Second-order dynamical systems of Lagrangian type with dissipation

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Abstract. We give a coordinate-independent version of the smallest set of necessary and sufficient conditions for a given system of second-order ordinary differential equations to be of Lagrangian form with additional dissipative forces.

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1 Introduction

Many mechanical systems are subject to conservative forces, i.e. forces $F = -\nabla V$ that can be derived from a potential $V(q)$. It is well-known that the equations of motion of such systems take the form of Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \tag{1}$$

where $L(q,\dot{q}) = T(q,\dot{q}) - V(q)$ is the Lagrangian of the system and where T stands for its kinetic energy. The equations of motion for the harmonic oscillator, $m\ddot{q} = -kq$, for example, are of this form, with $V(q) = \frac{1}{2}kq^2$ and $T(\dot{q},q) = \frac{1}{2}m\dot{q}^2$. Lagrangian systems have many interesting features. To name just one of them, it is easy to see that the kinetic energy,

$$E = T + V = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}kq^2,$$

remains constant along solutions.

Not all mechanical systems have an Euler-Lagrangian description. For example, the equations of motion for the damped oscillator are only a small modification of those of the harmonic oscillator, $m\ddot{q} = -kq - b\dot{q}$, but the additional force, $F_2 = -b\dot{q}$ is not conservative. It is not possible to add an extra potential to the Lagrangian, not even one that depends on velocities, to account for the force F_2. However, if we rewrite F_2 as $\partial D/\partial \dot{q}$, with $D(\dot{q},q) = -\frac{1}{2}b\dot{q}^2$, the equations of motion become of the (general) form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial D}{\partial \dot{q}^i}. \tag{2}$$

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Equations of the above type, although more general than the Euler-Lagrange equations, still exhibit interesting properties. For example, even though the energy $E = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2$ is no longer constant, we can still predict its qualitative behaviour along solutions. Indeed, now

$$\frac{dE}{dt} = 2D,$$

and one can conclude that the energy decays when time passes. This is a typical example of so-called dissipation, and equations of the form (2) are henceforward called Euler-Lagrangian equations with dissipative forces.

Both the equations of motion for the harmonic oscillator and for the damped oscillator are (systems of) differential equations of second-order, in general of the form $\Lambda_i(q, \dot{q}, \ddot{q}) = 0$. The inverse problem of the calculus of variations investigates whether or not a second-order system can be written as the Euler-Lagrange equations (1) of some regular Lagrangian L (see [6] for a recent review). In this paper we will investigate a generalization of this problem. Clearly, if the answer to the standard inverse problem is negative, it may still be of interest to know whether the given second-order equations can be brought in the form of Euler-Lagrange equations with dissipation (2). The goal of this paper is therefore to find necessary and sufficient conditions for a given set of functions $\Lambda_i(q, \dot{q}, \ddot{q})$ to take the form

$$\Lambda_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - \frac{\partial D}{\partial \dot{q}^i}$$

for functions $L(q, \dot{q})$ and $D(q, \dot{q})$.

In the literature one may sometimes find reference to 'Raleigh dissipation' (see e.g. [3]). We will reserve this terminology for a function D that is quadratic in the velocities with a positive or negative-definite coefficient matrix. It will not be necessary to make these additional assumptions in this paper.

2 The inverse problem for dissipative systems

The problem as we describe it above has, in fact, already been tackled in the paper [5] by Kielau and Maisser. In it, they give a set of necessary and sufficient conditions, but their result is not completely satisfactory, for the following three reasons. In the first place, as we shall show below, the conditions of [5] are not completely independent: some are even redundant. It is therefore natural to ask what is the smallest set of necessary and sufficient conditions.

Secondly, in a follow-up paper [4] a version of the conditions expressed in terms of quasi-velocities (nonholonomic velocities) is re-derived from scratch. This should be unnecessary: a truly satisfactory formulation of the conditions should be tensorial and thus independent of a choice of coordinates. It seems therefore better to use coordinate independent methods, i.e. methods that have their roots in differential geometry.

Thirdly, the conditions in [5] test whether a given system $\Lambda_i = 0$ is 'Lagrangian'. In case $D = 0$ (the standard inverse problem), there is, however, a more general approach. Remark first that, even in the dissipative case, it is obvious that for a function $\Lambda_i(\ddot{q}, \dot{q}, q)$ to be of the above described form it should be affine w.r.t. $\ddot{q}^j$, i.e. of the form

$$\Lambda_i(q, \dot{q}, \ddot{q}) = a_{ij}(q, \dot{q})\ddot{q}^j + B_i(q, \dot{q}).$$

Any non-singular matrix $M_j^i(q, \dot{q})$ will transform the set of equations $\Lambda_i = 0$ into the equivalent set $M_j^i \Lambda_i = 0$. Instead of wondering whether the given system $\Lambda_i = 0$ takes the Lagrangian form, it is therefore natural to ask the more general question whether we can find a system of Lagrangian type (1) within the class of all equivalent systems. Remark that, if the coefficient matrix a_{ij} is non-singular the second-order system $\Lambda_i = 0$ is even equivalent with a second-order system in normal form $\ddot{q}^i = f^i(\dot{q}, q)$. In that case, we can rephrase the more general problem as the search for a so-called multiplier g_{ij} (a non-singular matrix), which is such that the equivalent system $g_{ij}(\ddot{q}^j - f^j) = 0$ takes the form of the Euler-Lagrange equations (1) for some regular Lagrangian L. Obviously, when that is the case, $g_{ij}(q, \dot{q})$ will be the Hessian of the sought Lagrangian L with respect to differentiation by $\dot{q}$. The conditions for this to occur are well-known and are usually referred to as the Helmholtz conditions (see e.g. [6]). They form a mixed set of algebraic and PDE conditions for the unknown multiplier g_{ij}. In this paper we will show that the more general question can also be answered for systems with dissipative forces, i.e. with $D \neq 0$.

To conclude, starting from second-order equations in normal form

$$\ddot{q}^i = f^i(q, \dot{q}),$$

we wish to investigate the smallest set of (coordinate-independent) conditions for existence of a non-singular multiplier $(g_{ij}(q, \dot{q}))$ such that

$$g_{ij}(\ddot{q}^j - f^j) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - \frac{\partial D}{\partial \dot{q}^i}. \tag{3}$$

for some (regular) Lagrangian $L(q, \dot{q})$ and some $D(q, \dot{q})$.

Let $g_{ijk} = \partial g_{ij}/\partial \dot{q}^k$. It is easy to see that, as in the standard inverse problem, if a multiplier exists, it will be the Hessian of the desired Lagrangian. It is therefore natural to assume $g_{ij} = g_{ji}$ and $g_{ijk} = g_{jik}$, since then, there exists a function $K(q, \dot{q})$ such that

$$g_{ij} = \frac{\partial^2 K}{\partial \dot{q}^i \partial \dot{q}^j}.$$

All other functions L with that property are of the form $L = K + P_i \dot{q}^i + Q$, where $P_i(q)$ and $Q(q)$ are functions depending on q only.

We now derive conditions that will fix P_i and Q such that there exist functions L and D that will bring the equations in the desired form (2). Let, for now, Γ be short hand for the operator $\dot{q}^i \partial/\partial q^i + f^i \partial/\partial \dot{q}^i$. Let us define the functions $\kappa_i(q, \dot{q})$ by

$$\kappa_i = \Gamma\left(\frac{\partial K}{\partial \dot{q}^i}\right) - \frac{\partial K}{\partial q^i}.$$

Equation (3) is then equivalent with

$$\kappa_i + \left(\frac{\partial P_i}{\partial q^k} - \frac{\partial P_k}{\partial q^i}\right)v^k - \frac{\partial Q}{\partial q^i} = \frac{\partial D}{\partial \dot{q}^i}.$$

for some appropriate functions $P_i(q)$, $Q(q)$ and a $D(q, \dot{q})$. Obviously, if we set $\tilde{D} = D - (\partial Q/\partial q^i)\dot{q}^i$, we have incorporated the term in Q into D, so we can forget about it, from now on.

We will now solve the problem in two steps. A function $\tilde{D}(q, \dot{q})$ with the above property will exist if and only if there exists basic functions $P_i(q)$ such that

$$\frac{\partial \kappa_i}{\partial \dot{q}^j} + \frac{\partial P_i}{\partial q^j} - \frac{\partial P_j}{\partial q^i} = \frac{\partial \kappa_j}{\partial \dot{q}^i} + \frac{\partial P_j}{\partial q^i} - \frac{\partial P_i}{\partial q^j}. \tag{4}$$

If we define functions $S_{ij}(q, \dot{q})$ by

$$S_{ij} = \frac{\partial \kappa_i}{\partial \dot{q}^j} - \frac{\partial \kappa_j}{\partial \dot{q}^i},$$

equation (4) is equivalent with

$$S_{ij} = 2 \left(\frac{\partial P_j}{\partial q^i} - \frac{\partial P_i}{\partial q^j} \right).$$

The necessary and sufficient conditions for the existence of functions P_i (depending only on q!) with the above property are

$$\frac{\partial S_{ij}}{\partial \dot{q}^k} = 0 \quad \text{and} \quad \sum_{cyclic} \frac{\partial S_{ij}}{\partial q^k} = 0.$$

Remark that we have used Poincaré's Lemma twice, so the result will only hold locally. We conclude:

Proposition 1. *The necessary and sufficient condition for the second-order system $\ddot{q}^i = f^i(q, \dot{q})$ to be of the form (2) is that there exists a non-singular multiplier matrix $g_{ij}(q, \dot{q})$ satisfying:*

$$g_{ij} = g_{ji}, \quad g_{ijk} = g_{jik}, \quad \frac{\partial S_{ij}}{\partial \dot{q}^k} = 0, \quad and \quad \sum_{i,j,k} \frac{\partial S_{ij}}{\partial q^k} = 0.$$

$\sum_{i,j,k}$ stands for the cyclic sum over the indices. The above presented proof may be found with more details in [2].

We will show that also the last two conditions can be expressed in terms of the multiplier matrix $g_{ij}(q, \dot{q})$ and its derivatives. Moreover, we will cast the conditions into a coordinate independent form. For that reason, we will introduce the necessary geometric machinery in the next section.

3 Some elements of the calculus of forms along the tangent bundle projection

We will use the tangent bundle $\tau : TM \to M$ of the configuration manifold M and natural coordinates $(q, \dot{q})$ on it. One can associate a so-called SODE field Γ on TM to the equations $\ddot{q}^i = f^i(q, \dot{q})$, namely

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + f^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}.$$

Remark that we have already encountered this operator in the proof of Proposition 1. In the next paragraphs we recall some of the basics of SODE geometry.

The vertical and the complete lift are are two canonical ways to lift a vector field $X = X^i(q)\partial/\partial q^i$ on M to a vector field on TM. They are, respectively,

$$X^V = X^i(q) \frac{\partial}{\partial \dot{q}^i} \quad \text{and} \quad X^c = X^i(q) \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}.$$

A SODE Γ defines a third horizontal lift (i.e. a non-linear connection on TM):

$$X \in \mathcal{X}(M) \quad \mapsto \quad X^H \in \mathcal{X}(TM) = \frac{1}{2}(X^c + [X^V, \Gamma]),$$

$$X^i(q)\frac{\partial}{\partial q^i} \quad \mapsto \quad X^i H_i = X^i \left(\frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial \dot{q}^j}\right), \quad \text{where } \Gamma_i^j = -\frac{1}{2}\frac{\partial f^j}{\partial \dot{q}^i}.$$

Many objects of interest, although living on TM (a manifold with dimension $2n$), are fully determined by components in dimension n. One way to interpret these objects is to view them as tensor fields "along the map $\tau : TM \to M$". The idea of what follows is that more efficiency in the calculations should come from tools and operations which directly act on forms and vector fields along τ. The main references for this section are [7, 8].

So-called vector fields along τ are sections of the pullback bundle $\tau^*\tau : \tau^*TM \to TM$. A vector field X along τ can alternatively be defined as a map $X : TM \to TM$ with the property that $\tau \circ X = \tau$. Likewise, a 1-form along τ is a map $\alpha : T^*M \to TM$ such that $\tau^* \circ \alpha = \tau$. These definitions extend naturally to tensor fields along τ. In general, a vector field (resp. 1-form) along τ has the following coordinate representation:

$$X = X^i(q, \dot{q})\frac{\partial}{\partial q^i}, \qquad \alpha = \alpha_i(q, \dot{q})\, dq^i.$$

We will denote the set of vector fields along τ by $\mathcal{X}(\tau)$ and the set of 1-forms along τ by $\bigwedge(\tau)$. The canonical vector field along τ given by $\mathbf{T} = id = \dot{q}^i \partial/\partial q^i$, is a particular example. Of course, also vector fields on M can be interpreted as vector fields along τ. In that context, we will refer to them as 'basic' vector fields along τ. In the same spirit, we will also call 1-forms on M basic.

We will show below that, given the SODE field Γ, the calculus along τ has all the features that e.g. the ordinary calculus of forms on a Riemannian manifold has: exterior derivative, covariant derivative and curvature.

Obviously, the vertical and horizontal lift procedures extend naturally to vector fields along τ. In fact, every $\xi \in \mathcal{X}(TM)$ has a unique decomposition in a vertical and horizontal part:

$$\xi = \xi_v{}^V + \xi_h{}^H \qquad \text{for some } \xi_v, \ \xi_h \in \mathcal{X}(\tau).$$

With the aid of the SODE connection on TM, one can construct a linear connection on the pullback bundle $\tau^*\tau$, i.e. a map

$$\mathrm{D} : \mathcal{X}(TM) \times \mathcal{X}(\tau) \to \mathcal{X}(\tau),$$

by means of the expression

$$\mathrm{D}_\xi X = ([\xi_h{}^H, X^V])_v + ([\xi_v{}^V, X^H])_h, \qquad \text{with } \xi \in \mathcal{X}(TM), \ X \in \mathcal{X}(\tau).$$

This is said to be a connection of Berwald type. If we first set, for $X \in \mathcal{X}(\tau)$,

$$\mathrm{D}_{X^V} = \mathrm{D}_X^V, \qquad \mathrm{D}_{X^H} = \mathrm{D}_X^H,$$

and define, for functions F on TM,

$$\mathrm{D}_X^V F = X^V(F), \qquad \mathrm{D}_X^H F = X^H(F),$$

and then further extend by duality its action to 1-forms, the action of the two operators D^V and D^H extends to tensor fields along τ of arbitrary type. The relevant coordinate expressions for these operators are

$$D_X^V F = X^i \, V_i(F), \qquad\qquad D_X^V \frac{\partial}{\partial q^i} = 0,$$

$$D_X^H F = X^i \, H_i(F), \qquad\qquad D_X^H \frac{\partial}{\partial q^i} = X^j V_i(\Gamma_j^k)\frac{\partial}{\partial q^k}.$$

Here, and in what follows, V_i is short for $\partial/\partial\dot{q}^i$. For example, for a function $L \in C^\infty(TM)$, $D^V D^V L = V_i(V_j(L))dq^i \otimes dq^j$.

There are two more important operators which contain a great deal of information about the dynamics Γ: a degree 0 derivation, called the dynamical covariant derivative ∇ and a (1,1) tensor field Φ along τ, called the Jacobi endomorphism. They appear naturally in the decomposition of certain vector fields on TM:

$$[\Gamma, X^V] = -X^H + (\nabla X)^V, \qquad [\Gamma, X^H] = (\nabla X)^H + \Phi(X)^V.$$

If we set $\nabla F = \Gamma(F)$ for functions $F \in C^\infty(TM)$, and if we use self-duality again, the action of ∇ can be extended to arbitrary tensor fields along τ. For computational purposes, we list here the coordinate expressions of ∇ and Φ:

$$\nabla\left(\frac{\partial}{\partial q^i}\right) = \Gamma_i^k \frac{\partial}{\partial q^k} \,, \qquad \nabla(dq^i) = -\Gamma_k^i \, dq^k$$

$$\Phi_j^i = -\frac{\partial f^i}{\partial q^j} - \Gamma_j^k \Gamma_k^i - \Gamma(\Gamma_j^i).$$

Finally, the (1,2)-tensor field $R(X,Y) = [X^h, Y^h]_v$ along τ plays the role of the curvature of the nonlinear connection. The coefficients of R can be computed to be

$$R_{ij}^k = H_j(\Gamma_i^k) - H_i(\Gamma_j^k) = \tfrac{1}{3}(V_i(\Phi_j^k) - V_j(\Phi_i^k)).$$

4 Coordinate-independent conditions

Let's come back to the conditions we had found in Proposition 1. The multiplier matrix $g_{ij}(q,\dot{q})$ can be interpreted as a (0,2) tensor field g along τ, given by

$$g = g_{ij}dq^i \otimes dq^j.$$

The first condition simply says that this tensor field is symmetric, $g(X,Y) = g(Y,X)$ for all $X, Y \in \mathcal{X}(\tau)$. The second condition states that also the tensor field $D^V g$ along τ should be symmetric in all its arguments. One can easily compute that S_{ij} can be expressed as

$$S_{ij} = 2\left(\frac{\partial^2 K}{\partial q^j \partial \dot{q}^i} - \frac{\partial^2 K}{\partial q^i \partial \dot{q}^j} - \Gamma_j^k g_{ki} + \Gamma_i^k g_{kj}\right).$$

With that, the third condition becomes

$$H_i(g_{jk}) - g_{mk}\Gamma_{ij}^m = H_k(g_{ji}) - g_{mi}\Gamma_{kj}^m.$$

Equivalently, this says that the tensor field $D^H g$ along τ is symmetric.

Also the fourth condition can be expressed entirely in terms of g. It is easy to see that the inconvenient terms in e.g.

$$\frac{\partial^3 K}{\partial q^i \partial q^j \partial \dot{q}^k}$$

all add up to zero. A tedious calculation further shows that what remains can be related to the curvature R of the non-linear connection. In fact, the fourth condition is equivalent to

$$\sum_{X,Y,Z} g(R(X,Y),Z) = 0,$$

where $\sum_{X,Y,Z}$ stands for the cyclic sum over the indicated arguments.

Theorem 1. *The second-order field Γ represents a dissipative system of type*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial D}{\partial \dot{q}^i}$$

if and only if there exists a (non-singular) symmetric type $(0,2)$ tensor g along τ such that

$$
\begin{aligned}
D^V_X g(Y,Z) &= D^V_Z g(Y,X), \\
D^H_X g(Y,Z) &= D^H_Z g(Y,X), \\
\sum_{X,Y,Z} g(R(X,Y),Z) &= 0.
\end{aligned}
$$

What is remarkable in this result is that the symmetry of $D^H g$ and the curvature condition $\sum_{X,Y,Z} g(R(X,Y),Z) = 0$, which make their appearance here, are known as integrability conditions for the set of Helmholtz conditions in the standard inverse problem (see the remark after Theorem 2 for the Helmholtz conditions, and [10] for a discussion on the integrability conditions).

Let's compare our result with the version of the conditions that was obtained in [5]. Kielau and Maisser start from a system of the form $\Lambda_i(q,\dot{q},\ddot{q}) = 0$. Their condition (2.3d) simply says that the functions Λ_i should be linear in $\ddot{q}$, i.e. of the form $\Lambda_i = g_{ij}\ddot{q}^j + B_i$ (as we already remarked above). Their condition (2.3a) says that g_{ij} should be symmetric. They further introduce the objects

$$
\begin{aligned}
r_{ij} &= \frac{\partial \Lambda_i}{\partial q^j} - \frac{\partial \Lambda_j}{\partial q^i} + \tfrac{1}{2}\frac{d}{dt}\left(\frac{\partial \Lambda_j}{\partial \dot{q}^i} - \frac{\partial \Lambda_i}{\partial \dot{q}^j}\right) \\
s_{ij} &= \tfrac{1}{2}\left(\frac{\partial \Lambda_i}{\partial \dot{q}^j} + \frac{\partial \Lambda_j}{\partial \dot{q}^i}\right) - \frac{d}{dt}\left(\frac{\partial \Lambda_j}{\partial \ddot{q}^i}\right),
\end{aligned}
$$

which, in general, depend on $\ddot{q}$ and $\dddot{q}$. By not allowing that to be the case (condition (2.3b) and (2.3c)), one recovers our conditions on the symmetry of the tensor fields $D^V g$ and $D^H g$. If one takes that into account, the objects s_{ij} and r_{ij} become, in our notations,

$$r_{ij} = g_{ik}\Phi^k_j - g_{jk}\Phi^k_i + s_{ik}\Gamma^k_j - s_{jk}\Gamma^k_i \qquad \text{and} \qquad s_{ij} = (\nabla g)_{ij}.$$

Their condition (2.3e), which is

$$\frac{\partial s_{ij}}{\partial \dot{q}^k} = \frac{\partial s_{ik}}{\partial \dot{q}^j},$$

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is equivalent with the symmetry of $D^V \nabla g$. However, since also $D^V g$ is symmetric, and given that the commutator $[\nabla, D^V] = -D^H$ holds, this is equivalent with the already assumed symmetry of $D^H g$. Condition (2.3e) is therefore superfluous. A straightforward calculation further shows that our curvature condition is their condition (2.3f), namely

$$\frac{\partial r_{ij}}{\partial \dot{q}^k} = \frac{\partial s_{ik}}{\partial q^j} - \frac{\partial s_{jk}}{\partial q^i}.$$

Finally, there is an second superfluous condition (3.3g), given by

$$\sum_{cyclic} \frac{\partial r_{ij}}{\partial q^k} = 0.$$

One can show that the above can be written as

$$\nabla \left(\sum_{X,Y,Z} g(R(X,Y),Z) \right) = 0.$$

and that it is therefore a consequence of the other conditions.

If we find a solution g for the conditions in Theorem 1, the sought Lagrangian L will be such that its Hessian is g. Theorem 1 does, however, not help to recover the dissipation function D from the multiplier g. For that, we can use the next theorem.

Theorem 2. *The second-order field Γ represents a dissipative system if and only if there exists a function D and a (non-singular) symmetric type $(0,2)$ tensor g along τ such that*

$$\nabla g = D^V D^V D,$$
$$D_X^V g(Y,Z) = D_Z^V g(Y,X),$$
$$\Phi \lrcorner g - (\Phi \lrcorner g)^T = d^V d^H D.$$

Remark that by putting the 'dissipation function' D equal to zero in the conditions of Theorem 2, we recover the necessary and sufficient 'Helmholtz' conditions of the standard inverse problem, see e.g. [8].

The proof of the theorem can be found in [9]. The operators d^V and d^H in the statement of Theorem 2 are so-called exterior derivatives for the calculus of forms along τ. Indeed, there is a canonically defined vertical 'exterior' derivative d^V, determined by

$$d^V F = V_i(F)\,dq^i \quad \forall F \in C^\infty(TM)$$
$$d^V \alpha = 0 \quad \text{for } \alpha \in \bigwedge^1(M).$$

The definition of the horizontal derivative d^H requires a non-linear connection, however. Any choice of a basis of *horizontal* vector fields

$$H_i = \frac{\partial}{\partial q^i} - \Gamma_i^j(q,\dot{q})\frac{\partial}{\partial v^j},$$

allows to construct a horizontal exterior derivative d^H given by

$$d^H F = H_i(F)\,dq^i, \qquad F \in C^\infty(TM)$$
$$d^H \alpha = d\alpha \quad \text{for } \alpha \in \bigwedge^1(M).$$

In particular, one can take that connection to be the one that comes with the SODE vector field Γ.

It is now an easy calculation to show that a coordinate-independent representation of the equations of motion (2) is given by

$$\nabla \theta_L - d^H L = d^V D, \tag{5}$$

where $\theta_L = d^V L$. If one plugs in a vector field $X = \partial/\partial q^i$ in the above expression one gets back the coordinate expression (2) in the natural bundle coordinates $(q^i, \dot{q}^i)$ for the tangent bundle $\tau : TM \to M$. As is the case for all conditions we have encountered in the theorems above, there is, however, no need to restrict to expressions in natural bundle coordinates. In some applications it is appropriate to work with 'quasi-velocities' , i.e. fibre coordinates w^i in TQ w.r.t. a non-standard basis $\{X_i\}$ of vector fields on Q. For example, for a dynamical system on a Lie group which is invariant under left or right translations, it may be more convenient to work with a basis of left or right invariant vector fields (see e.g. [1]). It may even be convenient to choose a frame that does not consist of basic vector fields. In the analysis of the integrability conditions of the standard inverse problem in [10] it turned out to be useful to use a frame of eigenvectors of the tensor field Φ along τ.

We now give an expression of the equation (5) in terms of quasi-velocities. If $\{X_i\}$ is a basis of vector fields on Q, we have

$$0 = (\nabla \theta_L - d^H L - d^V D)(X_i) = \Gamma(\theta_L(X_i)) - \theta_L(\nabla X_i) - X_i^{\mathrm{H}}(L) - X_i^{\mathrm{V}}(D)$$
$$= \Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) - X_i^{\mathrm{V}}(D).$$

If $X_i = X_i^j \partial/\partial q^j$ and $[X_i, X_j] = A_{ij}^k X_k$, then

$$X_i^{\mathrm{C}} = X_i^j \frac{\partial}{\partial q^j} - A_{ik}^j w^k \frac{\partial}{\partial w^j} \quad \text{and} \quad X_i^{\mathrm{V}} = \frac{\partial}{\partial w^i},$$

and we get therefore

$$\Gamma\left(\frac{\partial L}{\partial w^i}\right) - X_i^j \frac{\partial L}{\partial q^j} + A_{ik}^j w^k \frac{\partial L}{\partial w^j} = \frac{\partial D}{\partial w^i}.$$

These last equations (without the dissipation term) are known as the Boltzmann-Hamel equations in the literature and they form the starting point for the analysis of the 'generalization' of the problem of [5] in [4]. For us, both the Euler-Lagrange equations as the Boltzmann-Hamel equations are just two manifestations of the same dynamical equation on a manifold. Needless to say, any of the other conditions in our theorems can also be recast in terms of quasi-velocities.

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