Lifting geometric objects to a cotangent bundle, and the geometry of the cotangent bundle of a tangent bundle

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Abstract

We show that the cotangent bundle $T^*T\mathcal{M}$ of the tangent bundle of any differentiable manifold $\mathcal{M}$ carries an integrable almost tangent structure which is generated by a natural lifting procedure from the canonical almost tangent structure (vertical endomorphism) of $T\mathcal{M}$. Using this almost tangent structure we show that $T^*T\mathcal{M}$ is diffeomorphic to a tangent bundle, namely $TT^*\mathcal{M}$. This provides a new and geometrically instructive proof of a result of Tulczyjew, which has applications in Lagrangian and Hamiltonian dynamics and in field theory. The requisite general definitions and results concerning liftings of geometric objects from a manifold to its cotangent bundle are given. As an application, we shed new light on the meaning of so-called adjoint symmetries of second-order differential equations.

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1 Introduction

In a recent paper [1] we have investigated the curious fact that an arbitrary system of second-order ordinary differential equations may be given both a Lagrangian and a Hamiltonian formulation, with essentially the same function serving as both Lagrangian and Hamiltonian for the system, by the introduction of additional variables. Our analysis began with the observation that the system of differential equations defined by a vector field on a differential manifold $\mathcal{N}$ may be embedded in a system of Hamiltonian equations by taking the complete lift of the vector field to the cotangent bundle $\mathrm{T}^*\mathcal{N}$. We applied this construction in the case where $\mathcal{N}$ is already the tangent bundle of a manifold $\mathcal{M}$ and the given vector field a second-order equation field. In order to obtain the Lagrangian formulation it was necessary to find an appropriate manifold with the structure of a tangent bundle related to both $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathrm{T}\mathcal{M}$. Now $\mathrm{T}^*\mathrm{T}\mathcal{M}$ is not itself a tangent bundle, of course; but it happens to be globally diffeomorphic to $\mathrm{T}\mathrm{T}^*\mathcal{M}$, and this fact played a key role in our further investigations.

The existence of the diffeomorphism $\mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathrm{T}^*\mathcal{M}$ was demonstrated by Tulczyjew in 1976 [16, 15], though it is not perhaps so well-known as it might be. Our principal aim in this paper is to give a new proof of this result using entirely different methods, ones which we believe are geometrically more instructive and appealing. The approach we take to the study of the geometry of tangent and cotangent bundles depends very much on exploiting the properties of the canonical geometric objects associated with them. The obvious example of such an object is the canonical 1-form on the cotangent bundle, from which its symplectic structure is derived. A somewhat analogous role is played in the case of a tangent bundle by its vertical endomorphism or almost tangent structure. This is a type $(1, 1)$ tensor field whose kernel, as a linear map of tangent vectors, coincides with its image and is just the vertical subspace at each point, and which is integrable in the sense that its Nijenhuis tensor vanishes [2, 3, 4]. In addition each of these manifolds carries a canonically defined vector field, as does any vector bundle: namely, its dilation field, which is the infinitesimal generator of dilations of the fibres.

As a prerequisite for our analysis, we need various techniques for lifting objects from a manifold $\mathcal{N}$ to its cotangent bundle $\mathrm{T}^*\mathcal{N}$. Most of these have been described by Yano and Ishihara [18]. However, we find their approach rather hard going and present therefore, in Sections 2 and 3, a new version of this theory, with different methods of proof which do not rely on coordinate calculations. In Section 4 we specialise to the case where $\mathcal{N} = \mathrm{T}\mathcal{M}$ and arrive at the main result of this paper: a geometrically constuctive proof of the diffeomorphism $\mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathrm{T}^*\mathcal{M}$. As a by-product of the techniques used in the proof, it becomes very simple to show that the fibration of $\mathrm{T}^*\mathrm{T}\mathcal{M}$ which gives rise to a tangent bundle structure, is actually Lagrangian and has a Lagrangian cross-section. We therefore thought it instructive to make a digression at this point; specifically, we construct in an appendix, relying in part on a theorem of Weinstein [17], a global symplectomorphism from $\mathrm{T}^*\mathrm{T}\mathcal{M}$ to $\mathrm{T}^*\mathrm{T}^*\mathcal{M}$. The main part of the paper continues with a section on dilation fields and coordinate expressions for most of the constructions we have been dealing with.

In Section 7 we comment on applications. It is fair to say that Tulczyjew's unification of Lagrangian and Hamiltonian mechanics [15, 16] relies for a great deal on the diffeomorphism $\mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathrm{T}^*\mathcal{M}$. Another application is our description of the Lagrangian extension of an arbitrary second-order system, referred to at the beginning. We believe, however, that there is more to be gained from lifting objects to a tangent or cotangent bundle: such operations may help to understand the cohesion between different results or may occasionally lead to the discovery of new properties. As an illustration of this idea we show that some rather surprising results on adjoint symmetries of second-order systems on $\mathrm{T}\mathcal{M}$, derived in [12], become perfectly plausible when one recognizes that they are manifestations of known properties concerning the Lagrangian extension on $\mathrm{T}^*\mathrm{T}\mathcal{M}$.

Our notation is more-or-less standard; but we should warn the reader that we do not distinguish notationally between the linear action of a type $(1,1)$ tensor on covectors and its action on vectors.

2 Lifts to the cotangent bundle

We begin by describing various lifts from an arbitrary differentiable manifold $\mathcal{N}$ to its cotangent bundle $\pi_{\mathcal{N}}\colon \mathrm{T}^*\mathcal{N} \to \mathcal{N}$.

We assume that the following lifting constructions are already known:

the vertical lift α^v of a 1-form α on $\mathcal{N}$

the complete lift $\tilde{X}$ of a vector field X on $\mathcal{N}$

together with the following formulae:

$$[\alpha^v, \beta^v] = 0$$

$$[\tilde{X}, \alpha^v] = (\mathcal{L}_X \alpha)^v$$

$$[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]}.$$

The vector fields obtained in this way span the vector fields on $\mathrm{T}^*\mathcal{N}$; we use this fact repeatedly in the sequel as a means of specifying geometric objects on $\mathrm{T}^*\mathcal{N}$ explicitly.

The vertical lifts of 1-forms and complete lifts of vector fields are related to the canonical 1-form $\theta_{\mathcal{N}}$ of $\mathrm{T}^*\mathcal{N}$ as follows:

$$\langle \alpha^v, \theta_{\mathcal{N}} \rangle = 0, \qquad \langle \tilde{X}, \theta_{\mathcal{N}} \rangle = h_X$$

where h_X denotes the fibre linear function on $\mathrm{T}^*\mathcal{N}$ determined by X: $h_X(x,p) = \langle X_x, p \rangle$. We also have $\mathcal{L}_{\alpha^v} \theta_{\mathcal{N}} = {\pi_{\mathcal{N}}}^* \alpha$ and $\mathcal{L}_{\tilde{X}} \theta_{\mathcal{N}} = 0$, from which the following formulae are easily derived:

$$d\theta_{\mathcal{N}}(\alpha^v, \beta^v) = 0$$

$$d\theta_{\mathcal{N}}(\alpha^v, \tilde{X}) = \alpha^v(h_X) = {\pi_{\mathcal{N}}}^* \langle X, \alpha \rangle$$

$$d\theta_{\mathcal{N}}(\tilde{X}, \tilde{Y}) = h_{[X,Y]}.$$

We shall be concerned with two different ways of lifting a type $(1,1)$ tensor field from $\mathcal{N}$ to $\mathrm{T}^*\mathcal{N}$ the first of which, now to be described, results in a vertical vector field. Let R be a type $(1,1)$ tensor field on $\mathcal{N}$. For each $x \in \mathcal{N}$, R determines a linear endomorphism R_x of the cotangent space $\mathrm{T}^*_x\mathcal{N}$. Let ρ_t be the one-parameter group of transformations of $\mathrm{T}^*\mathcal{N}$ given by

$$\rho_t(x,p) = \left(x, e^{tR_x} p\right).$$

The generator of ρ_t is a vertical vector field on $\mathrm{T}^*\mathcal{N}$ which we call the vertical lift of R and denote R^v.

The vertical lifts of 1-forms and type $(1,1)$ tensor fields to $\mathrm{T}^*\mathcal{N}$ are derived from the action of the affine group in the fibres, each of which is of course a vector space. The bracket relations between vertical lifts essentially reproduce the Lie algebra structure of the affine group:

$$[\alpha^v, \beta^v] = 0$$

$$[\alpha^v, R^v] = R(\alpha)^v$$

$$[Q^v, R^v] = [Q, R]^v$$

where $[Q, R]$ represents the commutator of Q and R derived from their actions on vectors. The bracket of a complete lift of a vector field and the vertical lift of a type $(1,1)$ tensor field may be computed from consideration of the flows they generate, and is given by

$$[\tilde{X}, R^v] = (\mathcal{L}_X R)^v.$$

By considering the action of ρ_t it is easy to show that

$$R^v(h_X) = h_{R(X)}.$$

Furthermore

$$\begin{aligned} d\theta_{\mathcal{N}}(R^v, \alpha^v) &= 0 \\ d\theta_{\mathcal{N}}(R^v, \tilde{X}) &= R^v\langle \tilde{X}, \theta_{\mathcal{N}}\rangle - \langle [R^v, \tilde{X}], \theta_{\mathcal{N}}\rangle \\ &= R^v(h_X) + \langle (\mathcal{L}_X R)^v, \theta_{\mathcal{N}}\rangle \\ &= h_{R(X)}. \end{aligned}$$

3 Complete lifts of type $(1,1)$ tensor fields

We next define another lift of a type $(1,1)$ tensor field on $\mathcal{N}$ to $\mathrm{T}^*\mathcal{N}$, which leads this time to a type $(1,1)$ tensor field again, rather than to a vector field. The construction is based on the non-degeneracy of $d\theta_{\mathcal{N}}$ and the possibility of using it, as a consequence, to convert 2-forms (or, in general, type $(0,2)$ tensor fields) into type $(1,1)$ tensor fields, in the manner of raising an index with a metric.

Any type $(1,1)$ tensor field R on $\mathcal{N}$ determines a fibre linear map $\tau_R \colon \mathrm{T}^*\mathcal{N} \to \mathrm{T}^*\mathcal{N}$, fibred over the identity, by

$$\tau_R(x, p) = (x, R_x p);$$

the complete lift $\tilde{R}$ of R to $\mathrm{T}^*\mathcal{N}$ is the type $(1,1)$ tensor field defined by

$$\iota_{\tilde{R}(\xi)} d\theta_{\mathcal{N}} = \iota_\xi(\tau_R^* d\theta_{\mathcal{N}})$$

where ι denotes the interior product.

The map τ_R is just the map whose exponential was used to define the vertical lift of R. It follows that $\tilde{R}$ satisfies

$$\iota_{\tilde{R}(\xi)} d\theta_{\mathcal{N}} = \iota_\xi(\mathcal{L}_{R^v} d\theta_{\mathcal{N}}).$$

The tensor $\tilde{R}$ may be specified explicitly by evaluating it on the vertical lift of a 1-form and on the complete lift of a vector field, as follows.

Theorem 1 *For any 1-form α and vector field X on $\mathcal{N}$*

$$\tilde{R}(\alpha^v) = R(\alpha)^v$$

$$\tilde{R}(\tilde{X}) = \widetilde{R(X)} + (\mathcal{L}_X R)^v.$$

Proof The proof consists essentially of repeated applications of the preceeding formula. To obtain the second result we use this formula with $\xi = \tilde{X}$.

$$\begin{aligned} d\theta_{\mathcal{N}}(\tilde{R}(\tilde{X}), \beta^v) &= \mathcal{L}_{R^v}(d\theta_{\mathcal{N}}(\tilde{X}, \beta^v)) \\ &\quad + d\theta_{\mathcal{N}}((\mathcal{L}_X R)^v, \beta^v) + d\theta_{\mathcal{N}}(\tilde{X}, R(\beta)^v) \\ &= d\theta_{\mathcal{N}}(\tilde{X}, R(\beta)^v) \\ &= -\pi_{\mathcal{N}}{}^*\langle X, R(\beta)\rangle \\ &= -\pi_{\mathcal{N}}{}^*\langle R(X), \beta\rangle \\ &= d\theta_{\mathcal{N}}(\widetilde{R(X)}, \beta^v). \end{aligned}$$

On the other hand

$$\begin{aligned} d\theta_{\mathcal{N}}(\tilde{R}(\tilde{X}), \tilde{Y}) &= \mathcal{L}_{R^v}(d\theta_{\mathcal{N}}(\tilde{X}, \tilde{Y})) \\ &\quad + d\theta_{\mathcal{N}}((\mathcal{L}_X R)^v, \tilde{Y}) + d\theta_{\mathcal{N}}(\tilde{X}, (\mathcal{L}_Y R)^v) \\ &= \mathcal{L}_{R^v}(h_{[X,Y]}) + d\theta_{\mathcal{N}}((\mathcal{L}_X R)^v, \tilde{Y}) - h_{\mathcal{L}_Y R(X)} \\ &= h_{R([X,Y])} + d\theta_{\mathcal{N}}((\mathcal{L}_X R)^v, \tilde{Y}) - h_{[Y,R(X)]} + h_{R([Y,X])} \\ &= d\theta_{\mathcal{N}}((\mathcal{L}_X R)^v, \tilde{Y}) + h_{[R(X),Y]} \\ &= d\theta_{\mathcal{N}}((\mathcal{L}_X R)^v, \tilde{Y}) + d\theta_{\mathcal{N}}(\widetilde{R(X)}, \tilde{Y}). \end{aligned}$$

The second assertion of the theorem now follows.
The first assertion is easily verified by similar considerations with $\xi = \alpha^v$. $\square$

Corollary 1 *The tensor fields $\tilde{R}$ and R are $\pi_{\mathcal{N}}$-related, in the sense that*

$$\pi_{\mathcal{N}*} \circ \tilde{R} = R \circ \pi_{\mathcal{N}*}.$$

Corollary 2 *If Q is another type $(1,1)$ tensor field on $\mathcal{N}$ then*

$$\tilde{R}(Q^v) = (Q \circ R)^v.$$

Proof

$$\begin{aligned} d\theta_{\mathcal{N}}(\tilde{R}(Q^v), \alpha^v) &= \mathcal{L}_{R^v} d\theta_{\mathcal{N}}(Q^v, \alpha^v) = 0 \\ d\theta_{\mathcal{N}}(\tilde{R}(Q^v), \tilde{X}) &= \mathcal{L}_{R^v}(d\theta_{\mathcal{N}}(Q^v, \tilde{X})) \\ &\quad - d\theta_{\mathcal{N}}([R,Q]^v, \tilde{X}) + d\theta_{\mathcal{N}}(Q^v, (\mathcal{L}_X R)^v) \\ &= h_{RQ(X)} - h_{[R,Q](X)} \\ &= h_{QR(X)} \\ &= d\theta_{\mathcal{N}}((Q \circ R)^v, \tilde{X}), \end{aligned}$$

from which the result follows. □

Finally, it is easy to prove the following result, by evaluating both sides on a vertical and a complete lift, using the formulae obtained in Theorem 1.

Corollary 3 *For any vector field X on $\mathcal{N}$*

$$\mathcal{L}_{\tilde{X}} \tilde{R} = \widetilde{\mathcal{L}_X R}.$$

We now derive some relationship between the Nijenhuis tensor of a type $(1,1)$ tensor on $\mathcal{N}$ and the Nijenhuis tensor of its complete lift to $\mathrm{T}^*\mathcal{N}$.

We first remind the reader that if R is any type $(1,1)$ tensor, its Nijenhuis tensor N_R is the type $(1,2)$ tensor given by

$$\begin{aligned} &N_R(X,Y) \\ &\quad = R^2([X,Y]) + [R(X), R(Y)] - R([R(X),Y]) - R([X,R(Y)]). \end{aligned}$$

This may be rewritten as $(\iota_X N_R)(Y) = N_R(X,Y)$, where $\iota_X N_R$ denotes the type $(1,1)$ tensor $\mathcal{L}_{R(X)} R - R \circ \mathcal{L}_X R$.

Lemma 1 *For any type $(1,1)$ tensor R on $\mathcal{N}$*

$$\begin{aligned} \tilde{R}^2(\alpha^v) &= R^2(\alpha)^v \\ \tilde{R}^2(\tilde{X}) &= \widetilde{R^2(X)} + (\mathcal{L}_X R^2)^v + (\iota_X N_R)^v. \end{aligned}$$

Proof These formulae follow directly from those in Theorem 1 and the definition of $\iota_X N_R$. $\square$

Lemma 2 *For any type* $(1,1)$ *tensor* R *on* $\mathcal{N}$

$$N_{\tilde{R}}(\alpha^v, \beta^v) = 0$$

$$N_{\tilde{R}}(\tilde{X}, \alpha^v) = (\iota_X N_R(\alpha))^v$$

$$N_{\tilde{R}}(\tilde{X}, \tilde{Y}) = N_R\widetilde{(X,Y)} + \Big(\iota_{[X,Y]}N_R + \mathcal{L}_Y(\iota_X N_R) - \mathcal{L}_X(\iota_Y N_R)\Big)^v .$$

Proof Again, the proof consists of calculations using the formulae given in Theorem 1, and in this case also the second formula in Lemma 1. We shall derive the last formula as an example.

$$\begin{aligned}
&N_{\tilde{R}}(\tilde{X}, \tilde{Y}) \\
&\quad = \quad \tilde{R}^2([\tilde{X}, \tilde{Y}]) + [\tilde{R}(\tilde{X}), \tilde{R}(\tilde{Y}] - \tilde{R}([\tilde{R}(\tilde{X}), \tilde{Y}]) - \tilde{R}([\tilde{X}, \tilde{R}(\tilde{Y})]) \\
&\quad = \quad \tilde{R}^2(\widetilde{[X,Y]}) + [\widetilde{R(X)} + (\mathcal{L}_X R)^v, \widetilde{R(Y)} + (\mathcal{L}_Y R)^v] \\
&\qquad\quad - \tilde{R}([\widetilde{R(X)} + (\mathcal{L}_X R)^v, \tilde{Y}]) - \tilde{R}([\tilde{X}, \widetilde{R(Y)} + (\mathcal{L}_Y R))^v]) \\
&\quad = \quad \tilde{R}^2(\widetilde{[X,Y]}) + [\widetilde{R(X)}, \widetilde{R(Y)}] - \tilde{R}([\widetilde{R(X)}, \tilde{Y}]) - \tilde{R}([\tilde{X}, \widetilde{R(Y)}]) \\
&\qquad\quad + [(\mathcal{L}_X R)^v, \widetilde{R(Y)}] + [\widetilde{R(X)}, (\mathcal{L}_Y R)^v] + [(\mathcal{L}_X R)^v, (\mathcal{L}_Y R)^v] \\
&\qquad\quad - \tilde{R}([(\mathcal{L}_X R)^v, \tilde{Y}]) - \tilde{R}([\tilde{X}, (\mathcal{L}_Y R)^v]) \\
&\quad = \quad N_R\widetilde{(X,Y)} + (\iota_{[X,Y]}N_R)^v + (Q_{(X,Y)})^v
\end{aligned}$$

where the type $(1,1)$ tensor $Q_{(X,Y)}$ on $\mathcal{N}$ is given by

$$\begin{aligned}
Q_{(X,Y)} \quad = \quad & \mathcal{L}_{[X,Y]}R^2 - \mathcal{L}_{[R(X),Y]}R - \mathcal{L}_{[X,R(Y)]}R \\
& - \mathcal{L}_{R(Y)}\mathcal{L}_X R + \mathcal{L}_{R(X)}\mathcal{L}_Y R \\
& + [\mathcal{L}_X R, \mathcal{L}_Y R] + \mathcal{L}_Y \mathcal{L}_X R \circ R - \mathcal{L}_X \mathcal{L}_Y R \circ R \\
= \quad & R \circ \mathcal{L}_{[X,Y]}R + \mathcal{L}_Y \mathcal{L}_{R(X)}R - \mathcal{L}_X \mathcal{L}_{R(Y)}R \\
& + \mathcal{L}_X R \circ \mathcal{L}_Y R - \mathcal{L}_Y R \circ \mathcal{L}_X R \\
= \quad & \mathcal{L}_Y(\mathcal{L}_{R(X)}R - R \circ \mathcal{L}_X R) - \mathcal{L}_X(\mathcal{L}_{R(Y)}R - R \circ \mathcal{L}_Y R).
\end{aligned}$$

Thus

$$N_{\tilde{R}}(\tilde{X}, \tilde{Y}) = N_R\widetilde{(X,Y)} + \Big(\iota_{[X,Y]}N_R + \mathcal{L}_Y(\iota_X N_R) - \mathcal{L}_X(\iota_Y N_R)\Big)^v$$

as asserted. ◻

From these two lemmas there follows

Theorem 2 *If a type $(1,1)$ tensor R on $\mathcal{N}$ satisfies $R^2 = 0$ and $N_R = 0$ then its complete lift has similar properties:*

$$\tilde{R}^2 = 0 \qquad\qquad N_{\tilde{R}} = 0.$$

4 The case where the base is a tangent bundle

We now specialise to the case where the base manifold $\mathcal{N}$ is already the tangent bundle $\tau_{\mathcal{M}}\colon \mathrm{T}\mathcal{M} \to \mathcal{M}$ of another manifold $\mathcal{M}$. We shall be concerned therefore with lifting geometrical objects from $\mathrm{T}\mathcal{M}$ to $\mathrm{T}^*\mathrm{T}\mathcal{M}$, and in particular with the complete lift of the canonical integrable almost tangent structure on $\mathrm{T}\mathcal{M}$. The almost tangent structure on $\mathrm{T}\mathcal{M}$ is a type $(1,1)$ tensor, which is often called the vertical endomorphism. We shall denote it S. It satisfies the conditions $\ker S = \operatorname{im} S$, whence $S^2 = 0$, (the condition for being an almost tangent structure); and $N_S = 0$ (the condition of integrability).

Theorem 3 *The complete lift $\tilde{S}$ of S to $\mathrm{T}^*\mathrm{T}\mathcal{M}$ is also an integrable almost tangent structure. With its aid, $\mathrm{T}^*\mathrm{T}\mathcal{M}$ may be given the structure of a tangent bundle, and is in fact diffeomorphic to $\mathrm{T}\mathrm{T}^*\mathcal{M}$.*

Proof By Theorem 2, $\tilde{S}^2 = 0$. It follows that $\operatorname{im}\tilde{S} \subseteq \ker\tilde{S}$. But $\dim\ker\tilde{S} + \dim\operatorname{im}\tilde{S} = 2\dim\mathrm{T}\mathcal{M}$, so that $\dim\operatorname{im}\tilde{S} \leq \dim\mathrm{T}\mathcal{M}$, with equality implying that $\operatorname{im}\tilde{S} = \ker\tilde{S}$. We show that $\dim\operatorname{im}\tilde{S} \geq \dim\mathrm{T}\mathcal{M}$.

For any 1-form α and vector field X on $\mathrm{T}\mathcal{M}$

$$\tilde{S}(\alpha^v) = S(\alpha)^v \qquad\qquad \tilde{S}(\tilde{X}) = \widetilde{S(X)} + (\mathcal{L}_X S)^v.$$

Now let β be any basic 1-form on $\mathrm{T}\mathcal{M}$: there is a 1-form α on $\mathrm{T}\mathcal{M}$ such that $\beta = S(\alpha)$, and therefore $\beta^v \in \operatorname{im}\tilde{S}$. Again, let Y be any vertical vector field on $\mathrm{T}\mathcal{M}$ which is the vertical lift of a vector field Z on $\mathcal{M}$. Then if Z^C denotes the complete lift of Z to $\mathrm{T}\mathcal{M}$, we have $Y = S(Z^C)$; moreover, $\mathcal{L}_{Z^C} S = 0$, and so $\tilde{Y} = \tilde{S}(\widetilde{Z^C}) \in \operatorname{im}\tilde{S}$. Thus if $\{E_a\}$ is a local basis of vector fields on $\mathcal{M}$, and $\{\omega^a\}$ the dual local basis of 1-forms, the vector fields $\{(\tau_{\mathcal{M}}^*\omega^a)^v, \widetilde{E_a^\nu}\}$

on $\mathrm{T}^*\mathrm{T}\mathcal{M}$, where the superscript ν denotes the vertical lift to the tangent bundle, belong to $\operatorname{im}\tilde{S}$. They are clearly linearly independent and $\dim \mathrm{T}\mathcal{M}$ in number, which proves that $\dim \operatorname{im}\tilde{S} \geq \dim \mathrm{T}\mathcal{M}$, as required.

Thus $\tilde{S}$ is an almost tangent structure. The fact that it is integrable follows from Theorem 2.

It is true for any integrable almost tangent structure that its image, or equivalently kernel, distribution is integrable in the sense of Frobenius's Theorem. In this case the image distribution has, over a suitable open subset of $\mathcal{M}$, a local basis consisting of complete, pairwise commuting vector fields: the vector fields given above have these properties, since vertical lifts to either a tangent or a cotangent bundle are necessarily complete as they are effectively affine vector fields in the fibres, and the complete lift of a complete vector field is necessarily complete; furthermore these vector fields do commute pairwise as a consequence of the bracket relations for complete and vertical lifts. Thus each leaf of the image distribution is diffeomorphic to $\mathbf{R}^{2m}$, where $m = \dim \mathcal{M}$, or at worst to a quotient space of it by a discrete group of translations. But the latter possibility is ruled out by the fact that the leaf projects onto a fibre of $\mathrm{T}\mathcal{M}$, and is itself fibered by vector subspaces of the fibres of $\mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}$. Thus the leaves of the image distribution are each diffeomorphic to $\mathbf{R}^{2m}$.

We next define an imbedding of $\mathrm{T}^*\mathcal{M}$ into $\mathrm{T}^*\mathrm{T}\mathcal{M}$. Consider the zero section $\mathcal{M}_0$ of $\mathrm{T}\mathcal{M}$, which we may identify with $\mathcal{M}$ itself. The tangent space to $\mathrm{T}\mathcal{M}$ at a point $(x, 0)$ in $\mathcal{M}_0$ has a direct sum decomposition $\mathrm{T}_{(x,0)}\mathrm{T}\mathcal{M} = \mathcal{Z} \oplus \mathcal{V}$ where $\mathcal{Z}$ is the subspace consisting of vectors tangent to the zero section and $\mathcal{V}$ the subspace consisting of vectors tangent to the fibre, that is, the vertical subspace. Each of these is a copy of $\mathrm{T}_x\mathcal{M}$. Define a map $\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}\mathcal{M}$ by mapping (x, p) to the covector $\hat{p}$ at $(x, 0) \in \mathrm{T}\mathcal{M}$ defined in terms of the direct sum decomposition by $\langle (z, v), \hat{p} \rangle = \langle v, p \rangle$. Thus $\mathrm{T}^*\mathcal{M}$ is identified by this map with $\mathrm{T}\mathcal{M}_0^\perp$, the annihilator, along the zero section $\mathcal{M}_0$ of $\mathrm{T}\mathcal{M}$, of the tangent spaces to $\mathcal{M}_0$. Under the projection $\pi_{\mathrm{T}\mathcal{M}} \colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}$ the submanifold $\mathrm{T}\mathcal{M}_0^\perp$ maps onto $\mathcal{M}_0$.

We now show that $\mathrm{T}\mathcal{M}_0^\perp$ is a cross-section of the distribution $\operatorname{im}\tilde{S}$, that is to say, that it intersects each of its leaves in exactly one point. The projection of any leaf onto $\mathrm{T}\mathcal{M}$ is a fibre of $\tau_{\mathcal{M}}$, and therefore intersects the projection $\mathcal{M}_0$ of $\mathrm{T}\mathcal{M}_0^\perp$ in exactly one point, say $(x, 0)$. The leaf therefore intersects the restriction of $\mathrm{T}^*\mathrm{T}\mathcal{M}$ to $\mathrm{T}\mathcal{M}_0^\perp$ in a subset of the fibre over

$(x, 0)$, that is, $\mathrm{T}^*_{(x,0)}\mathrm{T}\mathcal{M}$. Now the restriction of the distribution to this vector space is the subspace generated by the vertical lifts to $\mathrm{T}^*_{(x,0)}\mathrm{T}\mathcal{M}$ of basic covectors at $(x, 0)$. In terms of the direct sum decomposition used to construct $\mathrm{T}\mathcal{M}_0^{\perp}$ this is the annihilator of the tangent space to the fibre of $\tau_{\mathcal{M}}$, which is complementary to the annihilator of the tangent space to the zero section. Thus the leaf of the distribution $\operatorname{im}\tilde{S}$ intersects $\mathrm{T}^*_{(x,0)}\mathrm{T}\mathcal{M}$ in exactly one point, the zero covector; and so the leaf intersects $\mathrm{T}\mathcal{M}_0^{\perp}$ in exactly one point.

Thus $\mathrm{T}^*\mathrm{T}\mathcal{M}$ is the total space of a vector bundle $\hat{\tau}_{\mathrm{T}\mathcal{M}}\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}_0^{\perp}$ whose fibres are the leaves of the distribution $\operatorname{im}\tilde{S}$. Let ψ denote the imbedding $\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}\mathcal{M}$. Then at each point $(x, p) \in \mathrm{T}^*\mathcal{M}$ the map ψ_* is a linear isomorphism of $\mathrm{T}_{(x,p)}\mathrm{T}^*\mathcal{M}$ with the tangent space to the image $\mathrm{T}\mathcal{M}_0^{\perp}$ at $\psi(x, p)$. But $\mathrm{T}\mathcal{M}_0^{\perp}$ is a cross-section of the distribution $\operatorname{im}\tilde{S}$, and so its tangent space at any point is complementary to the image space of $\tilde{S}$ at that point. Thus $\tilde{S}_{\psi(x,p)}$ maps the tangent space to $\mathrm{T}\mathcal{M}_0^{\perp}$ at $\psi(x, p)$ linearly and isomorphically onto $\operatorname{im}\tilde{S}_{\psi(x,p)}$. Now $\operatorname{im}\tilde{S}_{\psi(x,p)}$ is the tangent space to the leaf of the image distribution through $\psi(x, p)$, and may be identified with the leaf itself, since it is a vector space. The map $\Psi = I \circ \tilde{S} \circ \psi_*$ defined in this way, where I represents the identification of the tangent space to a vector space at its origin with the vector space itself, is a diffeomorphism of $\mathrm{T}\mathrm{T}^*\mathcal{M}$ with $\mathrm{T}^*\mathrm{T}\mathcal{M}$ which is a linear bundle map with respect to the vector bundle structures $\tau_{\mathrm{T}^*\mathcal{M}}\colon \mathrm{T}\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathcal{M}$ and $\hat{\tau}_{\mathrm{T}\mathcal{M}}\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}_0^{\perp}$ and matches the integrable almost tangent structures on the two manifolds. $\square$

The maps introduced in the proof of Theorem 3 may be conveniently incorporated into a commutative diagram:

$$\begin{tikzcd} \mathrm{T}\mathrm{T}^*\mathcal{M} \arrow[r, "\Psi"] \arrow[d, "\tau_{\mathrm{T}^*\mathcal{M}}"'] & \mathrm{T}^*\mathrm{T}\mathcal{M} \arrow[d, "\hat{\tau}_{\mathrm{T}\mathcal{M}}"] \\ \mathrm{T}^*\mathcal{M} \arrow[r, "\psi"'] & \mathrm{T}\mathcal{M}_0^{\perp} \end{tikzcd}$$

It is further interesting to observe the following property of the fibration $\hat{\tau}_{\mathrm{T}\mathcal{M}}$.

Theorem 4 *The fibration* $\hat{\tau}_{\mathrm{T}\mathcal{M}}\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}_0^{\perp}$ *is Lagrangian with respect to the canonical symplectic structure of* $\mathrm{T}^*\mathrm{T}\mathcal{M}$*, and* $\mathrm{T}\mathcal{M}_0^{\perp}$ *is a Lagrangian cross-section.*

Proof For any basic 1-forms α and β on $\mathrm{T}\mathcal{M}$ and any vertical lifts V and W to $\mathrm{T}\mathcal{M}$ of vector fields on $\mathcal{M}$,

$$d\theta_{\mathrm{T}\mathcal{M}}(\alpha^v, \beta^v) = 0;$$

$$d\theta_{\mathrm{T}\mathcal{M}}(\alpha^v, \tilde{V}) = {\pi_{\mathrm{T}\mathcal{M}}}^* \langle V, \alpha \rangle = 0$$

because V is vertical and α basic; and

$$d\theta_{\mathrm{T}\mathcal{M}}(\tilde{V}, \tilde{W}) = h_{[V,W]} = 0$$

because the bracket of two vertical lifts to $\mathrm{T}\mathcal{M}$ is zero. Since these vector fields span the distribution whose leaves are the fibres of $\hat{\tau}_{\mathrm{T}\mathcal{M}}$ it follows that the fibration is Lagrangian.

A vertical lift α^v is tangent to $\mathrm{T}\mathcal{M}_0^{\perp}$ if and only if the 1-form α on $\mathrm{T}\mathcal{M}$ annihilates the tangent spaces to the zero section $\mathcal{M}_0$. A complete lift $\tilde{X}$, on the other hand, is tangent to $\mathrm{T}\mathcal{M}_0^{\perp}$ if and only X is tangent to $\mathcal{M}_0$. If α^v, β^v, $\tilde{X}$ and $\tilde{Y}$ are tangent to $\mathrm{T}\mathcal{M}_0^{\perp}$ then, on $\mathrm{T}\mathcal{M}_0^{\perp}$,

$$d\theta_{\mathrm{T}\mathcal{M}}(\alpha^v, \beta^v) = 0;$$

$$d\theta_{\mathrm{T}\mathcal{M}}(\alpha^v, \tilde{X}) = {\pi_{\mathrm{T}\mathcal{M}}}^*\langle X, \alpha\rangle = 0$$

because X is tangent to $\mathcal{M}_0$ and α annihilates vectors tangent to it; and

$$d\theta_{\mathrm{T}\mathcal{M}}(\tilde{X}, \tilde{Y}) = h_{[X,Y]} = 0$$

since $[X, Y]$ is tangent to $\mathcal{M}_0$, while the value of $h_{[X,Y]}$ at any point of $\mathrm{T}\mathcal{M}_0^\perp$ involves the pairing of $[X, Y]$ with a covector which annihilates vectors tangent to $\mathcal{M}_0$. Thus $\mathrm{T}\mathcal{M}_0^\perp$ is a Lagrangian submanifold of $\mathrm{T}^*\mathrm{T}\mathcal{M}$. □

According to a theorem of Weinstein [17], if a symplectic manifold has a Lagrangian foliation which admits a cross-section which is also a Lagrangian submanifold then a neighbourhood of the cross-section is symplectomorphic to a neighbourhood of the zero section in the cotangent bundle of the cross-section, in such a way that the fibres of the two manifolds correspond. Thus in the present case we are assured of the existence of a symplectic bundle diffeomorphism $\mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$, at least in neighbourhoods of the corresponding cross-sections. The fact that each of these manifolds is a vector bundle means that the diffeomorphism is global and indeed fibre linear. Actually, we are able to describe an explicit construction of this diffeomorphism, which is presented in the appendix.

5 Dilation fields

As well as the (almost) tangent structure of $\mathrm{T}^*\mathrm{T}\mathcal{M}$ we must consider the associated dilation field.

The dilation field Δ on $\mathrm{T}\mathcal{M}$, the infinitesimal generator of dilations $(x, u) \mapsto (x, e^t u)$, satisfies

$$S(\Delta) = 0$$

$$\mathcal{L}_\Delta S = -S$$

Δ vanishes on the zero section

and these properties determine it uniquely. From them follow certain properties of the complete lift $\tilde{\Delta}$ of Δ to $\mathrm{T}^*\mathrm{T}\mathcal{M}$.

Lemma 3 *The complete lift of* Δ *satisfies*

$$\tilde{S}(\tilde{\Delta}) = -S^v$$

$$\mathcal{L}_{\tilde{\Delta}}\tilde{S} = -\tilde{S}.$$

Proof

$$\tilde{S}(\tilde{\Delta}) = \widetilde{S(\Delta)} + (\mathcal{L}_\Delta S)^v = -S^v$$

$$\mathcal{L}_{\tilde{\Delta}}\tilde{S} = \widetilde{\mathcal{L}_\Delta S} = -\tilde{S}$$

as required. □

On the other hand, $\mathrm{T}^*\mathrm{T}\mathcal{M}$ carries a dilation field Δ^* by virtue of the fact that it is a cotangent bundle. When one takes the Lie derivative with respect to Δ^* of a geometric object on $\mathrm{T}^*\mathrm{T}\mathcal{M}$ which is homogeneous in the fibre coordinates one obtains a result which incorporates the homogeneity degree in the manner of Euler's Theorem on homogeneous functions. The vector field Δ^* has the following properties in relation to $\tilde{S}$:

Lemma 4 *The dilation field* Δ^* *satisfies*

$$\tilde{S}(\Delta^*) = S^v$$

$$\mathcal{L}_{\Delta^*}\tilde{S} = 0.$$

Proof For any 1-form α and vector field X on $\mathrm{T}\mathcal{M}$

$$\begin{aligned} d\theta_{\mathrm{T}\mathcal{M}}(\tilde{S}(\Delta^*), \alpha^v) &= \mathcal{L}_{S^v} d\theta_{\mathrm{T}\mathcal{M}}(\Delta^*, \alpha^v) = 0 \\ d\theta_{\mathrm{T}\mathcal{M}}(\tilde{S}(\Delta^*), \tilde{X}) &= \mathcal{L}_{S^v}(d\theta_{\mathrm{T}\mathcal{M}}(\Delta^*, \tilde{X})) \\ &\quad - d\theta_{\mathrm{T}\mathcal{M}}(\mathcal{L}_{S^v}\Delta^*, \tilde{X}) + d\theta_{\mathrm{T}\mathcal{M}}(\Delta^*, (\mathcal{L}_X S)^v) \\ &= \mathcal{L}_{S^v}(d\theta_{\mathrm{T}\mathcal{M}}(\Delta^*, \tilde{X})) \end{aligned}$$

since $\mathcal{L}_{S^v}\Delta^* = -\mathcal{L}_{\Delta^*}S^v = 0$, S^v being homogeneous of degree 0. Now

$$d\theta_{\mathrm{T}\mathcal{M}}(\Delta^*, \tilde{X}) = \Delta^*\langle \tilde{X}, \theta_{\mathrm{T}\mathcal{M}}\rangle = \Delta^* h_X = h_X$$

in view of the homogeneity of h_X. Consequently,

$$\mathcal{L}_{S^v}(d\theta_{\mathrm{T}\mathcal{M}}(\Delta^*, \tilde{X})) = \mathcal{L}_{S^v} h_X = h_{S(X)} = d\theta_{\mathrm{T}\mathcal{M}}(S^v, \tilde{X}).$$

The first result follows.

For the second we use the formula

$$(\mathcal{L}_{\Delta^*}\tilde{S})(\xi) = [\Delta^*, \tilde{S}(\xi)] - \tilde{S}([\Delta^*, \xi]).$$

With $\xi = \alpha^v$ the right hand side becomes

$$[\Delta^*, S(\alpha)^v] - \tilde{S}([\Delta^*, \alpha^v]) = -S(\alpha)^v + \tilde{S}(\alpha^v) = 0$$

since a vertical lift of a 1-form is, in effect, homogeneous of degree -1. With $\xi = \tilde{X}$ we obtain for the right hand side

$$[\Delta^*, \tilde{S}(\tilde{X})] = [\Delta^*, \widetilde{S(X)} + (\mathcal{L}_X S)^v] = 0$$

because complete lifts of vector fields and vertical lifts of type $(1,1)$ tensor fields are both homogeneous of degree 0. This completes the proof. $\square$

Using these results we can obtain the dilation field associated with $\tilde{S}$.

Theorem 5 *The dilation field D associated with $\tilde{S}$ is given by*

$$D = \tilde{\Delta} + \Delta^*.$$

Proof From Lemmas 3 and 4 we have

$$\tilde{S}(\tilde{\Delta} + \Delta^*) = 0$$

$$\mathcal{L}_{\tilde{\Delta}+\Delta^*}\tilde{S} = -\tilde{S}.$$

It remains to be shown that $D = \tilde{\Delta} + \Delta^*$ vanishes on the "zero" section $\mathrm{T}\mathcal{M}_0^\perp$ of the fibration of $\mathrm{T}^*\mathrm{T}\mathcal{M}$ determined by $\tilde{S}$. Now Δ generates the dilations $(x,u) \mapsto (x, e^t u)$ of $\mathrm{T}\mathcal{M}$, which leave $\mathcal{M}_0$ invariant. It follows that the one-parameter group generated by $\tilde{\Delta}$ maps $\mathrm{T}\mathcal{M}_0^\perp$ to itself. In constructing the complete lift of a vector field to a cotangent bundle one takes the inverse of the induced map of cotangent vectors. Bearing this in mind, as well as the linearity of the action, one sees that $\tilde{\Delta}$ generates on $\mathrm{T}\mathcal{M}_0^\perp$ a one-parameter group of transformations which may be written $(x,p) \mapsto (x, e^{-t}p)$ when that space is identified with $\mathrm{T}^*\mathcal{M}$. On the other hand, Δ^* generates the one-parameter group $(x,p) \mapsto (x, e^t p)$. Thus the one-parameter groups generated by $\tilde{\Delta}$ and Δ^*, both of which leave $\mathrm{T}\mathcal{M}_0^\perp$ invariant, are inverses of each other when restricted to that submanifold; so their generators satisfy

$\tilde{\Delta} = -\Delta^*$ there as required. $\square$

In the proof of the theorem in the appendix we shall use the following result.

Lemma 5 *The dilation field D satisfies*

$$\mathcal{L}_D \theta_{\mathrm{T}\mathcal{M}} = \theta_{\mathrm{T}\mathcal{M}}.$$

Proof We have

$$\mathcal{L}_{\tilde{\Delta}} \theta_{\mathrm{T}\mathcal{M}} = 0$$

because the Lie derivative of $\theta_{\mathrm{T}\mathcal{M}}$ along any complete lift vanishes, and

$$\mathcal{L}_{\Delta^*} \theta_{\mathrm{T}\mathcal{M}} = \theta_{\mathrm{T}\mathcal{M}}$$

by homogeneity. $\square$

6 Coordinate formulae

Before proceeding to an application, we collect together coordinate formulae for some of the quantities defined in the preceeding sections of the paper.

In the first place, given a manifold $\mathcal{N}$ with local coordinates (x^i), and coordinates (x^i, p_i) on $\mathrm{T}^*\mathcal{N}$ adapted to the cotangent bundle structure, we have the expressions

$$\alpha^v = \alpha_i \frac{\partial}{\partial p_i}$$

for the vertical lift of a 1-form $\alpha = \alpha_i dx^i$, and

$$\tilde{X} = X^i \frac{\partial}{\partial x^i} - p_i \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial p_j}$$

for the complete lift of a vector field $X = X^i \partial/\partial x^i$. The vertical lift of a type $(1,1)$ tensor field $R = R^i_j(\partial/\partial x^i) \otimes dx^j$ is

$$R^v = p_i R^i_j \frac{\partial}{\partial p_j}$$

and its complete lift is

$$\tilde{R} = R^i_j \left(\frac{\partial}{\partial x^i} \otimes dx^j + \frac{\partial}{\partial p_j} \otimes dp_i \right) + p_k \left(\frac{\partial R^k_i}{\partial x^j} - \frac{\partial R^k_j}{\partial x^i} \right) \frac{\partial}{\partial p_i} \otimes dx^j.$$

The following relations, similar in nature to the ones of Theorem 1, will be useful in the next section.

Lemma 6 *For a non-singular type* $(1,1)$ *tensor field* R *on* $\mathcal{N}$ *we have*

$$\tau_{R*}\alpha^v = R(\alpha)^v$$

$$\tau_{R*}\tilde{X} = \tilde{X} + (R^{-1} \circ \mathcal{L}_X R)^v.$$

Proof It is straightforward to verify these formulae in coordinates. □

We specialise now to the case $\mathcal{N} = \mathrm{T}\mathcal{M}$. We take tangent bundle coordinates (x^i, u^i) on $\mathrm{T}\mathcal{M}$ with corresponding coordinates (x^i, u^i, y_i, v_i) on $\mathrm{T}^*\mathrm{T}\mathcal{M}$. The vertical endomorphism on $\mathrm{T}\mathcal{M}$ is given by

$$S = \frac{\partial}{\partial u^i} \otimes dx^i$$

and its complete lift by

$$\tilde{S} = \frac{\partial}{\partial u^i} \otimes dx^i + \frac{\partial}{\partial y_i} \otimes dv_i.$$

Its image distribution is spanned by the coordinate vector fields $\partial/\partial u^i$ and $\partial/\partial y_i$. The imbedding $\psi\colon \mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}\mathcal{M}$ has the coordinate representation

$$(x^i, p_i) \mapsto (x^i, 0, 0, p_i).$$

The point with coordinates (x^i, p_i, r^i, s_i) in $\mathrm{T}\mathrm{T}^*\mathcal{M}$ corresponds to the vector

$$r^i \frac{\partial}{\partial x^i} + s_i \frac{\partial}{\partial p_i}$$

at the point (x^i, p_i) in $\mathrm{T}^*\mathcal{M}$; its image under ψ_* is the vector

$$r^i \frac{\partial}{\partial x^i} + s_i \frac{\partial}{\partial v_i}$$

at the point $(x^i, 0, 0, p_i)$ in $\mathrm{T}^*\mathrm{T}\mathcal{M}$. The result of applying $\tilde{S}$ to this is the vector

$$r^i \frac{\partial}{\partial u^i} + s_i \frac{\partial}{\partial y_i}$$

tangent to the fibre of $\hat{\tau}_{\mathrm{T}\mathcal{M}}$, which determines the point with coordinates (x^i, r^i, s_i, p_i) in $\mathrm{T}^*\mathrm{T}\mathcal{M}$. Thus the diffeomorphism $\Psi\colon \mathrm{T}\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}\mathcal{M}$ derived in Theorem 3 has the coordinate representation

$$(x^i, p_i, r^i, s_i) \mapsto (x^i, r^i, s_i, p_i).$$

(The simplicity of this coordinate representation makes a coordinate definition seem appealing: but it is worth pointing out that the confirmation that the map is actually well-defined, by consideration of the effects of coordinate transformations, is neither straightforward nor informative.)

The three dilation-related vector fields discussed in Section 5 are given by

$$\tilde{\Delta} = u^i \frac{\partial}{\partial u^i} - v_i \frac{\partial}{\partial v_i}$$

$$\Delta^* = y_i \frac{\partial}{\partial y_i} + v_i \frac{\partial}{\partial v_i}$$

$$D = u^i \frac{\partial}{\partial u^i} + y_i \frac{\partial}{\partial y_i}.$$

7 Application: adjoint symmetries and the Lagrangian extension of second-order systems

Let us first make some general comments about the possible relevance of lifting objects to a tangent or cotangent bundle of a manifold. When a manifold $\mathcal{N}$ is the natural carrier space for some dynamical system, it is perhaps not to be expected that essentially new features will be discovered by looking at lifted events, taking place say on $\mathrm{T}\mathcal{N}$ or on $\mathrm{T}^*\mathcal{N}$. Nevertheless, one is constantly forced to keep an eye on these spaces, if only as image spaces of vector fields and 1-forms on $\mathcal{N}$. It even occasionally can advance our understanding if we look at such images rather than at the objects on $\mathcal{N}$ themselves. A nice example in this respect is Tulczyjew's description of mechanics in terms of special symplectic structures [15], in which Lagrangian and Hamiltonian mechanica appear—loosely speaking—as two different manifestations of the same Lagrangian submanifold on $\mathrm{TT}^*\mathcal{M}$. A key role in this description was played by the diffeomorphism $\Psi\colon \mathrm{TT}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}\mathcal{M}$, the geometry of which we hope to have fully unravelled above. Note further that a similar approach later proved to be useful in field theory, in particular with respect to the energy-momentum tensor (see [7]).

Our present application is intended to shed new light on the meaning of so-called adjoint symmetries of an arbitrary second-order equation field Γ on $\mathrm{T}\mathcal{M}$. Among other things, it was shown in [12] that adjoint symmetries, which are related to invariant 1-forms, can give rise to first integrals or can generate, under appropriate circumstances, a Lagrangian for the system. Such occurrencies are much better understood when it concerns symmetry vector fields of a system which is a priori known to be Lagrangian. New insights in the role of adjoint symmetries can therefore be expected if we lift the relevant objects to $\mathrm{T}^*\mathrm{T}\mathcal{M}$ and relate them to the Lagrangian extension of Γ, referred to in the Introduction.

Let us first recall some definitions and results from [11, 12]. To every second-order equation field Γ on $\mathrm{T}\mathcal{M}$ we associate the following two sets, of 1-forms and vector fields respectively:

$$\mathcal{X}^*_\Gamma = \{\alpha \in \mathcal{X}^*(\mathrm{T}\mathcal{M}) \mid \mathcal{L}_\Gamma(S(\alpha)) = \alpha\},$$

$$\mathcal{X}_\Gamma = \{X \in \mathcal{X}(\mathrm{T}\mathcal{M}) \mid S([\Gamma, X]) = 0\}.$$

We have at our disposal the following projection operators:

$$\pi_\Gamma : \mathcal{X}^*(\mathrm{T}\mathcal{M}) \to \mathcal{X}^*_\Gamma \quad , \quad \alpha \to \pi_\Gamma(\alpha) = \mathcal{L}_\Gamma(S(\alpha)),$$

$$\pi_\Gamma : \mathcal{X}(\mathrm{T}\mathcal{M}) \to \mathcal{X}_\Gamma \quad , \quad X \to \pi_\Gamma(X) = X + S([\Gamma, X]).$$

A 1-form α on $\mathrm{T}\mathcal{M}$ is said to be an adjoint symmetry of Γ if $\alpha \in \mathcal{X}^*_\Gamma$ and $\mathcal{L}_\Gamma\alpha \in \mathcal{X}^*_\Gamma$. An equivalent formulation is that α is an adjoint symmetry if and only if $\mathcal{L}_\Gamma S(\alpha)$ is an invariant form for Γ. In coordinates, an adjoint symmetry is of the form $\alpha = \alpha_i\, du^i + \Gamma(\alpha_i)\, dx^i$, where the functions α_i satisfy the adjoint linear variational equations of Γ.

Concerning the Lagrangian extension of Γ to $\mathrm{T}^*\mathrm{T}\mathcal{M}$, this is obtained as follows. The vector field Γ on $\mathrm{T}\mathcal{M}$ induces a function on $\mathrm{T}^*\mathrm{T}\mathcal{M}$ (as does every vector field), namely the function $h_\Gamma = \langle \tilde{\Gamma}, \theta_{\mathrm{T}\mathcal{M}} \rangle$. Choosing $L \equiv h_\Gamma$, and using the tangent bundle structure of $\mathrm{T}^*\mathrm{T}\mathcal{M}$ we proceed to construct a Lagrangian vector field Γ_L in the usual way, that is, we introduce the Poincaré–Cartan 1-form $\theta_L = \tilde{S}(dL)$ and denote by Γ_L the vector field determined by

$$\iota_{\Gamma_L} d\theta_L = -dE_L,$$

where E_L is the energy function associated to L. The function h_Γ is fibre linear with respect to the cotangent bundle structure, but not with respect to the tangent bundle structure; in fact $d\theta_L$ is a symplectic form, as will become clear from the next argument. The type $(1,1)$ tensor field $\mathcal{L}_\Gamma S$ on $\mathrm{T}\mathcal{M}$ is non-singular: in fact $(\mathcal{L}_\Gamma S)^2 = 1$. Therefore, $\tau_{\mathcal{L}_\Gamma S}$, which henceforth will be abbreviated to τ_Γ, is a diffeomorphism of $\mathrm{T}^*\mathrm{T}\mathcal{M}$. The main result, reported in [1] states that

$$\tau_\Gamma{}^* d\theta_{\mathrm{T}\mathcal{M}} = -d\theta_L \quad , \quad \tau_\Gamma{}^* L = -E_L,$$

and

$$\tau_\Gamma{}^* \tilde{\Gamma} = \tau_\Gamma^{-1}{}_* \tilde{\Gamma} = \Gamma_L.$$

Since τ_Γ (which incidentally is equal to τ_Γ^{-1}) is fibred over the identity map of $\mathrm{T}\mathcal{M}$, it follows from the last relation that Γ_L projects onto Γ. We call Γ_L the Lagrangian extension of Γ. Naturally, Γ_L being a second-order equation field, we can also introduce the sets $\mathcal{X}^*_{\Gamma_L}$ and $\mathcal{X}_{\Gamma_L}$ on $\mathrm{T}^*\mathrm{T}\mathcal{M}$, with the corresponding projection operators π_{Γ_L}. The main point we wish to make now is that adjoint symmetries of Γ become symmetries of the Lagrangian extension Γ_L, when vertically lifted to $\mathrm{T}^*\mathrm{T}\mathcal{M}$.

Before proceeding, we should issue a little warning here. In those formulae of the previous section where a composition of $(1,1)$ tensors occurred, the tensors were regarded as linear maps on the module of vector fields. In some of the subsequent calculations, we will be dealing with the dual picture, where $(1,1)$ tensors which act linearly on 1-forms are composed. The order of composition is then of course the opposite of the order of composition in the action on vector fields. As an example, let us recall that, for the action on 1-forms on $\mathrm{T}\mathcal{M}$ we have the properties

$$S = S \circ \mathcal{L}_\Gamma S = -\mathcal{L}_\Gamma S \circ S.$$

Lemma 7 *For any 1-form* α *on* $\mathrm{T}\mathcal{M}$*, we have*

$$\pi_{\Gamma_L}(\alpha^v) = \pi_\Gamma(\alpha)^v.$$

Proof Using various results of the previous sections and the fact that $\tau_\Gamma = \tau_\Gamma^{-1}$, we have

$$\begin{aligned}
[\Gamma_L, \alpha^v] &= [\tau_\Gamma^{-1}{}_*\tilde{\Gamma}, \alpha^v] = \tau_\Gamma^{-1}{}_*[\tilde{\Gamma}, \tau_{\Gamma *}\alpha^v] \\
&= \tau_{\Gamma *}[\tilde{\Gamma}, (\mathcal{L}_\Gamma S(\alpha))^v] = \tau_{\Gamma *}(\mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(\alpha)))^v \\
&= \Big(\mathcal{L}_\Gamma S(\mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(\alpha)))\Big)^v.
\end{aligned}$$

It follows that

$$\begin{aligned}
\tilde{S}([\Gamma_L, \alpha^v]) &= \Big((S \circ \mathcal{L}_\Gamma S)(\mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(\alpha)))\Big)^v \\
&= \Big(S(\mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(\alpha)))\Big)^v \\
&= (\mathcal{L}_\Gamma(S \circ \mathcal{L}_\Gamma S(\alpha)) - (\mathcal{L}_\Gamma S)^2(\alpha))^v \\
&= (\mathcal{L}_\Gamma(S(\alpha)))^v - \alpha^v
\end{aligned}$$

whence

$$\pi_{\Gamma_L}(\alpha^v) = \alpha^v + \tilde{S}([\Gamma_L, \alpha^v]) = \mathcal{L}_\Gamma(S(\alpha))^v = \pi_\Gamma(\alpha)^v,$$

as asserted. □

Theorem 6 *Let* α *be a 1-form on* $\mathrm{T}\mathcal{M}$*, then*
(i) $\alpha \in \mathcal{X}_\Gamma^* \Longleftrightarrow \alpha^v \in \mathcal{X}_{\Gamma_L}$,
(ii) α *is an adjoint symmetry of* $\Gamma \iff \alpha^v$ *is a symmetry of* Γ_L.

Proof (i) This is an immediate consequence of Lemma 7.
(ii) It follows from the first calculation in the proof of Lemma 7 that

$$[\alpha^v, \Gamma_L] = 0 \Longleftrightarrow \mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(\alpha)) = 0,$$

which is exactly what we wished to show. □

Through the above stated equivalence, we are now in a position to reinterprete properties of adjoint symmetries of arbitrary second-order equations in terms of their more familiar counterparts for the Lagrangian system

Γ_L. Alternatively—and perhaps more importantly—one could arrive at new properties of adjoint symmetries by searching for those known properties of Lagrangian systems which project down to $\mathrm{T}\mathcal{M}$ with respect to the lifting process under consideration.

For a start, let us discuss the case where an adjoint symmetry α of Γ satisfies the additional requirement $\mathcal{L}_\Gamma S(\alpha) = dF$ for some function F. Then, clearly, $\Gamma(F)$ is a constant c (on connected components of $\mathrm{T}\mathcal{M}$) and for $c = 0$, F is a first integral. We have shown in [12] that if Γ itself happens to be Lagrangian, this is nothing but a dual version of Noether's theorem. The very simple relationship between firts integrals F and adjoint symmetries α, expressed by $\mathcal{L}_\Gamma S(\alpha) = dF$, may therefore come forward as something more fundamental than Noether's theorem, because it equally applies when Γ is not Lagrangian. This now should no longer surprise us because we are about to show that it is actually a manifestation of Noether's theorem for the Lagrangian extension Γ_L on $\mathrm{T}^*\mathrm{T}\mathcal{M}$. For the sake of clarity, we recall that a Noether symmetry of a Lagrangian system is a vector field which leaves both the Poincaré–Cartan 2-form and the energy function invariant.

Lemma 8 *For any 1-form α on $\mathrm{T}\mathcal{M}$, we have*

$$\mathcal{L}_\Gamma S(\alpha) = dF \Longleftrightarrow \iota_{\alpha^v} d\theta_L = -\pi^*_{\mathrm{T}\mathcal{M}}(dF)$$

$$\alpha^v(E_L) = -\pi^*_{\mathrm{T}\mathcal{M}}\langle \Gamma, \mathcal{L}_\Gamma S(\alpha)\rangle.$$

Proof We have

$$\begin{aligned}
\iota_{\alpha^v} d\theta_L &= -\iota_{\alpha^v} {\tau_\Gamma}^* d\theta_{\mathrm{T}\mathcal{M}} = -{\tau_\Gamma}^* \left(\iota_{\tau_{\Gamma *}\alpha^v} d\theta_{\mathrm{T}\mathcal{M}}\right) \\
&= -{\tau_\Gamma}^* \left(\iota_{\mathcal{L}_\Gamma S(\alpha)^v}\, d\theta_{\mathrm{T}\mathcal{M}}\right) = -{\tau_\Gamma}^*(\pi^*_{\mathrm{T}\mathcal{M}}(\mathcal{L}_\Gamma S(\alpha))) \\
&= -\pi^*_{\mathrm{T}\mathcal{M}}(\mathcal{L}_\Gamma S(\alpha)),
\end{aligned}$$

from which the result readily follows.
For the second part

$$\begin{aligned}
\alpha^v(E_L) &= -\alpha^v({\tau_\Gamma}^* L) = -{\tau_\Gamma}^*({\tau_\Gamma}_* \alpha^v(L)) \\
&= -{\tau_\Gamma}^*(\mathcal{L}_\Gamma S(\alpha)^v(h_\Gamma)) \\
&= -{\tau_\Gamma}^*(\pi^*_{\mathrm{T}\mathcal{M}}\langle \Gamma, \mathcal{L}_\Gamma S(\alpha)\rangle) \\
&= -\pi^*_{\mathrm{T}\mathcal{M}}\langle \Gamma, \mathcal{L}_\Gamma S(\alpha)\rangle,
\end{aligned}$$

as asserted. ◻

The content of the next statement is now obvious.

Theorem 7 *If α is an adjoint symmetry of Γ on $\mathrm{T}\mathcal{M}$ then $\mathcal{L}_\Gamma S(\alpha) = dF$ if and only if α^v is a symmetry of the Lagrangian extension Γ_L for which $\iota_{\alpha^v} d\theta_L$ is exact. Under these circumstances F is a first integral of Γ on $\mathrm{T}\mathcal{M}$ if and only if α^v is a Noether symmetry of Γ_L on $\mathrm{T}^*\mathrm{T}\mathcal{M}$.* ◻

Secondly, we wish to illustrate briefly how the reverse procedure of "projecting" down known results for a Lagrangian system such as Γ_L can, in principle, lead to the discovery of new properties concerning adjoint symmetries of Γ.

It is well known that a point symmetry of a Lagrangian system (which is not of Noether type) produces an alternative (or subordinate) Lagrangian (see [9]). The same thing may still happen for symmetries Y depending on the "velocities", provided an extra condition is satisfied, guaranteeing that the Lie derivative with respect to Y of the original Poincaré–Cartan 2-form is again a Poincaré–Cartan form (see form example [10]). To be specific, for the Lagrangian system Γ_L at hand, the extra condition amounts to the existence of functions L' and f such that

$$\mathcal{L}_Y \theta_L = \theta_{L'} + df.$$

The general idea now is to require that Y be a symmetry vector field of the form α^v for some 1-form α on $\mathrm{T}\mathcal{M}$ and that both L' and f are the pull back of functions on $\mathrm{T}\mathcal{M}$. Under these circumstances one can readily verify that the above requirement will be satisfied, provided α is an adjoint symmetry of Γ with the property

$$\alpha = \pi_\Gamma(dF)$$

for some function F. The expectation then is that the latter restriction on adjoint symmetries will lead to an interesting conclusion. And it surely does, because we know from [12] that any second-order equation field Γ (a priori not of Lagrangian type), for which an adjoint symmetry α exists of the form $\alpha = \pi_\Gamma(dF)$, turns out to be Lagrangian afterall with $\Gamma(F)$ as Lagrangian function.

There is a reasonable chance that the kind of techniques employed in this application would also be fruitful in field theory. For example, our notion of adjoint symmetry of a second-order equation was inspired by work of Gordon [6] which originates from field theory [5], where the adjoint linear equation of a partial differential equation appears to be more popular in defining the notion of symmetry. Of course, to even think of a further analogy, one would need a notion of almost tangent structure on jet bundles and some analogue of a second-order equation field. These are, however, exactly the kind of questions which have been tackled in recent work on jet fields by Saunders [13, 14], whereby it must be stipulated that Saunders's concept of jet field is directly related to the Cartan–Ehresmann connections in the work of Mangiarotti and Modugno [8].

Appendix

The construction below of a linear bundle diffeomorphism $\Phi\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$ follows the first stage of the general construction given by Weinstein. At one stage in the proof we use the fact that, given a vector at a point of $\mathrm{T}^*\mathrm{T}\mathcal{M}$ which is tangent to the fibre of $\hat{\tau}_{\mathrm{T}\mathcal{M}}$, one may always find a local vector field which generates fibre translations, leaves $d\theta_{\mathrm{T}\mathcal{M}}$ invariant and agrees with the given vector at its point of definition. To see this, observe first that translations in the fibre of $\hat{\tau}_{\mathrm{T}\mathcal{M}}$ leave $d\theta_{\mathrm{T}\mathcal{M}}$ invariant, provided, in the case of those coming from the vertical lifts of basic 1-forms, that these 1-forms are closed. This follows from the general fomulae $\mathcal{L}_{\alpha^v}\theta_{\mathcal{N}} = {\pi_{\mathcal{N}}}^*\alpha$ and $\mathcal{L}_{\tilde{X}}\theta_{\mathcal{N}} = 0$ quoted in Section 2. The remainder of the argument relies on the fact that a given covector at a point can always be regarded as originating from a closed 1-form defined on a neighbourhood of that point.

Theorem 8 $\mathrm{T}^*\mathrm{T}\mathcal{M}$ *is diffeomorphic to* $\mathrm{T}^*\mathrm{T}^*\mathcal{M}$ *by a map which is a linear bundle map with respect to the vector bundle structures defined by* $\hat{\tau}_{\mathrm{T}\mathcal{M}}$ *and* $\pi_{\mathrm{T}^*\mathcal{M}}$ *respectively, which is fibred over* $\psi^{-1}\colon \mathrm{T}\mathcal{M}_0^{\perp} \to \mathrm{T}^*\mathcal{M}$*, and which is a symplectomorphism with respect to the canonical symplectic structures of the two cotangent bundles.*

Proof We shall define a map $\Phi\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$ by using the vector space structure of the fibres of $\hat{\tau}_{\mathrm{T}\mathcal{M}}\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}_0^{\perp}$ and $\pi_{\mathrm{T}^*\mathcal{M}}\colon \mathrm{T}^*\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathcal{M}$. To any point $q \in \mathrm{T}^*\mathrm{T}\mathcal{M}$ there corresponds a vector ζ at $\hat{\tau}_{\mathrm{T}\mathcal{M}}(q) \in \mathrm{T}\mathcal{M}_0^{\perp}$

which is tangent to the fibre of $\hat{\tau}_{\mathrm{T}\mathcal{M}}$ there, by the identification of a vector space with its tangent space at the origin. The covector $\iota_\zeta d\theta_{\mathrm{T}\mathcal{M}}$ annihilates the tangent space to the fibre, since the fibration is Lagrangian, and may therefore be regarded as an element of $\mathrm{T}^*_{\hat{\tau}_{\mathrm{T}\mathcal{M}}(q)}\mathrm{T}\mathcal{M}_0^\perp$; in fact taking the interior product with $d\theta_{\mathrm{T}\mathcal{M}}$ at $\hat{\tau}_{\mathrm{T}\mathcal{M}}(q)$ is an isomorphism between the tangent space to the fibre and the cotangent space to the base, since the tangent spaces to the fibre and the base are complementary Lagrangian subspaces.

We may therefore define by this procedure a fibre linear diffeomorphism $\mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}^*\mathrm{T}\mathcal{M}_0^\perp$. Composing this with the map $\mathrm{T}^*\mathrm{T}\mathcal{M}_0^\perp \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$ induced by ψ gives the required map Φ. It is clear from the construction that Φ is a diffeomorphism, is fibre linear, and is fibred over ψ^{-1}.

It remains to be shown that Φ is symplectic. The proof is based on the observation that the vector ζ at $\hat{\tau}_{\mathrm{T}\mathcal{M}}(q)$ corresponding to the point $q \in \mathrm{T}^*\mathrm{T}\mathcal{M}$ is the translate to $\hat{\tau}_{\mathrm{T}\mathcal{M}}(q)$ of D_q (where D is the dilation field associated with $\tilde{S}$).

We show first that $\Phi^*\theta_{\mathrm{T}^*\mathcal{M}} = \iota_D d\theta_{\mathrm{T}\mathcal{M}}$. For any vector w at $q \in \mathrm{T}^*\mathrm{T}\mathcal{M}$ we have

$$\begin{aligned}
\langle w, \Phi^*\theta_{\mathrm{T}^*\mathcal{M}}\rangle_q &= \langle \Phi_* w, \theta_{\mathrm{T}^*\mathcal{M}}\rangle_{\Phi(q)} = \langle \pi_{\mathrm{T}^*\mathcal{M}*}\Phi_* w, \Phi(q)\rangle \\
&= \langle \psi_*^{-1}\hat{\tau}_{\mathrm{T}\mathcal{M}*} w, \Phi(q)\rangle = \langle \hat{\tau}_{\mathrm{T}\mathcal{M}*} w, \iota_\zeta d\theta_{\mathrm{T}\mathcal{M}}\rangle \\
&= d\theta_{\mathrm{T}\mathcal{M}}(\zeta, \hat{\tau}_{\mathrm{T}\mathcal{M}*} w)_{\hat{\tau}_{\mathrm{T}\mathcal{M}}(q)}.
\end{aligned}$$

By the remarks before the statement of the theorem, this final expression is equal to its translate back to q, along the vector ζ; moreover, the vector w and its translate to $\hat{\tau}_{\mathrm{T}\mathcal{M}}(q)$ have the same projection under $\hat{\tau}_{\mathrm{T}\mathcal{M}*}$. Thus

$$\langle w, \Phi^*\theta_{\mathrm{T}^*\mathcal{M}}\rangle_q = d\theta_{\mathrm{T}\mathcal{M}}(D_q, w),$$

from which the result follows.

It follows from Lemma 5 that

$$\Phi^* d\theta_{\mathrm{T}^*\mathcal{M}} = d(\iota_D d\theta_{\mathrm{T}\mathcal{M}}) = \mathcal{L}_D d\theta_{\mathrm{T}\mathcal{M}} = d\theta_{\mathrm{T}\mathcal{M}}$$

and so Φ is symplectic, as required. □

Notice that, although Φ is symplectic, it is not the case that $\Phi^*\theta_{\mathrm{T}^*\mathcal{M}} = \theta_{\mathrm{T}\mathcal{M}}$. In fact these two 1-forms differ by an exact form:

$$\Phi^*\theta_{\mathrm{T}^*\mathcal{M}} - \theta_{\mathrm{T}\mathcal{M}} = \iota_D d\theta_{\mathrm{T}\mathcal{M}} - \mathcal{L}_D\theta_{\mathrm{T}\mathcal{M}} = -d\langle D, \theta_{\mathrm{T}\mathcal{M}}\rangle.$$

As well as the diffeomorphism $\Phi\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$ defined above, there is a standard diffeomorphism $\mathrm{T}\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$ constructed using the symplectic structure of $\mathrm{T}^*\mathcal{M}$. It is not difficult to show that the two constructions are consistent, in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{T}\mathrm{T}^*\mathcal{M} \arrow[rr, "\Psi"] \arrow[dr] & & \mathrm{T}^*\mathrm{T}\mathcal{M} \arrow[dl, "\Phi"] \\
& \mathrm{T}^*\mathrm{T}^*\mathcal{M} &
\end{tikzcd}$$

In coordinates, the construction of the symplectomorphism $\Phi\colon \mathrm{T}^*\mathrm{T}\mathcal{M} \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$ proceeds as follows. The canonical 2-form on $\mathrm{T}^*\mathrm{T}\mathcal{M}$ is

$$d\theta_{\mathrm{T}\mathcal{M}} = dy_i \wedge dx^i + dv_i \wedge du^i.$$

The point (x^i, u^i, y_i, v_i) in $\mathrm{T}^*\mathrm{T}\mathcal{M}$ determines the vector

$$u^i \frac{\partial}{\partial u^i} + y_i \frac{\partial}{\partial y_i}$$

at $(x^i, 0, 0, v_i)$ in $\mathrm{T}\mathcal{M}_0^{\perp}$. Its interior product with $d\theta_{\mathrm{T}\mathcal{M}}$ is the covector

$$y_i dx^i - u^i dv_i.$$

The map Φ therefore has the coordinate representation

$$(x^i, u^i, y_i, v_i) \mapsto (x^i, v_i, y_i, -u^i).$$

The map $\mathrm{T}\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}^*\mathcal{M}$ based on the canonical 2-form $d\theta_{\mathcal{M}}$ is given by

$$(x^i, p_i, r^i, s_i) \mapsto (x^i, p_i, s_i, -r^i),$$

which is the composition of Φ with the diffeomorphism $\mathrm{T}\mathrm{T}^*\mathcal{M} \to \mathrm{T}^*\mathrm{T}\mathcal{M}$.

References

[1] F. Cantrijn, M. Crampin and W. Sarlet. Evading the Inverse Problem for Second Order Ordinary Differential Equations by Using Additional Variables. *Inverse Problems,* 3:51–63, 1987.

[2] R. S. Clark and M. R. Bruckheimer. Sur les Structures Presque Tangents. *Comptes Rendus de l'Academie des Sciences, Paris,* 251:627–629, 1960.

[3] M. Crampin. Tangent Bundle Geometry for Lagrangian Dynamics. *Journal of Physics A: Mathematical and General,* 16:3755–3772, 1983.

[4] M. Crampin and G. Thompson. Affine Bundles and Integrable Almost Tangent Structures. *Mathematical Proceedings of the Cambridge Philosophical Society,* 98:61–71, 1985.

[5] T. J. Gordon. Equivalent conserved currents and generalized Noether's theorem. *Annals of Physics,* 115:85–107, 1984.

[6] T. J. Gordon. On the symmetries and invariants of the harmonic oscillator. *Journal of Physics A: Mathematical and General,* 19:183–189, 1986.

[7] J. Kijowski and W. M. Tulczyjew. *A symplectic framework for field theories.* Lecture Notes in Physics 107, Springer–Verlag, Berlin, 1979.

[8] L. Mangiarotti and M. Modugno. Fibred spaces, jet spaces and connections. *Proceedings of the meeting "Geometry and Physics",(October 1982).* Pitagora Editrice, Bologna:135–165, 1983.

[9] G. Marmo and E. J. Saletan. Ambiguities in the Lagrangian and Hamiltonian formalism: transformation properties. *Nuovo Cimento,* 40B:67–89, 1977.

[10] G. E. Prince. Toward a classification of dynamical symmetries in classical mechanics. *Bulletin of the Australian Mathematical Society,* 27:53–71, 1983.

[11] W. Sarlet, F. Cantrijn and M. Crampin. A new look at second-order equations and Lagrangian mechanics. *Journal of Physics A: Mathematical and General,* 17:1999–2009, 1984.

[12] W. Sarlet, F. Cantrijn and M. Crampin. Pseudo-symmetries, Noether's theorem and the adjoint equation. *Journal of Physics A: Mathematical and General,* 20:1365–1376, 1987.

[13] D. J. Saunders. An alternative approach to the Cartan form in Lagrangian field theories. *Journal of Physics A: Mathematical and General,* 20:339–349, 1987.

[14] D. J. Saunders. Jet fields, connections and second-order differential equations. *Journal of Physics A: Mathematical and General,* 20:3261–3270, 1987.

[15] W. M. Tulczyjew. Hamiltonian systems, Lagrangian systems and the Legendre transformation. *Symposia Mathematica,* 16:247–258, 1974.

[16] W. M. Tulczyjew. Les Sous-variétés Lagrangiennes et la Dynamique Lagrangienne. *Comptes Rendus de l'Academie des Sciences, Paris,* 283:675–678, 1976.

[17] A. Weinstein. Symplectic Manifolds and their Lagrangian Submanifolds. *Advances in Mathematics,* 6:329–346, 1976.

[18] K. Yano and S. Ishihara. *Tangent and Cotangent Bundles.* Marcel Dekker, New York, 1973.