The inverse problem for Lagrangian systems with certain non-conservative forces

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Abstract. We discuss two generalizations of the inverse problem of the calculus of variations, one in which a given mechanical system can be brought into the form of Lagrangian equations with non-conservative forces of a generalized Rayleigh dissipation type, the other leading to Lagrangian equations with so-called gyroscopic forces. Our approach focusses primarily on obtaining coordinate-free conditions for the existence of a suitable non-singular multiplier matrix, which will lead to an equivalent representation of a given system of second-order equations as one of these Lagrangian systems with non-conservative forces.

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1 Introduction

The inverse problem of Lagrangian mechanics is the question: given a system of second-order ordinary differential equations, under what circumstances does there exist a regular Lagrangian function, such that the corresponding Lagrange equations are equivalent (i.e. have the same solutions) as the original equations. Locally, the question can be translated immediately into more precise terms as follows: considering a given second-order system in normal form

$$\ddot{q}^i = f^i(q, \dot{q}), \tag{1}$$

which (for the time being) we take to be autonomous for simplicity, what are the conditions for the existence of a symmetric, non-singular multiplier matrix $g_{ij}(q, \dot{q})$ such that

$$g_{ij}(\ddot{q}^j - f^j(q, \dot{q})) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}$$

for some L. Clearly (g_{ij}), if it exists, will become the Hessian of the Lagrangian L. The literature on this problem is extensive; the conditions for the existence of L are usually referred to as the Helmholtz conditions, but these can take many different forms depending on the mathematical tools one uses and on the feature one focusses on. For a non-exhaustive list of different approaches see [24], [5], [26], [22], [2], [9], [15], [1], [19], [3]. In this paper, the tools stem from differential geometry and therefore provide coordinate-free results. In addition, while we will actually study generalizations of the above problem which allow for certain classes of non-conservative forces, the attention will be mainly on conditions on the multiplier g.

We will consider two types of non-conservative forces, leading to Lagrangian equations of one of the following forms:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial D}{\partial \dot{q}^i}, \tag{2}$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \omega_{ki}(q)\dot{q}^k, \quad \omega_{ki} = -\omega_{ik}. \tag{3}$$

In the first case, when the function D is quadratic in the velocities (and $-D$ is positive definite) the classical terminology is that we have dissipation of Rayleigh type (see e.g. [13]); we will not put restrictions on the form of D, however. In the second case, in which the existence of a D as in (2) is excluded, the right-hand side is often referred to as a gyroscopic force (see e.g. [23]).

Perhaps we should specify first what we will not do in this paper. In older contributions to the inverse problem for dissipative systems, such as [10], the emphasis was on trying to recast a dissipative system into the form of genuine Euler-Lagrange equations, that is to say that in the case of given equations of type (2) one would try to find a different function L' such that the Euler-Lagrange equations of L' are equivalent to the given system. In contrast, our goal here is to study under what circumstances a given second-order system in normal form (1) can be recast into the form (2) (or (3)) for some functions L and D (or L and ω_{ki}).

In order to explain our objectives in more precise terms, let us recall first some of the different ways of characterizing the inverse problem conditions in the classical situation. The natural environment for a second-order system is a tangent bundle TQ, with coordinates (q, v) say, where it is represented by a vector field Γ of the form

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v)\frac{\partial}{\partial v^i}. \tag{4}$$

If $S = (\partial/\partial v^i) \otimes dq^i$ denotes the type $(1, 1)$ tensor field which characterizes the canonical almost tangent structure on TQ [6, 14], Γ represents a Lagrangian system provided there exists a regular Lagrangian function L such that (see e.g. [27])

$$\mathcal{L}_\Gamma(S(dL)) = dL; \tag{5}$$

$\theta_L := S(dL)$ is the Poincaré-Cartan 1-form. The above condition is perhaps the most compact formulation of the problem, but has little or no practical value when it comes to testing whether such an L exists for a given Γ. A shift of attention towards the existence of a multiplier leads to the following necessary and sufficient conditions [5]: the existence of a non-degenerate 2-form $\omega \in \bigwedge^2(TQ)$, such that

$$\mathcal{L}_\Gamma \omega = 0, \quad \omega(X^V, Y^V) = 0, \quad i_{Z^H} d\omega(X^V, Y^V) = 0, \quad \forall X, Y, Z \in \mathcal{X}(M). \tag{6}$$

Here X^V and X^H refer to the vertical and horizontal lift of vector fields, respectively. The latter makes use of the canonical Ehresmann connection on $\tau : TQ \to Q$ associated with a given second-order vector field Γ: in coordinates, the vertical and horizontal lift are determined by

$$V_i := \frac{\partial}{\partial q^i}^V = \frac{\partial}{\partial v^i}, \qquad H_i := \frac{\partial}{\partial q^i}^H = \frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial v^j}, \quad \text{where} \quad \Gamma_i^j = -\tfrac{1}{2}\frac{\partial f^j}{\partial v^i}. \tag{7}$$

Such a 2-form ω will be closed, hence locally exact, and as such will be the exterior derivative $d\theta_L$ for some Lagrangian L. At this point it is interesting to observe that the $2n \times 2n$ skew-symmetric component matrix of ω is completely determined by the $n \times n$ symmetric matrix

$$g_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j}.$$

The matrix (g_{ij}) geometrically represents the components of a $(0, 2)$ symmetric tensor field g along the tangent bundle projection τ, and the relationship between ω on TQ and g along τ has an intrinsic meaning as well: ω is the Kähler lift of g (see [21]). A more concise formulation of the Helmholtz conditions therefore, when viewed as conditions on the multiplier g, makes use of the calculus of derivations of forms along τ, as developed in [20, 21]. We will show in the next section how both the conditions (5) and (6) have an equivalent formulation in those terms, and this will be the basis for the generalization to Lagrangian systems with non-conservative forces, which will be the subject of the subsequent sections.

The first authors to discuss the inverse problem, in the sense of analyzing the conditions which a given representation of a second-order system must satisfy to be of the form (2), were Kielau and Maisser [18]. We showed in [8] how the results they obtained via an entirely analytical approach can in fact be reduced to a smaller set. But we also argued in the concluding remarks of that paper that the more important issue is the one we formulated above, which starts from a normal form representation of the dynamical system. For that purpose it is better to approach the problem in a coordinate-independent way, i.e. to make use of the tools of differential geometry already referred to. We will see that the methods we will develop for the dissipative case (2) apply equally to the gyroscopic case (3). To the best of our knowledge the latter problem has not been dealt with before in its entirety (though a relevant partial result has been published in [19]). An additional advantage of the coordinate-independence of our conditions is that they cover without extra effort results such as those derived in [17] for the description of Lagrangian systems in 'nonholonomic velocities'. In Section 3 we follow the lines of the construction of Helmholtz conditions on the multiplier g for the standard inverse problem, and arrive in this way at necessary and sufficient conditions which involve g and D in the dissipative case, and g and ω in the gyroscopic situation. At the end of this section we briefly discuss how the partial result mentioned above is related to our work. In Section 4 we succeed in eliminating the unknown D and ω altogether to arrive at necessary and sufficient conditions involving the multiplier g only. This is particularly interesting, because a given Γ may actually admit multiple representations of the form (2) for example. In other words, different choices of a multiplier g may exist, which each require an adapted (generalized) dissipation function D to match the required format. In fact it cannot be excluded that a given Γ may actually have representations in the form (2) and (3) at the same time, of course with different multipliers g (and thus different Lagrangians L). We will encounter such situations among the illustrative examples discussed in Section 5, where we also briefly indicate in the concluding remarks how the whole analysis can be carried over to the case of time-dependent systems. In an appendix we make an excursion to a different geometrical approach which in fact is essentially time-dependent: we use techniques from the theory of variational sequences to relate our results more closely, at least in the dissipative case, to those obtained in [18], which after all was the work which first brought this subject to our attention.

2 Basic set-up

In order to keep our analysis reasonably self-contained, we need to recall the basics of the calculus of derivations of forms along the tangent bundle projection $\tau : TQ \to Q$. Vector fields along τ are sections of the pull-back bundle $\tau^*TQ \to TQ$ and constitute a module over $C^\infty(TQ)$, denoted by $\mathcal{X}(\tau)$. Likewise, a k-form along τ assigns to every point $v_q \in TQ$ an exterior k-form at $q = \tau(v_q) \in Q$; we use the symbol $\bigwedge(\tau)$ for the $C^\infty(TQ)$-module of scalar forms along τ and $V(\tau)$ for the module of vector-valued forms. The theory of derivations of such forms, as established in [20, 21], follows closely the pioneering work of Frölicher and Nijenhuis [11]. The difference is that there is a natural vertical exterior derivative d^V available, but a full classification requires an additional horizontal exterior derivative d^H, which must come from a given connection: in our situation, this is the connection associated with Γ mentioned earlier. We limit ourselves here to a brief survey of the concepts and properties we will need. An elaborate version of the theory (with rather different notations) can also be found in [30].

Elements of $\bigwedge(\tau)$ in coordinates look like forms on the base manifold Q with coefficients which are functions on TQ. Thus they can be seen also as so-called semi-basic forms on TQ, and we will generally make no notational distinction between the two possible interpretations. It is clear that derivations of such forms are completely determined by their action on $C^\infty(TQ)$ and on $\bigwedge^1(Q)$. As such, the vertical and horizontal exterior derivatives are determined by

$$d^V F = V_i(F)dq^i, \qquad d^H F = H_i(F)dq^i, \qquad F \in C^\infty(TQ),$$

$$d^V dq^i = 0, \qquad d^H dq^i = 0.$$

Obviously, for $L \in C^\infty(TQ)$, $d^V L \in \bigwedge^1(\tau)$ has the same coordinate representation as $S(dL) \in \bigwedge^1(TQ)$; in line with the above remark therefore, we will also write $\theta_L = d^V L$ for the Poincaré-Cartan 1-form. Derivations of type i_* are defined as in the standard theory. For $A \in V(\tau)$, we put

$$d^V_A = [i_A, d^V], \qquad d^H_A = [i_A, d^H],$$

and call these derivations of type d^V_* and d^H_* respectively. The action of all such derivations can be extended to vector-valued forms and then another algebraic type derivation is needed for a classification, but we will introduce such extensions, which can all be found in [20, 21], only when needed. The horizontal and vertical lift operations, already referred to in the introduction, trivially extend to vector fields along τ and then every vector field on TQ has a unique decomposition into a sum of the form $X^H + Y^V$, with $X, Y \in \mathcal{X}(\tau)$. Looking in particular at the decomposition of the commutator $[X^H, Y^V]$ suffices to discover two important derivations of degree zero:

$$[X^H, Y^V] = (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H.$$

They extend to forms by duality and are called the horizontal and vertical covariant derivatives. In coordinates

$$\begin{aligned}
&\mathrm{D}^V_X F = X^i\, V_i(F), \quad \mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0, \quad \mathrm{D}^V_X dq^i = 0, \\
&\mathrm{D}^H_X F = X^i\, H_i(F), \quad \mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^j V_j(\Gamma^k_i) \frac{\partial}{\partial q^k}, \quad \mathrm{D}^H_X dq^i = -X^j V_j(\Gamma^i_k) dq^k.
\end{aligned}$$

For later use, we mention the following formulas for computing exterior derivatives of, for example, a 1-form α or a 2-form ρ along τ:

$$d^V\alpha\,(X,Y) = \mathrm{D}^V_X \alpha\,(Y) - \mathrm{D}^V_Y \alpha\,(X), \qquad \alpha \in \textstyle\bigwedge^1(\tau), \tag{8}$$

$$d^V\rho\,(X,Y,Z) = \sum_{X,Y,Z} \mathrm{D}^V_X \rho\,(Y,Z), \qquad \rho \in \textstyle\bigwedge^2(\tau), \tag{9}$$

and similarly for d^H. Here $\sum_{X,Y,Z}$ represents the cyclic sum over the indicated arguments. It is also of interest to list the decomposition of the other brackets of lifted vector fields:

$$\begin{aligned}
[X^V, Y^V] &= (\mathrm{D}^V_X Y - \mathrm{D}^V_Y X)^V, \\
[X^H, Y^H] &= (\mathrm{D}^H_X Y - \mathrm{D}^H_Y X)^H + (R(X,Y))^V.
\end{aligned}$$

The latter relation is just one of many equivalent ways in which the curvature tensor $R \in V^2(\tau)$ of the non-linear connection can be defined. The connection coming from Γ has no torsion (since (7) obviously implies that $V_i(\Gamma^j_k) = V_k(\Gamma^j_i)$): it follows that d^V and d^H commute. In fact the commutation table of the exterior derivatives, for their action on scalar forms, is given by

$$\tfrac{1}{2}[d^V, d^V] = d^V d^V = 0, \qquad d^V d^H = -d^H d^V, \qquad \tfrac{1}{2}[d^H, d^H] = d^H d^H = d^V_R. \tag{10}$$

Finally, the given dynamical system Γ comes canonically equipped with two other operators which are crucial for our analysis, namely the dynamical covariant derivative ∇, a degree zero derivation, and the Jacobi endomorphism $\Phi \in V^1(\tau)$. Again, the simplest way of introducing them comes from the decomposition of a Lie bracket: they are the uniquely determined operations for which, for each $X \in \mathcal{X}(\tau)$,

$$[\Gamma, X^H] = (\nabla X)^H + (\Phi X)^V.$$

The usual duality rule $\nabla\langle X, \alpha\rangle = \langle \nabla X, \alpha\rangle + \langle X, \nabla\alpha\rangle$ is used to extend the action of ∇ to 1-forms, and subsequently to arbitrary tensor fields along τ. In coordinates,

$$\nabla F = \Gamma(F), \quad \nabla\left(\frac{\partial}{\partial q^j}\right) = \Gamma^i_j \frac{\partial}{\partial q^i}, \quad \nabla(dq^i) = -\Gamma^i_j dq^j\,, \tag{11}$$

and

$$\Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^k_j\Gamma^i_k - \Gamma(\Gamma^i_j). \tag{12}$$

One clear indication of the importance of these operators is the following link with the curvature of the connection:

$$d^V\Phi = 3\,R, \qquad d^H\Phi = \nabla R. \tag{13}$$

We are now ready to go back to the generalities about the inverse problem discussed in the previous section. To begin with, using the tools which have just been established, the compact formulation (5) of the inverse problem is equivalent (see [21]) to the existence of a regular function $L \in C^\infty(TQ)$ such that

$$\nabla\theta_L = d^H L. \tag{14}$$

Secondly, the necessary and sufficient conditions (6) now really become conditions on the multiplier matrix; they are translated via the Kähler lift [21] into the existence of a non-degenerate, symmetric $(0,2)$-tensor g along τ satisfying the requirements

$$\nabla g = 0, \qquad g(\Phi X, Y) = g(X, \Phi Y), \qquad \mathrm{D}^V_X g(Y,Z) = \mathrm{D}^V_Y g(X,Z). \tag{15}$$

It is possible to prove directly that (14) implies (15) and vice versa (a sketch of such a proof was presented in [4]). We will not show how to do this here, however, as it can easily be seen later on as a particular case of the more general inverse problem studies we will start analyzing now.

3 Lagrangian systems with dissipative or gyroscopic forces

Consider first equations of type (2). It is obvious that, at the level of a characterization like (5), a given second-order field Γ will correspond to equations of type (2) if and only if there exist a regular function L and a function D such that

$$\mathcal{L}_\Gamma(S(dL)) = dL + S(dD). \tag{16}$$

We take the opportunity to illustrate first how such a relation, when stripped to its bare essentials, i.e. when one observes that it is in fact a condition on only n of the $2n$ components, is transformed into a corresponding generalization of (14). To this end, note first that there exists a dual notion of horizontal and vertical lifts of 1-forms, from $\bigwedge^1(\tau)$ to $\bigwedge^1(TQ)$, defined by $\alpha^H(X^H) = \alpha(X)$, $\alpha^H(X^V) = 0$, and likewise for α^V. We then have the following decompositions, for any $\alpha \in \bigwedge^1(\tau)$ and $L \in C^\infty(TQ)$:

$$\mathcal{L}_\Gamma \alpha^H = \alpha^V + (\nabla\alpha)^H, \tag{17}$$

$$dL = (d^H L)^H + (d^V L)^V. \tag{18}$$

Coming back to the notational remarks of the previous section: the horizontal lift is technically speaking the rigorous way of identifying a 1-form along τ with a semi-basic 1-form on TQ. So, when convenient, as will be the case in establishing the next result, we can also write $\theta_L = (d^V L)^H$, for example.

Proposition 1. *The second-order field Γ represents a dissipative system of type (2) if and only if there is a regular function $L \in C^\infty(TQ)$ and a function $D \in C^\infty(TQ)$ such that*

$$\nabla\theta_L = d^H L + d^V D. \tag{19}$$

Proof. We have that $S(dD) = (d^V D)^H$ for any function D, and in particular $S(dL) = (d^V L)^H = \theta_L$. Using the decompositions (17) and (18), the condition (16) then immediately translates into (19). $\square$

Corollary 1. *The condition (19) on the existence of functions L and D is equivalent to*

$$d^H\theta_L = 0, \tag{20}$$

which is a necessary and sufficient condition on L only.

Proof. Using the commutator property $[\nabla, d^V] = -d^H$ to re-express $\nabla\theta_L = \nabla d^V L$ in (19), we immediately get the expression

$$d^H L = \tfrac{1}{2}\, d^V(\Gamma(L) - D),$$

which is equivalent to saying that $d^H L = d^V G$ for some function G. This in turn, in view of (10) and the triviality of d^V-cohomology, is equivalent to $d^H\theta_L = -d^V d^H L = 0$. $\square$

We now want to translate these results into conditions on the multiplier g which generalize (15). As we observed earlier, this g will be the Hessian of L, so we look first at the relation between a function and its Hessian in intrinsic terms. To that end, we introduce covariant differentials D^V and D^H defined as follows: for any tensor field T along τ and $X \in \mathcal{X}(\tau)$,

$$\mathrm{D}^V T(X, \ldots) = \mathrm{D}^V_X T(\ldots),$$

and similarly for D^H. Then for any $F \in C^\infty(TQ)$ we can write for the corresponding Poincaré-Cartan 1-form $\theta_F = d^V F = \mathrm{D}^V F$, and define the Hessian tensor g_F of F as $g_F = \mathrm{D}^V\mathrm{D}^V F$, which means that

$$g_F(X,Y) = \mathrm{D}^V_X\mathrm{D}^V_Y F - \mathrm{D}^V_{\mathrm{D}^V_X Y} F = \mathrm{D}^V_X\theta_F(Y). \tag{21}$$

Lemma 1. *For any $F \in C^\infty(TQ)$, its corresponding Hessian tensor g_F is symmetric and satisfies $\mathrm{D}^V_X g_F(Y,Z) = \mathrm{D}^V_Y g_F(X,Z)$, i.e. $\mathrm{D}^V g_F$ is symmetric in all its arguments. Conversely, any symmetric g along τ for which $\mathrm{D}^V g$ is symmetric is the Hessian of some function F. Secondly, if Φ represents any type $(1,1)$ tensor field along τ, we have*

$$\Phi\lrcorner\, g_F - (\Phi\lrcorner\, g_F)^T = i_{d^V\Phi}\theta_F - d^V i_\Phi\theta_F, \tag{22}$$

where $(\Phi\lrcorner\, g_F - (\Phi\lrcorner\, g_F)^T)(X,Y) := g_F(\Phi X, Y) - g_F(X,\Phi Y)$.

Proof. The symmetry of g_F follows directly from $d^V\theta_F = d^V d^V F = 0$. The symmetry of $\mathrm{D}^V g_F$ can easily be shown by taking a further vertical covariant derivative of the defining relation of g_F and using the commutator property

$$[\mathrm{D}^V_X, \mathrm{D}^V_Y] = \mathrm{D}^V_{\mathrm{D}^V_X Y} - \mathrm{D}^V_{\mathrm{D}^V_Y X}. \tag{23}$$

The converse statement is obvious from the coordinate representation of the assumptions. Finally, making use of (8) we have

$$\begin{aligned} d^V i_\Phi\theta_F(X,Y) &= \mathrm{D}^V_X(\Phi(\theta_F))(Y) - \mathrm{D}^V_Y(\Phi(\theta_F))(X) \\ &= \langle \mathrm{D}^V_X\Phi(Y) - \mathrm{D}^V_Y\Phi(X), \theta_F\rangle + \mathrm{D}^V_X\theta_F(\Phi Y) - \mathrm{D}^V_Y\theta_F(\Phi X) \\ &= i_{d^V\Phi}\theta_F(X,Y) + g_F(X,\Phi Y) - g_F(Y,\Phi X), \end{aligned}$$

from which the last statement follows. $\square$

We are now ready to state and prove the first main theorem, which provides the transition of the single condition (19) to equivalent conditions involving a multiplier g, in precisely the same way as (14) relates to (15).

Theorem 1. *The second-order field Γ represents a dissipative system of type (2) if and only if there exists a function D and a symmetric type $(0,2)$ tensor g along τ such that $\mathrm{D}^V g$ is symmetric and g and D further satisfy*

$$\nabla g = \mathrm{D}^V\mathrm{D}^V D, \tag{24}$$

$$\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T = d^V d^H D, \tag{25}$$

where Φ is the Jacobi endomorphism of Γ.

Proof. Suppose Γ represents a system of type (2). Then we know there exist functions L and D such that (19) and (20) hold true. Define $g = \mathrm{D}^V\mathrm{D}^V L$ or equivalently $g(X,Y) = \mathrm{D}^V_X\theta_L(Y)$. Obviously, g and $\mathrm{D}^V g$ are symmetric by construction. Acting with ∇ on g and using the commutator property $[\nabla, \mathrm{D}^V] = -\mathrm{D}^H$, we get for a start

$$\begin{aligned}\nabla g &= \mathrm{D}^V\nabla\mathrm{D}^V L - \mathrm{D}^H\mathrm{D}^V L\\ &= \mathrm{D}^V\mathrm{D}^V D + \mathrm{D}^V\mathrm{D}^H L - \mathrm{D}^H\mathrm{D}^V L,\end{aligned}$$

where the last line follows from (19). Now the commutator of vertical and horizontal covariant differentials (see [21] or [9]) is such that, at least on functions, $\mathrm{D}^V\mathrm{D}^H L(X,Y) = \mathrm{D}^H\mathrm{D}^V L(Y,X)$. But

$$\mathrm{D}^H\mathrm{D}^V L(Y,X) - \mathrm{D}^H\mathrm{D}^V L(X,Y) = d^H\theta_L(Y,X) = 0,$$

in view of (20), so that (24) follows. When acting finally with d^H on (19), we have to appeal to the formula for $d^H d^H$ in (10) and further need the commutator of ∇ and d^H, which for the action on the module $\bigwedge(\tau)$ of scalar forms is given by

$$[\nabla, d^H] = 2\, i_R + d^V_\Phi. \tag{26}$$

It is then straightforward to check, using (20) and the first of (13), that we get

$$d^V i_\Phi\theta_L - i_{d^V\Phi}\theta_L = d^H d^V D, \tag{27}$$

from which (25) follows in view of the last statement in Lemma 1.

Conversely, assume that g and D satisfy the four conditions stated in the theorem. It follows from the symmetry of g and $\mathrm{D}^V g$ that g is a Hessian: $g = \mathrm{D}^V\mathrm{D}^V F$ say. The function F of course is not unique and the idea is to take advantage of the freedom in F to construct an L which will have the desired properties. This is not so difficult to do by a coordinate analysis. Keeping the computations intrinsic is a bit more technical, but will give us an opportunity to recall a few more features of interest of the calculus of forms along τ. Observe first that $\nabla\mathrm{D}^V g$ is obviously symmetric, and that the same is true for $\mathrm{D}^V\nabla g = \mathrm{D}^V\mathrm{D}^V\mathrm{D}^V D$. It follows from $[\nabla, \mathrm{D}^V] = -\mathrm{D}^H$ that $\mathrm{D}^H g$ is also symmetric. Hence

$$\mathrm{D}^H g(X,Y,Z) = \mathrm{D}^H\mathrm{D}^V\theta_F(X,Y,Z) = \mathrm{D}^H\mathrm{D}^V\theta_F(X,Z,Y).$$

If we interchange D^H and D^V in the last term, there is an extra term to take into account (since the action is on a 1-form this time, not a function). Indeed, we have

$$\mathrm{D}^H\mathrm{D}^V\theta_F(X,Z,Y) = \mathrm{D}^V\mathrm{D}^H\theta_F(Z,X,Y) + \theta_F(\theta(X,Z)Y). \tag{28}$$

Here θ is a type $(1,3)$ tensor along τ which is completely symmetric (and could in fact be defined by the above relation): its components in a coordinate basis are $\theta^k_{jml} = V_m V_l(\Gamma^k_j)$. Using the above two relations, expressing the symmetry of $\mathrm{D}^H g$ in its first two arguments now leads to

$$0 = \mathrm{D}^V\mathrm{D}^H\theta_F(Z,X,Y) - \mathrm{D}^V\mathrm{D}^H\theta_F(Z,Y,X) = \mathrm{D}^V_Z(d^H\theta_F)(X,Y).$$

This says that $d^H\theta_F$ is a basic 2-form, i.e. a 2-form on the base manifold Q. On the other hand, we have

$$\mathrm{D}^V\mathrm{D}^V D = \nabla g = \nabla \mathrm{D}^V\mathrm{D}^V F = \mathrm{D}^V\nabla\theta_F - \mathrm{D}^H\mathrm{D}^V F = \mathrm{D}^V(\nabla\theta_F - d^H F) - d^H\theta_F,$$

where we have used the property $\mathrm{D}^H\mathrm{D}^V F(X,Y) = \mathrm{D}^V\mathrm{D}^H F(Y,X)$ again in the transition to the last expression. But since $d^H\theta_F$ is basic, we can write it as $\mathrm{D}^V i_{\mathbf{T}} d^H\theta_F$, where $\mathbf{T}$ is the canonical vector field along τ (the identity map on TQ), which in coordinates reads

$$\mathbf{T} = v^i\frac{\partial}{\partial q^i}. \tag{29}$$

It follows that we can write the last relation in the form

$$\mathrm{D}^V\beta := \mathrm{D}^V(\nabla\theta_F - d^H F - i_{\mathbf{T}} d^H\theta_F - \mathrm{D}^V D) = 0,$$

which defines another basic form β. We next want to prove that the basic forms β and $d^H\theta_F$ are actually closed in view of the final assumption (25) or equivalently (27), which has not been used so far. Keeping in mind that d^H is the same as the ordinary exterior derivative for the action on basic forms, we easily find with the help of (10) that

$$d^H d^H\theta_F = \tfrac{1}{3} d^V i_{d^V\Phi}\theta_F = \tfrac{1}{3} d^V(d^V i_\Phi\theta_F - d^V d^H D) = 0.$$

Secondly, using also (26),

$$\begin{aligned}
d^H\beta &= d^H\nabla\theta_F - d^H d^H F - d^H i_{\mathbf{T}} d^H\theta_F - d^H d^V D \\
&= \nabla d^H\theta_F - 2i_R\theta_F - d^V_\Phi\theta_F - i_R d^V F - d^H_{\mathbf{T}} d^H\theta_F - d^H d^V D \\
&= \nabla d^H\theta_F - i_{d^V\Phi}\theta_F + d^V i_\Phi\theta_F - d^H_{\mathbf{T}} d^H\theta_F - d^H d^V D \\
&= \nabla d^H\theta_F - d^H_{\mathbf{T}} d^H\theta_F.
\end{aligned}$$

But this is zero also because the operators ∇ and $d^H_{\mathbf{T}}$ coincide when they are acting on basic (scalar) forms. It follows that, locally, $d^H\theta_F = d^H\alpha$ and $\beta = d^H f$, for some basic 1-form α and basic function f. The defining relation for β then further implies that

$$d^V D = \nabla(\theta_F - \alpha) - d^H(F - i_{\mathbf{T}}\alpha + f).$$

Putting $L = F - i_{\mathbf{T}}\alpha + f$, the difference between L and F is an affine function of the velocities, so both functions have the same Hessian g, and also $\theta_L = \mathrm{D}^V L = \theta_F - \alpha$. It now readily follows that the relation (19) holds true, which concludes our proof in view of Proposition 1. □

It is worthwhile listing the coordinate expressions for the necessary and sufficient conditions of Theorem 1. They call for a (non-singular) symmetric matrix $g_{ij}(q,v)$ and a function $D(q,v)$ such that

$$V_k(g_{ij}) = V_j(g_{ik}) \tag{30}$$

$$\Gamma(g_{ij}) - g_{ik}\Gamma^k_j - g_{jk}\Gamma^k_i = V_i V_j(D) \tag{31}$$

$$g_{ik}\Phi^k_j - g_{jk}\Phi^k_i = H_i V_j(D) - H_j V_i(D). \tag{32}$$

The classical Helmholtz conditions for the multiplier are recovered when we put $D = 0$, of course.

Let us now turn to the case of forces of gyroscopic type as in (3).

Proposition 2. *The second-order field Γ represents a gyroscopic system of type (3) if and only if there is a regular function $L \in C^\infty(TQ)$ and a basic 2-form $\omega \in \bigwedge^2(Q)$ such that*

$$\nabla\theta_L = d^H L + i_{\mathbf{T}}\omega. \tag{33}$$

Proof. The proof is straightforward, by a simple coordinate calculation or an argument like that in Proposition 1. □

As a preliminary remark: it is easy to verify in coordinates that for a basic 2-form ω, we have

$$\mathrm{D}^V i_{\mathbf{T}}\omega = \omega, \qquad d^V i_{\mathbf{T}}\omega = 2\,\omega, \qquad d^V i_{\mathbf{T}} d^H\omega = 3\,d^H\omega. \tag{34}$$

It follows by taking a vertical exterior derivative of (33) that this time $d^H\theta_L$ will not vanish but must be basic, specifically we must have

$$d^H\theta_L = \omega. \tag{35}$$

Theorem 2. *The second-order field Γ represents a gyroscopic system of type (3) if and only if there exists a basic 2-form $\omega \in \bigwedge^2(Q)$ and a symmetric type $(0,2)$ tensor g along τ such that $\mathrm{D}^V g$ is symmetric and g and ω further satisfy*

$$\nabla g = 0, \tag{36}$$

$$\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T = i_{\mathbf{T}} d^H\omega, \tag{37}$$

where Φ is the Jacobi endomorphism of Γ.

Proof. Assuming we are in the situation described by Proposition 2, we define g as before by $g = \mathrm{D}^V\mathrm{D}^V L$, or $g(X,Y) = \mathrm{D}^V_X\theta_L(Y) = \mathrm{D}^V_Y\theta_L(X)$, from which the usual symmetry of $\mathrm{D}^V g$ follows. Acting with ∇ on g and following the pattern of the proof of Theorem 1, we get $\nabla g(X,Y) = d^H\theta_L(Y,X) + \mathrm{D}^V i_{\mathbf{T}}\omega(X,Y)$, which is zero in view of (34) and (35). Finally, for the horizontal exterior derivative of (33), the modifications are that the left-hand side produces a term $\nabla\omega$ in view of (35), while the second term on the right gives $d^H i_{\mathbf{T}}\omega = d^H_{\mathbf{T}}\omega - i_{\mathbf{T}} d^H\omega$, and since $\nabla = d^H_{\mathbf{T}}$ on basic forms we end up with the relation

$$i_{d^V\Phi}\theta_L - d^V i_\Phi\theta_L = i_{\mathbf{T}} d^H\omega,$$

which is the desired result (37) in view of Lemma 1.

For the sufficiency, we observe as before that g is a Hessian, say $g = \mathrm{D}^V\mathrm{D}^V F$, and that also $\mathrm{D}^H g$ will be symmetric, which in exactly the same way implies that $d^H\theta_F$ is basic. Still following the pattern of Theorem 1, $\nabla g = 0$ will now imply that $\beta := \nabla\theta_F - d^H F - i_{\mathbf{T}} d^H\theta_F$ is a basic 1-form. In computing $d^H d^H\theta_F$, the modification is that

$$d^H d^H\theta_F = \tfrac{1}{3} d^V i_{d^V\Phi}\theta_F = \tfrac{1}{3} d^V (d^V i_\Phi\theta_F + i_{\mathbf{T}} d^H\omega) = d^H\omega,$$

in view of the last of (34). Since $d^H\theta_F$ and ω are basic, this expresses that their difference is closed and thus locally exact: $d^H\theta_F = \omega + d^H\alpha$ for some basic 1-form α. The computation of $d^H\beta$ leads as before to the conclusion that β is closed, thus locally $\beta = d^H f$ for some function f on Q. Using this double information, we find that

$$i_{\mathbf{T}} d^H\theta_F = i_{\mathbf{T}}\omega + \nabla\alpha - d^H i_{\mathbf{T}}\alpha,$$

and subsequently

$$\begin{aligned} 0 &= \nabla\theta_F - d^H F - i_{\mathbf{T}} d^H\theta_F - d^H f \\ &= \nabla(\theta_F - \alpha) - d^H(F - i_{\mathbf{T}}\alpha + f) - i_{\mathbf{T}}\omega. \end{aligned}$$

This is a relation of type (33), with $L = F - i_{\mathbf{T}}\alpha + f$, which concludes the proof. $\square$

In coordinates, in comparison with the dissipative case of Theorem 1, the conditions (31) and (32) are replaced in the gyroscopic case by

$$\Gamma(g_{ij}) = g_{ik}\Gamma^k_j + g_{jk}\Gamma^k_i \tag{38}$$

$$g_{ik}\Phi^k_j - g_{jk}\Phi^k_i = \tfrac{1}{2}\left(\frac{\partial\omega_{ij}}{\partial q^k} + \frac{\partial\omega_{jk}}{\partial q^i} + \frac{\partial\omega_{ki}}{\partial q^j}\right) v^k \tag{39}$$

with $\omega_{ij}(q) = -\omega_{ji}(q)$.

Remark: when $d\omega = 0$, the conditions of Theorem 2 reduce to the standard Helmholtz conditions for a multiplier g. This should not come as a surprise, since the local exactness of ω then implies that the gyroscopic forces are actually of the type of the Lorentz force of a magnetic field, for which it is known that a generalized potential can be introduced to arrive at a standard Lagrangian representation.

It is worth noting that in the sufficiency part of the proof the condition $\nabla g = 0$, given that g and $\mathrm{D}^V g$ are symmetric, is used to show the existence of a basic 1-form β such that $\nabla\theta_F = d^H F + i_{\mathbf{T}} d^H\theta_F + \beta$, where $d^H\theta_F$ is a basic 2-form. The condition involving Φ then has the role of ensuring that F can be modified by the addition of a function affine in the fibre coordinates so as to eliminate the β term. This suggests that it might be interesting to examine the effect of ignoring the Φ condition. When we do so we obtain the following result.

Proposition 3. *For a given second-order field Γ, the existence of a non-singulsr symmetric type $(0,2)$ tensor g along τ such that $\mathrm{D}^V g$ is symmetric and $\nabla g = 0$ is necessary and sufficient for there to be a regular function L, a basic 1-form β and a basic 2-form ω such that $\nabla\theta_L = d^H L + i_{\mathbf{T}}\omega + \beta$, that is to say, such that the equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q^i} = \omega_{ki}(q)\dot q^k + \beta_i(q), \quad \omega_{ki} = -\omega_{ik}.$$

are equivalent to those determined by Γ.

Proof. It remains to show that $\nabla g = 0$ still holds when $\nabla\theta_L = d^H L + i_{\mathbf{T}}\omega + \beta$. Since β is basic, $\mathrm{D}^V\beta = 0$, from which it follows easily that both of the formulas $d^H\theta_L = \omega$ and $\nabla g(X,Y) = d^H\theta_L(Y,X) + \mathrm{D}^V i_{\mathbf{T}}\omega(X,Y)$ continue to hold, so that $\nabla g = 0$ as before. □

One point of interest about this result is that it concerns a subset of the full Helmholtz conditions. Unlike Theorems 1 and 2 above, but like the full Helmholtz conditions, it involves conditions on the multiplier only, and in this respect it anticipates the results to be found in the following section.

An analogous result has been obtained by different methods in [19] (see Proposition 3.13 and the immediately following remarks). This is the partial result that we mentioned in the introduction.

4 Reduction to conditions on the multiplier only

We have seen in the previous section that Theorems 1 and 2 produce the direct analogues of the Helmholtz conditions (15) of the standard inverse problem of Lagrangian mechanics. It is quite natural that the extra elements in our analysis, namely the function D, respectively the 2-form ω, make their appearance in these covering generalizations. Quite surprisingly, however, one can go a step further in the generalizations and eliminate the dependence on D or ω all together, to arrive at necessary and sufficient conditions involving the multiplier g only. This is what we will derive now, but it is a rather technical issue, for which we will therefore prepare the stage by proving a number of auxiliary results first. We recall that, as in the relation (9), a notation like $\sum_{X,Y,Z}$ in what follows always refers to a cyclic sum over the indicated arguments.

Lemma 2. *If $F \in C^\infty(TQ)$, $\theta_F = d^V F$ and $g = \mathrm{D}^V\mathrm{D}^V F$ then*

$$d_R^V\theta_F(X,Y,Z) = \sum_{X,Y,Z} g(R(X,Y),Z), \tag{40}$$

$$d_R^H\theta_F(X,Y,Z) = \sum_{X,Y,Z} \mathrm{D}^H_{R(X,Y)}\theta_F(Z). \tag{41}$$

Proof. In view of the fact that $d^V d^V = 0$, $d_R^V\theta_F$ reduces to $d^V i_R d^V F$, and using (9) we then get

$$\begin{aligned} d_R^V\theta_F(X,Y,Z) &= \sum_{X,Y,Z} \mathrm{D}^V_X(i_R d^V F)(Y,Z) = \sum_{X,Y,Z} (i_{\mathrm{D}^V_X R} d^V F + i_R \mathrm{D}^V_X d^V F)(Y,Z) \\ &= \sum_{X,Y,Z} \Big[d^V F(\mathrm{D}^V_X R(Y,Z)) + g(X, R(Y,Z)) \Big] \\ &= \sum_{X,Y,Z} g(R(X,Y),Z) + d^V F(d^V R(X,Y,Z)). \end{aligned}$$

Taking into account the fact that $3d^V R = d^V d^V \Phi = 0$, the first result follows. For the second there are two terms to compute. For the first we have

$$
\begin{aligned}
d^H i_R \theta_F(X,Y,Z) &= \sum_{X,Y,Z} \mathrm{D}^H_X (i_R \theta_F)(Y,Z) \\
&= \sum_{X,Y,Z} (i_{\mathrm{D}^H_X R}\theta_F + i_R \mathrm{D}^H_X \theta_F)(Y,Z) = \sum_{X,Y,Z} \mathrm{D}^H_X \theta_F(R(Y,Z)),
\end{aligned}
$$

since the first term of the second line vanishes in view of the Bianchi identity $d^H R = 0$ [20]. Secondly,

$$
i_R d^H \theta_F(X,Y,Z) = \sum_{X,Y,Z} d^H \theta_F(R(X,Y),Z) = \sum_{X,Y,Z} \Big[\mathrm{D}^H_{R(X,Y)} \theta_F(Z) - \mathrm{D}^H_Z \theta_F(R(X,Y)) \Big].
$$

Adding these two expressions gives the desired result (41). □

Lemma 3. *If* $\mathrm{D}^H g$ *is symmetric then*

$$
d^H(\Phi \lrcorner\, g - (\Phi \lrcorner\, g)^T)(X,Y,Z) = \sum_{X,Y,Z} g(\nabla R(X,Y),Z). \tag{42}
$$

Proof. We have

$$
\begin{aligned}
d^H(\Phi \lrcorner\, g - (\Phi \lrcorner\, g)^T)(X,Y,Z) &= \sum_{X,Y,Z} \mathrm{D}^H_X(\Phi \lrcorner\, g - (\Phi \lrcorner\, g)^T)(Y,Z) \\
= \sum_{X,Y,Z} & (\mathrm{D}^H_X \Phi \lrcorner\, g + \Phi \lrcorner\, \mathrm{D}^H_X g - (\mathrm{D}^H_X \Phi \lrcorner\, g)^T - (\Phi \lrcorner\, \mathrm{D}^H_X g)^T)(Y,Z) \\
= \sum_{X,Y,Z} & \Big[g(\mathrm{D}^H_X \Phi(Y),Z) - g(\mathrm{D}^H_X \Phi(Z),Y) \Big] + \sum_{X,Y,Z} \Big[\mathrm{D}^H_X g(\Phi Y,Z) - \mathrm{D}^H_X g(\Phi Z,Y) \Big].
\end{aligned}
$$

Making use of the cyclic sum freedom in the second and fourth term, and of the symmetry of $\mathrm{D}^H g$ in the third, the right-hand side reduces to

$$
\sum_{X,Y,Z} g(\mathrm{D}^H_X \Phi(Y) - \mathrm{D}^H_Y \Phi(X), Z) = \sum_{X,Y,Z} g(d^H \Phi(X,Y),Z),
$$

which proves our statement in view of $d^H \Phi = \nabla R$. □

Lemma 4. *If* g *and* $\mathrm{D}^V g$ *are both symmetric then*

$$
\begin{aligned}
&[\nabla, \mathrm{D}^H] g\,(X,Y,Z) - [\nabla, \mathrm{D}^H] g\,(Y,X,Z) \\
&\qquad = \mathrm{D}^V_Z(\Phi \lrcorner\, g - (\Phi \lrcorner\, g)^T)(X,Y) - \sum_{X,Y,Z} g(R(X,Y),Z).
\end{aligned}
$$

Proof. The commutator $[\nabla, \mathrm{D}^H]$ is rather complicated when it comes to its action on a symmetric type $(0,2)$ tensor g. It reads (see for example [9] where it was already used):

$$\begin{aligned}[\nabla, \mathrm{D}^H]g\,(X,Y,Z) = \mathrm{D}^V_{\Phi X} g(Y,Z) - 2g(R(X,Y),Z) - 2g(R(X,Z),Y) \\ + g(\mathrm{D}^V_X\Phi(Y),Z) + g(\mathrm{D}^V_X\Phi(Z),Y).\end{aligned}$$

Subtracting the same expression with X and Y interchanged, it is however a fairly simple computation, using the symmetry of $\mathrm{D}^V g$ and the property $d^V\Phi = 3R$, to arrive at the desired result. $\square$

Lemma 5. *For all $F \in C^\infty(TQ)$ we have*

$$\mathrm{D}^H\mathrm{D}^V\mathrm{D}^V F(X,Y,Z) - \mathrm{D}^H\mathrm{D}^V\mathrm{D}^V F(Y,X,Z) = \mathrm{D}^V_Z d^H d^V F(X,Y). \tag{43}$$

Proof. This is in fact a variation of a formula which was already used in proving that $d^H\theta_F$ is basic in the second part of the proof of Theorem 1. We have to appeal again to the general formula (28), applied to $\mathrm{D}^V F = d^V F = \theta_F$. After swapping the last two arguments in each term on the left in (43), a direct application of this formula easily leads to the result. $\square$

Theorem 3. *The second-order field Γ represents a dissipative system of type (2) if and only if there exists a symmetric type $(0,2)$ tensor g along τ such that both $\mathrm{D}^V g$ and $\mathrm{D}^H g$ are symmetric and*

$$\sum_{X,Y,Z} g(R(X,Y),Z) = 0. \tag{44}$$

Proof. Assume first that the conditions of Theorem 1 hold true. So g and $\mathrm{D}^V g$ are symmetric and as before, since ∇ preserves the symmetry of $\mathrm{D}^V g$ and also $\mathrm{D}^V\nabla g = \mathrm{D}^V\mathrm{D}^V\mathrm{D}^V D$ is manifestly symmetric, we conclude that $\mathrm{D}^H g$ is symmetric. Moreover, if F is any function such that $g = \mathrm{D}^V\mathrm{D}^V F$, we know from Lemma 1 that

$$\Phi \lrcorner\, g - (\Phi \lrcorner\, g)^T = i_{d^V\Phi}\theta_F - d^V i_\Phi \theta_F.$$

It then follows from the last condition in Theorem 1 that

$$0 = d^V d^V d^H D = d^V i_{d^V\Phi}\theta_F = 3\, d^V_R \theta_F,$$

so that the first statement in Lemma 2 implies (44).

For the converse, symmetry of g and $\mathrm{D}^V g$ imply that ∇g and $\nabla\mathrm{D}^V g$ are symmetric, and since in addition $\mathrm{D}^H g$ is symmetric, we conclude that $\mathrm{D}^V\nabla g$ is symmetric, which means that ∇g is also a Hessian (see Lemma 1), say $\nabla g = \mathrm{D}^V\mathrm{D}^V D$ for some function D. Next, we look at the statement of Lemma 4 in which the last term vanishes here by assumption. We have that $\nabla\mathrm{D}^H g$ is symmetric, so that the left-hand side reduces to

$$-\mathrm{D}^H\mathrm{D}^V\mathrm{D}^V D(X,Y,Z) + \mathrm{D}^H\mathrm{D}^V\mathrm{D}^V g(Y,X,Z).$$

Combining the results of Lemma 4 and Lemma 5 we conclude that the 2-form

$$\beta := \Phi \lrcorner\, g - (\Phi \lrcorner\, g)^T + d^H d^V D$$

is basic. Now from the last of the properties (10) and Lemma 2 applied to ∇g, which is determined by θ_D, we know that

$$d^H d^H d^V D = d^V_R \theta_D = \sum_{X,Y,Z} \nabla g(R(X,Y),Z).$$

This in turn, making use also of the result of Lemma 3, gives rise to the following calculation:

$$\begin{aligned}
d^H\beta &= \sum_{X,Y,Z} g(\nabla R(X,Y),Z) + \sum_{X,Y,Z} \nabla g(R(X,Y),Z) \\
&= \nabla\Big(\sum_{X,Y,Z} g(R(X,Y),Z)\Big) \\
&\qquad - \sum_{X,Y,Z} g(R(\nabla X,Y),Z) - \sum_{X,Y,Z} g(R(X,\nabla Y),Z) - \sum_{X,Y,Z} g(R(X,Y),\nabla Z) \\
&= -\sum_{X,Y,Z} \left[g(R(\nabla Z,X),Y) + g(R(Y,\nabla Z),X) + g(R(X,Y),\nabla Z) \right].
\end{aligned}$$

The expression between square brackets in the last line is zero because of (44), with vector arguments X, Y and ∇Z; it follows that β is closed, thus locally $\beta = d^H\alpha$ for some basic 1-form α. Putting $\widetilde{D} = D - i_{\mathbf{T}}\alpha$, we have $\nabla g = \mathrm{D}^V\mathrm{D}^V D = \mathrm{D}^V\mathrm{D}^V \widetilde{D}$, and $\beta - d^H d^V D = -d^H d^V \widetilde{D}$, so that $\Phi \lrcorner\, g - (\Phi \lrcorner\, g)^T = d^V d^H \widetilde{D}$ and all conditions of Theorem 1 are satisfied. □

The results of Theorem 3 deserve some further comments. Establishing necessary and sufficient conditions for the existence of a Lagrangian is in a way the easy part of the inverse problem; the hard part is the study of formal integrability of these conditions, for which a number of different techniques exist (see for example [2], [15, 16], [28]). If we go back to the standard Helmholtz conditions (15), for example, two of the first integrability conditions one encounters are the symmetry of $\mathrm{D}^H g$ and the algebraic condition (44). So in the standard situation, if a g exists satisfying (15), these properties will automatically hold true: it seems to us noteworthy that these two integrability conditions make their appearance in the dissipative case as part of the starting set of necessary and sufficient conditions. It is further worth observing that the case of Rayleigh dissipation can be characterized by the further restriction that $\mathrm{D}^V\nabla g = 0$. Indeed, since $\nabla g = \mathrm{D}^V\mathrm{D}^V D$, this extra condition will imply that D must be quadratic in the velocities.

The coordinate expressions of the conditions in Theorem 3 are, apart from (30),

$$H_i(g_{jk}) - H_j(g_{ik}) + g_{il}\Gamma^l_{jk} - g_{jl}\Gamma^l_{ik} = 0 \tag{45}$$

$$g_{ij}R^j_{kl} + g_{lj}R^j_{ik} + g_{kj}R^j_{li} = 0, \tag{46}$$

where $\Gamma^l_{jk} = V_k(\Gamma^l_j)$ and $R^k_{ij} = H_j(\Gamma^k_i) - H_i(\Gamma^k_j) = \frac{1}{3}(V_i(\Phi^k_j) - V_j(\Phi^k_i))$.

We now proceed in the same way for the gyroscopic case.

Theorem 4. *If the second-order field* Γ *represents a gyroscopic system of type (3) then there exists a symmetric type* $(0,2)$ *tensor* g *along* τ *such that* $\mathrm{D}^V g$ *is symmetric,* $\nabla g = 0$ *and*

$$\left(\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T\right)(X,Y) = \sum_{X,Y,\mathbf{T}} g(R(X,Y),\mathbf{T}). \tag{47}$$

The converse is true as well, provided we assume that $\Phi\lrcorner\, g$ *is smooth on the zero section of* $TQ \to Q$.

Proof. Assume we have a g and ω satisfying the conditions of Theorem 2. Acting on the condition (37) with d^V, the left-hand side reduces, as in the proof of the preceding theorem, to $3d^V_R\theta_F$ for any F such that $g = \mathrm{D}^V\mathrm{D}^V F$. For the right-hand side, we get $d^V i_{\mathbf{T}} d^H\omega = 3d^H\omega$. Hence $d^H\omega = d^V_R\theta_F$, and (37) can be written as

$$\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T = i_{\mathbf{T}} d^V_R\theta_F. \tag{48}$$

Making use of Lemma 2 the result now immediately follows.

Conversely, (47) obviously implies (48) for any F such that $g = \mathrm{D}^V\mathrm{D}^V F$. $\mathrm{D}^V g$ symmetric and $\nabla g = 0$ imply that $\mathrm{D}^H g$ is symmetric and then also $[\nabla, \mathrm{D}^H]g$ is symmetric. It follows from Lemma 4 that

$$\mathrm{D}^V_Z(\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T)(X,Y) = \sum_{X,Y,Z} g(R(X,Y),Z), \qquad \forall X,Y,Z.$$

In particular, taking Z to be $\mathbf{T}$ and using Lemma 2 again plus (48), we obtain

$$\mathrm{D}^V_{\mathbf{T}}(\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T) = \Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T.$$

This asserts that $\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T$ is homogeneous of degree 1 in the fibre coordinates. The additional smoothness assumption then further implies linearity in the fibre coordinates, so that there exists a basic 3-form ρ such that $\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T = i_{\mathbf{T}}\rho$. There are two conclusions we can draw from this by taking appropriate derivatives. On the one hand, taking the horizontal exterior derivative and using Lemma 3 we obtain

$$\sum_{X,Y,Z} g(\nabla R(X,Y),Z) = (d^H i_{\mathbf{T}}\rho)(X,Y,Z) = (\nabla\rho - i_{\mathbf{T}} d^H\rho)(X,Y,Z).$$

On the other, knowing that $\mathrm{D}^V_Z i_{\mathbf{T}}\rho = i_Z\rho$ for any Z and appealing once more to the general conclusion of Lemma 4, we see that actually $\rho(X,Y,Z) = \sum_{X,Y,Z} g(R(X,Y),Z)$, from which it follows in view of $\nabla g = 0$ that $\nabla\rho(X,Y,Z) = \sum_{X,Y,Z} g(\nabla R(X,Y),Z)$. The conclusion from the last displayed equation is that $i_{\mathbf{T}} d^H\rho = 0$. But then $0 = \mathrm{D}^V_X i_{\mathbf{T}} d^H\rho = i_X d^H\rho$ for all X, so that $d^H\rho = 0$ and locally $\rho = d^H\omega$ for some basic ω. It follows that

$$\Phi\lrcorner\, g - (\Phi\lrcorner\, g)^T = i_{\mathbf{T}} d^H\omega,$$

which completes the proof. $\square$

In contrast with the preceding theorem, the condition (47) which makes its appearance here is not one which is directly familiar from the integrability analysis of the standard Helmholtz conditions. But indirectly, when $\omega = 0$, the left-hand side vanishes and the fact that this is also the case for the right-hand side follows from the integrability condition (44).

The coordinate expressions of the conditions in Theorem 4, in addition to (30) and (38), are

$$g_{lj}\Phi^j_k - g_{kj}\Phi^j_l = (g_{ij}R^j_{kl} + g_{lj}R^j_{ik} + g_{kj}R^j_{li})v^i. \tag{49}$$

Before embarking on examples, it is worth emphasizing a fundamental advantage of our intrinsic approach: we are not restricted to the coordinate expressions in natural bundle coordinates listed so far, if there are good reasons to work in a non-standard frame. This is the case, for example, in applications where it is appropriate to work with so-called quasi-velocities. Quasi-velocities are just fibre coordinates in TQ with respect to a non-standard frame $\{X_i\}$ of vector fields on Q (which also constitute a basis for the module of vector fields along τ). All conditions we have encountered so far may be projected onto such a frame and rewritten in terms of the quasi-velocities. For example, take the condition (19) we started from in the preceding section. It can be expressed as follows:

$$\begin{aligned}
0 &= (\nabla\theta_L - d^H L - d^V D)(X_i) \\
&= \Gamma(\theta_L(X_i)) - \theta_L(\nabla X_i) - X_i^{\mathrm{H}}(L) - X_i^{\mathrm{V}}(D) \\
&= \Gamma(X_i^{\mathrm{V}}(L)) - (\nabla X_i)^{\mathrm{V}}(L) - X_i^{\mathrm{H}}(L) - X_i^{\mathrm{V}}(D) \\
&= \Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) - X_i^{\mathrm{V}}(D),
\end{aligned}$$

where X_i^{C} stands for the complete lift of the vector field X_i. Quasi-velocities w^i can be thought of as the components of $\mathbf{T}$ with respect to some anholonomic frame $\{X_i\}$ of vector fields on Q. One can show (see e.g. [7]) that the complete and vertical lifts of such a frame, expressed in the coordinates (q, w), take the form

$$X_i^{\mathrm{C}} = X_i^j \frac{\partial}{\partial q^j} - A^j_{ik} v^k \frac{\partial}{\partial w^j}, \quad X_i^{\mathrm{V}} = \frac{\partial}{\partial w^i},$$

where $X_i = X_i^j \partial/\partial q^j$ and $[X_i, X_j] = A^k_{ij} X_k$. The condition (19) now becomes

$$\Gamma\left(\frac{\partial L}{\partial w^i}\right) - X_i^j \frac{\partial L}{\partial q^j} + A^j_{ik} w^k \frac{\partial L}{\partial w^j} = \frac{\partial D}{\partial w^i}.$$

These are the Boltzmann-Hamel equations referred to in [17], where, since the results the same authors obtained in [18] were expressed only in standard coordinates, all conditions had to be rederived from scratch. Needless to say, one can also recast any of the other coordinate-free conditions we have obtained in terms of quasi-velocities.

5 Illustrative examples and concluding remarks

We start with a simple linear system with two degrees of freedom, which will serve us well to illustrate a number of features of the results we have obtained. Consider the system

$$\ddot{q}_1 = -aq_1 - bq_2 - \omega\dot{q}_1, \tag{50}$$

$$\ddot{q}_2 = bq_1 - aq_2 + \omega\dot{q}_2, \tag{51}$$

where a, b and ω are constant, non-zero parameters. The only non-zero connection coefficients are

$$\Gamma^1_1 = \tfrac{1}{2}\omega = -\Gamma^2_2,$$

and we obtain

$$\Phi^1_1 = \Phi^2_2 = a - \tfrac{1}{4}\omega^2, \qquad \Phi^1_2 = b = -\Phi^2_1.$$

Since Φ is constant, the curvature tensor R is zero so that condition (44) is satisfied (in fact it is void anyway in view of the dimension). It follows from Theorem 3 that any constant symmetric g should be a multiplier for a representation of the given system in the form (2). We consider three such non-singular matrices:

$$g^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad g^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad g^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $g^{(1)}$, it is easy to verify that with

$$L_1 = \tfrac{1}{2}(\dot{q}_1^2 - \dot{q}_2^2) - \tfrac{1}{2}a(q_1^2 - q_2^2) - bq_1q_2,$$
$$D_1 = -\tfrac{1}{2}\omega(\dot{q}_1^2 + \dot{q}_2^2),$$

we have a representation of the given system in the form (2). In the case that $\omega = 0$, $g^{(1)}$ is still a multiplier for the standard inverse problem and L_1 then becomes a genuine Lagrangian. Also $g^{(2)}$, which changes the order of the equations, is a multiplier in that case, leading to an alternative Lagrangian for the same reduced system. But that Lagrangian cannot serve for a dissipative representation of the full system. Instead, we have to take

$$L_2 = \dot{q}_1\dot{q}_2 - aq_1q_2 - \tfrac{1}{2}b(q_2^2 - q_1^2) + \tfrac{1}{2}\omega(q_1\dot{q}_2 - q_2\dot{q}_1),$$

and then $D_2 = 0$. We discover here that the given system is variational, with L_2 as Lagrangian. For $g^{(3)}$ the situation is different again. This time, this is not a multiplier for the reduced system ($\omega = 0$), it violates the condition that $\Phi\lrcorner\, g$ must be symmetric. But for the full system, we can simply take a kinetic energy Lagrangian and then make a suitable adaptation for D. Explicitly,

$$L_3 = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2),$$
$$D_3 = -a(q_1\dot{q}_1 + q_2\dot{q}_2) + b(q_1\dot{q}_2 - q_2\dot{q}_1) + \tfrac{1}{2}\omega(\dot{q}_2^2 - \dot{q}_1^2).$$

Let us now look at the same system from the gyroscopic point of view. Since $R = 0$, the rather peculiar condition (47) of Theorem 4 reduces to the usual Φ-condition and Theorem 4 simply states the standard Helmholtz conditions for the existence of a multiplier. In other words, any multiplier for a representation in the form (3) will be a multiplier for a variational description as well. Of the non-singular, constant $g^{(i)}$ we considered before, only $g^{(2)}$ satisfies the conditions now, and we can take

$$L_4 = \dot{q}_1\dot{q}_2 - aq_1q_2 - \tfrac{1}{2}b(q_2^2 - q_1^2),$$

with the 2-form $\omega\, dq_1 \wedge dq_2$ to satisfy the requirements of Theorem 2. It should of course not come as a surprise that we must have a variational formulation here as well, since we are in the situation described in the remark towards the end of Section 3. In fact we have already found the Lagrangian for this variational formulation: it is the function L_2.

For a second example, with $n = 3$, consider the non-linear system

$$\ddot{q}_1 = q_2\dot{q}_1\dot{q}_3, \tag{52}$$

$$\ddot{q}_2 = \dot{q}_3^2, \tag{53}$$

$$\ddot{q}_3 = \dot{q}_1^2 - q_2^{-1}\dot{q}_2\dot{q}_3. \tag{54}$$

From (12) one easily verifies that

$$\left(\Phi^i_j\right) = \begin{pmatrix} -\frac{1}{4}q_2^2\dot{q}_3^2 & -\frac{3}{4}\dot{q}_1\dot{q}_3 & \frac{1}{4}q_2^2\dot{q}_1\dot{q}_3 + \frac{3}{4}\dot{q}_1\dot{q}_2 \\ -\dot{q}_1\dot{q}_3 & \frac{1}{2}q_2^{-1}\dot{q}_3^2 & -\frac{1}{2}q_2^{-1}\dot{q}_2\dot{q}_3 + \dot{q}_1^2 \\ \frac{1}{2}q_2\dot{q}_1\dot{q}_3 + \frac{1}{2}q_2^{-1}\dot{q}_1\dot{q}_2 & -\frac{1}{4}q_2^{-2}\dot{q}_2\dot{q}_3 - \frac{1}{2}q_2^{-1}\dot{q}_1^2 & -\frac{1}{2}q_2\dot{q}_1^2 + \frac{1}{4}q_2^{-2}\dot{q}_2^2 \end{pmatrix}$$

and the curvature tensor $R = \frac{1}{3}d^V\Phi$ is given by

$$\begin{aligned} R = &-\Big(\tfrac{1}{8}\dot{q}_3 dq_1\wedge dq_2 - (\tfrac{1}{8}\dot{q}_2 + \tfrac{1}{8}q_2^2\dot{q}_3)dq_1\wedge dq_3 - \tfrac{1}{4}\dot{q}_1 dq_2\wedge dq_3\Big)\otimes\frac{\partial}{\partial q_1} \\ &+\Big(\tfrac{1}{2}\dot{q}_1 dq_1\wedge dq_3 - \tfrac{1}{4}q_2^{-1}\dot{q}_1 dq_2\wedge dq_3\Big)\otimes\frac{\partial}{\partial q_2} \\ &-\Big(\tfrac{1}{4}q_2^{-1}\dot{q}_1 dq_1\wedge dq_2 + \tfrac{1}{4}q_2\dot{q}_1 dq_1\wedge dq_3 - \tfrac{1}{8}q_2^{-2}\dot{q}_2 dq_2\wedge dq_3\Big)\otimes\frac{\partial}{\partial q_3} \end{aligned}$$

The multiplier problem is already quite complicated for a system of this kind and it is not our intention here to explore all possible solutions. For simplicity, therefore, we limit ourselves in the dissipative case (2) to analyzing the existence of a diagonal multiplier g which depends on the coordinates q_i only. With such an ansatz, the curvature condition (44) in Theorem 3 reduces to

$$g_{33} = (g_{11} - 2g_{22})q_2,$$

and the requirement that $\mathrm{D}^H g$ should be symmetric subsequently imposes that $g_{11} = 4\,g_{22} =$ constant. Hence, up to a constant factor, we are reduced to the possibility that

$$g_{11} = 4, \qquad g_{22} = 1, \qquad g_{33} = 2q_2.$$

As was mentioned in the previous section, the conditions imposed so far are also integrability conditions in the standard inverse problem so that, starting from the same ansatz, this g would also be the only candidate for a standard Lagrangian representation of the system. When we compute ∇g now, we get

$$\Big((\nabla g)_{ij}\Big) = \begin{pmatrix} 4q_2\dot q_3 & 0 & 4q_2\dot q_1 \\ 0 & 0 & 0 \\ 4q_2\dot q_1 & 0 & 0 \end{pmatrix}.$$

Since $\nabla g \neq 0$, our candidate cannot lead to a variational formulation. On the other hand, Theorem 3 is satisfied, so there must exist a D for a dissipative representation. From the requirement (24) in Theorem 1, one easily verifies that such a D must satisfy

$$V_1(D) = 4q_2\dot q_1\dot q_3 + h_1, \qquad V_2(D) = h_2, \qquad V_3(D) = 2q_2\dot q_1^2 + h_3,$$

where the h_i are as yet arbitrary functions of the coordinates. The final requirement (25) of Theorem 1 then shows that the h_i can be taken to be zero. Thus,

$$L = \tfrac{1}{2}\Big(4\dot q_1^2 + \dot q_2^2 + 2q_2\dot q_3^2\Big) \quad \text{and} \quad D = 2q_2\dot q_1^2\dot q_3,$$

provide a solution for the inverse problem of type (2) for the given system.

Concerning the inverse problem of type (3), it is less appropriate to look for a diagonal g (as the example with $n = 2$ has shown), but even if we extend our search to a general g depending on the q_i only, the conditions of Theorem 4 have no non-singular solution.

Consider, finally, the system

$$\ddot q_1 = b\dot q_1\dot q_4, \tag{55}$$
$$\ddot q_2 = \dot q_2\dot q_4, \tag{56}$$
$$\ddot q_3 = (1-b)\dot q_1\dot q_2 + bq_2\dot q_1\dot q_4 - bq_1\dot q_2\dot q_4 + (b+1)\dot q_3\dot q_4, \tag{57}$$
$$\ddot q_4 = 0, \tag{58}$$

with $-1 < b < 1$ and $b \neq 0$. These equations can be interpreted as the geodesic equations of the canonical connection associated with a certain Lie group G, which is uniquely defined by $\nabla_X Y = \frac{1}{2}[X,Y]$, where X and Y are left-invariant vector fields. In the case of the above system, the Lie group is listed as $A_{4,9b}$ in [12], and it was shown (see also [1]) that the system does not have a variational formulation. This is a consequence of the integrability condition (44) which can only be satisfied by multipliers for which $g_{13} = g_{23} = g_{33} = 0$. But then, the Φ-condition in (15) leads automatically to $g_{34} = 0$ so that there is no non-singular solution.

Notice that the system is invariant for translations in the q_3 and q_4 direction; it is therefore reasonable that we limit ourselves in our search for non-conservative representations to multipliers with the same symmetry. In the dissipative case, after haven taken the same curvature condition into account, the $\mathrm{D}^H g$-condition leads to the further restrictions

$$\frac{\partial g_{34}}{\partial \dot q_3} = 0, \qquad \frac{\partial g_{34}}{\partial \dot q_3} - g_{34} = 0,$$

among others, from which again $g_{34} = 0$ follows, with the same negative conclusion. In the gyroscopic case, one can show that the condition (47) cannot be satisfied for a multiplier with coefficients depending on the coordinates q_i only.

Some final comments are in order. In the case of linear systems such as our first example, it frequently happens that a multiplier for the inverse problem exists which is a function of time only (see for example [25]), so we will briefly sketch here how our present theory can be extended to general, potentially time-dependent second-order systems. First of all, the extension of the calculus along $\tau : TQ \to Q$ to a time-dependent setting has been fully developed in [29], which also contains the analogues of the conditions (14) and (15) for the inverse problem. In all generality, we are then talking about a calculus of forms along the projection $\pi : \mathbb{R} \times TQ \to \mathbb{R} \times Q$ say. But as has been observed for example in [9], the extra time-component in this setting does not really play a role when it comes to studying the Helmholtz conditions and their integrability. That is to say: one has to use dt and the contact forms $dq^i - \dot{q}^i dt$ as local basis for forms along π and a suitable dual basis for the vector fields which includes the given second-order system Γ; important geometrical objects such as the Jacobi endomorphism Φ will pick up an extra term for sure, but when restricted to act on vector fields without Γ-component, all formulas of interest formally look the same. It is therefore not so hard to apply a suitably reformulated version of the present theory to time-dependent systems, when needed. That important formulas formally look the same will be seen also in the final observations in the appendix, where the setting is essentially time-dependent, though the approach adopted there is quite different again from the calculus along π we are referring to here.

Appendix

In this appendix we will relate our results to those obtained by Kielau et al. in [17, 18], especially the latter; but first we wish to derive those results anew, in a way which allows us to explain an interesting feature of them which was mentioned in [18] but not fully dealt with there.

The problem discussed in [17, 18] differs in several ways from the one which has been the subject of our paper, the most important of which is that it is assumed there that a system of second-order ordinary differential equations is given in implicit form $f_i(t, q, \dot{q}, \ddot{q}) = 0$, and the problem posed is to find necessary and sufficient conditions on the functions f_i such that the equations may be written in the form (2), where L and D are allowed to be time-dependent. That is, the question is whether the equations are of Lagrangian type with dissipation as they stand, rather then whether they may be made equivalent to such equations by a choice of multiplier.

It is probably most satisfactory to approach the inverse problem for a second-order system given in implicit form by using the methods associated with variational sequences, rather than the techniques employed in the body of the paper. Fortunately we will need only the rudiments of such methods, one version of which we now briefly describe; justification for the unsupported claims we make can be found in [31], for example.

We deal with the (trivial) fibred manifold $\pi : Q \times \mathbb{R} \to \mathbb{R}$, and its infinite jet bundle $J^\infty(\pi)$; this may seem a bit extravagant when we are interested only in second-order equations, but is convenient for technical reasons. However, all functions and forms under consideration will be of finite type (i.e. depend on finitely many variables). We take coordinates (t, q^i) on $\mathbb{R} \times Q$; the jet coordinates are written $\dot q^i$, $\ddot q^i$ and so on. We denote the contact 1-forms by

$$\theta^i = dq^i - \dot q^i dt, \quad \dot\theta^i = d\dot q^i - \ddot q^i dt, \quad \ddot\theta^i = d\ddot q^i - \dddot q^i dt, \quad \ldots .$$

We need two exterior-derivative-like operators on exterior forms on $J^\infty(\pi)$. The first is the vertical differential d_V (not to be confused with d^V), which is defined by

$$d_V f = \frac{\partial f}{\partial q^i}\theta^i + \frac{\partial f}{\partial \dot q^i}\dot\theta^i + \frac{\partial f}{\partial \ddot q^i}\ddot\theta^i + \cdots , \quad d_V dt = d_V\theta^i = d_V\dot\theta^i = d_V\ddot\theta^i = \ldots = 0.$$

The key properties of d_V are that $d_V^2 = 0$, and that d_V is locally exact. The second operator is the variational differential δ, about which we need to say just the following. First, for a Langrangian $L(t, q, \dot q)$

$$\delta(Ldt) = E_i(L)\theta^i \wedge dt$$

where the $E_i(L)$ are the Euler-Lagrange expressions. The 2-forms which, like $\delta(Ldt)$, are linear combinations of the $\theta^i \wedge dt$ are called source forms in [31] and dynamical forms in [19]. Secondly, for any source form $\varepsilon = f_i\theta^i \wedge dt$, $f_i = f_i(t, q, \dot q, \ddot q)$,

$$\delta\varepsilon = -\tfrac{1}{2}(r_{ij}\theta^i \wedge \theta^j + s_{ij}\theta^i \wedge \dot\theta^j + t_{ij}\theta^i \wedge \ddot\theta^j) \wedge dt,$$

where the coefficients are given by

$$\begin{aligned}
r_{ij} &= \frac{\partial f_i}{\partial q^j} - \frac{\partial f_j}{\partial q^i} - \tfrac{1}{2}\frac{d}{dt}\left(\frac{\partial f_i}{\partial \dot q^j} - \frac{\partial f_j}{\partial \dot q^i}\right) + \tfrac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial f_i}{\partial \ddot q^j} - \frac{\partial f_j}{\partial \ddot q^i}\right)\\
s_{ij} &= \frac{\partial f_i}{\partial \dot q^j} + \frac{\partial f_j}{\partial \dot q^i} - 2\frac{d}{dt}\left(\frac{\partial f_j}{\partial \ddot q^i}\right)\\
t_{ij} &= \frac{\partial f_i}{\partial \ddot q^j} - \frac{\partial f_j}{\partial \ddot q^i}.
\end{aligned}$$

Again, $\delta^2 = 0$ and δ is locally exact. With a source form $\varepsilon = f_i\theta^i \wedge dt$, $f_i = f_i(t, q, \dot q, \ddot q)$, one associates the second-order system $f_i = 0$, and conversely; so that $\delta\varepsilon = 0$ is necessary and sufficient for the second-order system $f_i = 0$ to be locally of Euler-Lagrange type. The vanishing of the coefficients r_{ij}, s_{ij} and t_{ij} are the (classical) Helmholtz conditions (see e.g. [18, 19, 24]).

By considering the transformation properties of the jet coordinates and the contact forms under transformations of the form $\bar q^i = \bar q^i(t, q)$, $\bar t = t$ one can show that the set of forms spanned by $\{dt, \theta^i, \dot\theta^i\}$ with coefficients which are functions of t, q and $\dot q$ is well-defined. We call forms like this first-order forms. Note that d_V maps first-order forms to first-order forms. Moreover, one proves exactness of d_V by using essentially the homotopy operator for the de Rham complex for the variables q, $\dot q$, ..., treating t as a parameter. It follows that if α is of first order and satisfies $d_V\alpha = 0$ then there is a *first-order* form β such that $\alpha = d_V\beta$.

The first step in applying these concepts to dissipative systems is to characterize dissipative force terms using them.

Proposition 4. *The first-order source form*

$$\Delta = \frac{\partial D}{\partial \dot q^i}\theta^i \wedge dt$$

satisfies $\delta\Delta = d_V\Delta$. Conversely, if ε is a first-order source form such that $\delta\varepsilon = d_V\varepsilon$ then $\varepsilon = \Delta$ for some first-order function D.

Proof. Applying the formula for δ acting on a source form one finds that

$$\begin{aligned}
\delta\Delta &= -\frac{\partial^2 D}{\partial q^j \partial \dot q^i}\theta^i \wedge \theta^j \wedge dt - \frac{\partial^2 D}{\partial \dot q^i \partial \dot q^j}\theta^i \wedge \dot\theta^j \wedge dt \\
&= \left(\frac{\partial}{\partial q^j}\left(\frac{\partial D}{\partial \dot q^i}\right)\theta^j + \frac{\partial}{\partial \dot q^j}\left(\frac{\partial D}{\partial \dot q^i}\right)\dot\theta^j\right) \wedge \theta^i \wedge dt \\
&= d_V\Delta.
\end{aligned}$$

Conversely, if $\varepsilon = f_i\theta^i \wedge dt$ is of first order then $t_{ij} = 0$ and

$$s_{ij} = \frac{\partial f_i}{\partial \dot q^j} + \frac{\partial f_j}{\partial \dot q^i}.$$

But

$$d_V\varepsilon = \frac{\partial f_j}{\partial q^i}\theta^i \wedge \theta^j \wedge dt - \frac{\partial f_i}{\partial \dot q^j}\theta^i \wedge \dot\theta^j \wedge dt,$$

and so if $\delta\varepsilon = d_V\varepsilon$ then

$$s_{ij} = \frac{\partial f_i}{\partial \dot q^j} + \frac{\partial f_j}{\partial \dot q^i} = 2\frac{\partial f_i}{\partial \dot q^j} \quad \text{or} \quad \frac{\partial f_j}{\partial \dot q^i} = \frac{\partial f_i}{\partial \dot q^j},$$

so that there is a function $D = D(t, q, \dot q)$ such that $f_i = \partial D/\partial \dot q^i$. The terms in $\theta^i \wedge \theta^j \wedge dt$ then agree. $\square$

Now take ε to be the source form representing the given equations. If they are of Euler-Lagrange type with a dissipative term then $\varepsilon = \delta(L dt) - \Delta$, so $\delta\varepsilon = \delta\Delta = d_V\Delta$. Then $\delta\varepsilon$ is of first order, and furthermore $d_V\delta\varepsilon = 0$. These are necessary conditions for the given system to take the desired form. They are in fact sufficient also, as we now show.

Theorem 5. *A system of second-order ordinary differential equations $f_i(t, q, \dot{q}, \ddot{q}) = 0$ may be written locally as*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial D}{\partial \dot{q}^i}$$

for some first-order functions L and D if and only if the corresponding source form ε is such that $\delta\varepsilon$ is of first order and satisfies $d_V\delta\varepsilon = 0$.

Proof. It remains to prove sufficiency. By the local exactness of d_V we may assume that $\delta\varepsilon = d_V\alpha$ for some first-order 2-form α (not necessarily a source form), which is determined only up to the addition of a d_V-exact form. Contact 2-forms β can be ignored in α since $d_V\beta$ then contains no dt terms and therefore cannot contribute to $\delta\varepsilon$. Moreover, if we put

$$\alpha = (\lambda_i\theta^i + \mu_i\dot{\theta}^i) \wedge dt,$$

since $\delta\varepsilon$ contains no $\dot{\theta}^i \wedge \dot{\theta}^j$ terms either, we must have

$$\frac{\partial \mu_j}{\partial \dot{q}^i} = \frac{\partial \mu_i}{\partial \dot{q}^j},$$

so that $\mu_i = \partial\psi/\partial\dot{q}^i$ for some function $\psi(t, q, \dot{q})$. Hence, up to the d_V-exact term $d_V(\psi dt)$, α is of the form

$$\alpha = \left(\lambda_i - \frac{\partial \psi}{\partial q^i}\right)\theta^i \wedge dt,$$

i.e. is a source form, say $\alpha = \nu_i\theta^i \wedge dt$. Since $\delta\varepsilon$ is of first order we must have $t_{ij} = 0$, whence s_{ij} is symmetric in i and j. It follows that the coefficient of $\theta^i \wedge \dot{\theta}^j$ in $d_V\alpha$ must be symmetric in i and j, whence $\nu_i = -\partial D/\partial\dot{q}^i$ for some function $D = D(t, q, \dot{q})$, and $\alpha = -\Delta$. Then $\delta\varepsilon = d_V\alpha = -d_V\Delta = -\delta\Delta$, and so there is some L such that $\varepsilon = \delta(Ldt) - \Delta$, as required. $\square$

The point of interest that we mentioned at the beginning of this appendix is that this version of the result represents the conditions in part as the closure (under d_V) of a certain form, namely $\delta\varepsilon$. That it might be possible to state the conditions in such a way was raised speculatively in [18], but the form and operator were not specifically identified there.

Under the assumption that $\delta\varepsilon$ is of first order, so that $t_{ij} = 0$, it is easy to verify that the d_V-closure conditions are

$$\begin{aligned} 0 &= \frac{\partial r_{ij}}{\partial q^k} + \frac{\partial r_{jk}}{\partial q^i} + \frac{\partial r_{ki}}{\partial q^j} \\ \frac{\partial r_{ij}}{\partial \dot{q}^k} &= \frac{\partial s_{ik}}{\partial q^j} - \frac{\partial s_{jk}}{\partial q^i} \\ \frac{\partial s_{ij}}{\partial \dot{q}^k} &= \frac{\partial s_{ik}}{\partial \dot{q}^j}. \end{aligned}$$

It must not be forgotten that r_{ij} and s_{ij} are supposed to be of first order; in addition, $t_{ij} = 0$ means that we have

$$\frac{\partial f_i}{\partial \ddot{q}^j} = \frac{\partial f_j}{\partial \ddot{q}^i}.$$

These are the generalized Helmholtz conditions as given in [18]. However, it turns out that the first and last of the closure conditions are consequences of the other conditions, as we showed in [8]. It then follows easily that the following conditions are equivalent to those given above: $f_i = g_{ij}\ddot{q}^j + h_i$ with g_{ij} symmetric, where g_{ij}, h_i are of first order and further satisfy

$$\begin{aligned}
&\frac{\partial g_{ij}}{\partial \dot{q}^k} = \frac{\partial g_{ik}}{\partial \dot{q}^k}\\
&\frac{\partial g_{ik}}{\partial q^j} - \tfrac{1}{2}\frac{\partial^2 h_i}{\partial \dot{q}^j \partial \dot{q}^k} = \frac{\partial g_{jk}}{\partial q^i} - \tfrac{1}{2}\frac{\partial^2 h_j}{\partial \dot{q}^i \partial \dot{q}^k}\\
&\sum_{i,j,k}\left(\frac{\partial^2 h_i}{\partial q^j \partial \dot{q}^k} - \frac{\partial^2 h_i}{\partial q^k \partial \dot{q}^j}\right) = 0,
\end{aligned}$$

where $\sum_{i,j,k}$ stands for the cyclic sum over the indices.

As the problem has been presented so far in this appendix, we must take g_{ij} and h_i as given; the equations above provide a test for determining whether the given second-order system can be put into the required form. However, it is now possible to regard these equations from the alternative point of view: we set $h_i = g_{ij}f^j$, where the second-order system is given in normal form $\ddot{q}^i = f^i(t, q, \dot{q})$; we regard the f^i as known but the g_{ij} as to be determined; the equations above now become partial differential equations for the unknowns g_{ij}. We leave it to the reader to verify that they correspond (take the autonomous case for simplicity), in the order written, to the conditions of Theorem 3, namely $\mathrm{D}^V g$ is symmetric, $\mathrm{D}^H g$ is symmetric, and $\sum_{X,Y,Z} g(R(X,Y),Z) = 0$.

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References

[1] J. E. Aldridge, G. E. Prince, W. Sarlet and G. Thompson, An EDS approach to the inverse problem in the calculus of variations, *J. Math. Phys.* **47** (2006) 103508.

[2] I. Anderson and G. Thompson, The inverse problem of the calculus of variations for ordinary differential equations, *Mem. Amer. Math. Soc.* **473** (1992).

[3] I. Bucataru and M. F. Dahl, Semi-basic 1-forms and Helmholtz conditions for the inverse problem of the calculus of variations, *J. Geom. Mech.* **1** (2009) 159–180.

[4] J. F. Cariñena and E. Martínez, Generalized Jacobi equation and inverse problem in classical mechanics, in *Group Theoretical Methods in Physics*, Proc. 18th Int. Colloquium 1990, Moscow, USSR, Vol. II, V. V. Dodonov and V. I. Manko eds. (Nova Science Publishers 1991) 59–64.

[5] M. Crampin, On the differential geometry of the Euler-Lagrange equations and the inverse problem of Lagrangian dynamics, *J. Phys. A: Math. Gen.* **14** (1981) 2567–2575.

[6] M. Crampin, Tangent bundle geometry for Lagrangian dynamics, *J. Phys. A: Math. Gen.* **16** (1983) 3755–3772.

[7] M. Crampin and T. Mestdag, Anholonomic frames in constrained dynamics, to appear in *Dynamical Systems* (2009). DOI: 10.1080/14689360903360888.

[8] M. Crampin, T. Mestdag and W. Sarlet, On the generalized Helmholtz conditions for Lagrangian systems with dissipative forces, to appear in *Z. Angew. Math. Mech.* (2010).

[9] M. Crampin, W. Sarlet, E. Martínez, G. B. Byrnes and G. E. Prince, Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations, *Inverse Problems* **10** (1994) 245–260.

[10] R. de Ritis, G. Marmo, G. Platania and P. Scudellaro, Inverse problem in classical mechanics: dissipative systems, *Int. J. Theor. Phys.* **22** (1983) 931–946.

[11] A. Frölicher and A. Nijenhuis, Theory of vector-valued differential forms, *Proc. Ned. Acad. Wetensch. Sér. A* **59** (1956) 338–359.

[12] R. Ghanam, G. Thompson, and E. J. Miller, Variationality of four-dimensional Lie group connections, *J. Lie Theory* **14** (2004) 395-425.

[13] H. Goldstein, *Classical Mechanics* (2nd. edition) (Addison-Wesley 1980).

[14] J. Grifone, Structure presque tangente et connexions I, *Ann. Inst. Fourier* **22** (1972) 287–334.

[15] J. Grifone and Z. Muzsnay, On the inverse problem of the variational calculus: existence of Lagrangians associated with a spray in the isotropic case, *Ann. Inst. Fourier* **49** (1999) 1387–1421.

[16] J. Grifone and Z. Muzsnay, *Variational Principles for Second-order Differential Equations*, (World Scientific 2000).

[17] U. Jungnickel, G. Kielau, P. Maisser and A. Müller, A generalization of the Helmholtz conditions for the existence of a first-order Lagrangian using nonholonomic velocities, *Z. Angew. Math. Mech.* **89** (2009) 44–53.

[18] G. Kielau and P. Maisser, A generalization of the Helmholtz conditions for the existence of a first-order Lagrangian, *Z. Angew. Math. Mech.* **86** (2006) 722–735.

[19] O. Krupková and G. E. Prince, Second order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations, in *Handbook of Global Analysis* D. Krupka and D. Saunders eds. (Elsevier 2008) 837–904.

[20] E. Martínez, J. F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Differential Geom. Appl.* **2** (1992) 17–43.

[21] E. Martínez, J. F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection II, *Differential Geom. Appl.* **3** (1993) 1–29.

[22] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo and C. Rubano, The inverse problem in the calculus of variations and the geometry of the tangent bundle, *Phys. Rep.* **188** (1990) 147–284.

[23] R. M. Rosenberg, *Analytical Dynamics of Discrete Systems* (Plenum Press 1977).

[24] R. M. Santilli, *Foundations of Theoretical Mechanics I. The Inverse Problem in Newtonian Mechanics*, (Spinger 1978).

[25] W. Sarlet, On linear nonconservative systems derivable from a variational principle, *Hadronic J.* **3** (1980) 765–793.

[26] W. Sarlet, The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics, *J. Phys. A: Math. Gen.* **15** (1982) 1503–1517.

[27] W. Sarlet, F. Cantrijn and M. Crampin, A new look at second-order equations and Lagrangian mechanics, *J. Phys. A: Math. Gen.* **17** (1984) 1999–2009.

[28] W. Sarlet, M. Crampin and E. Martínez, The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations, *Acta Appl. Math.* **54** (1998) 233–273.

[29] W. Sarlet, A. Vandecasteele, F. Cantrijn and E. Martínez, Derivations of forms along a map: the framework for time-dependent second-order equations, *Differential Geom. Appl.* **5** (1995) 171–203.

[30] J. Szilasi, A setting for spray and Finsler geometry, in *Handbook of Finsler Geometry* Vol. 2, P. L. Antonelli ed. (Kluwer 2003) 1185–1426.

[31] R. Vitolo, Variational sequences, in *Handbook of Global Analysis* D. Krupka and D. Saunders eds. (Elsevier 2008) 1115–1163.