Geometric calculus for second-order differential equations and generalizations of the inverse problem of Lagrangian mechanics

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Abstract. We review the main features of the geometric calculus which has been introduced over the past 15 years in the study of second-order ordinary differential equations and then explain how a recently introduced generalization of the inverse problem of Lagrangian mechanics can be very concisely dealt with by this calculus in an intrinsic way. This paper is an account of the lecture with the same title presented at the ICDVC-2010 conference in Hangzhou, May 12-14 (2010).

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1 Introduction

Differential geometry provides the tools for addressing qualitative questions about differential equations, which require a coordinate free answer. An example of such question is the following: when does a dynamical system decouple into subsystems of lower dimension, after an appropriate change of coordinates? Though this clearly concerns the search for special coordinates, the mere existence of such coordinates is a coordinate free property and the important issue then is to develop coordinate independent characteristics and tests, which will allow deciding about the possibility of decoupling prior to the identification of coordinates in which such decoupling effectively takes place. We can refer to [13] and [21], for example, for a study of complete separability of second-order ordinary differential equations, and to [20] for a more recent contribution to aspects of partial decoupling, but we shall not enter into this subject here. The main purpose of this paper is to present first a brief review of the geometric calculus which was developed for the study of second-order ordinary differential equations (SODEs for short), and has been used successfully in applications such as the one referred to above. We will illustrate secondly how this general theory works, and choose for this demonstration a more recent application which is about a generalization of the so-called inverse problem of Lagrangian mechanics. For most of our account, we restrict ourselves to autonomous systems for simplicity, which means that the natural environment for a differential geometric setting is a tangent bundle $\tau : TM \to M$.

The scheme of the paper is as follows. Basic elements of the geometry of a tangent bundle are recalled in section 2, including the additional structure coming from a SODE. In section 3, we try to explain the motivation for introducing the calculus of differential forms along the tangent bundle projection $\tau : TM \to M$ and sketch its main ingredients and the main classification theorem. This geometrical calculus was introduced in the PhD work of Eduardo Martínez [10] and was developed further in [11, 12]. In section 4, we see how the interplay between calculations on TM and the calculus along τ is extremely relevant to discover important derivations of degree zero. We briefly discuss the generalization of the calculus along τ to the framework for time-dependent SODEs in section 5, and give reference to some other generalizations. Section 6 provides an introduction to the *inverse problem of Lagrangian mechanics* and then passes to a sketch of 'the calculus at work', in the context of recent generalizations of this inverse problem.

2 Elements of the geometry of a tangent bundle

Consider a tangent bundle $\tau : TM \to M$, and let (q, v) denote natural coordinates on TM. There exist 'natural objects' on TM of the following kind: a dilation vector field $\Delta = v^i\, \partial/\partial v^i$ and, more importantly, a type $(1, 1)$ tensor field

$$S = dq^i \otimes \frac{\partial}{\partial v^i}$$

which characterizes the integrable almost tangent structure on TM and is commonly called the *vertical endomorphism* for its action on $\mathcal{X}(TM)$,

$$X = \mu^i \frac{\partial}{\partial q^i} + \nu^i \frac{\partial}{\partial v^i} \quad \rightsquigarrow \quad S(X) = \mu^i \frac{\partial}{\partial v^i}.$$

Furthermore, there are two ways of lifting vector fields on M canonically to vector fields on TM; they are called the *vertical* and *complete* lift (or prolongation) and are determined, in coordinates, by the following prescription: for $X = X^i(q)\partial/\partial q^i$,

$$X^V = X^i(q)\frac{\partial}{\partial v^i}, \quad X^c = X^i(q)\frac{\partial}{\partial q^i} + v^j \frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial v^i}.$$

More structure can only come from additional data, for example from a given SODE field Γ on TM.

A non-linear or Ehresmann connection on $\tau : TM \to M$ is a smooth procedure for defining at each point (q, v) of TM a 'horizontal subspace' of $T_{(q,v)}(TM)$, complementary to the space of vertical vectors. A SODE Γ is intrinsically determined by the condition $S(\Gamma) = \Delta$ and has a coordinate representation of the form

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v)\frac{\partial}{\partial v^i},$$

i.e. represents the second-order system of differential equations $\ddot{q}^i = f^i(q, \dot{q})$. The point is that such a SODE Γ canonically defines an Ehresmann connection on TM, as follows: from the property $(\mathcal{L}_\Gamma S)^2 = I$, one easily sees that

$$P_H = \frac{1}{2}(I - \mathcal{L}_\Gamma S), \quad P_V = \frac{1}{2}(I + \mathcal{L}_\Gamma S)$$

are complementary projection operators:

$$P_H{}^2 = P_H, \quad P_V{}^2 = P_V, \quad P_H \circ P_V = 0,$$

and their images determine the horizontal and vertical subspaces, respectively, of the tangent space at each point of TM. An alternative way of establishing a connection is to provide an intrinsic procedure for a *horizontal* lift of vector fields on M. This can be done here, with the aid of Γ, as follows

$$X \in \mathcal{X}(M) \quad \mapsto \quad X^H \in \mathcal{X}(TM) = \frac{1}{2}(X^c + [X^V, \Gamma]).$$

In coordinates, we have

$$X^H = X^i\, H_i, \quad \text{with} \quad H_i = \left(\frac{\partial}{\partial q^i}\right)^H = \frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial v^j}\,.$$

The functions

$$\Gamma_i^j = -\frac{1}{2}\frac{\partial f^j}{\partial v^i}$$

are the connection coefficients.

Some historical references for tangent bundle geometry and the SODE connection are [8], [5, 6] and [1].

3 The calculus of forms along $\tau : TM \to M$

The motivation for the introduction of the calculus along the tangent bundle projection stems from the following general observation. Many objects of interest in the study of second-order dynamical systems, although living on TM, a space with dimension $2n$, turn out to be fully determined by components which seem to come from the underlying base manifold which has only half of the dimension. Here are a few examples where this occurs.

Suppose Y is a symmetry vector field of a given SODE Γ, so that $[Y, \Gamma] = 0$. Then, if we represent Y in an arbitrary chart on TM by

$$Y = \mu^i(q, v)\frac{\partial}{\partial q^i} + \nu^i(q, v)\frac{\partial}{\partial v^i},$$

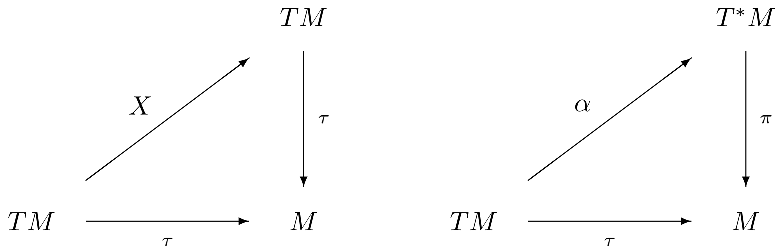
the symmetry requirement implies that $\nu^i = \Gamma(\mu^i)$ and the problem one has to solve in practice is to find solutions of second-order partial differential equations for the components μ^i, the so-called determining equations for symmetries, which are of the form $\Gamma^2(\mu^i) = Y(f^i)$. The object of interest in these equations, therefore, seems to be the 'differential operator' $X = \mu^i(q, v)\partial/\partial q^i$. This is neither a vector field on M nor on TM, however, it is a vector field along the tangent bundle projection τ, which needs to be handled with some care: as a derivation, X can only act on functions on M, but the set of such operators carries a module structure over the functions on TM. What the calculus along τ achieves in this context, is to provide an intrinsic, i.e. coordinate independent formulation of the equations $\Gamma^2(\mu^i) = Y(f^i)$; anticipating on concepts which will be developed in this section, the determining equations for symmetries will turn out to read $\nabla^2 X + \Phi(X) = 0$.

Another example exhibiting the same feature is the so-called Poincaré-Cartan 1-form θ_L of a Lagrangian second-order system with Lagrangian L. Its intrinsic definition, as a 1-form on TM reads $\theta_L = S(dL)$, so that in coordinates, $\theta_L = (\partial L/\partial v^i)\, dq^i$ and has only n components. So θ_L is a 1-form along τ and in that setting, it will have the representation $\theta_L = d^V L$. One may rightly come forward with the objection now that the more important object in modelling Lagrangian systems on TM is the Poincaré-Cartan 2-form $\omega_L := d\theta_L$, and the exterior derivative will of course introduce also dv-factors in the coordinate expression. However, it turns out that the $(2n \times 2n)$ skew-symmetric coefficient matrix of ω_L, at least when we express ω_L in the natural basis adapted to the SODE-connection, is completely determined by the $(n \times n)$-Hessian matrix $\partial^2 L/\partial v^i \partial v^j$; these are the components $g_{ij}(q, v)$

of a symmetric type $(0,2)$ tensor field along the tangent bundle projection, which can be defined (again anticipating on things to come) as $g := \mathrm{D}^V \mathrm{D}^V L$.

So, what we learn from these examples is that perhaps more efficiency in the calculations should come from tools and operations which directly act on forms and vector fields along τ. This does not mean, however, that the standard calculus of forms on TM has to be moved aside; on the contrary, as the survey of the theory below will amply demonstrate, progress most of the time comes from a close interaction between what happens along τ and appropriate lifts to corresponding standard calculations on TM.

We start our overview now by a summary of the classification of derivations of scalar and vector valued forms along τ (details can be found in [11, 12]). Intrinsically, a vector field X along τ (and similarly a 1-form α along τ) is a map defined by the following commutative scheme.



In coordinates:

$$X = X^i(q,v)\frac{\partial}{\partial q^i}, \qquad\qquad \alpha = \alpha_i(q,v)\, dq^i.$$

We denote by $\mathcal{X}(\tau)$ the set of vector fields along τ, $\bigwedge(\tau)$ the set of scalar forms and $V(\tau)$ the set of vector-valued forms along τ. They are modules over $C^\infty(TM)$. Typically, an element $L \in V^\ell(\tau)$ is of the form

$$L = \lambda^i \otimes \frac{\partial}{\partial q^i} \qquad \text{with} \quad \lambda^i = \lambda^i_{j_1\cdots j_\ell}(q,v)\, dq^{j_1} \wedge \cdots \wedge dq^{j_\ell} \in \textstyle\bigwedge^\ell(\tau).$$

Definition: $D : \bigwedge(\tau) \to \bigwedge(\tau)$ is a derivation of degree r if

1. $D(\bigwedge^p(\tau)) \subset \bigwedge^{p+r}(\tau)$
2. $D(\alpha + \lambda\,\beta) = D\alpha + \lambda\, D\beta, \qquad \lambda \in \mathbb{R}$
3. $D(\alpha \wedge \gamma) = D\alpha \wedge \gamma + (-1)^{pr}\alpha \wedge D\gamma, \qquad \alpha \in \bigwedge^p(\tau)$.

For the extension to vector valued forms, it suffices to know that a derivation D (of degree r) of $V(\tau)$ has an associated derivation of $\bigwedge(\tau)$, also denoted by D, such that in addition to the above rules, we have for $L \in V^\ell(\tau)$ and $\omega \in \bigwedge^p(\tau)$,

$$D(\omega \wedge L) = D\omega \wedge L + (-1)^{pr}\omega \wedge DL.$$

For all practical purposes, it is intuitively clear from the coordinate representation that every D of $\bigwedge(\tau)$ is in fact completely determined by its action on $C^\infty(TM)$ and on what we shall call *basic 1-forms*, which are 1-forms on M regarded as elements of $\bigwedge^1(\tau)$. Then, for a consistent extension of such a D to $V(\tau)$, one further has to specify an action on *basic vector fields*, i.e. vector fields on M regarded as elements of $\mathcal{X}(\tau)$.

For the standard calculus of forms on a manifold M, there exists a beautiful classification of derivations, established by Frölicher and Nijenhuis [4], which roughly goes as follows. Recall first of all that the algebra $\bigwedge(M)$ comes equipped with a canonical derivation of degree 1, namely the exterior derivative d. Then, two types of derivations can be distinguished:

- *derivations of type* $i_\star$, they are the ones which vanish on functions,
- *derivations of type* $d_\star$, which by definition commute with d.

For $L \in V^r(M)$, one defines the operator i_L, a derivation of degree $r-1$, by $i_L f = 0$ on functions $f \in C^\infty(M)$ and, for $\alpha \in \bigwedge^1(M)$,

$$i_L\alpha(X_1, \dots, X_r) = \alpha(L(X_1, \dots, X_r)).$$

The main theorems established in [4] then state that

- every type $i_\star$-derivation is of the form i_L for some L,
- every derivation of type $d_\star$ is the commutator of some i_L with the exterior derivative, i.e. is of the form $d_L := [i_L, d]$ for some L,
- finally, every derivation D has a unique decomposition in the form $D = i_{L_1} + d_{L_2}$, for some L_1, L_2.

For the calculus of forms along τ, the situation is fundamentally different because we do not have an overall exterior derivative at our disposal. What we do have is a canonically defined *vertical exterior derivative*, denoted by d^V and determined by

$$\begin{aligned} &d^V F = V_i(F)\, dq^i, \quad V_i := \frac{\partial}{\partial v^i}, \qquad F \in C^\infty(TM) \\ &d^V \alpha = 0 \quad \text{for } \alpha \in \textstyle\bigwedge^1(M), \qquad d^V\left(\frac{\partial}{\partial q^i}\right) = 0. \end{aligned}$$

But this can only be part of the story and is not sufficient to arrive at a classification theory. A full classification requires the availability of a connection on $\tau : TM \to M$. Indeed, any choice of a basis of *horizontal vector fields* on TM, say

$$H_i = \frac{\partial}{\partial q^i} - \Gamma^j_i(q, v)\frac{\partial}{\partial v^j},$$

allows us to construct a corresponding *horizontal exterior derivative* d^H, defined by

$$\begin{aligned} &d^H F = H_i(F)\, dq^i, \qquad F \in C^\infty(TM) \\ &d^H \alpha = d\alpha \quad \text{for } \alpha \in \textstyle\bigwedge^1(M), \quad d^H\left(\frac{\partial}{\partial q^i}\right) = V_i(\Gamma^k_j)\, dq^j \otimes \frac{\partial}{\partial q^k}. \end{aligned}$$

We are then led to consider four types of derivations:

- *derivations of type* $i_\star$: vanish on functions, are determined by some $L \in V(\tau)$ and are of the form i_L as before,
- *derivations of type* $d^V_\star$: are defined to be of the form $d^V_L := [i_L, d^V]$,
- *derivations of type* $d^H_\star$: similarly, defined to be of the form $d^H_L := [i_L, d^H]$,
- *derivations of type* $a_\star$: vanish on $\bigwedge(\tau)$ and thus are needed only for the extension to $V(\tau)$, they are written as a_Q for some $Q \in \bigwedge(\tau) \otimes V^1(\tau)$.

So, with the aid of a connection and the corresponding horizontal exterior derivative, the above four types of derivations again lead to a unique decomposition of arbitrary derivations of scalar and vector valued forms along τ. The statement is that every derivation D of $V(\tau)$, of degree r, has a unique decomposition in the form:

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3} + a_Q,$$

with $L_1 \in V^{r+1}(\tau)$, $L_2, L_3 \in V^r(\tau)$, $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$.

Note that the need for a connection is no handicap for our specific purposes, because a SODE Γ comes with its own canonical connection, as explained in the previous section. Perhaps we should make the remark also that such a classification theorem is not just an exercise for academical or aesthetical reasons. On the contrary, it is needed for very practical purposes. For example, in applications, there will always be equations involving different kinds of derivations; a study of the integrability of such equations prompts for knowledge about the commutators of such derivations, and this can only come from investigating the unique decomposition of such a commutator in the above sense. It is for this reason that a large portion of the analysis in [12] had to do with the study of commutators of the main ingredients of the theory of derivations. This is a quite technical matter, so that we will abstain from trying to summarize it here. But we will see in the sketch of the application to the generalized inverse problem later on, that one very quickly needs information about such commutators indeed.

In the next section, we discuss a particular class of derivations, namely those of degree zero which satisfy a duality property with respect to the pairing between vector fields and 1-forms along τ. They turn out to play a very important role in all applications.

4 Self-dual derivations of degree 0

Definition: A derivation D of $V(\tau)$, of degree 0, is said to be *self-dual* if $\forall X \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$

$$D\langle X, \alpha\rangle = \langle DX, \alpha\rangle + \langle X, D\alpha\rangle.$$

Such derivations immediately extend to tensor fields of arbitrary type by imposing a Leibnitz rule with respect to the tensor product. Observe also that they are completely known as soon as a consistent action on $C^\infty(TM)$ and $\mathcal{X}(\tau)$ is specified; the duality rule then determines the corresponding action on $\bigwedge^1(\tau)$.

Important self-dual derivations are the *vertical* and *horizontal covariant derivatives* D^V_X and D^H_X. In coordinates, they are determined by the following action on functions F and basic vector fields (and then further extend by duality):

$$\mathrm{D}^V_X F = X^i\, V_i(F), \qquad \mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0,$$

$$\mathrm{D}^H_X F = X^i\, H_i(F), \qquad \mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^j V_i(\Gamma^k_j) \frac{\partial}{\partial q^k}.$$

There are many ways to come to an intrinsic definition of these derivations. We want to focus on one of them, because it is related to a simple, yet very powerful procedure to discover interesting operators along τ. The point is simply this: the horizontal and vertical lifts from $\mathcal{X}(M)$ to $\mathcal{X}(TM)$ naturally extend to lifts of vector fields along τ; now every $\xi \in \mathcal{X}(TM)$ has a unique decomposition into a vertical and horizontal part which are necessarily lifts of vector fields along τ; we can write for example,

$$\xi = {\xi_v}^V + {\xi_h}^H \quad \text{for some } \xi_v,\ \xi_h \in \mathcal{X}(\tau).$$

Interesting computations involving vector fields on TM will in this way, by looking at the unique decomposition of the result of the computation, identify interesting operators on $\mathcal{X}(\tau)$. Applying this procedure to the brackets of horizontal and vertical lifts of vector fields along τ, we get

$$\begin{aligned}
[X^V, Y^V] &= ([X,Y]_V)^V, \\
[X^H, Y^V] &= (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H, \\
[X^H, Y^H] &= ([X,Y]_H)^H + R(X,Y)^V,
\end{aligned}$$

where

$$[X,Y]_V := \mathrm{D}^V_X Y - \mathrm{D}^V_Y X = \left(X^k V_k(Y^i) - Y^k V_k(X^i)\right) \frac{\partial}{\partial q^i},$$

$$[X,Y]_H := \mathrm{D}^H_X Y - \mathrm{D}^H_Y X = \left(X^k H_k(Y^i) - Y^k H_k(X^i)\right) \frac{\partial}{\partial q^i},$$

and R is the *curvature tensor* of the non-linear connection. It is the identification of the vertical and horizontal parts of $[X^H, Y^V]$ which unambiguously fixes the operators D^V_X and D^H_X.

There is more to D^V and D^H than meets the eye so far. Vector fields along τ are in fact sections of the so-called pullback bundle $\tau^*\tau : \ \tau^*TM \to TM$ and with the aid of the projection operators P_H and P_V of the connection on TM, one can construct in a direct way a corresponding linear connection on $\tau^*\tau$, i.e. an operator

$$\mathrm{D} : \mathcal{X}(TM) \times \mathcal{X}(\tau) \to \mathcal{X}(\tau),$$

which satisfies the conditions for a linear connection in the sense of Koszul. The defining relation reads: for $\xi \in \mathcal{X}(TM)$ and $X \in \mathcal{X}(\tau)$,

$$\mathrm{D}_\xi X = ([P_H(\xi), X^V])_v + ([P_V(\xi), X^H])_h.$$

This is said to be a connection of Berwald type [2], and we have

$$\mathrm{D}_{X^V} = \mathrm{D}^V_X, \qquad \mathrm{D}_{X^H} = \mathrm{D}^H_X.$$

Up to now, everything we said in this section is valid for any choice of a non-linear connection, needed to complete the classification of derivations of forms along τ. From now on, we assume that the horizontal distribution or non-linear connection is the canonical one associated to a given SODE Γ. Then, there are two important operators which contain a great deal of information about the given second-order dynamics:

- a degree 0 derivation, called the *dynamical covariant derivative* ∇,
- a type $(1,1)$ tensor $\Phi \in V^1(\tau)$, called the *Jacobi endomorphism.*

They manifest themselves via the same procedure as the one referred to above. In particular, computing the Lie bracket of the given SODE and an arbitrary horizontal lift and looking at the decomposition of the resulting vector field, we find that

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V,$$

where the operator in the horizontal part appears to have the properties of a derivation, while the vertical part depends tensorially on X. ∇ is further determined by self-duality and the fact that $\nabla F = \Gamma(F)$ on functions $F \in C^\infty(TM)$.

For computational purposes, we mention that

$$\nabla\left(\frac{\partial}{\partial q^i}\right) = \Gamma_i^k \frac{\partial}{\partial q^k}\,, \qquad \nabla(dq^i) = -\Gamma_k^i\, dq^k$$

and

$$\Phi_j^i = -\frac{\partial f^i}{\partial q^j} - \Gamma_j^k\Gamma_k^i - \Gamma(\Gamma_j^i).$$

Note that the relevance of Φ and ∇ is already obvious from the following properties about the curvature and its dynamical covariant derivative:

$$d^{\scriptscriptstyle V}\Phi = 3R, \qquad d^{\scriptscriptstyle H}\Phi = \nabla R.$$

For completeness, let us mention here also that the bracket of Γ and a vertical lift decomposes as follows:

$$\mathcal{L}_\Gamma X^{\scriptscriptstyle V} = -X^{\scriptscriptstyle H} + (\nabla X)^{\scriptscriptstyle V}.$$

5 Extension of the theory: time-dependent SODEs

The theory reviewed in the preceding two sections has been extended (at least partially) to other dynamical systems of interest, such as time-dependent SODEs (see [22]), mixed first- and second-order equations which includes the equations of non-holonomic mechanics (see [19]), and Lagrangian systems on Lie algebroids (see [14]). We limit ourselves here to a sketch of some features of the generalization to time-dependent SODEs. The basic underlying structure then is a manifold E which is fibred over $\mathbb{R}$ and its first-jet extension $J^1\pi$ (see the diagram below).

$$\begin{array}{ccccc}
\pi_1^{0*}(\tau_E) & \longrightarrow & TE & & \\
\downarrow & & \downarrow \tau_E & & \\
J^1\pi & \xrightarrow[\pi_1^0]{} & E & \xrightarrow[\pi]{} & \mathbb{R}
\end{array}$$

Geometrically, a SODE $\Gamma \in \mathcal{X}(J^1\pi)$ now is determined by the requirements $\langle \Gamma, dt\rangle = 1$ and $S(\Gamma) = 0$, where

$$S = \theta^i \otimes \frac{\partial}{\partial v^i}, \qquad \theta^i = dq^i - v^i dt.$$

In coordinates, Γ is of the form

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i} + f^i(t, q, v)\frac{\partial}{\partial v^i}.$$

As before, Γ defines a horizontal distribution on $J^1\pi$, corresponding to the projector

$$P_{\scriptscriptstyle H} = \tfrac{1}{2}(I - \mathcal{L}_\Gamma S + dt \otimes \Gamma).$$

We have

$$\operatorname{Im} P_{\scriptscriptstyle H} = \operatorname{sp}\left\{\Gamma, H_i = \frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial v^j}\right\} \qquad \Gamma_i^j = -\tfrac{1}{2}\frac{\partial f^j}{\partial v^i},$$

so a difference is that Γ is automatically horizontal here. In fact:

$$\Gamma = \mathbf{T}^H \quad \text{with} \quad \mathbf{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial q^i}.$$

What was $\mathcal{X}(\tau)$ before is now replaced by $\mathcal{X}(\pi_1^0) = \{$ vector fields along π_1^0 $\}$, and $\mathbf{T}$ is the canonical vector field along the projection π_1^0. We have a direct sum splitting $\mathcal{X}(\pi_1^0) \equiv \overline{\mathcal{X}}(\pi_1^0) \oplus \langle \mathbf{T} \rangle$, which entails that every $X \in \mathcal{X}(\pi_1^0)$ splits as follows

$$X = \overline{X} + \langle X, dt \rangle \mathbf{T}, \qquad \overline{X} = X^i(t,q,v)\frac{\partial}{\partial q^i}.$$

Everything said before can be extended to the present situation [22]. Roughly, the autonomous case is reflected in the $\overline{\mathcal{X}}(\pi_1^0)$-part of the calculus, and there often is a certain freedom in selecting the $\mathbf{T}$-component.

For example: for the 'linearization' of the SODE-connection, two versions have been advocated in the literature. One is

$$\mathrm{D}_\xi X = [P_H(\xi), X^V]_v + [P_V(\xi), X^H]_h + P_H(\xi)(\langle X, dt \rangle)\mathbf{T},$$

and was introduced in [3]. The other one was conceived [15] within the context of calculations on the full space $J^1\pi$, but reduced to its essential content in the language of vector fields along π_1^0, its defining relation reads

$$\mathrm{D}_\xi X = [P_H(\xi), \overline{X}^V]_v + [P_V(\xi), \overline{X}^H]_h + \xi(\langle X, dt \rangle)\mathbf{T}.$$

A comparative study of such Berwald and other connections can be found in [16].

6 Application: the inverse problem of the calculus of variations and some generalizations

Going back to the autonomous situation now, the inverse problem of the calculus of variations has many faces (see for example the recent review paper [9]), but the aspect which we want to discuss here is the search for existence of a non-singular multiplier $(g_{ij}(q,v))$ for a given SODE Γ, such that:

$$g_{ij}(\ddot{q}^j - f^j) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}$$

for some (regular) Lagrangian L. Geometrically, this can be captured into the single requirement of existence of a function L on TM, such that (see e.g. [18])

$$\mathcal{L}_\Gamma \theta_L = dL, \qquad \theta_L = S(dL).$$

As indicated before, the essence of this condition has only $n = \dim M$ components, however, so in the language of the calculus along $\tau : TM \to M$, $\theta_L = d^V L$ and the condition reads:

$$\nabla \theta_L = d^H L.$$

An equivalent formulation of the problem, which focusses on the existence of a suitable multiplier matrix g_{ij}, can be expressed as follows (see [12]): the SODE Γ has a variational formulation if and only if there exists a non-singular, symmetric type $(0,2)$ tensor field g along τ, such that

$$\begin{aligned} \nabla g &= 0, \\ \mathrm{D}^V_X g\,(Y,Z) &= \mathrm{D}^V_Z g\,(Y,X), \\ g(\Phi X, Y) &= g(\Phi Y, X). \end{aligned}$$

Observe the crucial role played by ∇ and Φ in these conditions again! Another way of expressing the third, algebraic condition is as: $\Phi \rfloor g = (\Phi \rfloor g)^T$.

Incidentally, if such a g exists, the Poincaré-Cartan 2-form $\omega_L = d\theta_L$ which creates a symplectic structure on TM is completely determined by g: it is its so-called Kähler lift, defined by the following action on horizontal and vertical lifts of elements of $\mathcal{X}(\tau)$,

$$\begin{aligned} g^K(X^H, Y^H) &= g^K(X^V, Y^V) = 0, \\ g^K(X^V, Y^H) &= g(X, Y) = -g^K(X^H, Y^V). \end{aligned}$$

In a recent generalization, inspired by related but somewhat incomplete work in [7], we have investigated for a given SODE the existence of an equivalent formulation in terms of Lagrangian equations with certain non-conservative forces [17]. The non-conservative forces we want to allow are of the form $\partial D/\partial v^i$ for some function D on TM, and we refer to them as forces of dissipative type, because they include the well known Rayleigh dissipation as a special case. So the question we address now becomes: given a SODE Γ, what are the conditions for existence of a non-singular, symmetric (g_{ij}) such that

$$g_{ij}(\ddot{q}^j - f^j) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - \frac{\partial D}{\partial \dot{q}^i}. \tag{1}$$

for some (regular) Lagrangian L and some D. Clearly, compared to the previous situation, Γ will have to satisfy a relation of the form:

$$\mathcal{L}_\Gamma \theta_L = dL + S(dD), \qquad \theta_L = S(dL),$$

and the more compact reformulation of this single requirement in terms of objects along τ reads

$$\nabla\theta_L = d^H L + d^V D. \tag{2}$$

Not surprisingly, the analogue of the g conditions in the standard problem, will contain here also the function D. The result is expressed by the following theorem, where for an arbitrary tensor field U say, $\mathrm{D}^V U(X, \ldots)$ stands for $\mathrm{D}^V_X U(\ldots)$.

Theorem 1: The second-order field Γ represents a dissipative system of type (1) if and only if there exists a function D and a (non-singular) symmetric type $(0,2)$ tensor g along τ such that

$$\begin{aligned} \nabla g &= \mathrm{D}^V \mathrm{D}^V D, \\ \mathrm{D}^V_X g\,(Y, Z) &= \mathrm{D}^V_Z g\,(Y, X), \\ \Phi \rfloor g - (\Phi \rfloor g)^T &= d^V d^H D. \end{aligned} \tag{3}$$

In order to give at least an idea of how the calculus along τ works in applications, in a way which keeps all steps coordinate independent, we now present a sketch of the proof of equivalence between (2) and (3).

1. Assume (2), i.e. $\nabla\theta_L = d^H L + d^V D$ and put $g := \mathrm{D}^V \mathrm{D}^V L$, which is equivalent to saying that $g(X, Y) := \mathrm{D}^V_X \theta_L(Y)$, with $\theta_L = d^V L = \mathrm{D}^V L$. Then g and $\mathrm{D}^V g$ are symmetric by construction. Acting with ∇ on g and using the commutator property $[\nabla, \mathrm{D}^V] = -\mathrm{D}^H$, we easily get that

$$\nabla g = \mathrm{D}^V \mathrm{D}^V D + \mathrm{D}^V \mathrm{D}^H L - \mathrm{D}^H \mathrm{D}^V L.$$

But D^H and D^V commute for their action on functions and it can be shown that (2) further implies that $d^H\theta_L = 0$. The required formula for ∇g then easily follows. Establishing the last relation in (3) is a bit more tricky. Essentially, one has to act on (2) with d^H, and make use of the commutator properties

$$d^H d^H = d^V_R, \qquad [\nabla, d^H] = 2\, i_R + d^V_\Phi.$$

The desired property will then be obtained, provided one recognizes further that for any g which is the Hessian of some function L, we have the identity

$$\Phi \lrcorner g - (\Phi \lrcorner g)^T = i_{d^V \Phi}\theta_L - d^V i_\Phi \theta_L.$$

2. For proving the converse, the first step is to observe that the symmetry of g and $\mathrm{D}^V g$ implies that g is a Hessian, say $g = \mathrm{D}^V \mathrm{D}^V F$ for some function F; such an F is not uniquely determined, so the rest of the reasoning inevitably will boil down to showing that the further assumptions (3) ensure that F can be suitably modified to provide the appropriate function L. This could be done by coordinate calculations, in principle, but we will illustrate in more intrinsic terms how one can reach the desired conclusion in a number of steps.

Since $\mathrm{D}^V g$ is symmetric, the same is true for $\nabla \mathrm{D}^V g$, and obviously also $\mathrm{D}^V \nabla g = \mathrm{D}^V \mathrm{D}^V \mathrm{D}^V D$ is symmetric. It follows that also $\mathrm{D}^H g$ is symmetric. Next, using the general commutator property of D^V and D^H (it is only on functions that this commutator is zero), one can quite easily show that $d^H \theta_F$ is a *basic 2-form*. With this new information, it is a matter of manipulating the assumption $\mathrm{D}^V \mathrm{D}^V D = \nabla g$ to show that also the 1-form β, determined by

$$\beta := \nabla \theta_F - d^H F - i_{\mathbf{T}} d^H \theta_F - \mathrm{D}^V D,$$

is a *basic 1-form*. Now, as we have seen in section 3, d^H coincides with the ordinary exterior derivative on basic forms, by construction. It then follows by taking a further d^H derivative that actually $d^H \theta_F$ and β are closed, hence locally exact, meaning that

$$d^H \theta_F = d^H \alpha \qquad \text{and} \qquad \beta = d^H f,$$

for some *basic* 1-form α and *basic* function f. It is finally easy to show that the function $L = F - i_{\mathbf{T}}\alpha + f$, which manifestly has the same Hessian as F, verifies the required relation $\nabla \theta_L = d^H L + d^V D$. □

One would expect that the story stops here, but quite surprisingly, one can eliminate the dependence on D all together and arrive at the following necessary and sufficient conditions involving the multiplier g only.

Theorem 2: The second-order field Γ represents a dissipative system of type

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial D}{\partial \dot{q}^i}$$

if and only if there exists a (non-singular) symmetric type $(0,2)$ tensor g along τ such that

$$\begin{aligned} \mathrm{D}^V_X g\,(Y,Z) &= \mathrm{D}^V_Z g\,(Y,X), \\ \mathrm{D}^H_X g\,(Y,Z) &= \mathrm{D}^H_Z g\,(Y,X), \\ \textstyle\sum_{X,Y,Z} g(R(X,Y),Z) &= 0, \end{aligned} \tag{4}$$

where $\sum_{X,Y,Z}$ stands for a cyclic sum over the indicated arguments. □

What is extra remarkable in this result is that the symmetry of $\mathrm{D}^H g$ and the curvature condition $\sum_{X,Y,Z} g(R(X,Y),Z) = 0$ which make their appearance here, are actually familiar conditions from the standard inverse problem. There, they come forward as further integrability conditions for the equations for g, whereas they turn up here as part of the package which has to hold true from the start.

The statement that (2) and (3) now are also equivalent to (4) thus looks fairly simple, but the proof requires the full machinery of the calculus along τ and makes use of a number of preliminary lemmas, so we omit it and refer to [17] for details.

In classifying non-conservative forces, there is a class which is complementary to the forces of dissipative type, namely gyroscopic forces. Lagrangian equations with non-conservative forces of gyroscopic type are equations of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \omega_{ki}(q)\dot{q}^k, \qquad \omega_{ki} = -\omega_{ik}. \tag{5}$$

Again, starting from second-order equations $\ddot{q}^i = f^i(q, \dot{q})$ in normal form (the SODE Γ), one may wonder whether a multiplier matrix $g_{ij}(q, \dot{q})$ may bring the system into the above form. We have succeeded in carrying out exactly the same analysis here as in the dissipative case. In the gyroscopic case, the starting, intrinsic geometrical formulation requires the existence of a function L and a basic 2-form ω, such that

$$\nabla\theta_L = d^H L + i_{\mathbf{T}}\omega. \tag{6}$$

The analogue of Theorem 1 of the dissipative case is found to read as follows.

Theorem 3: The second-order field Γ represents a gyroscopic system if and only if there exists a basic 2-form ω and a (non-singular) symmetric type $(0, 2)$ tensor g along τ such that

$$\begin{aligned} \nabla g &= 0, \\ \mathrm{D}^V_X g\,(Y, Z) &= \mathrm{D}^V_Z g\,(Y, X), \\ \Phi \,\lrcorner\, g - (\Phi \,\lrcorner\, g)^T &= i_{\mathbf{T}} d^H \omega. \end{aligned} \tag{7}$$

Again, one can eliminate the 2-form ω from the conditions, to arrive at conditions on the multiplier only.

Theorem 4: If the second-order field Γ represents a gyroscopic system, then there exists a symmetric type $(0, 2)$ tensor g along τ such that

$$\begin{aligned} \nabla g &= 0, \\ \mathrm{D}^V_X g\,(Y, Z) &= \mathrm{D}^V_Z g\,(Y, X), \\ \left(\Phi \,\lrcorner\, g - (\Phi \,\lrcorner\, g)^T\right)(X, Y) &= \textstyle\sum_{X,Y,\mathbf{T}} g(R(X, Y), \mathbf{T}). \end{aligned} \tag{8}$$

The converse is true as well, provided we assume that $\Phi \,\lrcorner\, g$ is smooth on the zero section of $TM \to M$.

Proofs of these results can be found in [17]. Let us illustrate the main theorems of this section on a couple of simple examples though.

Consider the following SODE

$$\begin{aligned} \ddot{q}_1 &= q_2 \dot{q}_1 \dot{q}_3, \\ \ddot{q}_2 &= \dot{q}_3^2, \\ \ddot{q}_3 &= \dot{q}_1^2 - q_2^{-1} \dot{q}_2 \dot{q}_3, \end{aligned}$$

which at first glance does not look like a system which would have a genuine Lagrangian description, but may be amenable to a representation as dissipative system of type (1). The best way to approach this question is to start from Theorem 2, with the conditions involving solely the multiplier g. Indeed, if a g can be found satisfying the conditions (4), we are guaranteed that a further function D will exist matching the conditions (3). The third of the conditions (4) in Theorem 2 is purely algebraic and reads,

$$g_{ij} R^j_{kl} + g_{lj} R^j_{ik} + g_{kj} R^j_{li} = 0,$$

where $R^k_{ij} = H_j(\Gamma^k_i) - H_i(\Gamma^k_j) = \frac{1}{3}(V_i(\Phi^k_j) - V_j(\Phi^k_i))$, the expressions of Γ^i_j and Φ^i_j having been listed before. Suppose we specifically look for a diagonal multiplier g which depends on the position variables q_i only. Then, the above requirement considerably simplifies and imposes that $g_{33} = (g_{11} - 2g_{22})q_2$. The second of the conditions (4) subsequently can be seen to require that $g_{11} = 4\,g_{22} =$ constant, which leaves us, up to an overall constant factor, with the only possibility that $g_{11} = 4,\ g_{22} = 1,\ g_{33} = 2q_2$. Theorem 2 is now taken care of and moving to Theorem 1, it is not so hard to verify that a function D which satisfies the first and third of the requirements (3) is found to be $D = 2q_2\dot{q}_1^2\dot{q}_3$. The Lagrangian corresponding to our g obviously is given by $L = \frac{1}{2}(4\dot{q}_1^2 + \dot{q}_2^2 + 2q_2\dot{q}_3^2)$, and the couple of functions (L, D) indeed provides a representation of the given system in the form (1).

For a simple example of the gyroscopic situation (5) and an illustration of the internal consistency of the statements in Theorems 3 and 4, we look at a system with two degrees of freedom and proceed in a different way. That is to say, we choose an L and extra gyroscopic forces, compute the corresponding SODE and then investigate what our inverse problem results tell us about this system. Take L to be

$$L = \tfrac{1}{2}(q_1\dot{q}_1^2 + \dot{q}_2^2) + \tfrac{1}{3}q_2^3,$$

and $\omega_{21} = -\omega_{12} = q_2$. Then, the resulting equations (5), written in normal form, read

$$\begin{aligned}\ddot{q}_1 &= -\tfrac{1}{2}q_1^{-1}\dot{q}_1^2 + q_2q_1^{-1}\dot{q}_2,\\ \ddot{q}_2 &= -q_2\dot{q}_1 + q_2^2.\end{aligned}$$

The algebraic condition in Theorem 4 is the third of equations (8) and requires that

$$g_{lj}\Phi^j_k - g_{kj}\Phi^j_l = (g_{ij}R^j_{kl} + g_{lj}R^j_{ik} + g_{kj}R^j_{li})v^i.$$

But in dimension 2, all terms in the right-hand side automatically cancel out in view of the symmetry of g and the skew-symmetry in the curvature tensor R. Comparison with Theorem 3 then reveals that we better have $i_{\mathbf{T}}d^H\omega = 0$ as well and this is of course true since $d^H\omega = 0$ by dimension. The net conclusion is that both theorems then simply reduce to the set of standard variationality conditions, as mentioned at the beginning of the section. This means that the gyroscopic system we started from should have a genuine Lagrangian representation as well. It is indeed easy to verify, and not in the least surprising, that by modifying the original L to

$$L' = \tfrac{1}{2}(q_1\dot{q}_1^2 + \dot{q}_2^2) + \tfrac{1}{3}q_2^3 - \tfrac{1}{2}q_2^2\dot{q}_1,$$

which amounts to adding to L a so-called generalized potential, the equations under consideration simply become the Euler-Lagrange equations of L'.

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