

Linear connections along the tangent bundle projection

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Abstract. An intrinsic characterization is given of the concept of linear connection along the tangent bundle projection $\tau : TM \rightarrow M$. The main observation thereby is that every such connection \mathcal{D} gives rise to a horizontal lift, which is needed to extend the action of the associated covariant derivative operator to tensor fields along τ in a meaningful way. The interplay is discussed between the given \mathcal{D} and various related connections, such as the canonical non-linear connection of the geodesic equations and certain linear connections on the pullback bundle $\tau^*\tau$. This is particularly relevant to understand similarities and differences between various notions of torsion and curvature. I further discuss aspects of variationality and metrizability of a linear \mathcal{D} along τ and let me guide for the selected topics by a very short, old paper of Krupka and Sattarov.

1 Introduction

On the occasion of celebrating a scientist's 65th birthday, it is respectable to look back at the history of the person's involvement in science and it is definitely a good sign if one can easily detect older work which still raises interesting questions or challenges. I recently laid my hands on what is in fact a rather minor contribution of Demeter Krupka [10], also one of the first reprints he gave me personally, and I was astonished to see that, looking at it now, it confronts me with questions I had not thought of before, even though they are directly related to my own research of the past 10 to 15 years.

Section 2 in [10] carries "connections on the tangent bundle" in the title, but is about maps from a tangent bundle TM into the fibre bundle $\Gamma M \rightarrow M$ of linear connections on M , which make the following diagram commutative:

$$\begin{array}{ccc} & & \Gamma M \\ & \nearrow \mathcal{D} & \downarrow \\ TM & \xrightarrow{\tau} & M \end{array}$$

In my opinion, the concept of a ‘connection on a manifold’ has an unambiguous meaning in the literature, and that is not what the above diagram is about. Instead, a much better name for the \mathcal{D} under consideration here is *linear connection along the tangent bundle projection*. The surprising observation for me, however, is that, having been involved in the development of a comprehensive theory of derivations of forms along the tangent bundle projection in [15, 16], and having made use of the calculus along τ in many applications since then, the idea of such a linear connection along τ never came up. Needless to say, other types of connections frequently play a role in my use of the calculus along τ , such as what are called *linear connections on the pullback bundle* $\tau^*\tau : \tau^*TM \rightarrow TM$, so it becomes intriguing to understand the difference or interplay between all such related, but different concepts. What is more, it turns out that (to the best of my knowledge) not much can be found in the literature about linear connections along τ and what is available all seems based on (sometimes rather untidy) ad hoc coordinate constructions, i.e. seems to lack a proper coordinate-free foundation. For example, going back to section 2 of [10] again, if (q, v) is taken as notation for coordinates on TM and $\gamma_{jk}^i(q, v)$ are connection coefficients of \mathcal{D} , the authors state, as though it should be common knowledge, that *geodesics* are curves in M , satisfying the equations

$$\ddot{q}^k + \gamma_{ij}^k(q, \dot{q})\dot{q}^i\dot{q}^j = 0, \quad (1)$$

and that there is a *covariant derivative operator* for tensor fields along τ , “defined in a standard manner”, which in the case of a metric tensor g along τ is given by

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{ij}}{\partial \dot{q}^s} \gamma_{rk}^s \dot{q}^r - g_{im} \gamma_{jk}^m - g_{jm} \gamma_{ik}^m. \quad (2)$$

But is that common knowledge? It seems to me that the term ‘geodesic’ should be used only if there is a clear notion of parallel transport first, leading subsequently to geodesics as auto-parallel curves. This, plus a coordinate-free backing for the “standard manner” in which $g_{ij;k}$ should be defined, I was unable to find in the literature on which the statements in [10] must have been based.

Linear connections along τ were probably introduced for the first time by Hanno Rund, who called them “direction-dependent connections” [21]. Unfortunately, what is probably a comprehensive account of Rund’s involvement in this theory, seems to have appeared only in extra chapters of the Russian translation (by Asanov) of his book on Finsler spaces (see the Math. Review MR0641695 (83i:53097)). So probably, the best source now (for illiterates in Russian) is Appendix A of Asanov’s own book [4]. There, one will find corresponding concepts of torsion, curvature, Bianchi identities, etcetera explained. The torsion tensor, for example, is defined as having components $\gamma_{ij}^k - \gamma_{ji}^k$, as one might expect, and the components of the curvature tensor are given by

$$\left(\frac{\partial \gamma_{ij}^k}{\partial q^l} - \frac{\partial \gamma_{ij}^k}{\partial v^m} \gamma_{rl}^m v^r \right) - \left(\frac{\partial \gamma_{il}^k}{\partial q^j} - \frac{\partial \gamma_{il}^k}{\partial v^m} \gamma_{rj}^m v^r \right) + \gamma_{ml}^k \gamma_{ij}^m - \gamma_{mj}^k \gamma_{il}^m. \quad (3)$$

But although an attempt is made to construct such tensors in an intrinsic way, the result is rather unsatisfactory for several reasons: to begin with, objects which should be regarded

as living along the tangent bundle projection are often subjected to operations (such as an exterior derivative) which act on the full tangent bundle; the result then is usually not a tensor along τ ; this in turn prompts the author to add corrective terms in a rather ad hoc manner, in order to arrive at a quantity with a proper tensorial meaning. Observe for later that such corrections always involve derivatives with respect to the fibre coordinates v^i on TM .

My aim is to shed a refreshing light on all such concepts by making use of the calculus along τ in a systematic way. This will lead to new questions which cannot all be exhaustively discussed in the course of the present paper. In selecting topics for discussion, therefore, I will let the further aspects treated in Krupka's paper [10] be my guidance.

It is unfortunate that the literature is full of rather strange terminology for things which are (closely or not) related to the topics under consideration here. The point is that in all such cases, the issue is about tensor fields along τ and operations on them, which have not been properly identified or recognized as such. In [2], for example, it is mentioned that what I would call objects along τ are sometimes called d -objects, or M -objects, or even Finsler objects (though they have nothing whatsoever to do with Finsler spaces). I dare hope that the present paper can inspire to more unification in this terminology as well.

2 Elements of the calculus of forms along τ

Let $\mathcal{X}(\tau)$ denote the $C^\infty(TM)$ -module of vector fields along τ , which are maps fitting in a similar commutative diagram as above, with TM replacing ΓM , or equivalently are sections of the pullback bundle $\tau^*\tau : \tau^*TM \rightarrow TM$. Likewise, $\bigwedge^k(\tau)$ and $V^k(\tau)$ will refer to scalar and vector-valued k -forms along τ respectively.

In coordinates, elements $X \in \mathcal{X}(\tau)$ and $\alpha \in \bigwedge^1(\tau)$ are of the form,

$$X = X^i(q, v) \frac{\partial}{\partial q^i}, \quad \alpha = \alpha_i(q, v) dq^i, \quad (4)$$

while, more generally, an element $L \in V^\ell(\tau)$ is of the form

$$L = \lambda^i \otimes \frac{\partial}{\partial q^i} \quad \text{with} \quad \lambda^i = \lambda_{j_1 \dots j_\ell}^i dq^{j_1} \wedge \dots \wedge dq^{j_\ell} \in \bigwedge^\ell(\tau), \quad (5)$$

where the $\lambda_{j_1 \dots j_\ell}^i$ again are functions on TM .

Definition: $D : \bigwedge(\tau) \rightarrow \bigwedge(\tau)$ is a derivation of degree r if

1. $D(\bigwedge^p(\tau)) \subset \bigwedge^{p+r}(\tau)$
2. $D(\alpha + \lambda\beta) = D\alpha + \lambda D\beta, \quad \lambda \in \mathbb{R}$
3. $D(\alpha \wedge \gamma) = D\alpha \wedge \gamma + (-1)^{pr} \alpha \wedge D\gamma, \quad \alpha \in \bigwedge^p(\tau).$

A derivation D of degree r of $V(\tau)$ has an associated derivation of $\bigwedge(\tau)$, also denoted by D , such that in addition to the above rules: for $L \in V^\ell(\tau)$ and $\omega \in \bigwedge^p(\tau)$,

$$D(\omega \wedge L) = D\omega \wedge L + (-1)^{pr}\omega \wedge DL.$$

For practical purposes, it is of interest to know that every D of $\bigwedge(\tau)$ is completely determined by its action on functions on TM and on basic 1-forms, i.e. 1-forms on M regarded as 1-forms along τ by composition with τ . For an extension to $V(\tau)$, it suffices to specify a consistent action on basic vector fields (vector fields on M).

The commutator of D_1 and D_2 (of degree r_1 and r_2 respectively) is the degree $r_1 + r_2$ derivation, defined by

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1, \quad (6)$$

and satisfies a graded Jacobi identity.

There is a canonically defined vertical exterior derivative d^V on $V(\tau)$. In the light of what was said above, it suffices to know that

$$d^V F = V_i(F) dq^i, \quad \text{with } V_i = \frac{\partial}{\partial v^i}, \quad \forall F \in C^\infty(TM), \quad (7)$$

$$d^V \alpha = 0 \quad \text{for } \alpha \in \bigwedge^1(M), \quad d^V \left(\frac{\partial}{\partial q^i} \right) = 0. \quad (8)$$

The classification of derivations of forms along τ and many related issues were discussed in great detail in [15, 16]. I shall limit myself here to recalling the essentials of this theory which will be needed further on. The first point to observe is that a classification requires the availability of a (non-linear or Ehresmann) connection on $\tau : TM \rightarrow M$. As a matter of fact, any choice of a connection, giving rise to a local basis of horizontal vector fields

$$H_i = \frac{\partial}{\partial q^i} - \Gamma_i^j(q, v) \frac{\partial}{\partial v^j}, \quad (9)$$

allows to construct a horizontal exterior derivative d^H as follows:

$$d^H F = H_i(F) dq^i, \quad F \in C^\infty(TM), \quad (10)$$

$$d^H \alpha = d\alpha \quad \text{for } \alpha \in \bigwedge^1(M), \quad d^H \left(\frac{\partial}{\partial q^i} \right) = V_i(\Gamma_j^k) dq^j \otimes \frac{\partial}{\partial q^k}. \quad (11)$$

Inspired by the standard Frölicher and Nijenhuis theory of derivations of (scalar) forms [8], one is then led to distinguish four types of derivations.

- Type i_\star derivations are those which vanish on functions; they are determined by some $L \in V(\tau)$, written as i_L , and defined exactly as in the standard theory. That is to say, for $L \in V^r(\tau)$, $\alpha \in \bigwedge^1(\tau)$,

$$i_L \alpha(X_1, \dots, X_r) = \alpha(L(X_1, \dots, X_r)),$$

which extends to a derivation of degree $r - 1$ on scalar forms, and is taken to be zero also on basic vector fields.

- Type d_\star^V derivations are those of the form $d_L^V = [i_L, d^V]$ for some L .
- Likewise, type d_\star^H derivations are those of the form $d_L^H = [i_L, d^H]$.
- Finally, the extension to vector-valued forms requires an extra class of derivations, said to be of type a_\star . By definition, these vanish on $\bigwedge(\tau)$, they are denoted as a_Q for some $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$ and (in view of what has been said before) are further completely determined by the following action on $X \in \mathcal{X}(\tau)$:

$$a_Q X(X_1, \dots, X_r) = Q(X_1, \dots, X_r)(X). \quad (12)$$

The classification theorem proved in [15] states that every derivation of $V(\tau)$ has a unique representation as the sum of one of each of the above four types of derivations.

The *torsion* T and *curvature* R of the non-linear connection we started from, make their appearance within this theory as vector-valued 2-forms along τ (as opposed to vertical-vector-valued semi-basic forms on TM in other approaches, for example). In fact, T and R are uniquely determined by the following commutators on $\bigwedge(\tau)$ (extra terms of type a_\star come in when the same commutators are regarded as derivations on $V(\tau)$):

$$[d^H, d^V] = d_T^V, \quad \frac{1}{2}[d^H, d^H] = -i_{d^V R} + d_R^V. \quad (13)$$

In coordinates,

$$T = \frac{1}{2} T_{ij}^k dq^i \wedge dq^j \otimes \frac{\partial}{\partial q^k}, \quad T_{ij}^k = V_j(\Gamma_i^k) - V_i(\Gamma_j^k), \quad (14)$$

$$R = \frac{1}{2} R_{ij}^k dq^i \wedge dq^j \otimes \frac{\partial}{\partial q^k}, \quad R_{ij}^k = H_j(\Gamma_i^k) - H_i(\Gamma_j^k). \quad (15)$$

A concise formulation of Bianchi identities then is obtained as follows:

$$d^H R = 0, \quad d^H T + d^V R = 0. \quad (16)$$

The connection we are using to develop these ideas of course also provides ways to pass from objects along τ to objects on the full tangent bundle and vice versa. This really works both ways: for example, if $X = X^i(q, v)\partial/\partial q^i$ is any vector field along τ , we have vertical and horizontal lifts to vector fields on TM , which in coordinates are given by

$$X^V = X^i V_i, \quad X^H = X^i H_i, \quad (17)$$

but conversely, every vector field on TM has a unique decomposition into a horizontal and vertical part and this may reveal new interesting objects along τ . To see this interplay at work, consider the brackets of horizontal and vertical lifts on TM . We have

$$[X^V, Y^V] = ([X, Y]_V)^V, \quad (18)$$

$$[X^H, Y^V] = (D_X^H Y)^V - (D_Y^V X)^H, \quad (19)$$

$$[X^H, Y^H] = ([X, Y]_H)^H + R(X, Y)^V. \quad (20)$$

We see, for example, that the decomposition of the bracket of a horizontal and a vertical lift inevitably leads to the identification of two important derivations of degree zero, the *horizontal* and *vertical covariant derivative*: D_X^V and D_X^H depend linearly on their vector field argument, are determined by the following action on functions $F \in C^\infty(TM)$ and coordinate vector fields

$$D_X^V F = X^V(F), \quad D_X^V \frac{\partial}{\partial q^i} = 0, \quad (21)$$

$$D_X^H F = X^H(F), \quad D_X^H \frac{\partial}{\partial q^i} = X^j V_i(\Gamma_j^k) \frac{\partial}{\partial q^k}, \quad (22)$$

extend to 1-forms along τ by the duality rule

$$D\langle X, \alpha \rangle = \langle DX, \alpha \rangle + \langle X, D\alpha \rangle, \quad (23)$$

and then further to arbitrary tensor fields along τ in the usual way. The other two brackets above identify the curvature tensor R again, plus *horizontal* and *vertical brackets* of vector fields along τ , which are given by

$$[X, Y]_V = (X^V(Y^i) - Y^V(X^i)) \frac{\partial}{\partial q^i}, \quad [X, Y]_H = (X^H(Y^i) - Y^H(X^i)) \frac{\partial}{\partial q^i}. \quad (24)$$

Note that the vertical bracket satisfies a Jacobi identity, but the horizontal one doesn't, unless R is zero.

The covariant derivative operators D^V and D^H in turn define a linear connection on the pull-back bundle $\tau^*TM \rightarrow TM$, as follows: every $\xi \in \mathcal{X}(TM)$ has its unique decomposition in the form $\xi = X^H + Y^V$, with $X, Y \in \mathcal{X}(\tau)$, define $\nabla_\xi : \mathcal{X}(\tau) \rightarrow \mathcal{X}(\tau)$ by (see [14])

$$\nabla_\xi = D_X^H + D_Y^V. \quad (25)$$

∇_ξ is said to be a *connection of Berwald type*. For more insight in the geometric features of such connections, see [7].

Since ∇_ξ acts on vector fields along τ , one may raise the question: should this be called a *linear connection along τ* ? As explained in the introduction, however, it is only after looking back at Krupka's old paper [10], that I realized that there is something else which corresponds better to this terminology, although there should be links with what has just been recalled. Before entering into the subtleties of this discussion, I need to say a few words about the special case that the non-linear connection which has been used so far, is the canonical one associated to a second-order equation field (SODE). To keep it well distinguished from the general case, I will do this in a separate section.

3 The case of a SODE connection

As is well known, the tangent bundle TM of a manifold comes equipped with an intrinsically defined type (1,1) tensor field S , usually called the *vertical endomorphism*, which

has the coordinate expression $S = dq^i \otimes \partial/\partial v^i$. A SODE Γ is characterized by the fact that $S(\Gamma)$ is the dilation vector field $\Delta = v^i \partial/\partial v^i$. It has the property $(\mathcal{L}_\Gamma S)^2 = I$ (the identity tensor), from which it follows that

$$P_H = \frac{1}{2}(I - \mathcal{L}_\Gamma S), \quad P_V = \frac{1}{2}(I + \mathcal{L}_\Gamma S)$$

are complementary projection operators and thus define a non-linear connection. If Γ has the coordinate representation

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v^i}, \quad (26)$$

the connection coefficients of the SODE connection are given by

$$\Gamma_i^j = -\frac{1}{2} \frac{\partial f^j}{\partial v^i}. \quad (27)$$

A SODE connection is characterized by the fact that it has zero torsion T .

Note that there exists a canonical vector field along τ , namely the identity map on TM , which will be denoted by

$$\mathbf{T} = v^i \frac{\partial}{\partial q^i}. \quad (28)$$

It is of some interest to point out that for any non-linear connection, \mathbf{T}^H is a SODE, let us call it the associated SODE, but if the connection we start from is a SODE connection, its associated \mathbf{T}^H will in general not coincide with the original Γ , as is obvious from the coordinate expressions.

In addition to the machinery developed for arbitrary connections, the case of a SODE connection has two very important extra tools to offer: one is the *dynamical covariant derivative* ∇ , which is a degree 0 derivation, the other is a (1,1) tensor $\Phi \in V^1(\tau)$, called the *Jacobi endomorphism*. They are forced upon us, for example, via the same sort of interplay between the calculus along τ and standard calculus on TM , by looking at the decomposition of $\mathcal{L}_\Gamma X^H$. Indeed, it turns out that the vertical part in this decomposition depends tensorially on X , while the horizontal part identifies a derivation. So, we can write,

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V, \quad (29)$$

which defines Φ and ∇ on $\mathcal{X}(\tau)$. ∇ is further determined by the duality rule (23) and the fact that $\nabla F = \Gamma(F)$ on functions $F \in C^\infty(TM)$. For computational purposes:

$$\nabla \left(\frac{\partial}{\partial q^i} \right) = \Gamma_i^k \frac{\partial}{\partial q^k}, \quad \nabla(dq^i) = -\Gamma_k^i dq^k, \quad (30)$$

and

$$\Phi = \Phi_j^i dq^j \otimes \frac{\partial}{\partial q^i}, \quad \text{with} \quad \Phi_j^i = -\frac{\partial f^i}{\partial q^j} - \Gamma_j^k \Gamma_k^i - \Gamma(\Gamma_j^i). \quad (31)$$

The relevance of Φ and ∇ is already obvious from the properties:

$$d^V \Phi = 3R, \quad d^H \Phi = \nabla R. \quad (32)$$

Since variationality will be one of the topics under discussion later on, I conclude this section with the very concise formulation of the so-called Helmholtz conditions within this geometric approach: the necessary and sufficient conditions for the existence of a (regular) Lagrangian formulation of a given SODE Γ are the existence of a non-degenerate, symmetric type (0,2) tensor field g along τ , satisfying the requirements [16]:

$$\nabla g = 0, \quad (33)$$

$$D_X^v g(Y, Z) = D_Z^v g(Y, X), \quad (34)$$

$$g(\Phi X, Y) = g(\Phi Y, X). \quad (35)$$

To close the sections about the main ingredients of the calculus along τ and its relevance in the study of second-order dynamics, I should say that the calculus along τ is being used systematically also in Szilasi's magnum opus [23].

4 Linear connections along τ

Let us go back now to Rund's direction-dependent connections, i.e. maps $\mathcal{D} : TM \rightarrow \Gamma M$ fitting in the commutative diagram of the introduction. My first aim is to find a coordinate-free justification for the equations (1), as geodesic equations, and for (2) as defining relation of the covariant derivative of a metric along τ .

The fibre of ΓM at $m \in M$ consist of maps

$$D : T_m M \times \mathcal{X}_m \rightarrow T_m M,$$

where \mathcal{X}_m denotes the module of vector fields on M defined in a neighbourhood of m , which have the properties

$$\begin{aligned} D_{\lambda v_m} Y &= \lambda D_{v_m} Y, \quad \lambda \in \mathbb{R} \\ D_{v_m}(fY) &= f(m)D_{v_m} Y + v_m(f)Y(m), \quad f \in C^\infty(M) \end{aligned}$$

plus linearity with respect to the sum in both arguments. So for each $w_m \in T_m M$, $\mathcal{D}(w_m)$ is such a map, and is locally defined by functions γ_{ij}^k on TM , such that

$$\mathcal{D}(w_m) \left. \frac{\partial}{\partial q^i} \right|_m = \gamma_{ij}^k(w_m) \left. \frac{\partial}{\partial q^k} \right|_m. \quad (36)$$

For an alternative view, given a \mathcal{D} in the above sense, define for all $X \in \mathcal{X}(\tau)$, and $Y \in \mathcal{X}(M)$, a map

$$D : \mathcal{X}(\tau) \times \mathcal{X}(M) \rightarrow \mathcal{X}(\tau),$$

by

$$(D_X Y)(w_m) = \mathcal{D}(w_m)_{X w_m} Y. \quad (37)$$

By construction, $D_X Y$ will be \mathbb{R} -linear in both arguments and further satisfies

$$D_{FX} Y = F D_X Y, \quad \forall F \in C^\infty(TM) \quad (38)$$

$$D_X(fY) = f D_X Y + X(f)Y, \quad \forall f \in C^\infty(M). \quad (39)$$

Proposition 1: A linear connection \mathcal{D} along τ is a map $D : \mathcal{X}(\tau) \times \mathcal{X}(M) \rightarrow \mathcal{X}(\tau)$ which is \mathbb{R} -linear in both arguments and has the properties (38, 39).

PROOF: Indeed, conversely, using the standard trick with a bump function, the above properties imply that the value of $D_X Y$ at a point $w_m \in T_m M$ only depends on the value of X at w_m . As a result, it makes sense to define a map $\mathcal{D} : TM \rightarrow \Gamma M$ by

$$\mathcal{D}(w_m)_{v_m} Y = (D_X Y)(w_m),$$

where X is any vector field along τ such that $X_{w_m} = v_m$, and this provides a linear connection along τ , in the sense of the commutative diagram we started from. \square

To extend D further as covariant derivative operator, we need to extend the second argument to $\mathcal{X}(\tau)$, which in turn requires some notion of horizontal lift for the action on functions on TM .

Given a curve $\sigma : t \mapsto q^i(t)$ in M , take a vector field along σ , i.e. a curve $\eta : t \mapsto (q^i(t), \eta^i(t))$ in TM which projects onto σ , and define another vector field $D_{\dot{\sigma}} \eta$ along σ by

$$(D_{\dot{\sigma}} \eta)(t) = \mathcal{D}(\eta(t))_{\dot{\sigma}(t)} Y. \quad (40)$$

Here, Y is any vector field, defined in a neighbourhood of $\sigma(t)$, such that $Y(\sigma(t)) = \eta(t)$. In coordinates:

$$D_{\dot{\sigma}} \eta(t) = (\dot{\eta}^k(t) + \gamma_{ij}^k(\eta(t)) \dot{q}^i(t) \eta^j(t)) \frac{\partial}{\partial q^k} \Big|_{\sigma(t)}, \quad (41)$$

where

$$\dot{\sigma}(t) = (q(t), \dot{q}(t)) \quad \text{and} \quad \dot{\eta}^k(t) = \frac{\partial Y^k}{\partial q^i}(q(t)) \dot{q}^i(t).$$

We can now come to a notion of parallel transport in the usual way: η is said to be *parallel* along σ if $D_{\dot{\sigma}} \eta(t) = 0$ for all t . As in the standard theory, for a given curve σ in M and an arbitrary point v in the fibre of $\sigma(t_0)$ say, there is a unique η along σ , which passes through v and is parallel. This η is called the *horizontal lift* of σ (through v): $\eta = \sigma^h$. Note, however, that the differential equations to be solved for the η^i are non-linear here.

Definition: A curve γ in M is said to be a *geodesic* of the linear connection \mathcal{D} along τ if $\gamma^h = \dot{\gamma}$, i.e.

$$D_{\dot{\gamma}} \dot{\gamma} = 0. \quad (42)$$

From (41), it is clear that in coordinates, $t \mapsto q^i(t)$ is a geodesic if it satisfies the SODE equations

$$\ddot{q}^k + \gamma_{ij}^k(q, \dot{q}) \dot{q}^i \dot{q}^j = 0,$$

which are indeed the equations (1). This resolves our first query.

In order to obtain a covariant derivative operator, acting on tensorial objects along τ , it suffices now to extend the horizontal lift construction to vector fields.

Definition: The *horizontal lift* of $X \in \mathcal{X}(M)$ is the vector field X^h on TM , which projects onto X and is further determined by the requirement that its integral curves are horizontal lifts of integral curves of X .

It follows that, in coordinates:

$$X^h = X^i(q) \left(\frac{\partial}{\partial q^i} - \gamma_{ij}^k(q, v) v^j \frac{\partial}{\partial v^k} \right) \stackrel{put}{=} X^i h_i, \quad (43)$$

and this horizontal lift naturally extends to $\mathcal{X}(\tau)$.

We can then, in the first place, extend the action of D to an operator

$$D : \mathcal{X}(\tau) \times \mathcal{X}(\tau) \rightarrow \mathcal{X}(\tau), \quad (44)$$

by putting

$$D_X F = X^h(F) \quad X \in \mathcal{X}(\tau), \quad F \in C^\infty(TM), \quad (45)$$

and

$$D_X(FY) = F D_X Y + X^h(F) Y \quad Y \in \mathcal{X}(M), \quad F \in C^\infty(TM). \quad (46)$$

Finally, D_X further extends to 1-forms along τ by duality, and subsequently to arbitrary tensor fields along τ . In particular, for $g \in \mathcal{T}_2^0(\tau)$, an intrinsic definition of $D_X g$ becomes:

$$(D_X g)(Y, Z) = X^h(g(Y, Z)) - g(D_X Y, Z) - g(Y, D_X Z). \quad (47)$$

This justifies the covariant derivative formula (2) of Rund (as found in [4] and [10], for example), except for a difference in convention! Indeed, the coordinate expression of (47) reads

$$g_{ij|k} = (D_{\partial/\partial q^k} g)_{ij} = \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{ij}}{\partial v^s} \gamma_{kr}^s v^r - \gamma_{ki}^m g_{jm} - \gamma_{kj}^m g_{im}, \quad (48)$$

and has a different order for the bottom indices of the connection coefficients, an issue of course which depends on the convention adopted at the very start, namely with the defining relations (36).

5 A variety of related connections

We have seen in the previous section that a linear connection \mathcal{D} along τ comes with an associated notion of horizontal lift, as it should, and defines a SODE, namely the equations for geodesics. But that inevitably means that there are a number of other connections around. Studying the relationship between those connections is a somewhat slippery domain, because it is easy to get dragged away into identifying all sorts of tensors, which are perhaps only marginally interesting. I shall try to limit myself here to what seem to me the bare essentials of such a discussion.

The horizontal lift (43) was indispensable to arrive at the covariant derivative operator D_X . It defines at the same time a non-linear connection, however, with connection coefficients

$${}^h\Gamma_i^k = \gamma_{ij}^k v^j, \quad (49)$$

and that in turn, in agreement with the general formulas (22), defines a horizontal covariant derivative D_X^h . Since D_X and D_X^h agree on functions, their difference, when acting on

some $Y \in \mathcal{X}(\tau)$, depends tensorially on Y . Hence, every linear connection \mathcal{D} along τ has an associated type (1,2) tensor field along τ , which I shall denote by \mathfrak{K} , and seems not to have been noticed or highlighted before in the literature.

Definition: *The fundamental tensor field \mathfrak{K} of a linear connection \mathcal{D} along τ is the type (1,2) tensor along τ determined by*

$$\mathfrak{K}(X, Y) = D_X Y - D_X^h Y. \quad (50)$$

In coordinates

$$\mathfrak{K} = \mathfrak{K}_{ij}^k dq^i \otimes dq^j \otimes \frac{\partial}{\partial q^k}, \quad \mathfrak{K}_{ij}^k = -\frac{\partial \gamma_{il}^k}{\partial v^j} v^l = -\frac{\partial}{\partial v^j} ({}^h \Gamma_i^k) + \gamma_{ij}^k. \quad (51)$$

In terms of the different types of derivations, discussed in section 2, we can write that $D_X = D_X^h + a_{X \lrcorner \mathfrak{K}}$ for the action on $\mathcal{X}(\tau)$ (with $X \lrcorner \mathfrak{K}(Y) = \mathfrak{K}(X, Y)$). By the duality rule (23), this implies that for the action on 1-forms $\alpha \in \Lambda^1(\tau)$, we will have $D_X \alpha = D_X^h \alpha - i_{X \lrcorner \mathfrak{K}} \alpha$. It follows that for the action on arbitrary tensor fields,

$$D_X = D_X^h + \mu_{X \lrcorner \mathfrak{K}}, \quad (52)$$

where for an arbitrary (1,1) tensor A , $\mu_A = a_A - i_A$ (see [16]).

There is, of course, also the SODE connection associated with the geodesic equations (1) coming from \mathcal{D} . Its connection coefficients, according to (27), are given by

$${}^H \Gamma_i^k = \frac{1}{2}(\gamma_{il}^k + \gamma_{li}^k) v^l + \frac{1}{2} \frac{\partial \gamma_{lm}^k}{\partial v^i} v^l v^m, \quad (53)$$

and it has its own horizontal covariant derivative D_X^H . From now on, I shall systematically use superscripts h (or subscripts $_h$) for everything that relates to the derivative D^h , while H (or $_H$) will refer to the canonical connection of the geodesic SODE Γ . The difference between the two horizontal distributions determines a type (1,1) tensor, for which I will not introduce a separate notation because it is derived from more fundamental tensors; its components are given by

$${}^H \Gamma_i^k - {}^h \Gamma_i^k = \frac{1}{2}(\gamma_{li}^k - \gamma_{il}^k) v^l + \frac{1}{2} \frac{\partial \gamma_{lm}^k}{\partial v^i} v^l v^m. \quad (54)$$

The second term on the right is easily seen to be $-\frac{1}{2}(\mathbf{T} \lrcorner \mathfrak{K})_i^k$, while the first term comes from the torsion tensor of \mathcal{D} . Indeed, as will be seen in the next section, one can give an intrinsic definition of the concept of torsion of \mathcal{D} , which then is found to have the coordinate representation

$${}^{\mathcal{D}} T = \frac{1}{2} {}^{\mathcal{D}} T_{ij}^k dq^i \wedge dq^j \otimes \frac{\partial}{\partial q^k}, \quad {}^{\mathcal{D}} T_{ij}^k = \gamma_{ij}^k - \gamma_{ji}^k. \quad (55)$$

So the first term on the right in (54) is $\frac{1}{2}(i_{\mathbf{T}} {}^{\mathcal{D}} T)_i^k$.

The subtle interplay between the different connections which are around can be seen from the following properties. From the general discussion in section 3, it should not come as

a surprise that \mathbf{T}^H is not the geodesic SODE Γ from which the horizontal lift H is derived. But we do have that $\mathbf{T}^h = \Gamma$. Recall further that an important degree zero derivation associated to a SODE is the dynamical covariant derivative ∇ . But by choosing the vector field X in the different covariant derivative operators we have so far considered to be the canonical \mathbf{T} , there are, so to speak, three more degree zero derivations available which should have some similarity to ∇ , namely $D_{\mathbf{T}}$, $D_{\mathbf{T}}^h$ and $D_{\mathbf{T}}^H$. The last one is quite different from the others, in general. Since $\mathbf{T}^h = \Gamma$, ∇ , $D_{\mathbf{T}}$ and $D_{\mathbf{T}}^h$ on the other hand all coincide on functions. Yet, they are not the same since, for example,

$$\nabla \frac{\partial}{\partial q^i} = {}^H\Gamma_i^k \frac{\partial}{\partial q^k}, \quad D_{\mathbf{T}} \frac{\partial}{\partial q^i} = v^l \gamma_{li}^k \frac{\partial}{\partial q^k}, \quad (56)$$

while, in accordance with (52), the difference between $D_{\mathbf{T}}$ and $D_{\mathbf{T}}^h$ is determined by $\mathbf{T} \lrcorner \mathfrak{K}$. An interesting property of the linear \mathcal{D} along τ is that

$$D_X \mathbf{T} = 0, \quad \forall X \in \mathcal{X}(\tau). \quad (57)$$

This can easily be verified in coordinates. It follows that

$$D_X^h \mathbf{T} = -\mathfrak{K}(X, \mathbf{T}). \quad (58)$$

In particular, we have $D_{\mathbf{T}} \mathbf{T} = 0$, a property which is in general not shared by the related derivations ∇ , $D_{\mathbf{T}}^h$ and $D_{\mathbf{T}}^H$.

A different aspect which needs some inspection in this section is the relationship between linear connections along τ and linear connections on the pullback bundle $\tau^*\tau : \tau^*TM \rightarrow TM$. This relationship is not a very strict one a priori as, again, there are many different elements one can bring into the picture. But I shall try to develop arguments which bring us to a kind of natural correspondence in the end. At the start of such a discussion, however, one has to make a clear distinction between the defining requirements of a linear connection along τ , as expressed for example by Proposition 1, and the extension to a map $D : \mathcal{X}(\tau) \times \mathcal{X}(\tau) \rightarrow \mathcal{X}(\tau)$ which was needed to arrive at a covariant derivative operator on tensors along τ .

The most immediate association between both concepts one can think of, goes as follows. Let \mathcal{D} be a linear connection along τ , locally determined by

$$D_{\partial/\partial q^i} \frac{\partial}{\partial q^j} = \gamma_{ij}^k(q, v) \frac{\partial}{\partial q^k}.$$

Define for each $\xi \in \mathcal{X}(TM)$ a map $\nabla_\xi : \mathcal{X}(\tau) \rightarrow \mathcal{X}(\tau)$ by

$$\nabla_{\partial/\partial q^i} \frac{\partial}{\partial q^j} = \gamma_{ij}^k \frac{\partial}{\partial q^k}, \quad \nabla_{\partial/\partial v^i} \frac{\partial}{\partial q^j} = 0, \quad \nabla_{F\xi} = F\nabla_\xi, \quad (59)$$

$$\nabla_\xi(FX) = F\nabla_\xi X + \xi(F)X, \quad F \in C^\infty(TM), \quad X \in \mathcal{X}(\tau). \quad (60)$$

It is easy to see that ∇ satisfies the requirements for a linear connection on $\tau^*\tau$, but this association calls for more intrinsic procedures and insights.

It seems to me, however, that in view of what precedes, it is appropriate to bring first the availability of an extra horizontal distribution in the discussion. This way, we can make a link also with yet another construction in the literature. Indeed, a pair (∇_ξ, P_H) , consisting of a linear connection on $\tau^*\tau$ and a horizontal projector on TM is essentially (possibly after identification of the pullback bundle with the bundle of vertical tangent vectors to TM) what is called a ‘‘Finsler connection’’ by Matsumoto [17] and in Bejancu’s book [5], for example, and indeed in many other sources.

Suppose we have such a pair (∇_ξ, P_H) , where P_H is general here (i.e. not necessarily the SODE connection of the geodesics), so that every ξ has its decomposition $\xi = X^H + Y^V$ for some $X, Y \in \mathcal{X}(\tau)$. Then, for each $X \in \mathcal{X}(\tau)$ we can define $D_X : \mathcal{X}(M) \rightarrow \mathcal{X}(\tau)$ as

$$D_X = \nabla_{X^H}|_{\mathcal{X}(M)}.$$

It is clear that D_X has all the right properties (Proposition 1), and if we put

$$\nabla_{\partial/\partial q^i} \frac{\partial}{\partial q^j} = \Gamma_{ij}^k \frac{\partial}{\partial q^k},$$

the connection coefficients of \mathcal{D} are $\gamma_{ij}^k = \Gamma_{ij}^k$. Observe, however, that the associated h -lift of \mathcal{D} is not the H we started from. Note also that $\nabla_{X^V}|_{\mathcal{X}(M)}$ defines a tensorial object which we disregard in this construction.

Conversely, suppose the data are a linear \mathcal{D} and an arbitrary horizontal lift H (not necessarily the h associated to \mathcal{D}). Then, a corresponding ∇_ξ can be constructed as follows: for $X \in \mathcal{X}(\tau)$, $Z \in \mathcal{X}(M)$ and $F \in C^\infty(TM)$, put

$$\begin{aligned} \nabla_{X^H} Z &= D_X Z, & \nabla_{X^V} Z &= 0, \\ \nabla_\xi(FZ) &= F \nabla_\xi Z + \xi(F) Z. \end{aligned}$$

The property $\nabla_{X^V} Z = 0$ expresses that ∇ acts on any fibre of TM by so-called complete parallelism (see [7]). The element of caution to mention here is that $\nabla_{X^H}(FZ) \neq D_X(FZ)$.

Now, it is clear that in both of the directions of the above construction, one of the data in fact can imply the availability of the other, so let us finally reduce the data again in that sense. This means that in the first construction, we now assume that only a horizontal projector is given. Then, there is a naturally associated linear connection on $\tau^*\tau$, namely the Berwald type connection (25). The linear \mathcal{D} along τ which then follows is determined by

$$D_X = D_X^H|_{\mathcal{X}(M)},$$

and if Γ_{ij}^k are the connection coefficients of the given non-linear connection, we have: $\gamma_{ij}^k = \partial\Gamma_{ij}^k/\partial v^j$. Again, one has to be careful, because the extension of this D_X to $\mathcal{X}(\tau)$, as discussed in the previous section is not the D_X^H we started from.

For the converse construction, it suffices to have \mathcal{D} as only data, because \mathcal{D} comes with its own horizontal lift h as discussed before. The above general construction did not depend on the choice of a horizontal lift anyway, so we can carry it out just as well with h . The connection coefficients of ∇_ξ are given by: $\Gamma_{ij}^k = \gamma_{ij}^k$ (and zero), i.e. we recover the ‘direct

association' we mentioned at the beginning, but in a more elegant way. This time, we do have that $\nabla_{X^h}(FZ) = D_X(FZ)$ for $F \in C^\infty(TM)$. However, the resulting ∇_ξ is generally not of Berwald type, which is essentially due to the fact that $D_X \neq D_X^h$, i.e. to the fundamental tensor \mathfrak{K} . Nevertheless, this association between \mathcal{D} and ∇_ξ is the most relevant point in our discussion, and we formalize it, therefore, in the following statement.

Proposition 2: *Let \mathcal{D} be a linear connection along τ , then there is a linear connection ∇ on $\tau^*\tau$, which is uniquely determined by the following prescriptions: for $X \in \mathcal{X}(\tau)$, $Z \in \mathcal{X}(M)$ and $F \in C^\infty(TM)$,*

$$\nabla_{X^h}Z = D_XZ, \quad \nabla_{X^v}Z = 0, \quad (61)$$

$$\nabla_\xi(FZ) = F\nabla_\xi Z + \xi(F)Z, \quad (62)$$

where h is the horizontal lift determined by \mathcal{D} .

We can now give an interesting alternative interpretation of the fundamental tensor \mathfrak{K} of \mathcal{D} . The point is the following: there are five tensors along τ which can be associated to a pair like (∇_ξ, P_h) . They can be regarded as torsions of ∇ and were called \mathcal{A} , \mathcal{R} , \mathcal{B} , \mathcal{P} and \mathcal{S} in [7]. The term torsion can be justified by the fact that any pair of a linear ∇ on $\tau^*\tau$ and a horizontal distribution on TM induces a linear connection on $T(TM)$ also, whose torsion has five components which (as by now familiar) are lifts of tensor fields along τ (see also [22] and [5] for a discussion about these five torsion tensors). Now, $\mathcal{B} = \mathcal{S} = 0$ here (as a result of the second condition in (61)), and \mathcal{P} , in particular is defined by

$$\mathcal{P}(X, Y) = \nabla_{X^h}Y - D_X^hY. \quad (63)$$

It follows from the first of (61) that $\mathcal{P} = \mathfrak{K}$. The two other tensors \mathcal{A} and \mathcal{R} will be encountered in the next section. Incidentally, the association between D_X and ∇_{X^h} in (61) is a kind of generalization of what are called h -basic covariant derivative operators by Szilasi [23].

The most obvious conclusion one can draw at the end of this section is that one should be extremely careful in comparing or using different types of connections which are around in this area!

6 Torsion and curvature of the linear \mathcal{D} along τ

There are different ways of approaching the concept of torsion of a connection. For a direct definition of the torsion tensor, one needs a bracket of vector fields. In the case of a linear connection \mathcal{D} along τ , since \mathcal{D} induces a horizontal distribution, it looks natural to think of the associated horizontal bracket, as defined in (24).

Definition: *The torsion ${}^{\mathcal{D}}T$ of a linear connection \mathcal{D} along τ is the vector-valued 2-form along τ , defined by*

$${}^{\mathcal{D}}T(X, Y) = D_XY - D_YX - [X, Y]_h, \quad X, Y \in \mathcal{X}(\tau). \quad (64)$$

It is easy to verify that ${}^{\mathcal{P}}T$ is indeed a tensor. In fact, we have (see [16]),

$$[X, Y]_h = D_X^h Y - D_Y^h X - {}^hT(X, Y), \quad (65)$$

where hT is the torsion of the non-linear connection h , which according to (14, 49) has components

$${}^hT_{ij}^k = V_j({}^h\Gamma_i^k) - V_i({}^h\Gamma_j^k), \quad {}^h\Gamma_i^k = \gamma_{il}^k v^l. \quad (66)$$

This implies that

$${}^{\mathcal{P}}T(X, Y) = \mathfrak{K}(X, Y) - \mathfrak{K}(Y, X) + {}^hT(X, Y). \quad (67)$$

Evaluating this expression in coordinates, all derivatives of the γ_{ij}^k cancel out, and one indeed obtains the previously cited formula (55), which agrees with the expression given by Asanov [4]. Note further that, in terms of the association expressed by Proposition 2, ${}^{\mathcal{P}}T$ is in fact the \mathcal{A} -torsion of the linear connection on $\tau^*\tau$.

As for curvature, it is best first of all to make a notational distinction: I shall use the notation *curv* when talking about curvature of any sort of linear connection and R as in section 2 for the curvature tensor of a non-linear connection. The natural definition of curvature of a linear \mathcal{D} along τ would seem to be

$${}^{\mathcal{P}}curv(X, Y)Z = (D_X D_Y - D_Y D_X - D_{[X, Y]_h}) Z. \quad (68)$$

Observe, however, that this is tensorial in X, Y , but not in Z , unless the Z -argument is restricted to $\mathcal{X}(M)$. With $F \in C^\infty(TM)$, it follows from (20) that

$${}^{\mathcal{P}}curv(X, Y)(FZ) = F({}^{\mathcal{P}}curv)(X, Y)Z + ({}^hR(X, Y))^V(F)Z, \quad (69)$$

where hR is the curvature of the non-linear connection associated to the h -lift. In coordinates, with h_i referring to the local basis of horizontal vector fields in (43), one readily finds that

$$({}^{\mathcal{P}}curv)_{ijk}^m = h_i(\gamma_{jk}^m) - h_j(\gamma_{ik}^m) + \gamma_{jk}^l \gamma_{il}^m - \gamma_{ik}^l \gamma_{jl}^m. \quad (70)$$

These are effectively (with a transposition of the lower indices in the connection coefficients, as reported before) the curvature components (3) mentioned in [4]. One should keep in mind, however, that they are, for the moment at least, components of a map ${}^{\mathcal{P}}curv : \mathcal{X}(\tau) \times \mathcal{X}(\tau) \times \mathcal{X}(M) \rightarrow \mathcal{X}(\tau)$. An interesting side observation is that

$${}^hR_{jk}^i = h_k({}^h\Gamma_j^i) - h_j({}^h\Gamma_k^i) = ({}^{\mathcal{P}}curv)_{kjl}^i v^l. \quad (71)$$

Note also that hR is, up to a sign, the so-called \mathcal{R} -torsion of the associated linear connection on $\tau^*\tau$, determined by Proposition 2.

At this point, it is worth referring to the comments in the introduction which follow equation (3), and to illustrate again now that there is a marked advantage in conceiving all objects and operations of interest as living along τ (as opposed to mixing calculations along τ with calculations on TM). Indeed, the property (69) provides the clue to remedy the non-tensorial aspect of ${}^{\mathcal{P}}curv$.

Definition: The curvature of a linear connection \mathcal{D} along τ is the type $(1, 1)$ tensor-valued 2-form ${}^{\mathcal{P}}\widetilde{curv}$ along τ , defined by

$${}^{\mathcal{P}}\widetilde{curv}(X, Y)Z = (D_X D_Y - D_Y D_X - D_{[X, Y]_h}) Z - D_{({}^h R)(X, Y)}^V Z. \quad (72)$$

The components of this tensor, which are obtained by taking coordinate vector fields for the arguments X, Y, Z , are the same as (70), since the extra term does not contribute to this computation. Note again that, as with the torsion (or in fact the covariant derivative D_X itself) the non-linear h -connection is needed to define the curvature of \mathcal{D} . It is clear then that one can define a related tensor field of curvature type as follows:

$${}^h curv(X, Y)Z = (D_X^h D_Y^h - D_Y^h D_X^h - D_{[X, Y]_h}^h) Z - D_{({}^h R)(X, Y)}^V Z. \quad (73)$$

One can verify that the relation between both curvature tensors is given by

$$\begin{aligned} {}^{\mathcal{P}}\widetilde{curv}(X, Y)Z &= {}^h curv(X, Y)Z + D_X^h \mathfrak{K}(Y, Z) - D_Y^h \mathfrak{K}(X, Z) \\ &\quad + \mathfrak{K}({}^h T(X, Y), Z) + \mathfrak{K}(X, \mathfrak{K}(Y, Z)) - \mathfrak{K}(Y, \mathfrak{K}(X, Z)), \end{aligned} \quad (74)$$

or equivalently

$$\begin{aligned} {}^{\mathcal{P}}\widetilde{curv}(X, Y)Z &= {}^h curv(X, Y)Z + D_X \mathfrak{K}(Y, Z) - D_Y \mathfrak{K}(X, Z) \\ &\quad + \mathfrak{K}({}^{\mathcal{P}}T(X, Y), Z) - \mathfrak{K}(X, \mathfrak{K}(Y, Z)) + \mathfrak{K}(Y, \mathfrak{K}(X, Z)). \end{aligned} \quad (75)$$

Concerning the geometrical interpretation of the tensor ${}^h curv$, it is worth observing that this is in fact the horizontal component of the curvature of the Berwald-type connection on $\tau^*\tau$, associated to the h -lift. Indeed, taking the general formula (20) for brackets of horizontal lifts into account, and denoting the Berwald-type covariant derivative by ∇_ξ as in (25), it is easy to see that

$${}^h curv(X, Y)Z = (\nabla_{X^h} \nabla_{Y^h} - \nabla_{Y^h} \nabla_{X^h} - \nabla_{[X^h, Y^h]}) Z. \quad (76)$$

Note finally that the property (71) has the following intrinsic content:

$${}^{\mathcal{P}}\widetilde{curv}(X, Y)\mathbf{T} = -{}^h R(X, Y). \quad (77)$$

I now briefly turn to the issue of Bianchi identities. In fact, with the insights we have gained now, we should not expect new features to appear here, because torsion and curvature of \mathcal{D} are after all closely related to ${}^h T$ and ${}^h R$, through (67) and (71) for example, and Bianchi identities for these tensors are known to be compactly represented by properties of the form (16). What looks appealing in this respect, however, is to explore the meaning of an exterior derivative associated to \mathcal{D} .

For the horizontal exterior derivative associated to the h -lift, we know [16] that on 1-forms $\alpha \in \Lambda^1(\tau)$,

$$d^h \alpha(X, Y) = D_X^h \alpha(Y) - D_Y^h \alpha(X) + \alpha({}^h T(X, Y)). \quad (78)$$

By analogy, therefore, it looks recommended to define the exterior derivative $d^{\mathcal{P}}$ on functions $F \in C^\infty(TM)$ by $d^{\mathcal{P}}F(X) = D_X F = X^h(F)$ and on 1-forms along τ by

$$d^{\mathcal{P}}\alpha(X, Y) = D_X \alpha(Y) - D_Y \alpha(X) + \alpha({}^{\mathcal{P}}T(X, Y)). \quad (79)$$

But then, in view of (50) and (67), it follows that $d^{\mathcal{D}} = d^h$ on scalar forms. For the action on vector fields along τ we have $d^h X(Y) = D_Y^h X$, hence by analogy put $d^{\mathcal{D}} X(Y) = D_Y X$. This implies that $(d^{\mathcal{D}} X - d^h X)(Y) = \mathfrak{K}(Y, X)$, meaning that for the full action on $V(\tau)$, we have

$$d^{\mathcal{D}} = d^h + a_{\mathfrak{K}}, \quad (80)$$

where \mathfrak{K} , in agreement with (12) is regarded here as a tensor in $\Lambda^1(\tau) \otimes V^1(\tau)$. However, since there is no difference between $d^{\mathcal{D}}$ and d^h on scalar forms, commutator properties of $d^{\mathcal{D}}$ of type (13), which could be regarded as defining torsion and curvature, will actually reproduce ${}^h T$ and ${}^h R$. As a result, the Bianchi identities essentially remain (see (16))

$$d^h({}^h R) = 0, \quad d^h({}^h T) + d^V({}^h R) = 0, \quad (81)$$

and can be re-formulated, using (80), (77) and (67) in terms of corresponding objects related to \mathcal{D} if needed.

7 Variationality versus metrizable

The two questions to be addressed here in fact have very little in common, but that is not always so clear in the literature. In my opinion, metrizable is a property that can be attributed to a connection, while variationality, in the context of a connection, can only be attributed to its geodesic equations, and as such is common to an equivalence class of connections, namely those which have the same geodesics.

A linear \mathcal{D} along τ is *variational* if its geodesic equations (1) are variational. The point to make is that studying this problem subsequently has very little to do with \mathcal{D} anymore; instead, the geometric tools of the SODE connection of the geodesics now enter the scene. And the most comprehensive formulation of the problem is simply the existence of a (non-singular) symmetric g along τ , satisfying the Helmholtz conditions (33, 34, 35). All operations of interest in this problem, such as the dynamical covariant derivative ∇ and Φ , have coordinate expressions in terms of the ‘forces’ f^i of the SODE Γ , which are given by

$$f^i(q, v) = -\gamma_{jk}^i(q, v) v^j v^k. \quad (82)$$

Hence, as observed in [12], nothing will change if we consider different \mathcal{D} , i.e. different γ_{jk}^i which produce the same f^i . In fact, variationality of a given \mathcal{D} was defined in [10] also as the existence of some set of γ_{jk}^i , possibly different from the given ones but giving rise to the same f^i , such that the Helmholtz conditions are satisfied.

Reference [12] contains another statement which is worth situating within our present analysis. As we discussed in section 5, there is a certain similarity between the dynamical covariant derivative ∇ of the geodesic SODE and the operator $D_{\mathbf{T}}$ (see (56) for example), so it is of some interest to investigate to what extent $\nabla g = 0$ differs from $D_{\mathbf{T}} g = 0$. The answer to this question is the result (8.14) in [12] which, translated into our present notations, states that, provided the linear \mathcal{D} along τ is taken to be torsion free,

$$\nabla g = 0 \quad \Leftrightarrow \quad D_{\mathbf{T}} g = \frac{1}{2} \left(g_{ik} \frac{\partial \gamma_{lm}^k}{\partial v^j} + g_{jk} \frac{\partial \gamma_{lm}^k}{\partial v^i} \right) v^l v^m. \quad (83)$$

Expressed differently, and in more intrinsic terms, we can say that for a torsion-free \mathcal{D} ,

$$(\nabla g = 0 \Leftrightarrow D_{\mathbf{T}}g = 0) \quad \text{if and only if} \quad \mu_{\mathbf{T} \rfloor \mathfrak{R}}g = 0. \quad (84)$$

In fact, in view of (52), we then also have $D_{\mathbf{T}}^h g = 0$.

A discussion of metrizable is more delicate, because here I differ in opinion with some of the references cited so far. In my view, a linear \mathcal{D} along τ is *metrizable* if there exists a (non-singular) symmetric g along τ , such that $D_X g = 0$ for all $X \in \mathcal{X}(\tau)$ (without any further assumptions on g). The requirements on g here are directly related to the given connection and are quite different from $\nabla g = 0$, for example. In other words, generally speaking, it doesn't make much sense to expect that metrizable might imply variationality or vice versa, unless of course you modify your definition until that works.

There are roughly two aspects one can consider: one is the strict and hard question of studying for a given fixed \mathcal{D} the existence of a suitable g ; the other and much more tangible question is to start from a given g and try to modify the connection to make it metric with respect to this g . I will argue, however, that even the second question does not have an entirely compatible solution in the case of a linear connection along τ .

Inspired by the concept of Cartan tensor which one can find, for example, in the work of Miron and co-workers (see e.g. [20] and [2]), the following looks like a natural concept to introduce here.

Definition: *The Cartan tensor associated to a given non-singular, symmetric g and a linear connection \mathcal{D} along τ , is the symmetric type (1,2) tensor $\mathcal{C}_{\mathcal{D}}$ along τ , determined by*

$$g(\mathcal{C}_{\mathcal{D}}(X, Y), Z) = D_X g(Y, Z) + D_Y g(X, Z) - D_Z g(X, Y), \quad X, Y, Z \in \mathcal{X}(\tau). \quad (85)$$

The coordinate expression of $\mathcal{C}_{\mathcal{D}}$ is found to be

$$\begin{aligned} \mathcal{C}_{\mathcal{D}ij}^k &= g^{kl} (h_i(g_{jl}) + h_j(g_{il}) - h_l(g_{ij})) - (\gamma_{ij}^k + \gamma_{ji}^k) \\ &\quad + g^{kl} [g_{jm}(\gamma_{li}^m - \gamma_{il}^m) + g_{im}(\gamma_{lj}^m - \gamma_{jl}^m)], \end{aligned} \quad (86)$$

and simplifies somewhat for a torsion-free connection.

Proposition 3: *We have*

- (i) $(D_X + \frac{1}{2}\mu_{X \rfloor \mathcal{C}_{\mathcal{D}}})g = 0, \quad \forall X \in \mathcal{X}(\tau),$
- (ii) $D_X g = 0, \quad \forall X \in \mathcal{X}(\tau) \Leftrightarrow \mathcal{C}_{\mathcal{D}} = 0.$

PROOF: The first property is a straightforward computation: we have

$$(D_X + \frac{1}{2}\mu_{X \rfloor \mathcal{C}_{\mathcal{D}}})g(Y, Z) = D_X g(Y, Z) - \frac{1}{2}[g(\mathcal{C}_{\mathcal{D}}(X, Y), Z) + g(\mathcal{C}_{\mathcal{D}}(X, Z), Y)],$$

from which the result immediately follows by using (85). Obviously then, if $\mathcal{C}_{\mathcal{D}} = 0$ it follows that $D_X g = 0, \forall X$, while the converse trivially follows from the definition of $\mathcal{C}_{\mathcal{D}}$. \square

So, metrizable of a given \mathcal{D} is (as usual) a matter of a vanishing Cartan tensor, in other words, the hard question then is to study under what circumstances, with given $\gamma_{ij}^k(q, v)$, the equations $g_{lk}C_{\mathcal{D}ij}^k = 0$ can have a solution for g .

It would seem that the statement (i) in the above proposition contains the (expected) answer about how to modify the given connection to make it metric with respect to g . There is a technical problem, however, which makes that this is not quite true! To see this, let's go back to the original concept of a linear connection along τ , in the interpretation of Proposition 1. This shows that the difference between two such connections is a type (1,2) tensorial object indeed, C say, but in the following sense:

$$C : \mathcal{X}(\tau) \times \mathcal{X}(M) \rightarrow \mathcal{X}(\tau).$$

Hence, starting from a D_X , putting $D'_X := D_X + \mu_{X \downarrow C}$ defines a new linear connection along τ , but as soon as one wants to extend the action of D'_X to $\mathcal{X}(\tau)$ by the rules (45, 46), there is a certain incompatibility, because the horizontal lift h' induced by \mathcal{D}' is different from the original h -lift, while the defining relation of D'_X , without modification, would imply that on functions $F \in C^\infty(TM)$: $D'_X F = X^h F$. As was the case in discussing the notion of curvature (see (72)), one has to correct with a vertical derivative term (which does not modify the connection coefficients) to remedy this deficiency, and the clue on how to do this here comes from the property (57) which every properly extended linear connection along τ should have.

Proposition 4: *If C is an arbitrary type (1,2) tensor along τ , then the following modification of a given \mathcal{D} defines a new linear connection along τ which is compatible with its induced horizontal lift:*

$$D'_X = D_X + \mu_{X \downarrow C} - D_{C(X, \mathbf{T})}^V. \quad (87)$$

PROOF: Recall that for $Y \in \mathcal{X}(\tau)$, we have: $\mu_{X \downarrow C} Y = C(X, Y)$ and $D_Y^V \mathbf{T} = Y$. It easily follows that $D_X \mathbf{T} = 0$ implies $D'_X \mathbf{T} = 0$, and one can verify in coordinates that this is the same as saying that for $F \in C^\infty(TM)$, $D'_X F = X^{h'}(F)$ where $\gamma_{ij}^k = \gamma_{ij}^k + C_{ij}^k$. \square

Coming back to the subject of Proposition 3 now, we see that the modified covariant derivative operator of the first statement necessarily has to be taken in its extended sense, since it acts on g (not just on basic vector fields). Therefore, the genuine modified linear connection $\tilde{\mathcal{D}}$ which is at stake here, reads

$$\tilde{D}_X = D_X + \frac{1}{2} \mu_{X \downarrow C_{\mathcal{D}}} - \frac{1}{2} D_{C_{\mathcal{D}}(X, \mathbf{T})}^V. \quad (88)$$

Unfortunately, however, the conclusion then is that we don't have $\tilde{D}_X g = 0$, $\forall X$, but rather

$$\tilde{D}_X g + \frac{1}{2} D_{C_{\mathcal{D}}(X, \mathbf{T})}^V g = 0. \quad (89)$$

One might consider to cover this technicality by defining a linear \mathcal{D} along τ to be metrical with respect to some g , if for all $X \in \mathcal{X}(\tau)$, $D_X g = 0$ modulo vertical derivatives of g .

It is interesting to look at the preceding technical problem still from a different perspective. By Proposition 2, we know how to associate with \mathcal{D} a linear connection ∇_ξ on the pullback

bundle $\tau^*\tau$. In turn, as was done (in a time-dependent set-up) in [18], for example, one can then define a horizontal Cartan tensor in that context by

$$g(C_h(X, Y), Z) = \nabla_{X^h}g(Y, Z) + \nabla_{Y^h}g(X, Z) - \nabla_{Z^h}g(X, Y), \quad X, Y, Z \in \mathcal{X}(\tau). \quad (90)$$

Actually $C_h = \mathcal{C}_{\mathcal{D}}$, but we can pose the problem of constructing a modified metrical connection at this level without any complications. Indeed,

$$\tilde{\nabla}_{X^h} = \nabla_{X^h} + \frac{1}{2} \mu_{X^h} C_h \quad (91)$$

will have the property that $\tilde{\nabla}_{X^h}g = 0$, for all X . Since it is only the horizontal component of $\tilde{\nabla}_{\xi}$ which becomes metrical in this process, one could call this a *connection of Chern-Rund type*. The problem we have encountered before comes from the fact (explained in detail in section 5) that in going back from $\tilde{\nabla}_{X^h}$ to a linear \tilde{D}_X along τ , the horizontal lift induced by this \tilde{D} is not the one we started from.

For completeness, coming back to the point I made at the very beginning of the section, I should mention the special case of a standard linear connection on M , where the connection coefficients do not depend on the v^i and the geodesics come from a spray. Then, the requirement that for some (quasi-Riemannian) metric g we have $g_{ij|k} = 0$ (for all k), obviously is equivalent to requiring that $g_{ij|k}v^k = 0$, in other words, in that situation we have

$$D_X g = 0, \quad \forall X \quad \Leftrightarrow \quad D_{\mathbf{T}}g = \nabla g = 0.$$

That is why variationality (where $\nabla g = 0$ is a key condition) and metrizable (which is about $D_X g = 0, \forall X$) are closely related then. By the way, it looks like an interesting question to investigate, for the general case of a linear \mathcal{D} along τ , is under what circumstances $D_{\mathbf{T}}g = 0$ will imply $D_X g = 0, \forall X \in \mathcal{X}(\tau)$ (some form of homogeneity is probably indispensable for that).

There is a final link I should explain to conclude this discussion. A very recent paper [6] carries the title ‘‘Metric nonlinear connections’’. So what is this about? The author takes a SODE Γ , plus a non-linear connection with connection coefficients N_j^i say, to build a covariant derivative operator, ∇ say, by putting (I am identifying vertical tangent vectors to TM with vectors along τ):

$$\nabla F = \Gamma(F), \quad F \in C^\infty(TM), \quad \nabla \frac{\partial}{\partial q^i} = N_i^j \frac{\partial}{\partial q^j}. \quad (92)$$

Here, the non-linear connection may or may not be the canonical one coming from Γ (although I don’t see the point really in taking a different one to start with). Anyhow, it is clear that this operator is of the type of a dynamical covariant derivative, and it can be of interest, of course, to study compatibility of ∇ with some metric g , in the sense that $\nabla g = 0$. Whether it is appropriate to classify this question under ‘metrizable’ problems is perhaps debatable here, if a dynamical covariant derivative is all one has. The main problem which is addressed in section 2 of [6] is to construct from ∇ a new ∇' such that $\nabla'g = 0$, where g is a given metric along τ . The solution to this problem is in fact quite

simple: taking ∇' and ∇ to be identical on functions, they must be related by a formula of the form:

$$\nabla' = \nabla + \mu_A,$$

for some $(1, 1)$ tensor A , which must be chosen in such a way that

$$g(AX, Y) + g(X, AY) = \nabla g(X, Y), \quad \forall X, Y.$$

It is clear that a solution for A is given by

$$A_j^i = \frac{1}{2}g^{il}(\nabla g)_{lj},$$

which means that the modified N_j^i are determined by

$$g_{li}N_j^i = \frac{1}{2}(\nabla g)_{lj} + g_{li}N_j^i = \frac{1}{2}\Gamma(g_{lj}) + \frac{1}{2}(g_{li}N_j^i - g_{ji}N_l^i).$$

In the case that the N_j^i we start from are the canonical Γ_j^i (see (27)), so that ∇ is precisely the dynamical covariant derivative of the SODE Γ , the above relation is exactly equation (12) in [6], and thus explains Theorem 2.2 of that paper.

8 Hessian metrics along τ and Finsler spaces

Let's go back to the general question of metrization of a linear \mathcal{D} along τ , meaning that we want a g such that $D_X g = 0$, $\forall X$. Since solving the vanishing of (48) for g , for example, is a hard problem, one will naturally be led to look at the effect of imposing extra restrictions on g . It seems to me that a good way to proceed would be to work in stages, as follows: first study the effect of imposing that g_{ij} be the Hessian matrix of some function (with respect to the fibre coordinates v^i), and secondly assume that g is homogeneous in the v^i , for example (but not exclusively) of degree zero. I will not enter into this study in any depth here. Though I think this has not been carried out yet in a systematic way, it is true of course that many aspects of this idea frequently turn up in the literature (sometimes rather related to one of the other types of connections discussed in section 5 though). In the terminology of the Miron-school, for example, the first step would correspond to passing from a "generalized Lagrange space" to a "Lagrange space".

In [11], a linear \mathcal{D} along τ is said to be metrizable if there exists a Finsler metric such that $D_X g = 0$, so homogeneity of g is taken to be part of the definition of metrization (and in fact also of the definition of variationality of \mathcal{D}). A general SODE Γ on the other hand is said to be metrizable if there exists a "variational" metric g , meaning it is a Hessian, such that $\nabla g = 0$. In other words, two of the three Helmholtz conditions (33, 34, 35) are incorporated in the definition of metrization here, which of course makes life easier. Incidentally, an interesting question which emerges in this context is: under what circumstances is the remaining Helmholtz condition redundant? It is well known that this is the case for Finsler metrics. I claim it is true also as soon as we have homogeneity of any order. It would take me too far away from the subject of this paper, however, to prove this statement here.

Instead, to finish, let's go back to the master we are celebrating in this volume. Specifically, let me return to the paper [10] I started from, because it contains a definition of metrizable, which is very surprising if you see it for the first time and leads to an equally surprising conclusion which is worth explaining in more intrinsic terms. Let me recall first the definition of a Hessian metric in an intrinsic way.

Definition: *The Hessian of a function L on TM is the symmetric type $(0, 2)$ tensor field g along τ , defined by*

$$g(X, Y) = D_Y^V D_X^V L - D_{D_Y^V X}^V L, \quad X, Y \in \mathcal{X}(\tau). \quad (93)$$

Now in [10], a linear \mathcal{D} along τ is said to be metrizable if there exists a non-singular, symmetric g along τ , such that

$$D_X g = 0, \quad \forall X, \quad \text{and} \quad \frac{\partial g_{ij}}{\partial v^k} v^j = 0. \quad (94)$$

In intrinsic terms, the surprise extra condition means that

$$\mathbf{T} \lrcorner D_X^V g = 0, \quad \forall X. \quad (95)$$

Lemma: *If a symmetric g along τ has the property (95), then it is a Hessian and is homogeneous of degree 0 in the fibre coordinates.*

PROOF: Put $L = \frac{1}{2}g(\mathbf{T}, \mathbf{T})$. Then, it follows from (95) and the general property $D_X^V \mathbf{T} = X$ that $D_X^V L = g(X, \mathbf{T})$. Taking a further vertical derivative with respect to Y and using the intermediate result plus (95) again, we obtain that g satisfies (93), i.e. is the Hessian of L . But a Hessian is characterized by the property (34), from which it follows, taking \mathbf{T} as one of the arguments and using (95) again, that $D_{\mathbf{T}}^V g = 0$. This precisely means that g is homogeneous of degree zero. \square

This result is far from new: (95) in one form or another is the defining relation for a g along τ to be what is called *normal*; it is well known as the necessary and sufficient condition for g to be a Finslerian metric (see e.g. [19], [13]) and as such is attributed to Hashiguchi [9]. A side remark: now that Finsler structures come into the picture, I will omit the technicalities about having to pass from the tangent bundle to the slit tangent bundle, and also not go into requirements about positive definiteness.

The main theorem in [10] states that under the conditions (94), the connection \mathcal{D} is variational and more precisely is the Cartan connection of a Finsler structure. The proof of this theorem leaves the reader a bit startled, however. First of all, it is clear that the calculations in the proof take for granted that the connection coefficients γ_{ij}^k of the given \mathcal{D} are symmetric: it is indeed a somewhat hidden assumption throughout the paper that the connection is torsion free. Secondly, what is explicitly shown is an equivalence of connections in the sense of variationality, that is to say: the geodesic SODE of \mathcal{D} is shown to be the same as the one coming from the Euler-Lagrange equations of the Finslerian g . There is no explicit verification, however, that the assumptions imply that the given γ_{ij}^k are effectively those of what is called (at least by some) the Cartan connection in that context.

I shall finish by presenting a slight generalization of this theorem which consists in obtaining roughly the same results from somewhat weaker conditions. This will give me a chance to illustrate some of the features discussed in the previous sections, while the details of the Krupka-Sattarov theorem will follow as a special case.

Theorem: *Let \mathcal{D} be a torsion-free, linear connection along τ , for which there exists a non-singular, symmetric g along τ , such that*

$$\mathcal{C}_{\mathcal{D}}(X, \mathbf{T}) = 0 \quad \text{and} \quad \mathbf{T} \lrcorner D_X^{\vee} g = 0, \quad \forall X. \quad (96)$$

Then the following assertions hold true:

- (i) \mathcal{D} is variational: its geodesic SODE in fact is the canonical spray of g which is a Finsler metric.
- (ii) There exists a variationally equivalent $\tilde{\mathcal{D}}$ along τ which is metric with respect to g : $\tilde{D}_X g = 0, \forall X$.
- (iii) The connection coefficients $\tilde{\gamma}_{ij}^k$ of $\tilde{\mathcal{D}}$ have an explicit expression in terms of g only and as such are those of the Cartan connection of g .

PROOF: (i) From the lemma we know that g is Finslerian: it is the Hessian of $L = \frac{1}{2}g(\mathbf{T}, \mathbf{T})$, and L is homogeneous of degree 2 in the v^i . From the coordinate expression (86), taking the symmetry of the connection into account, it is clear that $\mathcal{C}_{\mathcal{D}}(\mathbf{T}, \mathbf{T}) = 0$ implies that

$$\begin{aligned} \gamma_{ij}^k v^i v^j &= \frac{1}{2} g^{kl} (h_i(g_{jl}) + h_j(g_{il}) - h_l(g_{ij})) v^i v^j \\ &= \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial q^i} + \frac{\partial g_{il}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^l} \right) v^i v^j \stackrel{\text{set}}{=} (\gamma_g)_{ij}^k v^i v^j, \end{aligned} \quad (97)$$

where the reduction from the first to the second line follows from the property (95) again. This second line clearly reveals the force terms of the Euler-Lagrange equations of L (and should be read also as defining relations of the functions $(\gamma_g)_{ij}^k(q, v)$).

(ii) Since $\mathcal{C}_{\mathcal{D}}(X, \mathbf{T}) = 0$, the defining equation (88) of $\tilde{\mathcal{D}}$ reduces to

$$\tilde{D}_X = D_X + \frac{1}{2} \mu_{X \lrcorner} \mathcal{C}_{\mathcal{D}},$$

and it follows from Proposition 3 that $\tilde{D}_X g = 0, \forall X$. Moreover, the new $\tilde{\gamma}_{ij}^k$ are given by $\tilde{\gamma}_{ij}^k = \gamma_{ij}^k + \frac{1}{2} \mathcal{C}_{\mathcal{D}ij}^k$, so that $\mathcal{C}_{\mathcal{D}}(\mathbf{T}, \mathbf{T}) = 0$ implies that $\tilde{\gamma}_{ij}^k v^i v^j = \gamma_{ij}^k v^i v^j$, proving the variational equivalence of both connections.

(iii) We can now compute the $\tilde{\gamma}_{ij}^k$ from the vanishing of the Cartan tensor of $\tilde{\mathcal{D}}$. For simplicity in notations, let us omit the tildes, i.e. assume we are back in the Krupka-Sattarov situation now, so that $D_X g = 0$. Then $\mathcal{C}_{\mathcal{D}} = 0$ implies that

$$\gamma_{ij}^k = \frac{1}{2} g^{kl} (h_i(g_{jl}) + h_j(g_{il}) - h_l(g_{ij})), \quad (98)$$

but the right-hand sides in this expression contain in the vertical derivatives of the g -components factors of the form $\gamma_{ij}^k v^j$. Multiplying (98) with v^j (in other words, using $\mathcal{C}_{\mathcal{D}}(X, \mathbf{T}) = 0$) and making use of (95) again, we find that

$$\gamma_{ij}^k v^j = (\gamma_g)_{ij}^k v^j - \frac{1}{2} g^{kl} \gamma_{js}^r v^j v^s \frac{\partial g_{il}}{\partial v^r}. \quad (99)$$

Again, we still have γ 's in the right-hand side, but we can eliminate them now by using (97). Substituting these intermediate results back into the expression (98), we finally get

$$\begin{aligned} \gamma_{ij}^k &= (\gamma_g)_{ij}^k - \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial v^r} (\gamma_g)_{is}^r + \frac{\partial g_{il}}{\partial v^r} (\gamma_g)_{js}^r - \frac{\partial g_{ij}}{\partial v^r} (\gamma_g)_{ls}^r \right) v^s \\ &\quad + \frac{1}{4} g^{kl} g^{rt} \left(\frac{\partial g_{jl}}{\partial v^r} \frac{\partial g_{it}}{\partial v^u} + \frac{\partial g_{il}}{\partial v^r} \frac{\partial g_{jt}}{\partial v^u} - \frac{\partial g_{ij}}{\partial v^r} \frac{\partial g_{lt}}{\partial v^u} \right) (\gamma_g)_{sp}^u v^s v^p. \end{aligned} \quad (100)$$

This provides an explicit expression of the γ_{ij}^k in terms of the metric g , which is the same as equation (4.5) in [10], and is called the Cartan connection there. \square

Remark: the implicit specification of the γ_{ij}^k , from which the final result can be deduced, is also (for a symmetric connection) equation (A.27) in the previously cited Appendix of [4], where it is referred to as the Cartan connection too. The type of computation in part (iii) of the proof in fact is similar also to the way the Cartan connection is set up in [1]. We have seen in section 5, however, that when \mathcal{D} is mapped onto a linear connection on the pullback bundle according to Proposition 2, the metric nature of \mathcal{D} corresponds to ∇ being horizontally metric, so that the more common terminology in that context would be that we are talking about a connection of Chern-Rund type.

There are some interesting corollaries of the above theorem. Once we know that the geodesic SODE of \mathcal{D} is the canonical spray of a Finsler metric g , it follows that $\nabla g = 0$. Hence, we are in the situation (84) and know that $\mu_{\mathbf{T} \rfloor \mathfrak{K}} g = 0$. In fact, we can do a bit better and show that $\mathbf{T} \rfloor \mathfrak{K} = 0$.

Proposition 5: *Under the assumptions of the above theorem, the fundamental tensor \mathfrak{K} of the linear connection \mathcal{D} has the property $\mathfrak{K}(\mathbf{T}, X) = 0$, $\forall X \in \mathcal{X}(\tau)$. It follows that the horizontal lift h associated to \mathcal{D} , and the horizontal lift H of its geodesic SODE coincide.*

PROOF: For the components of $\mathbf{T} \rfloor \mathfrak{K}$, we can write

$$-\frac{\partial \gamma_{ij}^k}{\partial v^r} v^i v^j = -\frac{\partial}{\partial v^r} (\gamma_{ij}^k v^i v^j) + 2 \gamma_{rj}^k v^j.$$

But we know that $\mathcal{C}_{\mathcal{D}}(X, \mathbf{T}) = 0$, so that we can use the expressions (97) and (99) to compute the two terms in the right-hand side. It is straightforward to verify that making these substitutions, and using (95), all terms cancel out. The last statement then immediately follows from (54). \square

Finally, taking (52) and (56) into account, plus the fact that $\mathbf{T}^h = \Gamma$ always, it follows that under the assumptions of the theorem: $D_{\mathbf{T}} = D_{\mathbf{T}}^h = D_{\mathbf{T}}^H = \nabla$.

9 Concluding remarks

Having arrived in Finsler country at the end of our journey, where there is a vast literature, and where every household seems to foster its own notations, it is possible that there are still other results to which I could or should have compared aspects of what has been discussed in this paper. But I hope the reader will find the road to these results using [3], though this is not an easy navigating system.

The general construct of linear connections along τ , in the sense of Rund's direction-dependent connections, has been used *strictu senso* only occasionally in the literature, but a number of aspects about such operations remained unclear, specifically with respect to the intrinsic foundations of the theory. I hope I have managed to clarify such aspects in this paper. New, potentially interesting questions have come up in my analysis and there are undoubtedly many more one can think of. However, having identified also a number of technicalities and dangers for confusion with related concepts, there is one major question I would like to put forward, namely: "Do we actually need linear connections along τ ?" Isn't it possible for example that, with the geometrical calculus offered by linear connections on the pullback bundle $\tau^*\tau$, we have sufficient tools in our hand to analyse all theoretical questions one might wish to study with linear connections along τ ?

References

- [1] M. Abate and G. Patrizio, *Finsler metrics A Global Approach*, Lecture Notes in Math., **1591** (Springer-Verlag, Berlin) (1994).
- [2] M. Anastasiei, Finsler connections in generalized Lagrange spaces, *Balkan J. of Geom. and its Appl.* **1** (1996) 1–9.
- [3] P.L. Antonelli (Ed.), *Handbook of Finsler Geometry*, Vol.1 and Vol.2 (Kluwer Academic Publishers, Dordrecht) (2003).
- [4] G.S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, (D. Reidel Publ. Comp. Dordrecht) (1985).
- [5] A. Bejancu, *Finsler geometry and applications*, (Ellis Horwood) (1990).
- [6] I. Bucataru, Metric nonlinear connections, *Diff. Geometry and its Applications* **25** (2007) 335–343.
- [7] M. Crampin, Connections of Berwald type, *Publ. Math. (Debrecen)* **57** (2000) 455–473.
- [8] A. Frölicher and A. Nijenhuis, Theory of vector-valued differential forms, *Proc. Ned. Acad. Wetensch. Sér. A* **59** (1956) 338–359.
- [9] M. Hashiguchi, On parallel displacements in Finsler spaces, *J. Math. Soc. Japan* **10** (1958) 365–379.
- [10] D. Krupka and A.E. Sattarov, The inverse problem of the calculus of variations for Finsler structures, *Math. Slovaca* **35** (1985) 217–222.
- [11] O. Krupková, Variational metrics on $R \times TM$ and the geometry of nonconservative mechanics, *Math. Slovaca* **44** (1994) 315–335.
- [12] O. Krupková, Variational metric structures, *Publ. Math. Debrecen* **62** (2003) 461–495.
- [13] R.L. Lovas, J. Pék and J. Szilasi, Ehresmann connections, metrics and good metric derivatives, In: *Finsler Geometry, Sapporo 2005, Advanced Studies in Pure Mathematics* (Math. Soc. Japan) **48** (2007) 263–308.

- [14] E. Martínez and J.F. Cariñena, Geometric characterization of linearizable second-order differential equations, *Math. Proc. Camb. Phil. Soc.* **119** (1996) 373–381.
- [15] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Diff. Geometry and its Applications* **2** (1992) 17–43.
- [16] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection II, *Diff. Geometry and its Applications* **3** (1993) 1–29.
- [17] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press (Japan) (1986).
- [18] T. Mestdag and W. Sarlet, The Berwald-type connection associated to time-dependent second-order differential equations, *Houston J. of Math.* **27** (2001) 763–797.
- [19] T. Mestdag, J. Szilasi and V. Tóth, On the geometry of generalized metrics, *Publ. Math. (Debrecen)* **62** (2003) 511–545.
- [20] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Fundamental Theories of Physics Vol. 59, (Kluwer Academic Publishers, Dordrecht) (1994).
- [21] H. Rund, Direction-dependent connections and curvature forms, *Abh. Math. Sem. Univ. Hamburg* **50** (1980) 188–209.
- [22] J. Szilasi, Notable Finsler connections on a Finsler manifold, *Lect. Matemáticas* **19** (1998) 7–34.
- [23] J. Szilasi, A setting for spray and Finsler geometry, in: *Handbook of Finsler Geometry, vol. 2* (ed. P.L. Antonelli), (Kluwer Academic Publishers, Dordrecht) (2003) 1183–1426.