The canonical isomorphism between T^kT^*M and T^*T^kM

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Abstract — We discuss a natural symplectic structure on T^kT^*M and a natural kth-order almost tangent structure on T^*T^kM. The main result concerns the construction of a vector bundle isomorphism $\psi_k : T^kT^*M \longrightarrow T^*T^kM$ (over T^kM), which behaves naturally with respect to all structures of interest. We further use this result to prove that one can identify spaces such as T^rT^sM and T^sT^rM by a map which in coordinates simply consists in switching suffices.

1. Introduction. — The existence of a canonical isomorphism between TT^*M and T^*TM is well known and of fundamental importance in many applications. It is, for example, a key matter in Tulczyjew's description of Lagrangian and Hamiltonian theory, in terms of a Lagrangian submanifold which is shared by two special symplectic structures on TT^*M [1]. It is our belief that the canonical isomorphism between T^kT^*M and T^*T^kM, which we will discuss in the present note, has a number of interesting features in its own right and will be valuable for clarifying certain aspects of the approach to higher-order mechanics described in [2] and [3].

2. Notations and Preliminaries. — We seek conformity in notations with some of our earlier work (see e.g. [4] and [5]), but slight adaptations are always necessary. Let $\tau_k : T^kN \longrightarrow N$ denote the tangent bundle of order k of some manifold N. For $\ell < k$, we have projection operators $\tau_k^\ell : T^kN \longrightarrow T^\ell N$. It is known that a smooth map $\phi : N \longrightarrow N'$ induces a map $\phi^{(k)} : T^kN \longrightarrow T^kN'$. Thus, for example, the manifold T^kT^*M is fibred over T^kM (as a vector bundle) through the map $\pi_M^{(k)}$, induced by the cotangent bundle projection $\pi_M : T^*M \longrightarrow M$. As in [4] and [6], let $\mathbf{T}$ denote the canonical inclusion $\mathbf{T} : T^kN \longrightarrow TT^{k-1}N$ and $d_\mathbf{T}$ the total derivative operator which turns functions and forms on $T^\ell N$ into corresponding objects on $T^{\ell+1}N$. Repeated application of this operator on a 1-form α on N results in a 1-form ${d_\mathbf{T}}^k\alpha$ on T^kN. A tangent bundle of order k such as T^kN comes naturally equipped with a type (1,1) tensor field, the so-called vertical endomorphism, which we will denote by $S_N^{(k)}$. We recall from [4] the following commutation relations: ${\phi^{(k+1)}}^* \circ d_\mathbf{T} = d_\mathbf{T} \circ {\phi^{(k)}}^*$, ${\tau_{k+1}^{\ell+1}}^* \circ d_\mathbf{T} = d_\mathbf{T} \circ {\tau_k^\ell}^*$, and also, as a map on 1-forms,

$S_N^{(k+1)} \circ d_{\mathbf{T}} = d_{\mathbf{T}} \circ S_N^{(k)} + {\tau_{k+1}^k}^*$. From these properties it is easy to deduce that ${\phi^{(k)}}^* \circ {d_{\mathbf{T}}}^k = {d_{\mathbf{T}}}^k \circ \phi^*$ and $S_N^{(k)} \circ {d_{\mathbf{T}}}^k = k\, {\tau_k^{k-1}}^* \circ {d_{\mathbf{T}}}^{k-1}$ (where for $k = 1$, ${d_{\mathbf{T}}}^0$ is to be regarded as the identity operator). Other canonical objects which are soon to be discussed are the canonical 1-form of a cotangent bundle (notation: θ_M for T^*M), and dilation vector fields. For notational distinction we will write e.g. Δ_M^* for the dilation field on T^*M and $\Delta_N^{(k)}$ for the dilation field on T^kN. Finally, we will indicate the complete lift of a geometrical object to its cotangent bundle with a tilde (as in [5]), while for the complete lift of a vector field X on N to a vector field on T^kN we will write $\widetilde{X}^{(k)}$ (as in [4]).

3. Main Results. — Representing coordinates on T^*M as (q^a, p_a) ($a = 1, \dots, n = \dim M$), we will denote the corresponding natural coordinates on T^kT^*M by $(q_i^a, p_{a|i})$ ($i = 0, \dots, k$) and the coordinates on T^*T^kM by (q_i^a, p_a^i).

The space T^kT^*M obviously carries a kth-order tangent structure. On the other hand, starting from θ_M, its kth-order total derivative produces a globally defined 1-form on T^kT^*M, which in coordinates reads

$$ {d_{\mathbf{T}}}^k \theta_M = \sum_{i=0}^{k} \binom{k}{i} p_{a|k-i}\, dq_i^a $$

(with summation over a from 1 to n understood). It is clear from this expression that the exterior derivative yields a non-degenerate (exact) 2-form, which proves the following statement.

Theorem. — *$(T^kT^*M, d\, {d_{\mathbf{T}}}^k \theta_M)$ is a symplectic manifold.*

Next we turn to the space T^*T^kM which has a natural symplectic structure and can further be endowed with a type (1,1) tensor field via the complete lift of $S_M^{(k)}$. In coordinates:

$$ \widetilde{S_M^{(k)}} = \sum_{i=1}^{k} i\, \frac{\partial}{\partial q_i^a} \otimes dq_{i-1}^a + \sum_{i=1}^{k} i\, \frac{\partial}{\partial p_a^{i-1}} \otimes dp_a^i \,. $$

Theorem. — *$\widetilde{S_M^{(k)}}$ is an integrable kth-order almost tangent structure on T^*T^kM.*

Proof. — Inspection of the above coordinate expression shows that for $S = \widetilde{S_M^{(k)}}$, $\ker S^{j+1}$ and $\operatorname{im} S^{k-j}$ are both spanned by the coordinate vector fields

$$ \left\{ \frac{\partial}{\partial q_{k-j}^a}, \dots, \frac{\partial}{\partial q_k^a}, \frac{\partial}{\partial p_a^0}, \dots, \frac{\partial}{\partial p_a^j} \right\} \quad (\text{for } j = 0, \dots, k). $$

The vanishing of the Nijenhuis tensor $\mathcal{N}_S$ is obvious because the coefficients in that same expression are all constants. Thus, all requirements for an integrable kth-order almost tangent structure are verified.

We now come to the generalization of the diffeomorphism $TT^*M \longleftrightarrow T^*TM$.

THEOREM. — *There is a vector bundle isomorphism $\psi_k : T^kT^*M \longrightarrow T^*T^kM$, which is both a symplectomorphism and an isomorphism of kth-order almost tangent structures.*

The vector bundle structures we are referring to here are, respectively, the fibrations $\pi_M^{(k)} : T^kT^*M \longrightarrow T^kM$ and $\pi_{T^kM} : T^*T^kM \longrightarrow T^kM$. Observe first that the coordinate expression of $d_{\mathbf{T}}{}^k\theta_M$ clearly shows that this 1-form vanishes on vectors which are vertical with respect to $\pi_M^{(k)}$. This justifies the following intrinsic construction.

DEFINITION. — For each $Q \in T^kT^*M$ with projection $q = \pi_M^{(k)}(Q)$, we define $\psi_k(Q) \in T^*T^kM$ to be the covector at q, determined by the relation:

$$\forall\, \zeta_q \in T_qT^kM, \qquad \langle \zeta_q\,,\, \psi_k(Q)\rangle = \left\langle \xi_Q\,,\, (d_{\mathbf{T}}{}^k\theta_M)_Q \right\rangle,$$

where $\xi_Q \in T_QT^kT^*M$ is any vector with the property $T\pi_M^{(k)}(\xi_Q) = \zeta_q$.

It is clear from this definition that $\pi_{T^kM} \circ \psi_k = \pi_M^{(k)}$. Moreover, we have: $\forall\, \xi_Q \in T_QT^kT^*M$,

$$\begin{aligned}
\left\langle \xi_Q\,,\, ({\psi_k}^*\,\theta_{T^kM})_Q \right\rangle &= \left\langle T\psi_k(\xi_Q)\,,\, (\theta_{T^kM})_{\psi_k(Q)} \right\rangle \\
&= \langle T\pi_{T^kM} \circ T\psi_k(\xi_Q)\,,\, \psi_k(Q)\rangle \\
&= \left\langle T\pi_M^{(k)}(\xi_Q)\,,\, \psi_k(Q)\right\rangle \;=\; \left\langle \xi_Q\,,\, (d_{\mathbf{T}}{}^k\theta_M)_Q \right\rangle,
\end{aligned}$$

where we have used the definition of θ_{T^kM}. It follows that ${\psi_k}^*\theta_{T^kM} = d_{\mathbf{T}}{}^k\theta_M$. Since the coordinate expression for θ_{T^kM} reads $\theta_{T^kM} = \sum_{i=0}^k p_a^i\, dq_i^a$, we see rightaway that the map ψ_k in coordinates is given by

$$(q_i^a, p_{a|i}) \stackrel{\psi_k}{\longmapsto} \left(q_i^a, p_a^i = \binom{k}{i} p_{a|k-i} \right),$$

and is truly a vector bundle isomorphism. Finally, if ψ_{k*} stands for the push forward operation on tensor fields, we have

$$\psi_{k*}\frac{\partial}{\partial p_{a|i}} = \binom{k}{i}\frac{\partial}{\partial p_a^{k-i}}, \quad \psi_{k*}dp_{a|i-1} = \binom{k}{i-1}^{-1} dp_a^{k-i+1}.$$

The kth-order tangent structure on T^kT^*M is determined by

$$S_{T^*M}^{(k)} = \sum_{i=1}^k i\,\frac{\partial}{\partial q_i^a} \otimes dq_{i-1}^a + \sum_{i=1}^k i\,\frac{\partial}{\partial p_{a|i}} \otimes dp_{a|i-1}$$

and it is now easy to verify that $\psi_{k*}S_{T^*M}^{(k)} = \widetilde{S_M^{(k)}}$, which completes the proof of our main theorem.

The geometrical structure of a tangent bundle (of any order) is, in a way, fully determined by the almost tangent structure and its associated dilation field. The dilation field associated to $S^{(k)}_{T^*M}$ on T^kT^*M is the vector field

$$\Delta^{(k)}_{T^*M} = \sum_{i=1}^{k} i\, q^a_i \frac{\partial}{\partial q^a_i} + \sum_{i=1}^{k} i\, p_{a|i} \frac{\partial}{\partial p_{a|i}}.$$

Obviously, in view of the diffeomorphism ψ_k, the dilation field associated to the almost tangent structure $\widetilde{S^{(k)}_M}$ on T^*T^kM is the vector field

$$\psi_{k*}\, \Delta^{(k)}_{T^*M} = \sum_{i=1}^{k} i\, q^a_i \frac{\partial}{\partial q^a_i} + \sum_{i=0}^{k-1} (k-i)\, p^i_a \frac{\partial}{\partial p^i_a}.$$

It is of some interest to find the relation between this vector field and other dilation fields which naturally live on T^*T^kM. There are in fact two such dilation fields, namely the complete lift of the one on T^kM and the dilation field of T^*T^kM as a cotangent bundle. Their coordinate expressions read

$$\widetilde{\Delta^{(k)}_M} = \sum_{i=1}^{k} i\, q^a_i \frac{\partial}{\partial q^a_i} - \sum_{i=1}^{k} i\, p^i_a \frac{\partial}{\partial p^i_a}, \qquad \Delta^*_{T^kM} = \sum_{i=0}^{k} p^i_a \frac{\partial}{\partial p^i_a}\,.$$

The following correspondence now can easily be verified.

Theorem. — *The dilation field associated to the almost tangent structure $\widetilde{S^{(k)}_M}$ on T^*T^kM is given by $\psi_{k*}\, \Delta^{(k)}_{T^*M} = \widetilde{\Delta^{(k)}_M} + k\, \Delta^*_{T^kM}$.*

4. Remarks. — a) For $k = 1$ we recover the known results. Our present intrinsic definition of ψ_1 then coincides with the one in [7].

b) Note that the ψ_k-maps induce a projection of cotangent bundles of higher-order tangent bundles: we can define

$$\rho^{k-1}_k : T^*T^kM \longrightarrow T^*T^{k-1}M\,, \qquad \rho^{k-1}_k = \psi_{k-1} \circ \tau^{k-1}_k \circ {\psi_k}^{-1}.$$

It is easy to show, using the main theorem and results of section 2, that

$$\widetilde{S^{(k)}_M}(\theta_{T^kM}) = k\, \rho^{k-1\,*}_k\, \theta_{T^{k-1}M}.$$

c) We know that θ_M is uniquely determined by the property $\alpha^*\theta_M = \alpha$, for any 1-form α on M, regarded as a section of T^*M. The induced section $\alpha^{(k)}$ of $\pi^{(k)}_M$ is easily seen to have the property ${\alpha^{(k)}}^*\, d_{\mathbf{T}}{}^k\theta_M = d_{\mathbf{T}}{}^k(\alpha^*\theta_M) = d_{\mathbf{T}}{}^k\alpha$. It follows that $\psi_k \circ \alpha^{(k)} = d_{\mathbf{T}}{}^k\alpha$, regarded as a section of π_{T^kM}.

5. Generalization of the canonical involution of TTM. — Tulczyjew's construction of the map ψ_1 [1] was based on the canonical involution of TTM. Roughly speaking, we now want to turn the arguments around and use the fact

that we already have $\psi_k : T^kT^*M \longrightarrow T^*T^kM$ at our disposal, to define a canonical diffeomorphism between T^kTM and TT^kM. This will then be used to initialize an induction process.

A point $z \in T^kTM$ is the k-velocity of a curve $\gamma(t)$ in TM. Let $q \in T^kM$ denote the point $\tau_M^{(k)}(z)$, where $\tau_M : TM \longrightarrow M$ is the tangent bundle projection. The point of TT^kM which we want to associate to z is going to be the vector $\zeta_q \in T_qT^kM$, determined by the condition:

$$\forall\, \alpha_q \in T_q^*T^kM : \langle \zeta_q\,,\, \alpha_q \rangle = \frac{d^k}{dt^k} \langle \gamma(t)\,,\, \chi(t) \rangle \bigg|_{t=0} \,,$$

where $\chi(t)$ is a curve in T^*M, representing the k-velocity ${\psi_k}^{-1}(\alpha_q) \in T^kT^*M$ and satisfying $\tau_M(\gamma(t)) = \pi_M(\chi(t)) \ \ \forall\, t$.

To see what this means in coordinates, let us denote the coordinates of z as $(q^a_{0,i}\,,\, q^a_{1,i})$. A representative curve $\gamma(t)$ then is given by

$$\gamma(t) = \left(\sum_{i=0}^{k} \frac{1}{i!} q^a_{0,i}\, t^i \,,\, \sum_{i=0}^{k} \frac{1}{i!} q^a_{1,i}\, t^i \right) .$$

The element ζ_q we look for will have coordinates $(q^a_{i,0} = q^a_{0,i}\,,\, q^a_{i,1})$. Its pairing with an arbitrary $\alpha_q = (q^a_i, p^i_a)$ is given by $\sum_{i=0}^k q^a_{i,1}\, p^i_a$ and the defining relation will of course have to determine the $q^a_{i,1}$. We have ${\psi_k}^{-1}(\alpha_q) = \left(q^a_i \,,\, \binom{k}{i}^{-1} p^{k-i}_a \right)$ so that a representation of an appropriate $\chi(t)$ reads:

$$\chi(t) = \left(\sum_{i=0}^{k} \frac{1}{i!} q^a_i\, t^i \,,\, \sum_{i=0}^{k} \frac{(k-i)!}{k!} p^{k-i}_a\, t^i \right) .$$

The right-hand side of the defining relation is precisely the coefficient of $(1/k!)t^k$ in the product of the second components of $\gamma(t)$ and $\chi(t)$ and is easily found to be $\sum_{i=0}^k q^a_{1,i}\, p^i_a$. It follows that the map $T^kTM \longrightarrow TT^kM$ simply consists in switching suffices: $q^a_{r,i} \longmapsto q^a_{i,r} \quad (i = 0, \ldots, k;\ r = 0, 1)$.

For the induction, assume that we know about the identification $T^rT^sM = T^sT^rM$ for some s and all r and that it consists of switching suffices (the case $s = 1$ having just been proved). Using the canonical injection of $T^{s+1}M$ into TT^sM we then obtain the chain $T^rT^{s+1}M \subset T^rTT^sM = TT^rT^sM = TT^sT^rM$, which shows that there is an injective map $T^rT^{s+1}M \longrightarrow TT^sT^rM$. A schematic coordinate representation of this map is obtained as follows (the ranges of the different indices are $i = 0, \ldots, s+1;\ j = 0, \ldots, r;\ m = 0, \ldots, s;\ \ell = 0, 1$):

$$\begin{aligned} (q^a_{i,j}) \ &\longmapsto\ (q^a_{m,\ell,j}) \ \text{ where } q^a_{m,\ell,j} = q^a_{i,j} \ \text{ with } \ i = m + \ell \\ &\longmapsto\ (q^a_{m,j,\ell}) \ \longmapsto\ (q^a_{j,m,\ell}) \,. \end{aligned}$$

Since the image point in TT^sT^rM satisfies $q^a_{j,m,\ell} = q^a_{j,m',\ell'}$ when $m + \ell = m' + \ell'$, it is actually a point of the submanifold $T^{s+1}T^rM$, which means that there is a final identification: $(q^a_{j,m,\ell}) \longmapsto (q^a_{j,i})$.

We conclude that there is a natural identification of T^rT^sM and T^sT^rM for all r and s. For a derivation of this result in a more abstract setting, see e.g. [8].

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