Derivations of differential forms along the tangent bundle projection. Part II

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Abstract. The study of the calculus of forms along the tangent bundle projection τ, initiated in a previous paper with the same title, is continued. The idea is to complete the basic ingredients of the theory up to a point where enough tools will be available for developing new applications in the study of second-order dynamical systems. A list of commutators of important derivations is worked out and special attention is paid to degree zero derivations having a Leibnitz-type duality property. Various ways of associating tensor fields along τ to corresponding objects on TM are investigated. When the connection coming from a given second-order system is used in this process, two important concepts present themselves: one is a degree zero derivation called the dynamical covariant derivative; the other one is a type $(1, 1)$ tensor field along τ, called the Jacobi endomorphism. It is illustrated how these concepts play a crucial role in describing many of the interesting geometrical features of a given dynamical system, which have been dealt with in the literature.

KEYWORDS: derivations, forms along a map, second-order equations, nonlinear connections

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1 Introduction

In a previous paper [15] (in what follows referred to as part I), we have studied the algebra of derivations of scalar and vector-valued forms along the tangent bundle projection $\tau : TM \to M$. It is a truism to say that there are submodules of vector fields and differential forms which will play a special role on a tangent bundle (or indeed on any vector bundle), namely the vertical vector fields and the semi-basic forms. These can in a natural way be identified with corresponding objects along the projection τ. More specifically related to the structure of a tangent bundle is the canonical type (1,1) tensor field S: it has an associated degree 1 derivation d_S which also translates naturally to a derivation of forms along τ. This in itself may be a sufficient motivation for the investigations we started in part I. Additional impulses come from the study of second-order differential equations in general and of Lagrangian systems in particular, where many important concepts are somehow related to vertical fields and semi-basic forms. The systematic approach of part I revealed that a satisfactory classification of derivations of forms along τ can only be achieved with the help of a connection on the bundle $\tau : TM \to M$. This is a quite appealing feature of the theory because every second-order vector field Γ on TM comes naturally equipped with its own, generally non-linear connection. The relevance of the calculus of forms along τ was already exhibited in part I: we showed indeed that this new approach is capable to explain and complete the calculus (on TM) relative to a second-order differential equation, introduced in [18], where, for example, the need of bringing a connection into the picture was not recognized.

With the present paper we want to continue with the development of fundamental ingredients of the theory initiated in part I. It is not our purpose to come up with an exhaustive story of all possible relations between derivations one can think of. We rather wish to complete part I with all concepts and tools which are thought to be essential for bringing the theory on the verge of new applications in the study of second-order equations. Such applications will be the subject of future publications.

In Section 2 we list a number of commutators of the basic derivations which entered the classification and decomposition results of part I and are needed in later calculations. Section 3 focuses on properties of derivations of degree 0 which satisfy a Leibnitz-type rule with respect to the pairing between vector fields and 1-forms. Since the basic degree 0 derivations of part I do not have such a duality property, their extension by duality leads to new important derivations, among which we find derivations of the type of a Lie derivative and others of the type of a covariant derivative. In Section 4 we pay some attention to various processes of lifting tensor fields along τ to corresponding fields on TM. They are essential for the interplay between results obtained in the new formalism and the more traditional calculus on TM.

The remaining sections enter the heart of the matter which has to do with the study of second-order equations. There are two fundamental concepts in the calculus along τ which, in our opinion, contain most of the information about the dynamics of a given second-order vector field Γ. The first one is a degree 0 derivation with the above mentioned duality property which will be called the dynamical covariant derivative. It is introduced in Section 5. The second one, discussed in Section 6, is a vector-valued 1-form along τ, which is called the Jacobi endomorphism and relates to the familiar concept with that name in the case of a spray. The two together, for example, define equations which determine symmetries and adjoint symmetries of the given Γ. In Section 7 we discuss how other geometrical concepts of interest, such as tension and strong torsion, enter our approach. The final section gives an illustration of the way in which known problems and results, previously studied on the tangent bundle, acquire an elegant and transparent reformulation within the new framework.

2 Complement on commutators of basic derivations

Throughout this paper, we will of course preserve most of the notations of part I. Some simplifications will be made, however, where it is no longer felt to be necessary to keep track of all subtleties of a first exposure. For example, vector fields and forms on M, when thought of as fields along τ via their composition with τ, will no longer be indicated with a tilde. Also, we will omit the notational distinction between the usual vertical lift of a vector field on M to a vector field on TM and the similar operation for vector fields along τ. In other words, for $X \in \mathcal{X}(\tau)$, we will now write X^V for what previously was denoted as $X^\uparrow$. Similarly, assuming a connection has been selected, we will write X^H instead of the $\xi^H(X)$ of part I.

Recall that a general degree r derivation D of $V(\tau)$ (the module of vector-valued forms along τ) has a decomposition of the form

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3} + a_Q, \tag{1}$$

with $L_1 \in V^{r+1}(\tau)$, $L_2, L_3 \in V^r(\tau)$, $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$. The L_i are uniquely determined by the corresponding action of D on the scalar forms $\bigwedge(\tau)$; i_{L_1} and $d^V_{L_2}$, by definition, vanish on basic vector fields; d^H on the other hand was extended from $\bigwedge(\tau)$ to $V(\tau)$ by the following action on vector fields: for all $X, Z \in \mathcal{X}(\tau)$,

$$(d^H X(Z))^V = P_V([Z^H, X^V]), \tag{2}$$

where P_V is the vertical projector on TM. Eq. (2) is equivalent to the more elaborate construction of $d^H X$ in part I. By construction, the difference between D and the first three terms of the right-hand side of (1) vanishes on $\bigwedge(\tau)$ and so a_Q is the last part of D to be identified and acts on $\mathcal{X}(\tau)$ only. Strictly speaking, one should keep different notations for D, a derivation of $V(\tau)$, and $\overline{D}$, its corresponding derivation of $\bigwedge(\tau)$, but we will generally not do so and regard $\overline{D}$ loosely speaking as the restriction of D to $\bigwedge(\tau)$. The different ingredients in the decomposition (1) are called respectively: derivations of type $i_\star$, $d^V_\star$, $d^H_\star$ and $a_\star$. The derivations of type $i_\star$, $d^V_\star$ and $a_\star$ each constitute a subalgebra and as such can be used to define a bracket operation on $V(\tau)$ and on $\bigwedge(\tau) \otimes V^1(\tau)$. To avoid confusion, however, we do not wish to push this matter too far and, for the time being, only consider the most fundamental bracket of these, namely the one induced by $d^V_\star$ operations. We write

$$[d^V_L, d^V_M] = d^V_{[L,M]_V}, \tag{3}$$

where $[L,M]_V$ was computed in part I and reads

$$[L,M]_V = d^V_L M - (-1)^{\ell m} d^V_M L. \tag{4}$$

For the computation of other commutators which follows, two tools are very helpful. One is, of course, the Jacobi identity; the other one is the property (see Lemma 5.2 of part I) that for any D which vanishes on basic vector fields we have:

$$\forall \alpha \in \textstyle\bigwedge^1(M)\ ,\ L \in V(\tau): \qquad \overline{D}(i_L\alpha) = i_{DL}\alpha. \tag{5}$$

When it concerns a derivation of type $i_\star$, (5) remains valid for $\alpha \in \bigwedge^1(\tau)$ or even for a vector-valued 1-form along τ.

Using (5), it is easy to verify that

$$[i_L, i_M] = i_{i_L M} - (-1)^{(\ell-1)(m-1)} i_{i_M L}. \tag{6}$$

Next, concerning $[i_L, d^V_M]$, we observe that this is a derivation vanishing on basic functions (and basic vector fields) and thus decomposes into a part of type $i_\star$ and a part of type $d^V_\star$. The $d^V_\star$-part is most easily computed by looking at the action on fibre linear functions on TM. For any $\alpha \in \bigwedge^1(M)$ and the corresponding functions $\widehat{\alpha}$ on TM we find,

$$[i_L, d^V_M]\widehat{\alpha} = i_L d^V_M \widehat{\alpha} = i_L i_M \alpha = i_{i_L M}\alpha = d^V_{i_L M}\widehat{\alpha}.$$

Putting $[i_L, d^V_M] - d^V_{i_L M} = i_A$, we next make use of the Jacobi identity to compute

$$\begin{aligned} d^V_A &= [i_A, d^V] = [[i_L, d^V_M], d^V] \\ &= (-1)^{(m+\ell)}\,[[d^V, i_L], d^V_M] = (-1)^m [d^V_L, d^V_M]. \end{aligned}$$

It follows that

$$[i_L, d^V_M] = (-1)^m i_{[L,M]_V} + d^V_{i_L M}. \tag{7}$$

The commutators (3), (6) and (7) in fact could have been copied directly from the standard results of Frölicher and Nijenhuis [9] with d^V in the role of d. The situation becomes different when d^H comes into the picture. Note first that for any $F \in C^\infty(TM)$ we have,

$$[i_L, d^H_M]F = i_L d^H_M F = i_L i_M d^H F = i_{i_L M} d^H F = d^H_{i_L M} F,$$

where we have used the remark following equation (5). For the same reason, in fact, a relation of this type is still true for the action on any $X \in \mathcal{X}(\tau)$. It follows that $[i_L, d^H_M] - d^H_{i_L M}$ is of type $i_\star$. By analogy with (7), we therefore put

$$[i_L, d^H_M] = (-1)^m i_{[L,M]_H} + d^H_{i_L M}, \tag{8}$$

which defines $[L,M]_H$. If in coordinates, L is of the form $L^i \otimes (\partial/\partial q^i)$ with $L^i \in \bigwedge^\ell(\tau)$ (similar expression for M), then the coordinate expression of the "horizontal bracket" $[L,M]_H$ will result from the following computation:

$$\begin{aligned} i_{[L,M]_H} dq^i &= (-1)^m \left\{ i_L d^H_M dq^i - (-1)^{(\ell-1)m} d^H_M L^i - d^H_{i_L M} dq^i \right\} \\ &= i_L d^H M^i - (-1)^{\ell m} d^H_M L^i - (-1)^{\ell-1} d^H i_L M^i \\ &= d^H_L M^i - (-1)^{\ell m} d^H_M L^i. \end{aligned}$$

Hence,

$$[L,M]_{H} = \left(d^{H}_{L} M^i - (-1)^{\ell m} d^{H}_{M} L^i \right) \otimes \frac{\partial}{\partial q^i}. \tag{9}$$

Keep in mind, however, that this bracket will in general not satisfy a Jacobi identity. In particular, for $X, Y \in \mathcal{X}(\tau)$ we have

$$[X,Y]_{H} = (X^k H_k(Y^i) - Y^k H_k(X^i)) \frac{\partial}{\partial q^i}.$$

We now turn to some interesting commutators of $d^{V}_{\star}$ and $d^{H}_{\star}$ derivations. Recall from part I that on $\bigwedge(\tau)$ we had

$$\frac{1}{2}[d^{H}, d^{H}] = -i_{d^V R} + d^{V}_{R},$$

where R is the curvature, a vector-valued 2-form which in coordinates has the expression:

$$R = \frac{1}{2} R^i_{jk}\, dq^j \wedge dq^k \otimes \frac{\partial}{\partial q^i} \quad , \quad R^i_{jk} = H_k(\Gamma^i_j) - H_j(\Gamma^i_k),$$

Γ^i_j representing the connection coefficients. The more general decomposition on $V(\tau)$ therefore will be of the form

$$\frac{1}{2}[d^{H}, d^{H}] = -i_{d^V R} + d^{V}_{R} + a_{\mathrm{Rie}}, \tag{10}$$

for some element $\mathrm{Rie} \in \bigwedge^2(\tau) \otimes V^1(\tau)$. For practical computations, it is useful to know that for a general $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$ of the form $\omega \otimes U$ ($\omega \in \bigwedge^r(\tau)$, $U \in V^1(\tau)$), the definition of a_Q in part I simply means:

$$\forall X \in \mathcal{X}(\tau) \quad , \quad a_{\omega \otimes U}(X) = \omega \otimes U(X) \in V^r(\tau).$$

With this information, it is then easy to deduce the coordinate expression of Rie from the action of $d^{H} \circ d^{H}$ on $\partial/\partial q^j$. One obtains,

$$\begin{aligned} \mathrm{Rie} &= \frac{1}{2} R^i_{jk\ell}\, dq^k \wedge dq^\ell \otimes \left(\frac{\partial}{\partial q^i} \otimes dq^j \right), \\ R^i_{jk\ell} &= H_k(\Gamma^i_{\ell j}) - H_\ell(\Gamma^i_{kj}) + \Gamma^i_{km}\Gamma^m_{\ell j} - \Gamma^i_{\ell m}\Gamma^m_{kj}, \end{aligned} \tag{11}$$

where we have put $\Gamma^i_{kj} = \partial \Gamma^i_k / \partial v^j$. The origin of the notation should now be clear: in the case of a linear connection on M, the Γ^i_{kj} do not depend on the v^ℓ and Rie reduces to the classical Riemann tensor.

The torsion tensor

$$T = \frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji})\, dq^i \wedge dq^j \otimes \frac{\partial}{\partial q^k} \in V^2(\tau)$$

was introduced in part I via the commutator of d^{H} and d^{V} on $\bigwedge(\tau)$. From the action of this commutator on basic vector fields, it is easy to calculate the following more general relation on $V(\tau)$:

$$[d^{H}, d^{V}] = d^{V}_{T} - a_{\mathrm{D}^V T}. \tag{12}$$

Here, $\mathrm{D}^V T \in \bigwedge^2(\tau) \otimes V^1(\tau)$ for the time being is a formal notation for the tensor field

$$\mathrm{D}^V T = \frac{1}{2}(\Gamma^k_{ij\ell} - \Gamma^k_{ji\ell})\, dq^i \wedge dq^j \otimes \left(\frac{\partial}{\partial q^k} \otimes dq^\ell \right). \tag{13}$$

The meaning of the operator D^V (not a derivation of vector-valued forms!) will become clear in the next section. By $\Gamma^k_{ij\ell}$ we of course mean $\partial\Gamma^k_{ij}/\partial v^\ell = \partial^2\Gamma^k_i/\partial v^j \partial v^\ell$.

Commutators of d^H_L with d^V_M or d^H_M give rise to rather messy expressions. We therefore limit ourselves to the case of degree 0 derivations, which are the ones needed for what follows. From the Jacobi identity, applied to i_X, d^H and d^H, one gets

$$[d^H_X, d^H] = \frac{1}{2}\, [i_X, [d^H, d^H]]\, .$$

Taking account of this in the Jacobi identity for i_Y, d^H_X and d^H and using also (8), there results:

$$[d^H_X, d^H_Y] = d^H_{[X,Y]_H} + \frac{1}{2}\, [i_Y, [i_X, [d^H, d^H]]]\, . \tag{14}$$

The commutators of i_X with (10) can be computed with the help of (6),(7) and (4), plus the fact that $[i_X, a_Q] = a_{i_X Q}$, whereby $i_X Q$ stands for contraction of the $\bigwedge(\tau)$-part of Q with X. We thus obtain,

$$\begin{aligned} \frac{1}{2}\, [i_X, [d^H, d^H]] &= -i_{i_X d^V R} + d^V_{i_X R} + i_{[X,R]_V} + a_{i_X \mathrm{Rie}} \\ &= i_{d^V i_X R - i_R d^V X} + d^V_{i_X R} + a_{i_X \mathrm{Rie}}, \end{aligned}$$

and substitution into (14) ultimately leads to

$$[d^H_X, d^H_Y] = d^H_{[X,Y]_H} + d^V_{R(X,Y)} + a_{\mathrm{Rie}(X,Y)} + i_A, \tag{15}$$

where $A \in V^1(\tau)$ is found (using also (5)) as follows:

$$\begin{aligned} A &= i_Y d^V i_X R - i_Y i_R d^V X - [Y, i_X R]_V \\ &= -d^V i_Y i_X R - i_{i_Y R} d^V X + d^V_{i_X R} Y \\ &= -d^V(R(X,Y)) + d^V_{i_X R} Y - d^V_{i_Y R} X. \end{aligned}$$

Remark: We will observe in the next section that $\mathrm{Rie} = -\mathrm{D}^V R$. The relations (10) and (15) then show that $R = 0$ is the necessary and sufficient condition for having $d^H \circ d^H = 0$ and for the H-bracket of vector fields to satisfy the Jacobi identity. One can indeed, more generally show that $d^H_\star$-derivations under those circumstances consitute a subalgebra of the algebra of derivations of $V(\tau)$.

Replacing one d^H by d^V at the starting point of the preceding calculation and following a similar procedure, one is led to the following relation:

$$\begin{aligned} [d^H_X, d^V_Y] + [d^V_X, d^H_Y] &= d^V_{[X,Y]_H + T(X,Y)} + d^H_{[X,Y]_V} \\ &\quad - a_{\mathrm{D}^V T(X,Y)} + i_{d^V_{i_X T} Y - d^V_{i_Y T} X - d^V(T(X,Y))}. \end{aligned} \tag{16}$$

However, we also need an expression for a single term of the left-hand side and this requires a separate calculation. Note first that we have

$$[d^H, d^V_X] = i_{d^V d^H X} + d^V_{d^H X} - d^H_{d^V X} - a_B. \tag{17}$$

As a matter of fact, the action on $\bigwedge(\tau)$ of this commutator was mentioned in part I (eq. (16)). A coordinate calculation for the action on $\partial/\partial q^i$ reveals that the tensor $B \in \bigwedge^1(\tau) \otimes V^1(\tau)$ is given by

$$B = \Gamma^k_{jm\ell} X^\ell \, dq^j \otimes \left(\frac{\partial}{\partial q^k} \otimes dq^m \right).$$

Subsequently, from the Jacobi identity for i_Y, d^V_X and d^H, one obtains

$$[d^V_X, d^H_Y] = d^H_{[X,Y]_V} - [i_Y, [d^H, d^V_X]].$$

The desired result then will follow from (17) and an appropriate use of (6),(7),(8) and (4). It reads,

$$\begin{aligned} [d^V_X, d^H_Y] &= d^H_{d^V_X Y} - d^V_{d^H_Y X} + a_{\theta(X,Y)} + i_A, \\ A &= d^V d^H_Y X - d^V_{d^H X} Y - [Y, d^V X]_H. \end{aligned} \tag{18}$$

The tensor field θ (not in $\bigwedge(\tau) \otimes V(\tau)$!) here is introduced in such a way that $i_Y B = \theta(X,Y)$ and has the following coordinate expression,

$$\theta = \Gamma^k_{jm\ell} \, dq^\ell \otimes dq^j \otimes \left(\frac{\partial}{\partial q^k} \otimes dq^m \right). \tag{19}$$

It is instructive to interchange X and Y in (18) and subtract the resulting expression from (18). Comparison with (16) then gives rise to the following interesting properties

$$[X,Y]_H + T(X,Y) = d^H_X Y - d^H_Y X, \tag{20}$$

$$\theta(X,Y) - \theta(Y,X) = -\mathrm{D}^V T(X,Y). \tag{21}$$

A few remarks to conclude this section. We have occasionally used the property $[i_X, a_Q] = a_{i_X Q}$. In fact, derivations of type $a_\star$ constitute an ideal of the full algebra. This property could be used to extend the action of the different types of derivations to tensor fields $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$. It is at the moment, however, not very appropriate to discuss this because for $r = 0$, this new action is different from the original one on vector-valued 1-forms. For practical purposes, it often suffices to know that for $U_1, U_2 \in V^1(\tau)$:

$$[a_{U_1}, a_{U_2}] = a_{[U_1,U_2]} \quad \text{with} \quad [U_1, U_2] = U_1 \circ U_2 - U_2 \circ U_1, \tag{22}$$

while on the other hand

$$[i_{U_1}, i_{U_2}] = -i_{[U_1,U_2]}. \tag{23}$$

3 Extending derivations of degree zero by duality

Definition 3.1 *A derivation D of $V(\tau)$ of degree 0 is said to be self-dual if $\forall X \in \mathcal{X}(\tau)$, $\forall \alpha \in \bigwedge^1(\tau)$:*

$$D(\alpha(X)) = D\alpha(X) + \alpha(DX). \tag{24}$$

Theorem 3.2 *The following characterizations are equivalent:*

(i) D *is self-dual*
(ii) $[D, i_X] = i_{DX} \quad , \quad \forall X \in \mathcal{X}(\tau)$
(iii) $[D, i_L] = i_{DL} \quad , \quad \forall L \in V(\tau)$
(iv) For $\omega \in \bigwedge^p(\tau)$ $(p > 0)$ *and* $X_1, \ldots, X_p \in \mathcal{X}(\tau)$*:*

$$D\omega(X_1, \ldots, X_p) = D(\omega(X_1, \ldots, X_p)) - \sum_{i=1}^{p} \omega(X_1, \ldots, DX_i, \ldots, X_p) \tag{25}$$

(v) For $L \in V^\ell(\tau)$ $(\ell > 0)$ *and* $X_1, \ldots, X_\ell \in \mathcal{X}(\tau)$*:*

$$DL(X_1, \ldots, X_\ell) = D(L(X_1, \ldots, X_\ell)) - \sum_{i=1}^{\ell} L(X_1, \ldots, DX_i, \ldots, X_\ell). \tag{26}$$

Proof:
1. It is obvious that (ii) implies (i). Conversely, $[D, i_X]$ is a derivation of degree -1 and therefore of type $i_{X'}$ for some $X' \in \mathcal{X}(\tau)$. The action on $\bigwedge^1(\tau)$ completely determines X' so that (i) implies $X' = DX$.
2. Property (iii) implies (ii). For the converse, $[D, i_L]$ is a derivation of degree $\ell - 1$, vanishing on functions and on basic vector fields, and therefore is of the form $i_{L'}$ for some $L' \in V^\ell(\tau)$, which will be completely determined by the action on $\bigwedge^1(\tau)$. It suffices to look at the case where L is of the form $\omega \otimes X$ with $\omega \in \bigwedge^\ell(\tau)$ and to remember that in such a case: $i_L\alpha = \omega \wedge i_X\alpha$ for any $\alpha \in \bigwedge(\tau)$. We then have using (ii),

$$\begin{aligned}
[D, i_{\omega\otimes X}] &= [D, \omega \wedge i_X] = D\omega \wedge i_X + \omega \wedge [D, i_X] \\
&= D\omega \wedge i_X + \omega \wedge i_{DX} = i_{D\omega\otimes X + \omega\otimes DX},
\end{aligned}$$

which shows that $L' = DL$.
3. We show that (ii) implies (iv) by induction. The property is obviously true for $p = 1$. Assume that (25) is valid for a $(p-1)$-form and let ω be a p-form. Then,

$$\begin{aligned}
D\omega(X_1, \ldots, X_p) &= (i_{X_1} D\omega)(X_2, \ldots, X_p) \\
&= (Di_{X_1}\omega - i_{DX_1}\omega)(X_2, \ldots, X_p) \qquad \text{(by (ii))} \\
&= D(\omega(X_1, \ldots, X_p)) - \sum_{i=2}^{p} \omega(X_1, X_2, \ldots, DX_i, \ldots, X_p) \\
&\quad - \omega(DX_1, X_2, \ldots, X_p) \qquad \text{(induction hypothesis)}
\end{aligned}$$

and the result now readily follows. The converse is obvious since (25) implies (24).

4. Concerning (v) it suffices again to look at the case $L = \omega \otimes X$, for which $DL = D\omega \otimes X + \omega \otimes DX$. Then,

$$DL(X_1, \ldots, X_\ell) = D\omega(X_1, \ldots, X_\ell)X + \omega(X_1, \ldots, X_\ell)DX,$$

while the right-hand side of (26) becomes,

$$D(\omega(X_1, \ldots, X_\ell)X) - \sum_{i=1}^{\ell} \omega(X_1, \ldots, DX_i, \ldots, X_\ell)X.$$

Comparison of both expressions clearly shows that (iv) implies (v) and vice versa. □

One of the important features of self-dual degree 0 derivations is that they obviously extend to tensor fields of arbitrary type by a Leibnitz-type of rule.

There are two different ways for obtaining a self-dual derivation by a proper extension. Let D be a derivation of degree 0 on $\bigwedge(\tau)$. We define a derivation $D^\star$ on $V(\tau)$ as follows: $D^\star = D$ on $\bigwedge(\tau)$ and for $X \in \mathcal{X}(\tau)$,

$$\langle D^\star X, \alpha \rangle = D(\alpha(X)) - D\alpha(X) \qquad , \qquad \forall \alpha \in \textstyle\bigwedge^1(\tau). \tag{27}$$

It easily follows from the defining relation that $D^\star X$ is $C^\infty(TM)$-linear over $\bigwedge^1(\tau)$ and therefore belongs to $\mathcal{X}(\tau)$, and also that for $F \in C^\infty(TM)$, $D^\star(FX) = (DF)X + FD^\star X$, ensuring that the extension is compatible with D. The new $D^\star$ is self-dual by construction.

Suppose on the other hand that a derivation D of $\mathcal{X}(\tau)$ is given, i.e. $D : \mathcal{X}(\tau) \to \mathcal{X}(\tau)$ is an $\mathbb{R}$-linear map, satisfying $D(FX) = (DF)X + F\,DX$ and $D(FG) = F\,DG + G\,DF$ for $F, G \in C^\infty(TM)$. Then we define $D^\dagger$ on $V(\tau)$ by: $D^\dagger = D$ on $C^\infty(TM)$ and on $\mathcal{X}(\tau)$ and for $\alpha \in \bigwedge^1(\tau)$,

$$\langle X, D^\dagger \alpha \rangle = D(\alpha(X)) - \alpha(DX), \qquad \forall X \in \mathcal{X}(\tau). \tag{28}$$

As above, $D^\dagger\alpha$ is a 1-form and $D^\dagger$ has the right derivation properties for a degree 0 derivation. Its action thus extends to all of $\bigwedge(\tau)$ and subsequently also to $V(\tau)$ and $D^\dagger$ will be self-dual by construction.

If D is a given derivation of degree 0 on $V(\tau)$ which is not self-dual, then it should be emphasized that the above two procedures will lead to different results, i.e. $(D|_{\bigwedge(\tau)})^\star \neq (D|_{\mathcal{X}(\tau)})^\dagger$ and both are of course different from D. Obviously, if D was self-dual from the outset, the above constructions will not change it. This implies, with a slight abuse of notations, that we have

$$(D^\star)^\dagger = D^\star \qquad \text{and} \qquad (D^\dagger)^\star = D^\dagger. \tag{29}$$

As an example, it readily follows from the defining relations (27) and (28) that for $A \in V^1(\tau)$:

$$i_A^\star = i_A - a_A \qquad , \qquad a_A^\star = 0, \tag{30}$$

$$i_A^\dagger = 0 \qquad , \qquad a_A^\dagger = a_A - i_A. \tag{31}$$

More generally, for given D of degree 0, $D - D^\star$ vanishes on $\bigwedge(\tau)$ and hence is of type $a_\star$, while $D - D^\dagger$ vanishes on functions and on vector fields and thus is of type $i_\star$. Putting $D - D^\star = a_A$, $D - D^\dagger = i_B$ and applying the $\star$-operation on the latter, it follows from (29) and (30) that $B = A$. So, we generally have

$$D - D^\star = a_A \quad , \quad D - D^\dagger = i_A \quad , \quad D^\star - D^\dagger = i_A - a_A, \tag{32}$$

for some $A \in V^1(\tau)$.

A case of special interest arises, when we start from an arbitrary derivation of degree 1, say $d^{(1)}$, on $V(\tau)$ and consider for $X \in \mathcal{X}(\tau)$, the degree 0 derivation $d^{(1)}_X = [i_X, d^{(1)}]$. Then, the defining relation (27) for $d^{(1)\star}_X$ can be written in the form: $\forall Y \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$,

$$\left\langle d^{(1)\star}_X Y, \alpha \right\rangle = d^{(1)}_X(\alpha(Y)) - d^{(1)}_Y(\alpha(X)) - d^{(1)}\alpha(X, Y). \tag{33}$$

This shows that $d^{(1)\star}_X Y = -d^{(1)\star}_Y X$, so we propose to introduce the notation $[X, Y]_{(1)} = d^{(1)\star}_X Y$. Moreover, since this situation is clearly reminiscent of the usual Lie derivative operations, we will also write $\mathcal{L}^{(1)}_X$ instead of $d^{(1)\star}_X$.

Proposition 3.3 *Let σ be a scalar or vector-valued p-form along τ. Then,*

$$\begin{aligned} d^{(1)}\sigma(X_0, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i \, d^{(1)}_{X_i}(\sigma(X_0, \ldots, \widehat{X_i}, \ldots, X_p)) \\ &\quad + \sum_{0 \le i < j \le p} (-1)^{i+j} \sigma\left([X_i, X_j]_{(1)}, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p\right). \end{aligned} \tag{34}$$

Proof:
1. Consider first the case $\sigma \in \bigwedge^p(\tau)$. The property is trivially true for $p = 0$ and also for $p = 1$, where it reduces precisely to the definition (33) of $[\,,\,]_{(1)}$. Assume then the validity of (34) for $p - 1$. We have,

$$d^{(1)}\sigma(X_0, \ldots, X_p) = [i_{X_0}, d^{(1)}]\sigma(X_1, \ldots, X_p) - (d^{(1)} i_{X_0}\sigma)(X_1, \ldots, X_p). \tag{35}$$

Regarding the derivation in the first term of the right-hand side as being $\mathcal{L}^{(1)}_{X_0}$, we can compute this first term via the property (25) of a self-dual derivation. For the second term, we invoke the induction hypothesis applied to the $(p-1)$-form $i_{X_0}\sigma$. The calculation then proceeds in exactly the same way as in the proof of the standard cocycle identity for the Lie derivative on a manifold (keeping in mind, at the end, that $\mathcal{L}^{(1)}_X$ and $d^{(1)}_{X_i}$ coincide on functions).
2. The situation is slightly more complicated for $\sigma = L \in V^p(\tau)$. Again, the property is true for $p = 0$ and for the induction process, we have to start from the identity (35). Obviously, we would like to appeal to (26) now for the action of a self-dual derivation on vector-valued forms, but this time $d^{(1)}_{X_0} L$ is not the same as $\mathcal{L}^{(1)}_{X_0} L$. Re-writing the right-hand side of (35) as,

$$\left(\mathcal{L}^{(1)}_{X_0} L - d^{(1)} i_{X_0} L + (d^{(1)}_{X_0} - \mathcal{L}^{(1)}_{X_0}) L\right)(X_1, \ldots, X_p),$$

we can use (26) for the first term and the induction hypothesis for the second and in the end go back from $\mathcal{L}^{(1)}_{X_0}$ to $d^{(1)}_{X_0}$ by adding and subtracting the appropriate expression. The net result is the relation we want to prove, apart from the following extra terms:

$$(\mathcal{L}^{(1)}_{X_0} - d^{(1)}_{X_0})(L(X_1,\ldots,X_p)) + ((d^{(1)}_{X_0} - \mathcal{L}^{(1)}_{X_0})L)(X_1,\ldots,X_p).$$

But as argued before, $d^{(1)}_{X_0} - \mathcal{L}^{(1)}_{X_0}$ is a derivation of type $a_\star$ and thus only acts on the vector field part of L, which shows that the extra terms will vanish. $\square$

Remark: Although (34) is valid for scalar and vector-valued forms, it is worth emphasizing the difference between both situations again. In the former case, all derivations in the right-hand side are the self-dual ones of Lie-derivative type, whereas this is not true for the first term when it concerns a vector-valued form.

Next, we look at the extension $d^{(1)\dagger}_X$ which has another interesting feature. We know that $d^{(1)}_X$, restricted to act on vector fields is $C^\infty(TM)$-linear in its dependence on X. The defining relation (28) then shows that this property will be preserved by the extension $d^{(1)\dagger}_X$ on the whole of $V(\tau)$. In this way, $d^{(1)\dagger}_X$ is a covariant-derivative type derivation and we will also write it as $\mathrm{D}^{(1)}_X$. Using (33), it is easy to show that the expression

$$[X,Y]_{(1)} - \mathrm{D}^{(1)}_X Y + \mathrm{D}^{(1)}_Y X$$

is $C^\infty(TM)$-linear in both X and Y. Therefore, there exists a vector-valued 2-form $T_{d^{(1)}}$ (which can be thought of as a torsion form), such that

$$[X,Y]_{(1)} = \mathrm{D}^{(1)}_X Y - \mathrm{D}^{(1)}_Y X - T_{d^{(1)}}(X,Y). \tag{36}$$

There is no direct way of proving a relation like (34) involving the covariant derivative extension. However, the $d^{(1)}_{X_i}$ in the first term can be read as $\mathrm{D}^{(1)}_{X_i}$ and the bracket in the second term can, if desired, be replaced by (36).

Let us finally look at the $A \in V^1(\tau)$ which, according to (32), will determine the difference between $d^{(1)}_X$ and its extensions $\mathcal{L}^{(1)}_X$ and $\mathrm{D}^{(1)}_X$. Acting with the first relation in (32) on an arbitrary $Y \in \mathcal{X}(\tau)$, we find

$$\begin{aligned} A(Y) &= d^{(1)}_X Y - [X,Y]_{(1)} \\ &= d^{(1)}_Y X + T_{d^{(1)}}(X,Y) \\ &= (d^{(1)}X + i_X T_{d^{(1)}})(Y). \end{aligned}$$

We conclude that,

$$\mathcal{L}^{(1)}_X = d^{(1)}_X - a_{d^{(1)}X + i_X T_{d^{(1)}}} \quad , \quad \mathrm{D}^{(1)}_X = d^{(1)}_X - i_{d^{(1)}X + i_X T_{d^{(1)}}}. \tag{37}$$

Remark: Most of what precedes in this section is not really typical for the calculus of forms along τ; similar properties will be encountered in an entirely different context (cfr. [16]). Our presentation may shed some new light on related theories.

We now turn our attention more specifically to the basic degree 0 derivations in our theory. Naturally, the most important self-dual derivations will arise from extensions of d^{V}_X and d^{H}_X. In agreement with the notations and terminology introduced above, we distinguish the following self-dual derivations: $\mathcal{L}^V_X$ and $\mathcal{L}^H_X$, the vertical and horizontal Lie derivative; D^V_X and D^H_X, the vertical and horizontal covariant derivative. From the characterization (iii) of a self-dual derivation in theorem 3.2 and comparison with (7) and (8), it immediately follows that for all $L \in V(\tau)$,

$$\mathcal{L}^V_X L = [X, L]_V \qquad , \qquad \mathcal{L}^H_X L = [X, L]_H, \tag{38}$$

and more specifically,

$$\mathcal{L}^V_X Y = [X, Y]_V \qquad , \qquad \mathcal{L}^H_X Y = [X, Y]_H. \tag{39}$$

Since (4) further tells us that

$$[X, Y]_V = \mathrm{D}^V_X Y - \mathrm{D}^V_Y X, \tag{40}$$

(36) indicates that the torsion related to the vertical covariant derivative is zero. Comparison of (20) with (36) on the other hand shows that the torsion related to the horizontal covariant derivative is just the torsion of the given connection on $\tau : TM \to M$,

$$[X, Y]_H = \mathrm{D}^H_X Y - \mathrm{D}^H_Y X - T(X, Y). \tag{41}$$

The general relations (37) thus imply,

$$\mathcal{L}^V_X = d^V_X - a_{d^V X} \qquad , \qquad \mathrm{D}^V_X = d^V_X - i_{d^V X}, \tag{42}$$

$$\mathcal{L}^H_X = d^H_X - a_{d^H X + i_X T} \qquad , \qquad \mathrm{D}^H_X = d^H_X - i_{d^H X + i_X T}. \tag{43}$$

If D is an arbitrary derivation of degree 0 and its decomposition reads

$$D = i_A + d^V_X + d^H_Y + a_B, \tag{44}$$

for some $X, Y \in \mathcal{X}(\tau)$, $A, B \in V^1(\tau)$, then, putting $\mu_A = a_A - i_A$, the extensions read

$$D^\star = \mathcal{L}^V_X + \mathcal{L}^H_Y - \mu_A, \tag{45}$$

$$D^\dagger = \mathrm{D}^V_X + \mathrm{D}^H_Y + \mu_B. \tag{46}$$

Proposition 3.3 tells us that for both scalar and vector-valued p-forms,

$$\begin{aligned} d^V\sigma(X_0, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i d^V_{X_i}(\sigma(X_0, \ldots, \widehat{X_i}, \ldots, X_p)) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \sigma\left([X_i, X_j]_V, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p\right), \end{aligned} \tag{47}$$

and a similar relation holds for $d^H\sigma$.

For future applications, it is useful to have the following complete list of commutators of all self-dual derivations entering (45) and (46). Knowing that the extension by duality of a commutator is the commutator of the extensions (an easily verifyable property), they follow trivially from results of the preceding section, except for the last two, which are easy to check by direct calculation of the action on vector fields and 1-forms

$$
\begin{aligned}
&[\mu_A, \mu_B] = \mu_{[A,B]} \quad , \quad [\mathcal{L}^V_X, \mu_A] = \mu_{[X,A]_V}, \\
&[\mathcal{L}^H_X, \mu_A] = \mu_{[X,A]_H} \quad , \quad [\mathcal{L}^V_X, \mathcal{L}^V_Y] = \mathcal{L}^V_{[X,Y]_V}, \\
&[\mathcal{L}^V_X, \mathcal{L}^H_Y] = \mathcal{L}^H_{d^V_X Y} - \mathcal{L}^V_{d^H_Y X} - \mu_{d^V d^H_Y X - d^V_{d^H X} Y - [Y, d^V X]_H}, \\
&[\mathcal{L}^H_X, \mathcal{L}^H_Y] = \mathcal{L}^H_{[X,Y]_H} + \mathcal{L}^V_{R(X,Y)} + \mu_{d^V(R(X,Y)) + d^V_{i_Y R} X - d^V_{i_X R} Y}, \\
&[\mathrm{D}^V_X, \mathrm{D}^V_Y] = \mathrm{D}^V_{[X,Y]_V} \quad , \quad [\mathrm{D}^V_X, \mathrm{D}^H_Y] = \mathrm{D}^H_{d^V_X Y} - \mathrm{D}^V_{d^H_Y X} + \mu_{\theta(X,Y)}, \\
&[\mathrm{D}^H_X, \mathrm{D}^H_Y] = \mathrm{D}^H_{[X,Y]_H} + \mathrm{D}^V_{R(X,Y)} + \mu_{\mathrm{Rie}(X,Y)}, \\
&[\mathrm{D}^V_X, \mu_A] = \mu_{\mathrm{D}^V_X A} \quad , \quad [\mathrm{D}^H_X, \mu_A] = \mu_{\mathrm{D}^H_X A}.
\end{aligned}
$$

Another point of practical interest concerns coordinate expressions. As argued before, the self-dual covariant and Lie-type derivations extend to arbitary tensor fields and since they are of degree zero, it suffices to have a table for their action on functions and on basic vector fields and 1-forms. Recalling that $H_i = \partial/\partial q^i - \Gamma^i_k\, \partial/\partial v^k$ and writing V_i for $\partial/\partial v^i$, we have

$$
\begin{aligned}
\mathrm{D}^V_X F = \mathcal{L}^V_X F = X^i V_i(F) \quad &, \quad \mathrm{D}^H_X F = \mathcal{L}^H_X F = X^i H_i(F), \\
\mathcal{L}^V_X \left(\frac{\partial}{\partial q^i}\right) = -V_i(X^k) \frac{\partial}{\partial q^k} \quad &, \quad \mathcal{L}^V_X\, dq^i = V_k(X^i)\, dq^k, \\
\mathcal{L}^H_X \left(\frac{\partial}{\partial q^i}\right) = -H_i(X^k) \frac{\partial}{\partial q^k} \quad &, \quad \mathcal{L}^H_X\, dq^i = H_k(X^i)\, dq^k, \\
\mathrm{D}^V_X \left(\frac{\partial}{\partial q^i}\right) = 0 \quad &, \quad \mathrm{D}^V_X\, dq^i = 0, \\
\mathrm{D}^H_X \left(\frac{\partial}{\partial q^i}\right) = X^j \Gamma^k_{ji} \frac{\partial}{\partial q^k} \quad &, \quad \mathrm{D}^H_X\, dq^i = -X^j \Gamma^i_{jk}\, dq^k.
\end{aligned}
$$

We are also in a position now to explain the origin of the operator D^V which was forced upon us in computing the commutator $[d^H, d^V]$. Since D^V_X and D^H_X depend linearly on X, two operators D^V and D^H, mapping an arbitrary tensor field $U \in \mathcal{T}^p_q(\tau)$ into an element of $\mathcal{T}^p_{q+1}(\tau)$, can be defined by the rule:

$$
X \rfloor \mathrm{D}^V U = \mathrm{D}^V_X U \quad , \quad X \rfloor \mathrm{D}^H U = \mathrm{D}^H_X U. \tag{48}
$$

In particular, an element $L = L^i \otimes \partial/\partial q^i \in V^\ell(\tau)$ is mapped under D^V or D^H into an element of $\bigwedge^\ell(\tau) \otimes V^1(\tau)$. The coordinate expression for $\mathrm{D}^H L$ is rather involved, but if $L^i = a^i_I\, dq^I$ (where I is a multi-index), $\mathrm{D}^V L$ simply reads

$$
\mathrm{D}^V L = V_j(a^i_I)\, dq^I \otimes \left(\frac{\partial}{\partial q^i} \otimes dq^j\right). \tag{49}
$$

It is then clear that the tensor field in (13) is indeed $\mathrm{D}^V T$.

Proposition 3.4 *The tensor fields $R \in V^2(\tau)$ and* Rie $\in \bigwedge^2(\tau) \otimes V^1(\tau)$ *are related by*

$$
\mathrm{Rie} = -\mathrm{D}^V R. \tag{50}
$$

Proof: A coordinate calculation is by far the simplest way of proving this property. We have

$$\mathrm{D}^{V} R = \frac{1}{2} V_\ell (H_k(\Gamma^i_j) - H_j(\Gamma^i_k)) dq^j \wedge dq^k \otimes \left(\frac{\partial}{\partial q^i} \otimes dq^\ell \right).$$

Making use of the property $[H_j, V_\ell] = \Gamma^m_{j\ell} V_m$, comparison with (11) immediately produces the desired result. □

To finish this section, we would like to indicate briefly that there is a deeper reason for calling D^{H}_X and D^{V}_X the horizontal and vertical covariant derivative. One can show that a (non-linear) connection on the bundle $\tau : TM \to M$ determines in a natural way a linear connection on the pull-back bundle $\tau^\star\tau : \tau^\star(TM) \to TM$ in the following sense. In terms of the given connection, every vector field on TM has a unique decomposition in the form $X^{H} + Y^{V}$, with $X, Y \in \mathcal{X}(\tau)$. Putting then

$$\widetilde{D}_{X^{H}+Y^{V}} = \mathrm{D}^{H}_X + \mathrm{D}^{V}_Y, \tag{51}$$

$\widetilde{D}$ is an operation which to each vector field on TM associates a derivation of the $C^\infty(TM)$-module of sections of $\tau^\star\tau$, satisfying the requirements of a covariant derivative. A full study of this interrelationship will shed new light on the meaning of the tensor fields Rie and θ in our present analysis and will be the subject of a separate paper by two of us. It will moreover lead to an application to second-order equations of direct practical relevance, which afterall remains the main motivation for the theory under development.

4 Horizontal and vertical lifts

It is important, for the interpretation of results, to have procedures by which tensor fields along τ can be put into correspondence with fields on TM. Such procedures in fact exist at three different levels. First, without appealing to extra tools, there is the natural identification $\imath_0$ between $\bigwedge(\tau)$ and the semi-basic forms on TM and, dually, the natural map ${}^{V} : \mathcal{X}(\tau) \to \mathcal{X}^{V}(TM)$, whose inverse is denoted by ${}^{\downarrow}$ and which also extends to the whole of $V(\tau)$. Secondly, with the aid of the given connection, we have horizontal and vertical lifts at our disposal. A third level, where the dynamics of a given SODE gets involved, was used in Section 7 of part I and comes back into the picture further on.

Concerning horizontal and vertical lifts, the difference between the present framework and the more familiar process in the case of a linear connection on M is merely that coefficient functions here are functions on TM. Recall that for $X \in \mathcal{X}(\tau)$, X^{H} and $X^{V} \in \mathcal{X}(TM)$ are given by $X^{H} = X^j H_j$, $X^{V} = X^j V_j$ and we have

$$X^{H}(F) = d^{H}_X F \quad , \quad X^{V}(F) = d^{V}_X F \quad , \quad F \in C^\infty(TM). \tag{52}$$

Note that there is no conflict in notation between this X^{V} and the one referred to above.

The set $\{H_j, V_j\}$ constitutes for many purposes a handy basis of vector fields on TM. The dual basis of 1-forms will be denoted as $\{H^i, V^i\}$, with

$$H^i = dq^i \qquad , \qquad V^i = dv^i + \Gamma^i_k \, dq^k.$$

Other tensor fields on TM can often be characterized by their action on horizontal and vertical lifts, so it is of some interest to know expressions for the brackets of such vector fields.

Lemma 4.1 *For $X, Y \in \mathcal{X}(\tau)$, we have*

$$[X^V, Y^V] = ([X,Y]_V)^V \tag{53}$$
$$[X^H, Y^V] = (d^H_X Y)^V - (d^V_Y X)^H \tag{54}$$
$$[X^H, Y^H] = ([X,Y]_H)^H + R(X,Y)^V. \tag{55}$$

Proof: Expressing the action of these brackets on arbitrary functions F via (52), the desired relations follow immediately from the commutators in (3), (15) and (18). $\square$

For 1-forms $\alpha \in \bigwedge^1(\tau)$, it seems natural to define the horizontal and vertical lifts as follows:

$$\alpha^H(X^H) = \alpha(X) \quad , \quad \alpha^H(X^V) = 0, \tag{56}$$
$$\alpha^V(X^H) = 0 \quad , \quad \alpha^V(X^V) = \alpha(X), \tag{57}$$

which in coordinates means that $\alpha^H = \alpha_j H^j$, $\alpha^V = \alpha_j V^j$. Note, however, that our definition is just the opposite of the one adopted by Yano and Ishihara [22]. We have already used the fact that every $Z \in \mathcal{X}(\tau)$ has a unique decomposition in the form

$$Z = X^H + Y^V \quad , \quad \text{with } X, Y \in \mathcal{X}(\tau).$$

To be precise, if Z is given, X and Y can be determined as follows:

$$X = S(Z)^{\downarrow} \quad , \quad Y = (Z - X^H)^{\downarrow}.$$

Similarly, every 1-form $\rho \in \bigwedge^1(TM)$ has a unique decomposition in the form

$$\rho = \alpha^H + \beta^V,$$

whereby $\alpha, \beta \in \bigwedge^1(\tau)$ are determined by

$$\beta = \imath_0^{-1} S^\star(\rho) \quad , \quad \alpha = \imath_0^{-1}(\rho - \beta^V),$$

$S^\star$ denoting the (transposed) action of the $(1,1)$-tensor S on 1-forms. As an example, one can easily verify that

$$dF = (d^H F)^H + (d^V F)^V. \tag{58}$$

The story of lifting can easily be continued for other types of tensor fields, but for our present needs we can limit ourselves to vector-valued 1-forms and symmetric $(0,2)$ tensor fields. For $U \in V^1(\tau)$, we define $U^V, U^H \in V^1(TM)$ by

$$U^V(X^V) = 0 \quad , \quad U^V(X^H) = U(X)^V \tag{59}$$
$$U^H(X^V) = U(X)^V \quad , \quad U^H(X^H) = U(X)^H. \tag{60}$$

This time we are in agreement with Yano and Ishihara. The notation U^V is further justified by the fact that this U^V coincides with the natural identification of $V^1(\tau)$ with a class of vertical vector-valued forms on TM, referred to at the beginning. From (56) and (57) it follows that the transposed action on 1-forms is characterized by

$$U^{V\star}(\alpha^V) = U^\star(\alpha)^H \quad , \quad U^{V\star}(\alpha^H) = 0, \tag{61}$$

$$U^{H\star}(\alpha^H) = U^\star(\alpha)^H \quad , \quad U^{H\star}(\alpha^V) = U^\star(\alpha)^V. \tag{62}$$

As an example, if I is the identity in $V^1(\tau)$, we have

$$I^V = S \quad , \quad I^H = I_{TM}. \tag{63}$$

It is quite obvious, however, that elements of $V^1(TM)$ do not generally decompose into a horizontal plus a vertical lift: there are 4 $(n \times n)$-blocks in the coefficient matrix (with respect to the adapted basis), which individually can be related to an element of $V^1(\tau)$. Explicitly, for $U \in V^1(\tau)$, we can define $U^{H;H}, U^{H;V}, U^{V;H}, U^{V;V} \in V^1(TM)$ by

$$\begin{aligned} U^{H;H}(X^H) &= U(X)^H \quad , \quad U^{H;H}(X^V) = 0, \\ U^{H;V}(X^H) &= U(X)^V \quad , \quad U^{H;V}(X^V) = 0, \\ U^{V;H}(X^H) &= 0 \quad , \quad U^{V;H}(X^V) = U(X)^H, \\ U^{V;V}(X^H) &= 0 \quad , \quad U^{V;V}(X^V) = U(X)^V. \end{aligned}$$

In a more visual form (with slight abuse of notation), these definitions amount to:

$${U_1}^{H;H} + {U_2}^{V;H} + {U_3}^{H;V} + {U_4}^{V;V} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},$$

the right-hand side representing the coefficient matrix of the tensor field on TM in question. We have,

$$U^V = U^{H;V} \quad , \quad U^H = U^{H;H} + U^{V;V},$$

while the projectors determined by the connection can be interpreted as:

$$P_H = I^{H;H} \quad , \quad P_V = I^{V;V}.$$

It is useful to know the following commutator properties (ordinary commutator of endomorphisms), which can easily be obtained from the action on horizontal and vertical lifts with the aid of (59) and (60):

$$[P_H, U^H] = 0 \quad , \quad [P_H, U^V] = -U^V, \tag{64}$$

$$[P_V, U^H] = 0 \quad , \quad [P_V, U^V] = U^V. \tag{65}$$

Another interesting type $(1,1)$ tensor field on TM is the one defining an almost complex structure. It can be written as

$$J = I^{H;V} - I^{V;H}.$$

Finally, concerning covariant tensor fields, we limit ourselves to the following lifting procedures, which will be relevant for later discussions. Let g be a *symmetric* type $(0,2)$ tensor field along τ.

Definition 4.2 *The Sasaki lift of g, denoted by $g^S \in \mathcal{T}_2^0(TM)$ is the symmetric tensor field determined by*

$$g^S(X^H, Y^H) = g^S(X^V, Y^V) = g(X, Y),$$
$$g^S(X^V, Y^H) = 0 \quad , \qquad \forall X, Y \in \mathcal{X}(\tau).$$

It is easy to verify that g^S is "Hermitian with respect to J", by which we mean:

$$g^S(JZ_1, JZ_2) = g^S(Z_1, Z_2) \qquad \qquad \forall Z_1, Z_2 \in \mathcal{X}(TM).$$

Definition 4.3 *The Kähler lift of g, denoted by $g^K \in \bigwedge^2(TM)$ is the 2-form on TM, defined by*

$$g^K(Z_1, Z_2) = g^S(Z_1, JZ_2) \qquad \qquad \forall Z_1, Z_2 \in \mathcal{X}(TM).$$

Alternatively, g^K is characterized by the properties

$$g^K(X^H, Y^H) = g^K(X^V, Y^V) = 0,$$
$$g^K(X^V, Y^H) = g(X, Y) = -g^K(X^H, Y^V).$$

We also have:

$$g^K(JZ_1, JZ_2) = g^K(Z_1, Z_2).$$

The terminology adopted above is inspired by existing constructions in the theory of linear connections: if in particular g is a Riemannian metric on M, then g^S is known as the Sasaki metric on TM (see e.g. [7]) and gives TM the structure of an almost Hermitian manifold. The 2-form g^K then is the corresponding fundamental or Kähler form. Note that in our present generalized construction, g^K need not be closed. For related material we can refer to [11], [13], [17].

5 The dynamical covariant derivative associated to a SODE

Let now $\Gamma \in \mathcal{X}(TM)$ be the vector field determining a SODE $\ddot{q}^i = f^i(q, \dot{q})$. It is well-known that Γ defines a connection on $\tau : TM \to M$, such that

$$P_V = \frac{1}{2}(I_{TM} + \mathcal{L}_\Gamma S) \quad , \quad P_H = \frac{1}{2}(I_{TM} - \mathcal{L}_\Gamma S) \tag{66}$$

(see e.g. [11],[12],[4]), the connection coefficients being given by

$$\Gamma^i_j = -\frac{1}{2}\frac{\partial f^i}{\partial v^j}. \tag{67}$$

As described in part I, there is then another way of mapping forms and vector fields along τ into corresponding objects on TM. We refer to part I for the definition and properties of the maps $J_\Gamma : \mathcal{X}(\tau) \to \mathcal{X}_\Gamma$ and $I_\Gamma : \bigwedge(\tau) \to \bigwedge_\Gamma$ and here simply recall that in coordinates:

$$X = X^i \frac{\partial}{\partial q^i} \quad \xrightarrow{J_\Gamma} \quad J_\Gamma X = X^i \frac{\partial}{\partial q^i} + \Gamma(X^i)\frac{\partial}{\partial v^i},$$
$$\alpha = \alpha_j \, dq^j \quad \xrightarrow{I_\Gamma} \quad I_\Gamma \alpha = \alpha_j \, dv^j + \Gamma(\alpha_j) \, dq^j.$$

In agreement with the previous section, $J_\Gamma X$ and $I_\Gamma\alpha$ uniquely decompose into a horizontal and vertical part; X^H is the horizontal part of $J_\Gamma X$ because $S(J_\Gamma X)^\downarrow = \tau_\star \circ J_\Gamma X = X$; α^V is the vertical part of $I_\Gamma\alpha$ because ${I_\Gamma}^{-1} = \imath_0^{-1} S\,\lrcorner$ (see part I).

Definition 5.1 *For $X \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$, $\nabla X \in \mathcal{X}(\tau)$ and $\nabla\alpha \in \bigwedge^1(\tau)$ are implicitly defined by*

$$J_\Gamma X = X^H + (\nabla X)^V, \tag{68}$$
$$I_\Gamma\alpha = (\nabla\alpha)^H + \alpha^V. \tag{69}$$

Explicitly, we have

$$\nabla X = P_V(J_\Gamma X)^\downarrow \quad , \quad \nabla\alpha = \imath_0^{-1} {P_H}^\star(I_\Gamma\alpha). \tag{70}$$

From $J_\Gamma(FX) = F\,J_\Gamma X + \Gamma(F)S(J_\Gamma X) = (FX)^H + (F\,\nabla X + \Gamma(F)X)^V$, it follows that

$$\nabla(FX) = \Gamma(F)X + F\,\nabla X. \tag{71}$$

Hence, the operator $\nabla : \mathcal{X}(\tau) \to \mathcal{X}(\tau)$ is a derivation of vector fields, provided we define for $F \in C^\infty(TM)$:

$$\nabla F = \Gamma(F). \tag{72}$$

On the one hand, we then have $\langle J_\Gamma X, I_\Gamma\alpha\rangle = \mathcal{L}_\Gamma\langle X,\alpha\rangle = \nabla\langle X,\alpha\rangle$, while on the other hand, from (68) and (69): $\langle J_\Gamma X, I_\Gamma\alpha\rangle = \langle X,\nabla\alpha\rangle + \langle\nabla X,\alpha\rangle$. We thus see that

$$\nabla\langle X,\alpha\rangle = \langle\nabla X,\alpha\rangle + \langle X,\nabla\alpha\rangle, \tag{73}$$

showing, from the results of Section 3, that ∇ extends to a self-dual derivation on the whole of $V(\tau)$ and further has a consistent action on tensor fields of arbitrary type. In coordinates, we have

$$\nabla X = \Big(\Gamma(X^i) + \Gamma^i_j X^j\Big)\frac{\partial}{\partial q^i} \quad , \quad \nabla\alpha = \Big(\Gamma(\alpha_j) - \alpha_k\Gamma^k_j\Big)\,dq^j, \tag{74}$$

which is clearly reminiscent of a covariant-derivative type derivation.

Definition 5.2 *The self-dual degree 0 derivation ∇, associated to a given* SODE *Γ is called the dynamical covariant derivative.*

Remark: This notion of covariant derivative was first introduced in [2]. It is of some importance to understand how it differs from or generalizes other notions of covariant derivative in the literature. There exists a concept of covariant derivative of a map $Y : N \to TM$ with respect to a vector field X on N (see e.g. [11] or [10]). One might expect our ∇ to correspond to this concept for the case that $N = TM$ and X is the given SODE Γ. There is, however, a striking difference. Indeed, if ψ denotes the canonical involution on $T(TM)$, then one easily verifies (in coordinates for example) that for $X \in \mathcal{X}(\tau)$,

$$J_\Gamma X = \psi \circ TX \circ \Gamma. \tag{75}$$

Hence, $\nabla X = P_V(\psi \circ TX \circ \Gamma)^\downarrow$ and the interference of ψ is what makes the difference with respect to the concept referred to above. There is another reason why this extra

ψ-action is important: one can generalize the familiar notion of parallel transport to the case of a non-linear connection and the involution on $T(TM)$ is inevitably present in this generalization. That subject, of interest in its own right, will be discussed elsewhere.

To conclude this section, we look at the $(1,1)$ tensors preserving $\mathcal{X}_\Gamma$, which are of the form $J_\Gamma U$ for some $U \in V^1(\tau)$. As repeatedly used in part I, such tensor fields are in fact determined by their action on $\mathcal{X}_\Gamma$ and we have $J_\Gamma U(J_\Gamma X) = J_\Gamma(U(X))$. It follows from (68),(59) and (60) that,

$$\begin{aligned} J_\Gamma U(J_\Gamma X) &= U(X)^H + \nabla(U(X))^V \\ &= U^H(X^H) + (\nabla U(X))^V + (U(\nabla X))^V \\ &= U^H(X^H + \nabla X^V) + (\nabla U)^V(X^H + \nabla X^V), \end{aligned}$$

which implies that

$$J_\Gamma U = U^H + (\nabla U)^V. \tag{76}$$

If U has coefficients u^i_j, the coefficient matrix of ∇U is given by

$$(\nabla U)^j_k = \Gamma(u^j_k) + \Gamma^j_\ell u^\ell_k - u^j_\ell \Gamma^\ell_k. \tag{77}$$

From (76) and (64) we obtain

$$[P_H, J_\Gamma U] = -(\nabla U)^V.$$

This enables us to give an interpretation on TM of a vanishing covariant derivative: $\nabla U = 0$ is equivalent to $J_\Gamma U$ commuting with $\mathcal{L}_\Gamma S$. For vector fields, $\nabla X = 0$ obviously means that $J_\Gamma X$ belongs to the horizontal distribution, i.e. $\mathcal{L}_\Gamma S(J_\Gamma X) = -J_\Gamma X$. In the same way, for $\alpha \in \bigwedge^1(\tau)$, $\nabla\alpha = 0$ is equivalent to $(\mathcal{L}_\Gamma S)^\star(I_\Gamma \alpha) = I_\Gamma \alpha$. The dynamical covariant derivative in itself, therefore, is insufficient to characterize more interesting objects on TM such as invariant vector fields or 1-forms and recursion operators for symmetries.

6 The Jacobi endomorphism, symmetries and adjoint symmetries of a SODE

If we want to investigate how different objects on TM evolve under the flow of Γ, the preceding section shows that we need the decomposition of the Lie derivative of all different lifts with respect to Γ. This will be the central theme in this section.

Proposition 6.1 *For $X \in \mathcal{X}(\tau)$, we have*

$$\mathcal{L}_\Gamma X^V = -X^H + (\nabla X)^V \tag{78}$$

and there exists an endomorphism Φ of $\mathcal{X}(\tau)$ such that

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V. \tag{79}$$

Proof: Recall that by definition $Z \in \mathcal{X}_\Gamma$ iff $S(\mathcal{L}_\Gamma Z) = 0$, which is equivalent to $\mathcal{L}_\Gamma(S(Z)) = \mathcal{L}_\Gamma S(Z)$ or (using $(\mathcal{L}_\Gamma S)^2 = I_{TM}$) $\mathcal{L}_\Gamma S \circ \mathcal{L}_\Gamma(S(Z)) = Z$. In particular, for $Z = J_\Gamma X = X^H + (\nabla X)^V$, this becomes

$$\mathcal{L}_\Gamma S(\mathcal{L}_\Gamma X^V) = X^H + (\nabla X)^V,$$

from which (78) immediately follows. The same formula, taking into account that $\mathcal{L}_\Gamma X^V = \mathcal{L}_\Gamma(S(X^H)) = \mathcal{L}_\Gamma S(X^H) + S(\mathcal{L}_\Gamma X^H)$ and using the property $\mathcal{L}_\Gamma S \circ S = S$, also shows us that $S(\mathcal{L}_\Gamma X^H) = (\nabla X)^V = S((\nabla X)^H)$, which means that $(\nabla X)^H$ is the horizontal part of $\mathcal{L}_\Gamma X^H$. Consequently, there exists a map $\Phi : \mathcal{X}(\tau) \to \mathcal{X}(\tau)$, implicitly defined by the relation (79). Now for $F \in C^\infty(TM)$ we have

$$\begin{aligned} \mathcal{L}_\Gamma(FX^H) &= \Gamma(F)X^H + F(\nabla X)^H + F(\Phi(X))^V \\ &= (\nabla(FX))^H + (F\Phi(X))^V, \end{aligned}$$

which shows that $\Phi \in V^1(\tau)$. □

Applying (79) to $\partial/\partial q^j$ we find

$$[\Gamma, H_j] - \Gamma^k_j H_k = \Phi^i_j V_i, \tag{80}$$

from which it is easy to calculate the coordinate expression for Φ, which reads

$$\Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^k_j \Gamma^i_k - \Gamma(\Gamma^i_j). \tag{81}$$

Corollary 6.2 *For $\alpha \in \bigwedge^1(\tau)$, we have*

$$\mathcal{L}_\Gamma \alpha^H = \alpha^V + (\nabla\alpha)^H \tag{82}$$

$$\mathcal{L}_\Gamma \alpha^V = (\nabla\alpha)^V - (\Phi^\star(\alpha))^H. \tag{83}$$

Proof: The proof proceeds by duality from (78),(79),(56) and (57). We have,

$$\begin{aligned} \langle X^H, \mathcal{L}_\Gamma \alpha^H \rangle &= \mathcal{L}_\Gamma \langle X^H, \alpha^H \rangle - \langle \mathcal{L}_\Gamma X^H, \alpha^H \rangle \\ &= \nabla\langle X, \alpha\rangle - \langle \nabla X, \alpha \rangle \\ &= \langle X, \nabla\alpha \rangle = \langle X^H, (\nabla\alpha)^H \rangle; \end{aligned}$$

$$\begin{aligned} \langle X^V, \mathcal{L}_\Gamma \alpha^H \rangle &= -\langle \mathcal{L}_\Gamma X^V, \alpha^H \rangle \\ &= \langle X, \alpha \rangle = \langle X^V, \alpha^V \rangle, \end{aligned}$$

which proves (82). The proof of (83) is entirely similar. □

In exactly the same way, from (78) and (79) and the defining relations of the various lifts of a $U \in V^1(\tau)$, one obtains the following results.

Corollary 6.3 *For $U \in V^1(\tau)$, the Lie derivative with respect to Γ of the different lifts decomposes as follows:*

$$\begin{aligned} \mathcal{L}_\Gamma U^{H;H} &= U^{V;H} + (\nabla U)^{H;H} + (\Phi \circ U)^{H;V}, \\ \mathcal{L}_\Gamma U^{V;H} &= (\nabla U)^{V;H} + (\Phi \circ U)^{V;V} - (U \circ \Phi)^{H;H}, \\ \mathcal{L}_\Gamma U^{H;V} &= -U^{H;H} + U^{V;V} + (\nabla U)^{H;V} = \mathcal{L}_\Gamma U^V, \\ \mathcal{L}_\Gamma U^{V;V} &= -U^{V;H} + (\nabla U)^{V;V} - (U \circ \Phi)^{H;V}, \end{aligned} \tag{84}$$

$$\mathcal{L}_\Gamma U^H = (\nabla U)^H + [\Phi, U]^V. \tag{85}$$

As an application of these results one easily computes, for example, the Lie derivative of the almost complex structure:

$$\mathcal{L}_\Gamma J = (I - \Phi)^{V;V} - (I - \Phi)^{H;H}.$$

Corollary 6.4 *An explicit definition of the endomorphism Φ is given by*

$$\Phi = -\frac{1}{2}(\mathcal{L}_\Gamma \mathcal{L}_\Gamma S \circ P_H)^{\downarrow}. \tag{86}$$

Proof: From (84) with $U = I$ we find $\mathcal{L}_\Gamma S = -I^{H;H} + I^{V;V}$ and taking a second Lie derivative, it follows that

$$\mathcal{L}_\Gamma \mathcal{L}_\Gamma S = -2(I^{V;H} + \Phi^{H;V})$$

and composition with $P_H = I^{H;H}$ subsequently gives

$$\mathcal{L}_\Gamma \mathcal{L}_\Gamma S \circ P_H = -2\,\Phi^{H;V} \circ I^{H;H} = -2\,\Phi^{H;V} = -2\,\Phi^{V},$$

which is equivalent to (86). □

Following the pattern of the preceding sections, we next move to the Lie derivative of images under J_Γ or I_Γ and immediately hit an interesting characterization of symmetries and adjoint symmetries (as defined in [20]) of Γ.

Proposition 6.5 *For $X \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$,*
(i) $J_\Gamma X$ is a symmetry of Γ if and only if

$$\mathcal{J}(X) := \nabla\nabla X + \Phi(X) = 0. \tag{87}$$

(ii) $I_\Gamma \alpha$ is an adjoint symmetry of Γ if and only if

$$\mathcal{J}^\star(\alpha) := \nabla\nabla\alpha + \Phi^\star(\alpha) = 0. \tag{88}$$

Proof: From (68) and (78), (79) we easily find $\mathcal{L}_\Gamma(J_\Gamma(X)) = (\nabla\nabla X + \Phi(X))^V$, which proves (87). Similarly, (69) plus (82),(83) leads to $\mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(I_\Gamma\alpha)) = (\nabla\nabla\alpha + \Phi^\star(\alpha))^H$, which implies (88). □

Remarks: The notation $\mathcal{J}(X)$ for the operator defined in (87) is reminiscent of the concept of a Jacobi field (see e.g. [5]). In the case of a linear connection, it is well known (see [6]) that a Jacobi field can be obtained as the restriction to a geodesic of a symmetry of the geodesic spray on TM. One can regard (87) as a generalized Jacobi equation, which justifies the following terminology (compare in this respect also with the work of Foulon [8]).

Definition 6.6 *The element $\Phi \in V^1(\tau)$ defined by (79) or (86) is called the Jacobi endomorphism associated to the given vector field Γ.*

With a slight abuse of terminology, solutions of (87) will be called symmetries of Γ and solutions of the adjoint equation (88) will be referred to as adjoint symmetries of Γ. That $\mathcal{J}^\star$ is truly the adjoint operator of $\mathcal{J}$ is seen from the following identity which is easily obtained using the self-duality of the covariant derivative ∇:

$$\langle \mathcal{J}(X), \alpha \rangle - \langle X, \mathcal{J}^\star(\alpha) \rangle = \nabla \left(\langle \nabla X, \alpha \rangle - \langle X, \nabla \alpha \rangle \right). \tag{89}$$

Thinking about recursion-type operators for symmetries, the constructions which obviously first come into one's mind are: type $(1,1)$ tensor fields mapping symmetries into symmetries and type $(0,2)$ tensor fields mapping symmetries into adjoint symmetries.

Proposition 6.7 *$U \in V^1(\tau)$ maps symmetries into symmetries if and only if $\nabla U = 0$ and $[U, \Phi] = 0$.*

Proof: It is straightforward to check that

$$\mathcal{J}(U(X)) - U(\mathcal{J}(X)) = (\nabla\nabla U + [\Phi, U])(X) + 2\,\nabla U(\nabla X).$$

Hence, U will have the desired property if and only if the right-hand side of this equality vanishes for all symmetries (i.e. solutions of (87)). Thinking for a moment of the corresponding symmetries on TM, it has been argued before (see part I) that the set of symmetries of Γ locally span the whole of $\mathcal{X}_\Gamma$. This in turn means, in view of the decomposition (68) of $J_\Gamma X$, that X and ∇X in the present context can be regarded as independent arguments. An alternative way for clarifying this point is to replace the second-order PDE's (87) by the equivalent first-order system $\nabla X = Y$, $\nabla Y = -\Phi(X)$ and to appeal, for example, to existence theorems of Cauchy-Kowalewski type (first away from the zero section of TM, then extended by continuity). The conclusion is that U will have the desired property if and only if the right-hand side of the above expression vanishes for all X and all ∇X, treated as independent variables. The result then immediately follows. □

Proposition 6.8 *A symmetric type $(0,2)$ tensor field (along τ) g maps symmetries into adjoint symmetries, if and only if $\nabla g = 0$ and $\Phi \lrcorner g$ is symmetric.*

Proof: The method of proof is exactly the same as for the preceding result. □

In the calculus on TM, recursion operators for symmetries are elements of V^1_Γ (see [17] and references therein). Covariant tensor fields on TM mapping symmetries into adjoint symmetries were discussed in [20] and more generally in [1]; they more naturally appear there as 2-forms. The link with this existing literature is expected to come forward if we complete the line of thought in this section by looking at $\mathcal{L}_\Gamma(J_\Gamma U)$ and $\mathcal{L}_\Gamma g^K$. This involves computing the action on horizontal and vertical lifts again and thus connects with the series of corollaries of Proposition 6.1.

Corollary 6.9 *For $U \in V^1(\tau)$, the decompostion of $\mathcal{L}_\Gamma(J_\Gamma U)$ is given by*

$$\mathcal{L}_\Gamma(J_\Gamma U) = 2(\nabla U)^{V;V} + (\nabla\nabla U + [\Phi, U])^V.$$

We conclude, not unexpectedly, that the conditions $\nabla U = 0$ and $[U, \Phi] = 0$, identified in Proposition 6.7 are equivalent to $\mathcal{L}_\Gamma(J_\Gamma U) = 0$. Observe from (85) that they are also equivalent to $\mathcal{L}_\Gamma U^H = 0$.

From the characterizing properties of g^K and repeated use of Proposition 6.1, one easily finds,

$$\mathcal{L}_\Gamma g^K(X^H, Y^H) = g(X, \Phi(Y)) - g(\Phi(X), Y),$$
$$\mathcal{L}_\Gamma g^K(X^V, Y^V) = 0 \quad , \quad \mathcal{L}_\Gamma g^K(X^V, Y^H) = \nabla g(X, Y),$$

Corollary 6.10 *$\mathcal{L}_\Gamma g^K = 0 \Longleftrightarrow \nabla g = 0$ and $\Phi \lrcorner g$ is symmetric.*

7 Other tensorial quantities of interest

We return for a while to the case of an arbitrary non-linear connection, e.g. not necessarily the one coming from a given SODE. Recall that there exists a canonical vector field along τ, namely $\mathbf{T} = v^i\, \partial/\partial q^i$. It is clear that $\mathbf{T}^V$ is the dilation or Liouville vector field on TM, while

$$\mathbf{T}^H = v^i \frac{\partial}{\partial q^i} - \Gamma^i_k v^k \frac{\partial}{\partial v^i},$$

is a second-order vector field on TM which, following Grifone, will be called the *associated semispray* of the given connection.

Definition 7.1 *The tension of the connection is the vector-valued 1-form $-d^H\mathbf{T}$.*

In coordinates we have

$$-d^H\mathbf{T} = \left(\Gamma^j_i - v^k \Gamma^j_{ik}\right) dq^i \otimes \frac{\partial}{\partial q^j},$$

from which one can see that $-(d^H\mathbf{T})^V$ corresponds to the tension in [11] (see also [7]). A different terminology which has been used for this concept is *homogeneity torsion* [21]. Concerning the relevance of this object, observe first that $d^H\mathbf{T} = 0$ expresses that the connection coefficients are homogeneous of degree 1 in the fibre coordinates and since we are assuming everything to be smooth on the zero section, $d^H\mathbf{T} = 0$ actually is equivalent to the connection being linear. More generally, if $d^H\mathbf{T}$ is basic, say $d^H\mathbf{T} = U \in V^1(M)$, then

$$\Gamma^j_i - v^k \frac{\partial \Gamma^j_i}{\partial v^k} = u^j_i(q)$$

implies that the connection coefficients are of the form

$$\Gamma^j_i(q, v) = \Gamma^j_{ij}(q) v^k + u^j_i(q),$$

for some functions $\Gamma^j_{ik}(q)$, i.e. that the connection is affine. Note in passing that an equivalent characterization of the connection being affine is the vanishing of the tensor field θ introduced in Section 2 (see equation (19)).

Definition 7.2 *The strong torsion of the connection is the element $T^s \in V^1(\tau)$ defined by*

$$T^s = d^H\mathbf{T} + i_{\mathbf{T}}T. \tag{90}$$

In coordinates,

$$T^s = \left(v^k \frac{\partial \Gamma^j_k}{\partial v^i} - \Gamma^j_i \right) dq^i \otimes \frac{\partial}{\partial q^j},$$

from which one can again see the analogy with the similar concept in the references cited above.

Proposition 7.3 *Torsion and strong torsion are related by the property*

$$d^V T^s = 2\,T. \tag{91}$$

Proof: Observe first that $d^V\mathbf{T} = I$, $d^H I = T$ (see Part I) and that (as for any vector-valued 2-form): $i_T I = T$, $i_I T = 2\,T$. Using these relations and the commutator (12), we obtain

$$\begin{aligned}
d^V T^s &= d^V d^H \mathbf{T} + d^V i_{\mathbf{T}} T \\
&= [d^V, d^H]\mathbf{T} - d^H d^V \mathbf{T} + d^V_{\mathbf{T}} T \\
&= d^V_T \mathbf{T} - a_{\mathrm{D}^V T}\mathbf{T} - T + d^V_{\mathbf{T}} T \\
&= -\mathrm{D}^V_{\mathbf{T}} T + d^V_{\mathbf{T}} T \\
&= i_{d^V \mathbf{T}} T = 2\,T,
\end{aligned}$$

where the final step is based on (42). □

The following statement is an immediate consequence of (90) and (91).

Corollary 7.4 $T^s = 0 \Longleftrightarrow T = 0$ *and* $d^H\mathbf{T} = 0$.

Since $T = 0$ is the condition for the connection being defined by a SODE and $d^H\mathbf{T} = 0$ means that the connection is linear, $T^s = 0$ is equivalent to saying that the connection comes from a (quadratic) spray.

Let us then go back to the case of a SODE, for which the concepts of strong torsion and tension coincide up to a sign. Since $\Gamma = J_\Gamma \mathbf{T}$, we know from (68) that

$$\Gamma = \mathbf{T}^H + (\nabla \mathbf{T})^V. \tag{92}$$

Hence, $\nabla\mathbf{T}$ can rightly be called the *deviation*, since it characterizes the difference between the given SODE and the *associated semispray* of the connection. We have learnt in the preceding section that the dynamical covariant derivative ∇ and the Jacobi endomorphism Φ somehow contain all the information of the given dynamics. We are now able to complete the picture by computing the decomposition of ∇ and will arrive at the same time at another interesting characterization of Φ.

Proposition 7.5 *The decomposition (1) of the dynamical covariant derivative is given by*

$$\nabla = -i_{d^H\mathbf{T}} + d^V_{\nabla\mathbf{T}} + d^H_{\mathbf{T}} - a_{d^H\mathbf{T}}. \tag{93}$$

The Jacobi endomorphism Φ can be written as

$$\Phi = i_{\mathbf{T}}R - d^H\nabla\mathbf{T}. \tag{94}$$

Proof: From (79) and (92), using the results of Lemma 4.1 and (39), we get

$$\begin{aligned}(\nabla X)^H + \Phi(X)^V &= \mathcal{L}_\Gamma X^H = [\mathbf{T}^H + (\nabla\mathbf{T})^V, X^H] \\ &= ([\mathbf{T}, X]_H)^H + R(\mathbf{T}, X)^V - (d^H_X\nabla\mathbf{T})^V + (d^V_{\nabla\mathbf{T}}X)^H \\ &= (d^V_{\nabla\mathbf{T}}X + \mathcal{L}^H_{\mathbf{T}}X)^H + ((i_{\mathbf{T}}R - d^H\nabla\mathbf{T})(X))^V.\end{aligned}$$

The vertical part produces (94). From the horizontal part, we see that the action of ∇ on vector fields is given by

$$\nabla X = d^V_{\nabla\mathbf{T}}X + \mathcal{L}^H_{\mathbf{T}}X.$$

This in turn implies that the dual action on 1-forms must read:

$$\nabla\alpha = \mathrm{D}^V_{\nabla\mathbf{T}}\alpha + d^H_{\mathbf{T}}\alpha.$$

Using (43) to replace $\mathcal{L}^H_{\mathbf{T}}$ and (42) to replace $\mathrm{D}^V_{\nabla\mathbf{T}}$, the first expression reveals that the $a_\star$-part of ∇ is $-a_{d^H\mathbf{T}}$ (remember that $T = 0$); the second expression similarly reveals that the $i_\star$-part is $-i_{d^V\nabla\mathbf{T}}$. However, if we do the same type of calculation starting from the expression (78) for $\mathcal{L}_\Gamma X^V$, we find that

$$\nabla X = d^H_{\mathbf{T}}X + \mathcal{L}^V_{\nabla\mathbf{T}}X$$

and hence that the $a_\star$-part of ∇ is also $-a_{d^V\nabla\mathbf{T}}$. Therefore,

$$d^V\nabla\mathbf{T} = d^H\mathbf{T} \tag{95}$$

and the decomposition (93) now directly follows. □

Remarks: It is of some interest to write the self-dual ∇ also as a sum of self-dual ingredients. We thus have,

$$\nabla = \mathrm{D}^H_{\mathbf{T}} + \mathrm{D}^V_{\nabla\mathbf{T}} - \mu_{d^H\mathbf{T}} = \mathcal{L}^H_{\mathbf{T}} + \mathcal{L}^V_{\nabla\mathbf{T}} + \mu_{d^H\mathbf{T}}. \tag{96}$$

Observe also that the expression (94) for Φ exhibits more clearly the relationship between equation (87) and the classical equation for Jacobi fields.

Proposition 7.6 *The curvature 2-form R and the Jacobi endomorphism Φ are related by the properties*

$$d^V\Phi = 3R, \qquad d^H\Phi = \nabla R. \tag{97}$$

Proof: We start from (94) and in the calculations below make use of the following facts and properties: $T = 0$ implies that d^H and d^V commute and that the Bianchi identities read $d^V R = 0$ and $d^H R = 0$ (see part I); property (95), the expression (10) for $d^H \circ d^H$ together with the property $\mathrm{Rie} = -\mathrm{D}^V R$; the decomposition of ∇.

$$
\begin{aligned}
d^V \Phi &= d^V i_{\mathbf{T}} R - d^V d^H \nabla \mathbf{T} \\
&= d^V_{\mathbf{T}} R + d^H d^H \mathbf{T} \\
&= d^V_{\mathbf{T}} R + d^V_R \mathbf{T} + a_{\mathrm{Rie}} \mathbf{T} \\
&= d^V_{\mathbf{T}} R + i_R d^V \mathbf{T} - \mathrm{D}^V_{\mathbf{T}} R \\
&= i_R I + i_I R = R + 2R = 3R.
\end{aligned}
$$

$$
\begin{aligned}
d^H \Phi &= d^H i_{\mathbf{T}} R - d^H d^H \nabla \mathbf{T} \\
&= d^H_{\mathbf{T}} R - d^V_R \nabla \mathbf{T} - a_{\mathrm{Rie}}(\nabla \mathbf{T}) \\
&= d^H_{\mathbf{T}} R - i_R d^H \mathbf{T} + \mathrm{D}^V_{\nabla \mathbf{T}} R \\
&= d^H_{\mathbf{T}} R - a_{d^H \mathbf{T}} R + d^V_{\nabla \mathbf{T}} R - i_{d^H \mathbf{T}} R = \nabla R.
\end{aligned}
$$

In the last step, we have also used the general property that for $L \in V(\tau)$ and $U \in V^1(\tau)$

$$
i_L U = a_U L,
$$

which is a direct consequence of the definitions in part I. □

We finally list the following interesting commutators,

$$
[d^V, \nabla] = d^H \quad , \quad [d^H, \nabla] = -2\, i_R - d^V_{\Phi} + a_{\mathrm{D}^V \Phi - \overline{R}}, \tag{98}
$$

where $\overline{R} \in \bigwedge^1(\tau) \otimes V^1(\tau)$ is implicitly defined by $a_{\overline{R}}(X) = i_X R$. They can be proved, for example, by looking at the Jacobi identity for Γ, X^H and Y^V, or perhaps just as simply by a coordinate calculation.

8 Applications and outlook for further study

It would take too much space to develop truly new applications at the end of this paper. What we can do is pick out a couple of known results from the literature and illustrate how these acquire a natural and elegant formulation within the framework of the calculus of forms along τ. Generally speaking, the advantage of the present formulation is that conditions and results are stated in their most economical form. For example, a symmetry of Γ, being a vector field on TM, has $2n$ components, but only n of them are relevant and have to satisfy the condition $\mathcal{J}(X) = 0$, which has a natural place within the present theory. Similarly, the usual geometrical formulation of the inverse problem of Lagrangian mechanics is about the existence of a Cartan 2-form on TM with a $(2n \times 2n)$-coefficient matrix. The theorem below gives a geometrical version of the Helmholtz conditions, which involves only the essential $(n \times n)$-part of the Cartan 2-form.

Theorem 8.1 *A* SODE Γ *is locally Lagrangian if and only if there exists a non-degenerate symmetric* $(0,2)$ *tensor field* g *(along* τ*), such that*

$$\nabla g = 0 \quad , \quad \Phi \,\lrcorner\, g \text{ is symmetric}$$
$$\mathrm{D}^V g \text{ is symmetric.}$$

Proof: We know from Corollary 6.10 that the first two conditions are equivalent to $\mathcal{L}_\Gamma g^K = 0$. By definition of the Kähler lift of g, we further know that $g^K(X^V, Y^V) = 0$. Following Crampin's conditions in their weakest form [3], g^K will therefore satisfy all requirements for the local construction of a Cartan form, provided we have

$$i_{X^H}\, dg^K(Y^V, Z^V) = 0 \qquad \forall X, Y, Z \in \mathcal{X}(\tau).$$

Now, using the classical formula of type (34) for the computation of $dg^K({X_0}^H, {X_1}^V, {X_2}^V)$, the bracket relations (53),(54) and the defining equations of g^K, we get

$$\begin{aligned} dg^K({X_0}^H, {X_1}^V, {X_2}^V) &= \mathcal{L}_{{X_1}^V}(g(X_0, X_2)) - \mathcal{L}_{{X_2}^V}(g(X_0, X_1)) + g^K((\mathrm{D}^V_{X_1} X_0)^H, {X_2}^V) \\ &\quad - g^K((\mathrm{D}^V_{X_2} X_0)^H, {X_1}^V) - g^K(([X_1, X_2]_V)^V, {X_0}^H) \\ &= \mathrm{D}^V_{X_1}(g(X_0, X_2)) - \mathrm{D}^V_{X_2}(g(X_0, X_1)) - g(\mathrm{D}^V_{X_1} X_0, X_2) \\ &\quad + g(\mathrm{D}^V_{X_2} X_0, X_1) - g(\mathrm{D}^V_{X_1} X_2 - \mathrm{D}^V_{X_2} X_1, X_0) \\ &= (\mathrm{D}^V_{X_1} g)(X_0, X_2) - (\mathrm{D}^V_{X_2} g)(X_0, X_1). \end{aligned}$$

Hence, in view of the symmetry of g, the remaining condition of Crampin's formulation is equivalent to $\mathrm{D}^V g$ being a symmetric tensor field. □

Observe by the way that this last condition will automatically imply that g^K is closed. For a sketch of the way this theorem can be proved without recurring to the results on TM, see [2]. Such a more direct proof requires of course a direct way of expressing when a function L is a Lagrangian for the given SODE Γ. With θ_L standing for $d^V L$, we can state the following characterization.

Proposition 8.2 *A regular function* $L \in C^\infty(TM)$ *is a Lagrangian for* Γ *if and only if*

$$\nabla \theta_L = d^H L. \tag{99}$$

Proof: Recalling from [19] that L is a Lagrangian iff $\mathcal{L}_\Gamma(S^\star(dL)) = dL$ and using (58) and (82), the result immediately follows. □

REMARK: Using the first of equations (98), one easily verifies that (99) is equivalent to $d^H L = \frac{1}{2} d^V(\Gamma(L))$. Translated to TM, this is a characterization of a Lagrangian SODE which is frequently used in the work of Klein (see [13] and [14]).

We finally reformulate a result from [20] about a class of adjoint symmetries which produce a Lagrangian (not necessarily) for Γ.

Proposition 8.3 *Let* α *be an adjoint symmetry of* Γ *which is* d^V*-closed and therefore* d^V*-exact, say* $\alpha = d^V F$*, then* $L = \Gamma(F)$ *is a Lagrangian for* Γ*. Conversely, if* L *is a Lagrangian of the form* $L = \Gamma(F)$ *for some function* F*, then* $\alpha = d^V F$ *is an adjoint symmetry.*

Proof: Making use of the commutators (98), we have

$$\begin{aligned}\mathcal{J}^\star(d^V F) &= \nabla\nabla d^V F + \Phi^\star(d^V F)\\ &= \nabla(d^V\nabla - d^H)(F) + d^V_\Phi F\\ &= \nabla(d^V(\nabla F)) - d^H(\nabla F),\end{aligned}$$

from which it follows that ∇F satisfies the criterion (99) if and only if $\mathcal{J}^\star(d^V F) = 0$. □

The calculus of forms along τ has now sufficiently been developed. There is no doubt that every result which can be obtained within this framework will have a translation in terms of objects living on TM and vice versa. The tools for making such a translation have also been established here (and in part I). Obviously, the relevance of the new theory will become fully evident, when it leads to new results in the study of second-order equations, which would have been very hard to detect or prove by staying within the framework of forms and fields on TM. In a forthcoming paper, we will show that this is indeed happening when we develop a theory for testing whether a given system of second-order equations can be completely decoupled within a suitable set of coordinates.

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