Comment on ‘Conservation laws of higher-order nonlinear PDEs and the variational conservation laws in the class with mixed derivatives’

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Abstract. In a recent paper [2], the authors claim to be applying Noether's theorem to higher-order PDEs and state that in a large class of examples “the resultant conserved flows display some previously unknown interesting ‘divergence properties’ owing to the presence of the mixed derivatives” (citation from their abstract). It turns out that what this obscure sentence is meant to say is that the vector whose divergence must be zero (according to Noether's theorem) turns out to have non-zero divergence and subsequently must be modified to obtain a true conservation law. Clearly, that cannot be right: we explain in detail what is the main source of the error.

1 Introduction

Noether's famous theorem about symmetries and conservation laws is now almost a century old and has never been challenged. It has been discussed in literally hundreds of papers and is covered in many textbooks, for example in [3] and [1]. The proof actually is a quite straightforward calculation within the framework of the calculus of variations. Admittedly, when it concerns field equations, i.e. systems of PDEs, coming from a Lagrangian density depending on higher-order derivatives of the field variables, the calculations become a bit tricky, not so much because of conceptual difficulties, but rather due to notational complications. The point is essentially that one has to be careful in working out multiple summations, not to do a double counting of the appropriate jet coordinates: after all, variables referring to mixed derivatives such as u_{tx} and u_{xt} are of course the same.

But here is an excerpt of what the authors in [2] tell us in their introduction: “When considering the construction of conservation laws via Noether's theorem … an interesting situation arises when the equations under investigation are such that the highest derivative term is mixed . . . . When substituting the conserved flow back into the divergence relationship, a number of 'extra' terms (on which the Euler operator vanishes) arise. Thus, we have essentially 'trivial' conserved quantities that need to be fed back into the conserved vectors that are computed initially via Noether's theorem – these are necessary terms that may guarantee the notion of 'association' between conserved flows and symmetries – otherwise, the total divergence of the conserved flows is the equations modulo the trivial part."

In fact, this sort of prose still goes on for a while in the introduction with the accumulation of phrases such as 'substituting a conserved flow back into a divergence relationship'. When one looks a bit further, specifically at section 3, it turns out that the real story roughly goes as follows. Consider, for example, the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}u_{tx}^2 - \tfrac{1}{2}u_t u_x - \tfrac{1}{2}u_t^2, \tag{1}$$

with corresponding field equation

$$u_{xxtt} + u_{tt} + u_{tx} = 0. \tag{2}$$

It is clear that $\mathcal{L}$ has time-translation symmetry and according to [2], "the Noether conserved vector components are"

$$T^1 = -\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2 + u_t u_{txx}, \qquad T^2 = \tfrac{1}{2}u_t^2 + u_t u_{ttx} - u_{tt}u_{tx}.$$

But in the next line (equations (3.41) and (3.42) in [2]), we see that $D_tT^1 + D_xT^2$ is not zero, and that we have to redefine

$$\tilde{T}^1 = T^1 - u_t u_{txx}, \qquad \tilde{T}^2 = T^2 + u_{tt}u_{tx},$$

in order to get a vector whose divergence is zero. Since this self-contradicting effect is attributed by the authors to the presence of higher-order mixed derivatives, it is clear that this must be due to not understanding how to do the calculations properly.

To the defense of the authors, it is true that the good general reference works such as those already cited [3, 1] do not really contain explicit examples where you can learn how to do this. The theoretical results are written, for very understandable reasons of course, in a rather compact format, with summations running over multi-indices for example, whereby one has to be well aware of the conventions which the use of such notations involve (for a thorough explanation of the use of multi-indices, see section 6.1 in [4]). In order to point out all the errors of the paper under discussion, I feel obliged therefore to be very detailed and explicit about the calculations in concrete examples. For this reason, and at the risk of sounding very pedantic, I will re-derive Noether's theorem in the next section (for Lagrangian densities depending on second-order derivatives to fix the idea), and pay particular attention to the pitfalls which lie ahead in actual applications.

2 Noether's theorem

We consider field functions $\psi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, (x^i) \mapsto (\psi^\alpha(x))$ and a functional of the form

$$J(\psi) = \int_\Omega \mathcal{L}\Big(x^i, \psi^\alpha(x), \psi^\alpha_{x^i}(x), \psi^\alpha_{x^i x^j}(x)\Big)\,dx. \tag{3}$$

So, Latin indices are used for the independent variables and Greek ones for the field variables. If no special summation sign is used, summations over repeated indices are understood to run over the whole range. The corresponding Euler-Lagrange equations are

$$\sum_{k\le l} \mathrm{D}_{x^k}\mathrm{D}_{x^l}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}}\right) - \mathrm{D}_{x^k}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}}\right) + \frac{\partial \mathcal{L}}{\partial \psi^\alpha} = 0, \tag{4}$$

where

$$\mathrm{D}_{x^i} = \frac{\partial}{\partial x^i} + \psi^\alpha_{x^i}\frac{\partial}{\partial \psi^\alpha} + \psi^\alpha_{x^i x^j}\frac{\partial}{\partial \psi^\alpha_{x^j}} + \sum_{j\le k} \psi^\alpha_{x^i x^j x^k}\frac{\partial}{\partial \psi^\alpha_{x^j x^k}}. \tag{5}$$

Notice that we are using here already some restricted summations to avoid double counting of variables. Consider next an infinitesimal transformation of the form

$$\bar{x}^i = x^i + \varepsilon\,\tau^i(x,\psi), \tag{6}$$
$$\bar{\psi}^\alpha = \psi^\alpha + \varepsilon\,\xi^\alpha(x,\psi), \tag{7}$$

with its natural extension or prolongation to second-jet variables:

$$\bar{\psi}^\alpha_{\bar{x}^k} = \psi^\alpha_{x^k} + \varepsilon\,\eta^\alpha_k, \tag{8}$$
$$\bar{\psi}^\alpha_{\bar{x}^k\bar{x}^l} = \psi^\alpha_{x^k x^l} + \varepsilon\,\eta^\alpha_{kl}, \tag{9}$$

where

$$\eta^\alpha_k = \mathrm{D}_{x^k}\xi^\alpha - \psi^\alpha_{x^j}\mathrm{D}_{x^k}\tau^j, \tag{10}$$
$$\eta^\alpha_{kl} = \mathrm{D}_{x^l}\eta^\alpha_k - \psi^\alpha_{x^k x^j}\mathrm{D}_{x^l}\tau^j = \mathrm{D}_{x^k}\eta^\alpha_l - \psi^\alpha_{x^l x^j}\mathrm{D}_{x^k}\tau^j. \tag{11}$$

Such an infinitesimal transformation is said to be a Noether symmetry with respect to $\mathcal{L}$ if for all field functions $\psi^\alpha(x)$ and for all (sufficiently regular) subdomains $\mathcal{D} \subset \Omega$ (which transform into some $\overline{\mathcal{D}}$), there exist functions $f^i(x,\psi)$ such that

$$\int_{\overline{\mathcal{D}}} \mathcal{L}\Big(\bar{x}^i, \bar{\psi}^\alpha(\bar{x}), \bar{\psi}^\alpha_{\bar{x}^i}(\bar{x}), \bar{\psi}^\alpha_{\bar{x}^i\bar{x}^j}(\bar{x})\Big)\,d\bar{x} =$$
$$\int_{\mathcal{D}} \mathcal{L}\Big(x^i, \psi^\alpha(x), \psi^\alpha_{x^i}(x), \psi^\alpha_{x^i x^j}(x)\Big)\,dx + \varepsilon \int_{\mathcal{D}} \mathrm{D}_{x^i}\Big(f^i(x,\psi(x)\Big)\,dx + \mathcal{O}(\varepsilon^2). \tag{12}$$

Making a substitution in the left-hand side to express both sides in the same integration variables, since the equality then has to hold for all $\mathcal{D} \subset \Omega$, the integrands have to be identified, up to order ε^2. From a Taylor expansion in the left-hand side, it is then straightforward to derive the following necessary and sufficient condition, often referred to as the Noether identity,

$$\tau^i \frac{\partial \mathcal{L}}{\partial x^i} + \xi^\alpha \frac{\partial \mathcal{L}}{\partial \psi^\alpha} + \eta^\alpha_k \frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}} + \sum_{k\le l} \eta^\alpha_{kl} \frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} + \mathcal{L}\Big(\mathrm{D}_{x^i}\tau^i\Big) = \mathrm{D}_{x^i} f^i, \tag{13}$$

which, as a first interpretation, has to hold true for arbitrary field functions $\psi^\alpha(x)$ still. An equivalent interpretation is that this must hold for arbitrary values of the $(x^i, \psi^\alpha, \psi^\alpha_{x^i}, \psi^\alpha_{x^i x^j})$ regarded as independent variables. In the second interpretation, the Noether identity gives rise to PDEs for the unknown generators (τ^i, ξ^α), coupled with the associated undetermined functions f^i. In the first interpretation, it is a matter of a straightforward manipulation of terms involving the D_{x^i} operators, to rewrite the identity in a form which unambiguously identifies the components F^i of a conservation law. This, I will now carry out in great detail. We have

$$\eta^\alpha_k \frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}} = \mathrm{D}_{x^k}\left[(\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}}\right] - (\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\,\mathrm{D}_{x^k}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}}\right) + \psi^\alpha_{x^k x^j}\tau^j \frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}}, \tag{14}$$

likewise

$$\mathcal{L}\Big(\mathrm{D}_{x^i}\tau^i\Big) = \mathrm{D}_{x^k}(\mathcal{L}\tau^k) - \tau^i\left(\frac{\partial \mathcal{L}}{\partial x^i} + \psi^\alpha_{x^i}\frac{\partial \mathcal{L}}{\partial \psi^\alpha} + \psi^\alpha_{x^i x^j}\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^j}} + \sum_{k\le j}\psi^\alpha_{x^k x^j x^i}\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^j}}\right), \tag{15}$$

and

$$\begin{aligned}
\sum_{k\le l}\eta^\alpha_{kl}\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} &= \sum_{k\le l}\mathrm{D}_{x^k}\left[(\eta^\alpha_l - \psi^\alpha_{x^l x^j}\tau^j)\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}}\right] \\
&\quad - \sum_{k\le l}(\eta^\alpha_l - \psi^\alpha_{x^l x^j}\tau^j)\mathrm{D}_{x^k}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}}\right) + \sum_{k\le l}\psi^\alpha_{x^k x^l x^j}\tau^j\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} \\
&= \sum_{k\le l}\mathrm{D}_{x^k}\left[(\eta^\alpha_l - \psi^\alpha_{x^l x^j}\tau^j)\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}}\right] + \sum_{k\le l}\psi^\alpha_{x^k x^l x^j}\tau^j\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} \\
&\quad - \sum_{k\le l}\left[\mathrm{D}_{x^l}(\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\right]\mathrm{D}_{x^k}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}}\right).
\end{aligned}$$

Interchanging the names of the indices k and l in the last term and then bringing the derivation D_{x^k} to the forefront, we have to add a correction term again, in which it does not harm to swap the names of k and l once more. Thus we obtain

$$\begin{aligned}
\sum_{k\le l}\eta^\alpha_{kl}\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} &= \sum_{k\le l}\mathrm{D}_{x^k}\left[(\eta^\alpha_l - \psi^\alpha_{x^l x^j}\tau^j)\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}}\right] + \sum_{k\le l}\psi^\alpha_{x^k x^l x^j}\tau^j\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} \\
&\quad - \sum_{l\le k}\mathrm{D}_{x^k}\left[(\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\mathrm{D}_{x^l}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^l x^k}}\right)\right] + \sum_{k\le l}(\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\mathrm{D}_{x^k}\mathrm{D}_{x^l}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}}\right).
\end{aligned} \tag{16}$$

An important thing to observe, of course, is that $k \leq l$ has been transformed into $l \leq k$ in the first term of this last operation. When we substitute these three results in the Noether identity, all we have essentially done is adding terms and subtracting them again, but the condition which still must hold for arbitrary field functions $\psi^\alpha(x)$ now reads, after some obvious cancellations,

$$(\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\left[\frac{\partial\mathcal{L}}{\partial\psi^\alpha} - \mathrm{D}_{x^k}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^k}}\right) + \sum_{k\leq l}\mathrm{D}_{x^k}\mathrm{D}_{x^l}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^kx^l}}\right)\right] + \mathrm{D}_{x^k}F^k = 0, \tag{17}$$

where (for each fixed k)

$$\begin{aligned} F^k = \mathcal{L}\tau^k + (\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^k}} - \sum_{l=1}^{k}\mathrm{D}_{x^l}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^lx^k}}\right)\right) \\ + \sum_{l=k}^{n}(\eta^\alpha_l - \psi^\alpha_{x^lx^j}\tau^j)\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^kx^l}} - f^k. \end{aligned} \tag{18}$$

This implies that $\mathrm{D}_{x^k}F^k = 0$ along solutions of the Euler-Lagrange equations, which is the meaning of a conservation law in this context.

To facilitate the computations in the next section and to underscore once more the importance of the restricted summations in this result, here are the explicit expressions of the F^k in the case that $n = 2$ (summation over j from 1 to 2 and over α from 1 to m understood):

$$\begin{aligned} F^1 &= \mathcal{L}\tau^1 + (\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\left[\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^1}} - \mathrm{D}_{x^1}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^1x^1}}\right)\right] \\ &\quad + (\eta^\alpha_1 - \psi^\alpha_{x^1x^j}\tau^j)\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^1x^1}} + (\eta^\alpha_2 - \psi^\alpha_{x^2x^j}\tau^j)\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^1x^2}} - f^1, \end{aligned} \tag{19}$$

$$\begin{aligned} F^2 &= \mathcal{L}\tau^2 + (\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\left[\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^2}} - \mathrm{D}_{x^1}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^1x^2}}\right) - \mathrm{D}_{x^2}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^2x^2}}\right)\right] \\ &\quad + (\eta^\alpha_2 - \psi^\alpha_{x^2x^j}\tau^j)\frac{\partial\mathcal{L}}{\partial\psi^\alpha_{x^2x^2}} - f^2. \end{aligned} \tag{20}$$

3 Examples

We return to the example (1) now and look at the four simple Noether symmetries which are discussed in [2]. So we have one field variable ($m = 1$), called u here, and two independent variables ($n = 2$), called t and x.

(i) Time translations are generated by taking $\tau^1 = 1, \tau^2 = \xi = 0$, resulting in $\eta_1 = \eta_2 = \eta_{12} = 0$. The Noether condition (13) is obviously satisfied for the choice

$f^1 = f^2 = 0$. From (19) and (20), the resulting components F^i for a conservation law are easily found to be

$$F^1 = -\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2, \qquad F^2 = \tfrac{1}{2}u_t^2 + u_t u_{ttx}. \tag{21}$$

It is easy to verify that, along solutions of the equation (2), $D_t F^1 + D_x F^2 = 0$ indeed, and our F^i are in fact the $\tilde{T}^i$ obtained after uncanny maneuvers in [2].

(ii) There is obviously also space translation invariance, corresponding to $\tau^1 = 0, \tau^2 = 1, \xi = 0$. Proceeding in the same way, we obtain

$$F^1 = \tfrac{1}{2}u_x^2 + u_x u_t - u_{xx}u_{tx}, \qquad F^2 = \tfrac{1}{2}u_{tx}^2 - \tfrac{1}{2}u_t^2 + u_x u_{ttx}. \tag{22}$$

Notice that this time F^i is not the $\tilde{T}^i$ obtained in [2]; in fact their $\tilde{T}^i$ by far is not a vector with zero divergence.

(iii) Since $\mathcal{L}$ does not depend on u either, a further trivial Noether symmetry is determined by $\tau^1 = \tau^2 = 0, \xi = 1$ (with $f^i = 0$). We get the conservation law determined by

$$F^1 = -\tfrac{1}{2}u_x - u_t, \qquad F^2 = -\tfrac{1}{2}u_t - u_{ttx}. \tag{23}$$

(iv) A somewhat less trivial Noether symmetry listed in [2] is generated by taking $\tau^1 = \tau^2 = 0, \xi = t$. It then follows that $\eta_1 = 1, \eta_2 = \eta_{12} = 0$ and the Noether condition (13) can easily be satisfied by choosing $f^1 = -u, f^2 = -\frac{1}{2}u$. The conservation law is determined by

$$F^1 = -\tfrac{1}{2}tu_x - tu_t + u, \qquad F^2 = -\tfrac{1}{2}tu_t - tu_{ttx} + \tfrac{1}{2}u. \tag{24}$$

Again, this is not in agreement with the claims in [2], but this time there is only a sign error to blame.

4 Some final comments

There is one aspect of this paper I have not yet commented on: the authors occasionally make an attempt to sort of generalize the application of Noether's theorem to so-called 'partial Lagrangians' and, judging from the list of references, this has actually been done before! The point is that also this concept of partial Lagrangians is rather meaningless if there is no specific underlying structural idea which would justify what part of the equations is associated to a Lagrangian. What I mean is: the way the authors consider partial Lagrangians, one can simply take any system of differential equations (odes or pdes), select a function L to be a 'partial Lagrangian', with the only restriction that it has to generate a part of the given system which contains the highest-order derivatives (possibly after multiplication by an appropriate non-singular matrix factor) and then rewrite the obtained equivalent system as the sum of that part and the remainder terms. What benefit can one expect from such rather random manipulations?

The authors have seen these criticisms of their approach in another forum and have chosen to ignore them. As a result, I have little choice than to make them public in such an unequivocal way through the pages of this journal.

References

[1] G.W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer (New York) (1989).

[2] R. Narain and A.H. Kara, Conservation laws of higher-order nonlinear PDEs and the variational conservation laws in the class with mixed derivatives, *J. Phys. A: Math. Theor.* **43** (2010) 085205.

[3] P. Olver, *Applications of Lie groups to differential equations*, Grad. Texts in Math. **107**, Springer (New York) (1986).

[4] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lecture Note Series **142**, Cambridge University Press (1989).