regular if the matrix $(C^Z_{\sigma\nu})$ is regular. In this case we can write

$$\ddot q^\sigma = F^{Z\sigma}, \tag{4.21}$$

where $F^Z = -(C^Z)^{-1}A^Z$.

Note that condition $F^Z = F$, i.e., that the transformed dynamical form $\partial_Z \varepsilon$ defines the same semispray connection Γ as ε, means that

$$C^Z F + A^Z = 0. \tag{4.22}$$

4.2. Variational vector fields for non-variational equations

Let $Y = \mathbb{R}^3$, $X = \mathbb{R}$. Let us consider, in the canonical coordinates on $J^2\mathbb{R}^3$, equations

$$\ddot{q}^1 = -\dot{q}^2, \qquad \ddot{q}^2 = -q^2. \tag{4.23}$$

Eqs. (4.23) were studied by Douglas [3] in connection with his analysis of variational multipliers for systems of two second-order ordinary differential equations. He proved that these equations have no variational multiplier (cf. also [1] and [19] for a geometric analysis of the problem).

We shall show that Eqs. (4.23) possess variational vector fields which transform them to variational equations.

Consider the canonical dynamical form $\varepsilon = (\varepsilon_1\omega^1 + \varepsilon_2\omega^2) \wedge dt$ for (4.23), given by

$$\varepsilon_1 = \dot{q}^2 + \ddot{q}^1, \qquad \varepsilon_2 = q^2 + \ddot{q}^2. \tag{4.24}$$

We wish to find vertical contact symmetries Z of the Helmholtz form $H(\varepsilon)$ of ε. Then the Lie derivative $\partial_Z\varepsilon$ will be a variational dynamical form. Since $\dim Y = 3$, vertical contact symmetries on J^2Y are vector fields

$$Z = \zeta^\sigma(t,q)\frac{\partial}{\partial q^\sigma} + \frac{d\zeta^\sigma}{dt}\frac{\partial}{\partial\dot{q}^\sigma} + \frac{d^2\zeta^\sigma}{dt^2}\frac{\partial}{\partial\ddot{q}^\sigma}. \tag{4.25}$$

Substituting into (4.14) we have

$$\begin{aligned}
A_1^Z &= \partial_Z\dot{q}^2 + \frac{\partial\zeta^1}{\partial q^1}\dot{q}^2 + \frac{\partial\zeta^2}{\partial q^1}q^2 + \frac{d'}{dt}\frac{d\zeta^1}{dt}\\
&= \frac{\partial\zeta^2}{\partial t} + \frac{\partial\zeta^2}{\partial q^\nu}\dot{q}^\nu + \frac{\partial\zeta^1}{\partial q^1}\dot{q}^2 + \frac{\partial\zeta^2}{\partial q^1}q^2 + \frac{\partial^2\zeta^1}{\partial t^2} + 2\frac{\partial^2\zeta^1}{\partial t\,\partial q^\nu}\dot{q}^\nu + \frac{\partial^2\zeta^1}{\partial q^\nu\partial q^\tau}\dot{q}^\nu\dot{q}^\tau,\\
A_2^Z &= \partial_Z q^2 + \frac{\partial\zeta^1}{\partial q^2}\dot{q}^2 + \frac{\partial\zeta^2}{\partial q^2}q^2 + \frac{d'}{dt}\frac{d\zeta^2}{dt}\\
&= \zeta^2 + \frac{\partial\zeta^1}{\partial q^2}\dot{q}^2 + \frac{\partial\zeta^2}{\partial q^2}q^2 + \frac{\partial^2\zeta^2}{\partial t^2} + 2\frac{\partial^2\zeta^2}{\partial t\,\partial q^\nu}\dot{q}^\nu + \frac{\partial^2\zeta^2}{\partial q^\nu\partial q^\tau}\dot{q}^\nu\dot{q}^\tau,\\
C_{11}^Z &= 2\frac{\partial\zeta^1}{\partial q^1}, \qquad C_{12}^Z = C_{21}^Z = \frac{\partial\zeta^2}{\partial q^1} + \frac{\partial\zeta^1}{\partial q^2}, \qquad C_{22}^Z = 2\frac{\partial\zeta^2}{\partial q^2}.
\end{aligned} \tag{4.26}$$

Eqs. (4.19) for Z be a symmetry of $H(\varepsilon)$, i.e., a variational vector field, take the form

$$\begin{aligned}
&\frac{\partial A_1^Z}{\partial\dot{q}^1} - \frac{d'C_{11}^Z}{dt} = \frac{\partial\zeta^2}{\partial q^1} = 0,\\
&\frac{\partial A_2^Z}{\partial\dot{q}^2} - \frac{d'C_{22}^Z}{dt} = \frac{\partial\zeta^1}{\partial q^2} = 0,
\end{aligned} \tag{4.27}$$

and (with help of the above)

$$\begin{aligned}
&\frac{\partial A_2^Z}{\partial\dot{q}^1} + \frac{\partial A_1^Z}{\partial\dot{q}^2} - 2\frac{dC_{12}^Z}{dt} = \frac{\partial\zeta^2}{\partial q^2} + \frac{\partial\zeta^1}{\partial q^1} = 0,\\
&\frac{\partial A_2^Z}{\partial q^1} - \frac{\partial A_1^Z}{\partial q^2} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial A_2^Z}{\partial\dot{q}^1} - \frac{\partial A_1^Z}{\partial\dot{q}^2}\right) = \frac{d}{dt}\frac{\partial\zeta^1}{\partial q^1} = 0.
\end{aligned} \tag{4.28}$$

Solving this system we obtain

$$\zeta^1 = A + Cq^1, \qquad \zeta^2 = B - Cq^2, \tag{4.29}$$

where $A = A(t)$, $B = B(t)$ are arbitrary functions, and $C \in \mathbb{R}$. Take for simplicity $A, B = \text{const}$. Then the dynamical form $\partial_Z\varepsilon$, with $Z = J^2\Xi$ defined by (4.29) is given as $\partial_Z\varepsilon = (\tilde{\varepsilon}_1\omega^1 + \tilde{\varepsilon}_2\omega^2) \wedge dt$, where

$$\tilde{\varepsilon}_1 = 2C\ddot{q}^1, \qquad \tilde{\varepsilon}_2 = B - 2Cq^2 - 2C\ddot{q}^2, \tag{4.30}$$

and it is variational ($H(\partial_Z \varepsilon) = 0$).

Note that in the family of vector fields $Z = J^2\Xi$ where Ξ is defined by (4.29) there exist vector fields such that the systems $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\tilde{\varepsilon}_1 = 0$, $\tilde{\varepsilon}_2 = 0$ have common solutions. Indeed, setting $A = \text{const.}$, $B = 0$, we can easily check that the functions $q^1 = at + b$, $q^2 = 0$, in which $a, b \in \mathbb{R}$, verify both the systems.

The same results as above were obtained in [12] by a direct computation of symmetries of the Helmholtz form

$$H(\varepsilon) = \frac{1}{2}(\dot{\omega}^2 \wedge \omega^1 + \dot{\omega}^1 \wedge \omega^2) \wedge dt \tag{4.31}$$

of the dynamical form ε (4.24).

In the paper by Prince [16] a similar and possibly related phenomena was reported: the generic dynamical symmetry of the vector field Γ of (4.1) maps a variational system to another sharing common solutions with the first.

4.3. Variational vector fields for variational equations

We shall consider equations

$$\ddot{q}^1 = \dot{\rho} q^2 - \rho^2 q^1, \qquad \ddot{q}^2 = -\dot{\rho} q^1 - \rho^2 q^2, \tag{4.32}$$

where $\rho = \rho(t)$ is a function, $\dot{\rho} \neq 0$. The canonical dynamical form $\varepsilon = \varepsilon_1 \omega^1 \wedge dt + \varepsilon_2 \omega^2 \wedge dt$, where

$$\varepsilon_1 = \ddot{q}^1 - \dot{\rho} q^2 + \rho^2 q^1, \qquad \varepsilon_2 = \ddot{q}^2 + \dot{\rho} q^1 + \rho^2 q^2, \tag{4.33}$$

is not variational. However, the semispray connection Γ corresponding to Eqs. (4.33) is variational, since, obviously, e.g., the equivalent dynamical form ε' with

$$\varepsilon'_1 = -\ddot{q}^1 + \dot{\rho} q^2 - \rho^2 q^1, \qquad \varepsilon'_2 = \ddot{q}^2 + \dot{\rho} q^1 + \rho^2 q^2, \tag{4.34}$$

has a Lagrangian

$$L = \frac{1}{2}\big((\dot{q}^1)^2 - (\dot{q}^2)^2\big) + \dot{\rho} q^1 q^2 - \frac{1}{2}\rho^2\big((q^1)^2 - (q^2)^2\big) \tag{4.35}$$

(this means that Eqs. (4.32) have a variational multiplier). For the corresponding Helmholtz forms we get the formulas

$$H(\varepsilon) = 2\dot{\rho}\, \omega^1 \wedge \omega^2 \wedge dt, \quad H(\varepsilon') = 0. \tag{4.36}$$

Let us find vertical contact symmetries of the Helmholtz form (4.36). By a direct computation we have

$$\partial_Z H(\varepsilon) = 2\dot{\rho}\left(\frac{\partial \zeta^1}{\partial q^1} + \frac{\partial \zeta^2}{\partial q^2}\right)\omega^1 \wedge \omega^2 \wedge dt = 0. \tag{4.37}$$

Hence, $Z = J^1\Xi$, where

$$\Xi = \zeta^1 \frac{\partial}{\partial q^1} + \zeta^2 \frac{\partial}{\partial q^2}, \qquad \frac{\partial \zeta^1}{\partial q^1} + \frac{\partial \zeta^2}{\partial q^2} = 0. \tag{4.38}$$

We obtain variational dynamical forms $\tilde{\varepsilon} = \partial_{J^2\Xi}\varepsilon$, where

$$\begin{aligned}
\tilde{\varepsilon}_1 &= \partial_{J^2\Xi}\varepsilon_1 + \varepsilon_1 \frac{\partial \zeta^1}{\partial q^1} + \varepsilon_2 \frac{\partial \zeta^2}{\partial q^1} \\
&= 2\frac{\partial \zeta^1}{\partial q^1}\ddot{q}^1 + \left(\frac{\partial \zeta^1}{\partial q^2} + \frac{\partial \zeta^2}{\partial q^1}\right)\ddot{q}^2 + \frac{d}{dt}\frac{\partial \zeta^1}{\partial t} + \dot{q}^1 \frac{d}{dt}\frac{\partial \zeta^1}{\partial q^1} + \dot{q}^2 \frac{d}{dt}\frac{\partial \zeta^1}{\partial q^2} \\
&\quad + \rho^2\left(\zeta^1 + q^1 \frac{\partial \zeta^1}{\partial q^1} + q^2 \frac{\partial \zeta^2}{\partial q^1}\right) - \dot{\rho}\left(\zeta^2 + q^2 \frac{\partial \zeta^1}{\partial q^1} - q^1 \frac{\partial \zeta^2}{\partial q^1}\right), \\
\tilde{\varepsilon}_2 &= \partial_{J^2\Xi}\varepsilon_2 + \varepsilon_1 \frac{\partial \zeta^1}{\partial q^2} + \varepsilon_2 \frac{\partial \zeta^2}{\partial q^2} \\
&= \left(\frac{\partial \zeta^1}{\partial q^2} + \frac{\partial \zeta^2}{\partial q^1}\right)\ddot{q}^1 - 2\frac{\partial \zeta^1}{\partial q^1}\ddot{q}^2 + \frac{d}{dt}\frac{\partial \zeta^2}{\partial t} + \dot{q}^1 \frac{d}{dt}\frac{\partial \zeta^2}{\partial q^1} - \dot{q}^2 \frac{d}{dt}\frac{\partial \zeta^1}{\partial q^1}
\end{aligned}$$

$$+\rho^2\left(\zeta^2 - q^2\frac{\partial\zeta^1}{\partial q^1} + q^1\frac{\partial\zeta^1}{\partial q^2}\right) + \dot\rho\left(\zeta^1 - q^1\frac{\partial\zeta^1}{\partial q^1} - q^2\frac{\partial\zeta^1}{\partial q^2}\right). \tag{4.39}$$

Lagrangians for the transformed dynamical forms can be easily computed using the well-known Tonti formula (see e.g. [6,7])

$$\tilde L = q^1\int_0^1 \tilde\varepsilon_1(t, uq^\nu, u\dot q^\nu, u\ddot q^\nu)\,du + q^2\int_0^1 \tilde\varepsilon_2(t, uq^\nu, u\dot q^\nu, u\ddot q^\nu)\,du. \tag{4.40}$$

In particular, the following vector fields studied in [17] are symmetries of the Helmholtz form $H(\varepsilon)$:

$$\begin{aligned} Z^{(1)} &= J^1\Xi^{(1)} = \frac12 q^1\frac{\partial}{\partial q^1} - \frac12 q^2\frac{\partial}{\partial q^2} + \frac12\dot q^1\frac{\partial}{\partial\dot q^1} - \frac12\dot q^2\frac{\partial}{\partial\dot q^2},\\ Z^{(2)} &= J^1\Xi^{(2)} = \frac12 q^2\frac{\partial}{\partial q^1} + \frac12 q^1\frac{\partial}{\partial q^2} + \frac12\dot q^2\frac{\partial}{\partial\dot q^1} + \frac12\dot q^1\frac{\partial}{\partial\dot q^2}. \end{aligned} \tag{4.41}$$

The corresponding variational dynamical forms are $\tilde\varepsilon^{(1)}$,

$$\tilde\varepsilon_1^{(1)} = \ddot q^1 + \rho^2 q^1, \qquad \tilde\varepsilon_2^{(1)} = -\ddot q^2 - \rho^2 q^2, \tag{4.42}$$

and $\tilde\varepsilon^{(2)}$,

$$\tilde\varepsilon_1^{(2)} = \ddot q^2 + \rho^2 q^2, \qquad \tilde\varepsilon_2^{(2)} = \ddot q^1 + \rho^2 q^1, \tag{4.43}$$

respectively. Notice that both the symmetries $Z^{(1)}$, $Z^{(2)}$ give rise to the same semispray connection, arising from the Lagrangian $\tilde\lambda = \partial_{Z^{(1)}}\lambda$,

$$\tilde L = \frac12\big((\dot q^1)^2 + (\dot q^2)^2\big) - \frac12\rho^2\big((q^1)^2 + (q^2)^2\big). \tag{4.44}$$

It is worth comparing all of this with the way the same ingredients enter into the application of the theory of pseudo-symmetries, as developed in [17]. Eqs. (4.32) are regarded in [17] as representing a non-conservative mechanical system, the conservative part of which is governed by the function $\tilde L$ which is generated here in (4.44). Pseudo-symmetries are defined with respect to a certain 2-form σ which has the given semispray distribution in its kernel. It is shown in [17] that pseudo-symmetries of point type generically produce a Lagrangian for the given system (which may not be known to be variational at the outset). In the case at hand, the 2-form σ is precisely the ρ of (4.6) (up to a sign). The vector fields $Z^{(i)}$ of (4.41) are indeed pseudo-symmetries of point type with respect to this σ, and thus give rise to a Lagrangian. For $Z^{(1)}$, this Lagrangian is the function L of (4.35), $Z^{(2)}$ on the other hand gives a different Lagrangian, namely

$$\hat L = \dot q^1\dot q^2 - \frac12\dot\rho\big((q^1)^2 - (q^2)^2\big) - \rho^2 q^1 q^2. \tag{4.45}$$

It may be of interest to investigate the similarities between both theories in more general terms in the future. Better still, since the theory of pseudo-symmetries was later reformulated and put into a more general form by the introduction of the concept of adjoint symmetries (see e.g. [18]), the above observations suggest a comparative study of variational vector fields and adjoint symmetries.

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