New aspects of integrability of generalized Hénon-Heiles systems

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Abstract. The class of so-called Hénon-Heiles systems is slightly broadened by allowing for the existence of non-standard Hamiltonians. The extra parameter in the equations of motion is shown to give rise to a generalization of the three known integrability cases. In addition, three degenerate cases are detected, characterized by a partial decoupling of the equations. For these cases, we still obtain two independent first integrals, but their involutiveness can only be understood in terms of a non-standard Poisson structure.

1. Introduction

There is a vast literature on case studies of complete integrability of Hamiltonian systems in general and of so-called Hénon-Heiles systems in particular. For a recent revision of the present state of the art and a link between integrable cases of the Hénon-Heiles system and a class of integrable 5th-order PDE's, see Fordy (1991). The purpose of this note is to indicate that there are still certain aspects of the problem which have been overlooked so far and which lead to a larger class of integrable cases. Our case study concerns the following system of second-order ODE's,

$$\ddot{q}_1 = -c_1\, q_1 + b\, q_1^2 - a\, q_2^2, \tag{1}$$

$$\ddot{q}_2 = -c_2\, q_2 - 2m\, q_1 q_2. \tag{2}$$

The original system studied by Hénon and Heiles (1964) corresponds to the case where all parameters are equal to one. Its non-integrability is now well understood. In all other searches for special parameter values which entail integrability, one has so far a priori set $m = a$. The reason for this is of course very simple : it is the necessary and sufficient requirement for the existence of a potential function $V(q_1, q_2)$ such that the right-hand sides of (1) and (2) are of the form $-\partial V/\partial q_i$, and one naturally wants the system to be Hamiltonian from the outset. What is being overlooked, however, is the possible existence of a non-standard Lagrangian or Hamiltonian for the system, one which arises after multiplication of the given set of equations with a non-singular matrix. For example, it is easy to verify that for m and a different from zero, the given system always admits the multiplier matrix diag(m, a), leading to the Lagrangian

$$L = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}a\dot{q}_2^2 - \frac{1}{2}mc_1q_1^2 + \frac{1}{3}mbq_1^3 - maq_1q_2^2 - \frac{1}{2}ac_2q_2^2. \tag{3}$$

Note that a rescaling of the coordinates can bring the given system back into the more familiar form only if we have $am > 0$. Our system (1),(2) therefore is a generalization of the usual Hénon-Heiles system and provides the additional option of letting one of the coupling parameters be zero (we of course exclude the trivially decoupled case $a = m = 0$).

Among the various techniques which have been developed in the hunt for integrable cases we prefer to use one which will produce the two independent first integrals in the process. First integrals can be constructed via a direct search, as in Grammaticos *et al* (1982) and Leach (1980), or via the study of Hamiltonian symmetries or Noether symmetries, as in Fordy (1983) and Sahadevan and Lakshmanan (1986). The latter would seem to be excluded here as our extension is meant to cover cases where a Lagrangian or Hamiltonian is not a priori known. Fortunately, a theory exists which can cope exactly with this complication and amounts to searching, not for symmetries, but for so-called adjoint symmetries of the given differential equations. We refer to Sarlet *et al* (1987) for the theory of adjoint symmetries of autonomous second-order equations (the case at hand) and to Sarlet *et al* (1990) for its extension to time-dependent systems. What we need of this theory in the present context can be summarized as follows.

Let Γ denote the vector field associated to the given system, i.e. $\Gamma = v_i(\partial/\partial q_i) + f_i(\partial/\partial v_i)$, where the functions f_i represent the right-hand sides of (1),(2). Ad adjoint symmetry is a 1-form of the type $\alpha = \alpha_i(q,v)\,dv_i + \Gamma(\alpha_i)\,dq_i$, where the leading coefficients α_i satisfy the set of second-order PDE's,

$$\Gamma^2(\alpha_i) + \Gamma\left(\alpha_j \frac{\partial f_j}{\partial v_i}\right) - \alpha_j \frac{\partial f_j}{\partial q_i} = 0, \tag{4}$$

which are the adjoints of the equations for a symmetry vector field of Γ. Suppose we have a solution of (4), matching the additional requirement $\alpha_i = \partial F/\partial v_i$ for some function F. Then, the function $L = \Gamma(F)$ is a Lagrangian for the given system in the sense that we have the identities $\Gamma(\partial L/\partial v_i) - \partial L/\partial q_i \equiv 0$. This function is not always terribly interesting as a Lagrangian, because it may be of some degenerate nature. In particular, we may have $\Gamma(F) = 0$, in which case, of course, F is a first integral. It is further important to know that every first integral can be obtained this way. If a regular Lagrangian is a priori known, then there is a map between adjoint symmetries and symmetries, which makes that we are then essentially talking about Noether's theorem. In the other event, while Noether's theorem is no longer available, the adjoint symmetry technique still survives with the same level of ease (or complexity).

2. First integrals of degree 2 and 4

Having the classical results about the Hénon-Heiles system in mind, we now want to explore the existence of two independent first integrals of (1) and (2), which are quadratic or at most of degree 4 in the velocities. For the quadratic case, the leading coefficients of the corresponding adjoint symmetry will be linear in the velocities. With this ansatz, the usual proces of splitting up equations (4) into the different parts coming from independent monomials, gives rise to a set of 20 defining equations. These and all subsequent computations are ... straightforward but tedious, but fortunately computer algebra can be of great assistance (see further). In the generic case (i.e. no special assumptions on the parameters), only one adjoint symmetry emerges and it produces a first integral which is the Hamiltonian corresponding to (3). In the course of the solution process that we followed, we encountered the following list of special cases that needed a separate investigation (often with a considerable number of subbranches) : $b = 0$; $b = -2m$; $b = -m$; $m = 0$; $b = -8m/3$; $a = 0$; $c_2 = c_1$; $b(c_1 + c_2) + 2mc_1 = 0$; $b = -6m$. It is not excluded, however, that somebody else, following a different path of solution, would manage to avoid some of these subcases. We will not list all the cases which led to two or more adjoint symmetries, because these need not always result in two first integrals. So here is a survey of the interesting parameter values with the corresponding first integrals.

case 1 : $b = -6m \quad (m \neq 0)$

$$F_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}av_2^2 + \frac{1}{2}c_1 mq_1^2 + \frac{1}{2}c_2 aq_2^2 + amq_1q_2^2 + 2m^2q_1^3, \tag{5}$$

$$F_2 = q_2v_1v_2 - q_1v_2^2 + \frac{4c_2 - c_1}{4m}v_2^2 + c_2q_1q_2^2 + mq_1^2q_2^2 + \frac{c_2}{4m}(4c_2 - c_1)q_2^2 - \frac{a}{4}q_2^4. \tag{6}$$

case 2 : $b = -m,\ c_2 = c_1 \quad (m \neq 0)$

$$F_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}av_2^2 + \frac{1}{2}c_1 mq_1^2 + \frac{1}{2}c_1 aq_2^2 + amq_1q_2^2 + \frac{1}{3}m^2q_1^3, \tag{7}$$

$$F_2 = v_1v_2 + c_1q_1q_2 + mq_1^2q_2 + \frac{1}{3}aq_2^3. \tag{8}$$

case 3 : $a = 0,\ b = -2m/5,\ c_2 = 4c_1 \quad (m \neq 0)$

$$F_1 = \frac{1}{2}v_1^2 + \frac{1}{2}c_1q_1^2 + \frac{2}{15}mq_1^3, \tag{9}$$

$$F_2 = q_1v_1v_2 - q_2v_1^2 + c_1q_1^2q_2 + \frac{2}{5}mq_2q_1^3. \tag{10}$$

case 4 : $a = 0,\ b = -2m \quad (m \neq 0)$

$$F_1 = \frac{1}{2}v_1^2 + \frac{1}{2}c_1q_1^2 + \frac{2}{3}mq_1^3, \tag{11}$$

$$\begin{aligned} F_2 = {} & mq_2^2v_1^2 - 2mq_1q_2v_1v_2 + mq_1^2v_2^2 + (c_1 - c_2)q_1v_2^2 - (c_1 - c_2)q_2v_1v_2 \\ & + \frac{1}{4m}(c_1 - c_2)(c_1 - 4c_2)v_2^2 - c_2(c_1 - c_2)q_1q_1^2 + \frac{c_2}{4m}(c_1 - c_2)(c_1 - 4c_2)q_2^2. \end{aligned} \tag{12}$$

case 5 : $m = 0,\ b = 0$

$$F_1 = \frac{1}{2}v_2^2 + \frac{1}{2}c_2 q_2^2 \tag{13}$$

$$F_2 = \frac{1}{2}(c_1 - 4c_2)v_1^2 + 2aq_2v_1v_2 - 2aq_1v_2^2 + \frac{1}{2}c_1(c_1-4c_2)q_1^2 + a(c_1-2c_2)q_1q_2^2 + \frac{1}{2}a^2q_1^4. \tag{14}$$

Before entering into a discussion of these results, let us move on to the case of first integrals of degree 4 in the velocities. With the ansatz that the leading coefficients α_i of adjoint symmetries this time can be of degree 3, the determining equations ensuing from (4) are already quite horrendous and tend to make the computer algebra package we have been using run out of memory. However, if we concentrate on the generation of first integrals, the requirements $\alpha_i = \partial F/\partial v_i$ impose certain relations between the various coefficients of the α_i and one can easily deduce further that the coefficients of the highest-order terms must be of a certain polynomial nature (as functions of the q_i). For example, the coefficient of $v_1v_2^2$ in α_1 will necessarily have to be a polynomial of degree 2 in q_1 with coefficients which are again polynomial of degree 2 in q_2. With this extra knowledge built into the starting equations, we were able to master the situation and we have further reduced the algebra by investigating this time only the cases where none of the parameters of the nonlinear terms (a, b or m) is zero. The special parameter relations which showed up (in order op appearance) read : $b=-m$; $b=-4m/3$; $b=-10m/3$; $b=-6m$; $b=-2m$; $c_2=c_1$; $c_2=4c_1$; $c_2=9c_1$; $c_1=9c_2$; $c_2=16c_1$; $b=-3m/5$; $b=-16m/5$; $b=-16m$ (again, of course, with numerous subbranches requiring separate investigation). Not quite surprisingly, only one additional favourable case was detected and is listed below.

case 6 : $b=-16m$, $c_1=16c_2$ $\quad (m \neq 0)$

$$F_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}av_2^2 + 8c_2mq_1^2 + \frac{1}{2}c_2aq_2^2 + amq_1q_2^2 + \frac{16}{3}m^2q_1^3 \tag{15}$$

$$F_2 = \frac{1}{4}v_2^4 + mq_1q_2^2v_2^2 + \frac{1}{2}c_2q_2^2v_2^2 - \frac{1}{3}mq_2^3v_1v_2 + \frac{1}{4}c_2^2q_2^4 - \frac{1}{3}mc_2q_1q_2^4 - \frac{1}{3}m^2q_1^2q_2^4 - \frac{1}{18}amq_2^6. \tag{16}$$

A number of comments and interpretations are in order now. Clearly, the cases 1, 2 and 6 correspond to the three well-known integrability cases of the Hénon-Heiles system which can be found in many publications. One must keep in mind, however, that we are still looking at a more general situation here : the standard cases are recovered for the additional requirement $m=a$! For an example of an integrable case which is not discussed in the literature so far, we could take e.g. $c_1 = c_2 = m = 1$, $b = a = -1$. The first integral F_1 in the cases 1, 2 and 6 is of course the Hamiltonian corresponding to (3). It is easy to verify that the formal relationship between integrable Hénon-Heiles systems and the stationary flow of a class of integrable 5th-order PDE's, as recently discussed by Fordy (1991), is not affected by our present extension.

3. Degenerate cases and complete integrability

Let us turn now to the cases 3, 4 and 5, which in some sense are degenerate cases, to the best of our knowledge not discussed before. Clearly, in each of these cases, there is a partial decoupling of the given second-order equations (but a nonlinear coupling term remains). At first glance, there is no reason why, for eaxmple, the first integrals (9) and (10) would be less valuable than the two first integrals of the cases 1, 2 or 6. Yet, there is an important difference, because we apparently lost our Hamiltonian : the first integrals (9), (11) and (13) correspond in each of these cases to a Hamiltonian for the decoupled equation only. It is accordingly no longer clear to what extent the two first integrals in these degenerate cases could be regarded as being in involution. Obviously, if this new question can be resolved, the symplectic form (or the Poisson bracket) cannot be the standard one and one would perhaps prefer that the first integral F_2 would take over the role of Hamiltonian. To be precise, we are addressing here the following problem : given the first integral F_2, find a non-degenerate 2-form ω, such that

$$i_\Gamma \omega + dF_2 = 0 \qquad \text{and} \qquad d\omega = 0. \tag{17}$$

We will sketch how this problem, at least locally, has a fairly elegant solution for case 3, the other two cases being similar.

As a preliminary remark, using the techniques of the so-called inverse problem of Lagrangian mechanics, as described e.g. in Sarlet (1982) or Morandi *et al* (1990), one can verify that the differential equations for each of these degenerate cases do not admit a Lagrangian description. This means that there will be no solution for the symplectic form ω in (17) with vanishing $dv_1 \wedge dv_2$-part. It is further known to be trivial that every system can locally be cast into a Hamiltonian form, and one can even do this with a preassigned Hamiltonian as we wish to achieve here. Writing ω in the form $\frac{1}{2}\omega_{ij}\, dx_i \wedge dx_j$, where the x_i collectively denote the variables (q_1, q_2, v_1, v_2), and choosing F_2 in accordance with (10), the algebraic part of (17) gives rise to a set of linear equations, whose coefficient matrix has rank 3. We choose to solve the last three equations for ω_{12}, ω_{13} and ω_{14} in terms of ω_{23}, ω_{24} and ω_{34}. Imposing next the requirement $d\omega = 0$, we end up with the conditions,

$$\frac{\partial \omega_{24}}{\partial v_1} - \frac{\partial \omega_{23}}{\partial v_2} - \frac{\partial \omega_{34}}{\partial q_2} = 0 \qquad , \qquad \Gamma(\omega_{24}) = 0,$$
$$v_1\, \Gamma(\omega_{34}) = v_2\, \omega_{24} + v_1 (q_1 + \omega_{23}) + \Gamma(v_1)\, \omega_{34},$$
$$v_1\, \Gamma(\omega_{23}) = \Gamma(v_1)\, (q_1 + \omega_{23}) + \Gamma(v_2)\, \omega_{24} + v_1^2 - 2v_1 (m q_1 + 2c_1)\, \omega_{34}.$$

This is a system of 4 linear PDE's for only 3 unknowns, but one can show that formal integrability conditions are satisfied. In fact, the whole problem can be reduced to finding a particular solution of just one partial differential equation as follows. Choosing $\omega_{24} = 0$ and putting $\omega_{23} = -q_1 + \tilde{\omega}_{23}$, the first equation implies $\tilde{\omega}_{23} = -\partial f / \partial q_2$, $\omega_{34} = \partial f / \partial v_2$ for some function f. Putting further $f = v_1 g$, it is easy to verify that the remaining equations are satisfied, provided g is a particular solution of the equation

$$\Gamma(g) = -2q_2. \tag{18}$$

In terms of such a solution, perhaps difficult to construct but certainly existing locally, a symplectic form with respect to which F_2 is a Hamiltonian for our problem is given by

$$\omega = dg \wedge (\Gamma(v_1)\, dq_1 - v_1\, dv_1) + v_1\, dq_1 \wedge dq_2 - q_1 (dv_1 \wedge dq_2 + dq_1 \wedge dv_2).$$

Inverting the coefficient matrix of this symplectic form, one obtains a Poisson bracket structure with respect to which (9) and (10) are in involution. Similar constructions can be made for the cases 4 and 5.

4. Discussion and outlook for future studies

The approch we have followed puts the emphasis on constructing independent first integrals of given second-order equations without worrying about a possible

Hamiltonian structure from the outset. Involutiveness of these first integrals, as we have seen, is an aspect that can be brought into the picture at a later stage, if desired. An interesting question for further study thus emerges : "Is it possible to find criteria for verifying complete integrability directly at the level of the second-order equations?". In this context, we can announce forthcoming work with E. Mart´inez and J.F. Cariñena on a somewhat related question. A theory has been developed which enables testing given second-order equations for the existence of a suitable coordinate system in which the equations completely decouple. Case 2 is such a separable case and the fact that it is slightly broader than the standard case of separability of the Hénon-Heiles system actually motivated the present paper.

It is worth observing that our results seem to give more ground to the suggestion of Chang *et al* (1982) that more cases of integrability of the Hénon-Heiles system may exist. Translated to the broader system (1),(2), their Painlevé analysis would point to integrability whenever $\sqrt{1-48m/b}$ is a rational number. Many of the subbranches in our analysis, which needed separate investigation, actually correspond to this profile. More importantly, the degenerate cases 3 and 4 (where $b=-2m$ or $b=-2m/5$) are exactly of this type. We know of other examples of such ratio's, namely $b=-2m/15$, $b=-m/11$, $b=-3m/5$, $b=-16m/5$, which would certainly turn up (among others?) if the study of first integrals of degree 4 were completed to include the degenerate case $a=0$. Maybe, the conjecture of Chang *et al* therefore only is true if such degenerate cases are allowed into the picture. A word of caution about the search for higher-order invariants is perhaps appropriate here. Some authors referred to before, in trying to reduce the algebra, have restricted the structure of the 4th-degree invariant they were looking for by making use of the already known energy integral. One cannot do this, however, without loss of generality and they were simply lucky not to miss out a case.

Finally, we would like to describe briefly what kind of computer algebra assistance we have been able to use. The first part of the problem in our approach concerns setting up the defining equations for adjoint symmetries of a second-order system. REDUCE-procedures have been developed by Sarlet and Vanden Bonne (1991) for the automation of this process. Once an adjoint symmetry has been found, the same package is able to test whether it matches the additional requirement for producing a Lagrangian or a first integral and will generally automatically compute this function.

The hard part, of course, concerns solving the defining equations. For that problem, one should be able to exploit the know-how which has been put into various programs for computing Lie symmetries of differential equations. Unfortunately, not many of these programs offer the possibility of entering this process with one's own set of linear, homogeneous, overdetermined PDE's. A very nice program which has such an interface is the MUMATH-package LIE, developed by Head (version 2.1, 1990). We used a slightly customized version of this program for being able to detect the special parameter relations which require separate investigation. This way, the whole procedure has to be monitored much more interactively than in the original setup, but it still remains a great tool.

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