# A direct geometrical construction of the dynamics of non-holonomic Lagrangian systems

W Sarlet

Theoretical Mechanics Division, University of Ghent,
Krijgslaan 281-S9, B-9000 Ghent, Belgium

**Abstract.** A geometrical framework is discussed for the treatment of a class of Lagrangian systems with non-holonomic constraints. The starting point for the model is a bundle $\pi : E \to M$, where both $E$ and $M$ are fibred over $\mathbb{R}$, with projections $\tau_1$ and $\tau_0$. Linear constraint equations correspond to a connection on $\pi$, which can be viewed as defining a section $\sigma$ of $J^1\tau_1$ over the pull-back bundle $\pi^* J^1\tau_0$. The dynamical system of interest is governed by a vector field $\Gamma$ on $J^1_\sigma$, the image of $\sigma$, and defines in itself a connection on the bundle $\rho : J^1_\sigma \to E$. We present an intrinsic procedure by which the correct reduced dynamics on $J^1_\sigma$ can be constructed out of a given Lagrangian on $J^1\tau_1$ and the constraint connection $\sigma$. We further discuss how the scheme can be generalized to the case of non-linear non-holonomic constraints.

## 1 Introduction: the classical approach to non-holonomic constraints

Consider a mechanical system described by a Lagrangian $L(t, q^A, \dot{q}^A)$ (where the index $A$ runs from 1 to $n = k+m$), which is subject to linear non-holonomic constraints, expressed in a solved form with respect to $m$ of the velocities $\dot{q}^a$ in terms of the $k$ remaining $\dot{q}^\alpha$:

$$\dot{q}^a = B^a_\alpha(t, q^A)\dot{q}^\alpha + B^a(t, q^A), \qquad a = 1, \dots, m.$$

Combining ideas coming from the notion of virtual velocities and d'Alembert's principle, with techniques borrowed from the calculus of variations, the classical way of arriving at what are believed to be the right equations of motion (see e.g. [10]) consists in introducing Lagrange multipliers for expressing the non-independence of the variations $\delta q^A$ in Hamilton's integral principle. For the type of constraints under consideration — which might be described as characterizing "generalized Čaplygin systems" in the terminology of [7] — one obtains the following equations for the unknown $q^A(t)$ and multipliers $\lambda_a(t)$:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} &= -\lambda_a\, B^a_\alpha, \qquad \alpha = 1,\ldots,k,\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^a}\right) - \frac{\partial L}{\partial q^a} &= \lambda_a, \qquad a = 1,\ldots,m.
\end{aligned}$$

Elimination of the multipliers is very easy here, and results in second-order equations for the $q^\alpha$, which can be written in the following form:

$$\frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial \dot q^\alpha}\right) = X_\alpha(\overline{L}) + C^a_\alpha\,\frac{\partial L}{\partial \dot q^a},$$

where

$$\begin{aligned}
\overline{L}(t, q^A, \dot q^\alpha) &\equiv L(t, q^A, \dot q^\alpha, B^a_\beta \dot q^\beta + B^a)\\
X_\alpha &= \frac{\partial}{\partial q^\alpha} + B^a_\alpha\,\frac{\partial}{\partial q^a}\\
C^a_\alpha &= \dot B^a_\alpha - X_\alpha(B^a_\beta \dot q^\beta + B^a).
\end{aligned}$$

Assuming regularity in the sense that $\det\left(\partial^2\overline{L}/\partial\dot q^\alpha\partial\dot q^\beta\right) \neq 0$, the ultimate result is a mixed system of the form

$$\begin{aligned}
\ddot q^\alpha &= f^\alpha(t, q^A, \dot q^\beta), \qquad \alpha = 1,\ldots,k\\
\dot q^a &= B^a_\alpha(t, q^A)\,\dot q^\alpha + B^a(t, q^A), \qquad a = 1,\ldots,m.
\end{aligned}$$

The question we want to address here is: "How can we model these equations geometrically?", or more precisely "Can one give a direct geometrical (coordinate free) construction of this reduced system, without needing the intermediate process of introducing and eliminating multipliers?"

The answer to this question is contained in joint work with Frans Cantrijn and David Saunders, recently reported in [8]. We will briefly outline this construction and give a sketch of the generalization to the case of non-linear non-holonomic constraints.

## 2 The constraint submanifold and its intrinsic structures

The immediate suggestion coming from the way the constraints are expressed is that the configuration space should be regarded as a bundle $\pi : E \to M$, with both $E$ and $M$ fibred over $\mathbb{R}$:

$$
\begin{array}{ccccc}
(t, q^\alpha, q^a) & E & \xrightarrow{\ \pi\ } & M & (t, q^\alpha) \\
 & \downarrow{\scriptstyle \tau_1} & & {\scriptstyle \tau_0}\downarrow & \\
 & \mathbb{R} & \xrightarrow{\ \mathrm{id}\ } & \mathbb{R} &
\end{array}
$$

Coordinates on the first jet extension $J^1\pi$ could then be denoted as $(t, q^\alpha, q^a, q^a_t, q^a_\alpha)$. An Ehresmann connection on $\pi : E \to M$ is a section $\tilde{\sigma}$ of $J^1\pi \to E$, i.e. an assignment of the form $q^a_t = B^a(t, q^A)$, $q^a_\alpha = B^a_\alpha(t, q^A)$. Alternatively, it can be interpreted as a splitting $\sigma$ of the sequence below,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V\tau_1 & \longrightarrow & J^1\tau_1 & \overset{\sigma}{\underset{}{\rightleftarrows}} & \pi^* J^1\tau_0 & \longrightarrow & 0 \\
 & & (t, q^A, \dot{q}^a) & & (t, q^A, \dot{q}^\alpha, \dot{q}^a) & & (t, q^A, \dot{q}^\alpha) & &
\end{array}
$$

so that $\sigma$ has a coordinate representation of the form:

$$\sigma : (t, q^A, \dot{q}^\alpha) \longmapsto (t, q^A, \dot{q}^\alpha, \dot{q}^a = B^a_\beta \dot{q}^\beta + B^a).$$

Solution curves of our reduced system will be curves $q^A(t)$ in $E$ satisfying the constraints, i.e. whose prolongation to $J^1\tau_1$ lies in the image of $\sigma$. We are thus led to define the constraint manifold as being $J^1_\sigma = \sigma(\pi^* J^1\tau_0)$, and our problem is reduced to constructing the reduced dynamics as an appropriate vector field on $J^1_\sigma$.

The manifold $J^1_\sigma$ is equipped with two interesting type (1,1) tensor fields. One is inherited from the so-called canonical "vertical endomorphism" on $J^1\tau_0$:

$$S = \theta^\alpha \otimes \frac{\partial}{\partial \dot{q}^\alpha}, \quad \text{with} \quad \theta^\alpha = dq^\alpha - \dot{q}^\alpha dt.$$

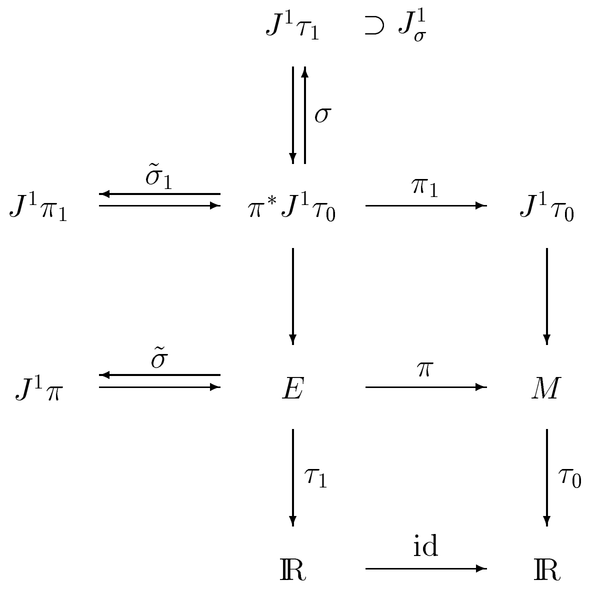
The other one comes from the connection and has coordinate expression

$$N = \eta^a \otimes \frac{\partial}{\partial q^a}, \quad \text{with} \quad \eta^a = dq^a - B^a_\alpha \, dq^\alpha - B^a \, dt.$$

Originally, $N$ is the "vertical projector" of the connection on $\pi$ and as such lives on the space $E$. It is, however, well defined also on $\pi^* J^1\tau_0$ (and therefore also on $J^1_\sigma$), because $\tilde{\sigma} : E \to J^1\pi$ has a canonical lift to a connection $\tilde{\sigma}_1 : \pi^* J^1\tau_0 \to J^1\pi_1$ (where $\pi_1 : \pi^* J^1\tau_0 \to J^1\tau_0$), defined, in the coordinates $(t, q^\alpha, \dot{q}^\alpha, q^a, q^a_t, q^a_\alpha, q^a_{\dot{\alpha}})$ of $J^1\pi_1$, as follows:

$$q^a_t = B^a, \qquad q^a_\alpha = B^a_\alpha, \qquad q^a_{\dot{\alpha}} = 0.$$

A diagram of all spaces and maps involved in this discussion is presented below:




Remark: the geometry of non-holonomic mechanics is a rather popular subject nowadays (see e.g. [6] and references therein); the idea of modelling non-holonomic constraints via a connection can be found also in other approaches, see e.g. [2, 1]. We wish, however, to arrive in addition at a model for the dynamical vector field directly living on $J^1_\sigma$. To that end, we now first introduce the class of vector fields to which our model will belong and discuss the additional structure which is entailed by such vector fields.

# 3 Second-order vector fields on $J^1_\sigma$

A second-order vector field $\Gamma$ (Sode for shorthand) on $J^1_\sigma$ is a vector field characterized by the following properties:

$$\langle \Gamma, dt\rangle = 1, \qquad \langle \Gamma, \theta^\alpha\rangle = 0, \qquad \langle \Gamma, \eta^a\rangle = 0.$$

In coordinates, such a $\Gamma$ is of the form

$$\Gamma = \frac{\partial}{\partial t} + \dot q^\alpha \frac{\partial}{\partial q^\alpha} + (B^a_\beta \dot q^\beta + B^a)\frac{\partial}{\partial q^a} + f^\alpha(t, q^A, \dot q^\beta)\frac{\partial}{\partial \dot q^\alpha},$$

for some $f^\alpha \in C^\infty(J^1_\sigma)$, so that the corresponding differential equations do indeed constitute a system of the mixed form introduced before.

$\Gamma \in \mathcal{X}(J^1_\sigma)$ comes with its own connection on the bundle $\rho : J^1_\sigma \to E$. One way to discover it, is to observe that

$$(\mathcal{L}_\Gamma S)^2 = I - \Gamma \otimes dt - N,$$

from which it follows that

$$P_H \;=\; \frac{1}{2}(I - \mathcal{L}_\Gamma S + \Gamma \otimes dt + N),$$

$$P_V = \frac{1}{2}(I + \mathcal{L}_\Gamma S - \Gamma \otimes dt - N),$$

are two complementary projectors. The associated "horizontal lift" operation, which displays the corresponding connection coefficients, reads

$$\begin{aligned}
\frac{\partial}{\partial t} &\longmapsto \frac{\partial}{\partial t} + (f^\alpha + \dot{q}^\beta \Gamma^\alpha_\beta)\frac{\partial}{\partial \dot{q}^\alpha} \\
\frac{\partial}{\partial q^\alpha} &\longmapsto \frac{\partial}{\partial q^\alpha} - \Gamma^\beta_\alpha \frac{\partial}{\partial \dot{q}^\beta}, \quad \text{with} \quad \Gamma^\beta_\alpha = -\frac{1}{2}\frac{\partial f^\beta}{\partial \dot{q}^\alpha} \\
\frac{\partial}{\partial q^a} &\longmapsto \frac{\partial}{\partial q^a}.
\end{aligned}$$

An interesting tensor field associated to a connection is its curvature tensor. The two connections which are at our disposal now will each give rise to an important type (1,1) tensor field along $\rho : J^1_\sigma \to E$, coming from a component of their curvature. In order to introduce these tensors, let us first observe that there exists a canonical vector field along $\rho$, given by

$$\mathbf{T} = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + (B^a_\alpha \dot{q}^\alpha + B^a)\frac{\partial}{\partial q^a}.$$

Consider then the Nijenhuis tensor $\mathcal{N}_N$ of the projector $N$ on $E$, which is precisely the curvature of $\tilde{\sigma}$. It is a vector-valued 2-form on $E$, and as such can be regarded also as a vector-valued 2-form along $\rho$. The curvature tensor $R$ of the SODE connection on the other hand, can directly be defined as being a vector-valued 2-form along $\rho$. For background on tensor fields along a projection, in the context of standard second-order equations, see [4, 5, 9]. Define now:

$$\begin{aligned}
\Psi = i_{\mathbf{T}}\mathcal{N}_N &= C^a_\beta\, \theta^\beta \otimes \frac{\partial}{\partial q^a} \\
\Phi \;=\; i_{\mathbf{T}} R &= -\Phi^\beta_\alpha \theta^\alpha \otimes X_\beta - \frac{\partial f^\beta}{\partial q^a}\, \eta^a \otimes X_\beta
\end{aligned}$$

whereby,

$$\begin{aligned}
C^a_\beta &= \mathbf{T}(B^a_\beta) - X_\beta(B^a_\alpha \dot{q}^\alpha + B^a), \\
\Phi^\beta_\alpha &= X_\alpha(f^\beta) + \Gamma^\gamma_\alpha \Gamma^\beta_\gamma + \Gamma(\Gamma^\beta_\alpha).
\end{aligned}$$

We will in fact not need the tensor $\Phi$ in this survey, but it is an important tool, to name only one of its applications, in the study of symmetry properties of the system (see [8]). Concerning the tensor $\Psi$, we need one further manipulation before we can proceed to construct the vector field we want. The point is that $\Psi$ has a "lift" to a tensor field $\dot{\Psi}$ on $J^1_\sigma$, given by

$$\dot{\Psi} = C^a_\alpha\, \theta^\alpha \otimes \frac{\partial}{\partial \dot{q}^a}.$$

# 4 Direct construction of the reduced non-holonomic dynamics

For a Lagrangian system with given non-holonomic constraints, we have seen that the data are: a function $L \in C^\infty(J^1\tau_1)$ and a connection $\sigma$ on $\pi$. From these data, we introduce new objects, according to the following steps:

- Put $\overline{L} = i^*L$, with $i : J^1_\sigma \to J^1\tau_1$ (injection).
- Define two 1-forms on $J^1_\sigma$:

$$\begin{aligned} \theta_{\overline{L}} &= \overline{L}dt + S(d\overline{L}), \\ \psi_{(L,\sigma)} &= i^*(\dot{\Psi}(dL)) - N(d\overline{L}). \end{aligned}$$

- Define the "fundamental 2-form of non-holonomic Lagrangian mechanics" to be:

$$\Omega = d\theta_{\overline{L}} + \psi_{(L,\sigma)} \wedge dt.$$

Finally then, define the dynamics of the reduced non-holonomic Lagrangian system to be governed by the unique Sode $\Gamma$, for which

$$i_\Gamma \Omega = 0.$$

It is clear that such a construction may look like playing a game of magic, which is justified only by the fact that it produces the correct equations! The benefit is, however, that this game has revealed interesting tensorial objects, which may well turn out to add substantially to our knowledge and understanding of non-holonomic mechanics in the future. For example, a topic which looks worthwhile being investigated now is the study of the role of symmetries of the fundamental 2-form $\Omega$. Also, having obtained a geometrical model for the basic situation of non-holonomic Lagrangian systems, this model may well be fruitful for exploring possible generalizations. As a matter of fact, there is still some controversy in the classical literature about the way the theory should be generalized for the case of non-linear constraints. Needless to say, the ultimate test for the validity of a more general model will always come from physics, more specifically from experiments. But the structure of the geometry behind the model can be a very useful guide for detecting appropriate candidates for the generalization. This is exactly what we propose to do in the next section. More precisely, assuming for a moment that attempts to generalize the theory to non-linear constraints would never have been made before, we will simply explore where a direct imitation of the simple model will bring us.

# 5 Generalization: non-linear constraints

Suppose now that we are thinking of a Lagrangian system subject to non-linear non-holonomic constraints, still in a solved form with respect to $m$ of the velocities. Assume that it will again be possible, under some regularity condition as yet to be detected, to obtain a reduced dynamics on some constraint submanifold. This will then lead to a mixed system of first- and second-order equations of the form

$$\begin{aligned} \ddot{q}^\alpha &= f^\alpha(t, q^A, \dot{q}^\beta), \qquad & \alpha = 1, \dots, k \\ \dot{q}^a &= g^a(t, q^A, \dot{q}^\beta), \qquad & a = 1, \dots, m. \end{aligned}$$

Geometrically, the first-order equations clearly correspond to a general section $\sigma$ of $J^1\tau_1 \to \pi^* J^1\tau_0$.

This time, the section $\sigma$ is not coming from a connection on $\pi : E \to M$. It does define, however, a connection $\tilde{\sigma}_1$ on $\pi_1 : \pi^* J^1\tau_0 \to J^1\tau_0$, determined by the following horizontal lift construction from $\mathcal{X}(J^1\tau_0)$ to $\mathcal{X}(\pi^* J^1\tau_0)$:

$$\begin{aligned} \frac{\partial}{\partial t} &\longmapsto \frac{\partial}{\partial t} + \left( g^a - \dot{q}^\alpha \frac{\partial g^a}{\partial \dot{q}^\alpha} \right) \frac{\partial}{\partial q^a} \\ \frac{\partial}{\partial q^\alpha} &\longmapsto \frac{\partial}{\partial q^\alpha} + \frac{\partial g^a}{\partial \dot{q}^\alpha} \frac{\partial}{\partial q^a}, \\ \frac{\partial}{\partial \dot{q}^\alpha} &\longmapsto \frac{\partial}{\partial \dot{q}^\alpha}. \end{aligned}$$

As before, we set $J^1_\sigma = \sigma(\pi^* J^1\tau_0)$. It turns out that, putting

$$B^b_\alpha = \frac{\partial g^b}{\partial \dot{q}^\alpha}, \qquad B^a = g^a - \dot{q}^\alpha B^a_\alpha,$$

many of the previous constructions formally remain unaltered. For example, with $\eta^a = dq^a - B^a_\alpha dq^\alpha - B^a dt$, a SODE $\Gamma \in \mathcal{X}(J^1_\sigma)$ is defined, as before, to satisfy:

$$\langle \Gamma, dt \rangle = 1, \quad \langle \Gamma, \theta^\alpha \rangle = 0, \quad \langle \Gamma, \eta^a \rangle = 0,$$

which in coordinates means that $\Gamma$ will be of the form

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + g^a(t, q^A, \dot{q}^\beta) \frac{\partial}{\partial q^a} + f^\alpha(t, q^A, \dot{q}^\beta) \frac{\partial}{\partial \dot{q}^\alpha}.$$

Formulas such as those which determine the projectors $P_H$ and $P_V$ of the SODE connection remain the same. Note, however, that $N$ is now a tensor field living on $\pi^* J^1\tau_0$ or $J^1_\sigma$ (via the diffeomorphism $\sigma$) and so is the curvature of $\tilde{\sigma}_1$, namely $\mathcal{N}_N$. We cannot contract such a tensor field with the canonical vector field $\mathbf{T}$ along $\rho : J^1_\sigma \to E$. Therefore, for any given SODE $\Gamma$, we are led to introduce the following $\Gamma$-dependent type (1,1) tensor field on $J^1_\sigma$:

$$\Psi_\Gamma = i_\Gamma \mathcal{N}_N \; = \; C^a_\beta \, \theta^\beta \otimes \frac{\partial}{\partial q^a}$$

with

$$C^a_\beta = \Gamma(B^a_\beta) - X_\beta(g^a).$$

As before, there is a corresponding lifted tensor field on $J^1\tau_1$:

$$\dot{\Psi}_\Gamma = C^a_\alpha\,\theta^\alpha \otimes \frac{\partial}{\partial \dot{q}^a}.$$

These are constructions which very closely resemble the ones we had in Section 3. So, we dare push the analogy further by imitating as follows the direct construction of Section 4.

Let there be given a function $L \in C^\infty(J^1\tau_1)$ and a section $\sigma$ of $J^1\tau_1 \to \pi^* J^1\tau_0$, the image of which determines the constraint submanifold $J^1_\sigma \subset J^1\tau_1$.

- Put $\overline{L} = i^* L$, with $i : J^1_\sigma \to J^1\tau_1$.
- Define two 1-forms on $J^1_\sigma$:

$$\begin{aligned} \theta_{\overline{L}} &= \overline{L}dt + S(d\overline{L}), \\ \psi_{(\Gamma,L,\sigma)} &= i^*(\dot{\Psi}_\Gamma(dL)) - N(d\overline{L}). \end{aligned}$$

  where $\Gamma \in \mathcal{X}(J^1_\sigma)$ is, for the time being, any Sode, to be determined later.

- Define for each such $\Gamma$ the 2-form

$$\Omega_\Gamma = d\theta_{\overline{L}} + \psi_{(\Gamma,L,\sigma)} \wedge dt,$$

  which of course (as in the previous section) also depends on $L$ and $\sigma$.

Under some appropriate regularity assumption, to be discovered in a moment, our game of magic now gives rise to the following prediction.

**Definition:** The reduced dynamics on $J^1_\sigma$ of the non-holonomic Lagrangian system is governed by the unique Sode $\Gamma$, determined by: $i_\Gamma \Omega_\Gamma = 0$.

Fixing a Sode $\Gamma$ on $J^1_\sigma$ boils down, in coordinates, to give a prescription for fixing the functions $f^\alpha(t, q^A, \dot{q}^\beta)$. A coordinate calculation reveals that the condition $i_\Gamma\Omega_\Gamma = 0$ requires that we have

$$\Gamma\left(\frac{\partial \overline{L}}{\partial \dot{q}^\alpha}\right) \equiv X_\alpha(\overline{L}) + \left(i^*\frac{\partial L}{\partial \dot{q}^a}\right)[\Gamma(B^a_\alpha) - X_\alpha(g^a)].$$

The idea that our prescription should fix $\Gamma$ clearly means that we should be able to solve the above relations unequivocally for the functions $f^\alpha$. This will be the case, if and only if the following regularity condition holds:

$$\det\left(\frac{\partial^2 \overline{L}}{\partial \dot{q}^\beta \partial \dot{q}^\alpha} - \left(i^*\frac{\partial L}{\partial \dot{q}^a}\right)\frac{\partial^2 g^a}{\partial \dot{q}^\beta \partial \dot{q}^\alpha}\right) \neq 0.$$

Let us finally see how our conclusions relate to whatever differential equations might be obtained from classical procedures after elimination of auxiliary functions such as Lagrange multipliers. To that end, we first make the following observation: the fact that the above identities (under the indicated regularity assumption) uniquely define the $f^\alpha$ is the same as saying that the reduced differential equations $\ddot{q}^\alpha = f^\alpha(t, q^A, \dot{q}^\beta)$ are equivalent to the equations

$$\frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial \dot{q}^\alpha}\right) = X_\alpha(\overline{L}) + \left(i^* \frac{\partial L}{\partial \dot{q}^a}\right)\left[\frac{d}{dt}(B^a_\alpha) - X_\alpha(g^a)\right].$$

These are exactly the equations which would follow from the classical procedure (introducing and subsequently eliminating Lagrangian multipliers), provided one adopts the generally accepted point of view that the 'variations' (or 'virtual velocities') $\delta q^A$ must satisfy

$$\delta q^a = \frac{\partial g^a}{\partial \dot{q}^\alpha}\,\delta q^\alpha\,,$$

which are often referred to as Četaev's conditions.

Details and proofs of the statements of this section will be published elsewhere, in the context of a general approach to the geometrical description of mixed systems of first- and second-order differential equations. As a final remark, an interesting topic for further research would be to investigate the possibility of transition to a reduced Hamiltonian dynamics under the regularity condition which our approach has revealed. Usually, the transition to a Hamiltonian picture is discussed at the level of the original description of the system on $J^1\tau_1$ and requires the regularity of the free Lagrangian $L$ (see e.g. [3]).

**Acknowledgements.** This research was partially supported by a NATO Collaborative Research Grant (CRG 940195). We further thank the Belgian National Fund for Scientific Research for continuing support.

# References

[1] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden and R.M. Murray, Nonholonomic mechanical systems with symmetry, (1994) Preprint, University of California, Berkeley.

[2] J. Koiller, Reduction of some classical non-holonomic systems with symmetry, *Arch. Rational Mech. Anal.* **118** (1992) 113–148.

[3] C.-M. Marle, Reduction of constrained mechanical systems and stability of relative equilibria, (1995) Preprint, Inst. de Math. de Jussieu.

[4] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Diff. Geometry and its Applications* **2** (1992) 17–43.

[5] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection II, *Diff. Geometry and its Applications* **3** (1993) 1–29.

[6] E. Massa and E. Pagani, Classical dynamics of non-holonomic systems: a geometric approach, *Ann. Inst. Henri Poincaré* **55** (1991) 511–544.

[7] Ju.I. Neimark and N.A. Fufaev, *Dynamics of non-holonomic systems*, Translations of Mathematical Monographs **33**, Am. Math. Society (1972).

[8] W. Sarlet, F. Cantrijn and D.J. Saunders, A geometrical framework for the study of non-holonomic Lagrangian systems, *J. Phys. A: Math. Gen.* **28** (1995) 3253–3268.

[9] W. Sarlet, A. Vandecasteele, F. Cantrijn and E. Martínez, Derivations of forms along a map: the framework for time-dependent second-order equations, *Diff. Geometry and its Applications*, **5** (1995) 171–203.

[10] E.T. Whittaker, *A treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press (1964).