A geometrical framework for the study of non-holonomic Lagrangian systems: II

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Abstract. A Lagrangian system subject to linear non-holonomic constraints may be represented in several different geometrical frameworks. We describe one such framework, involving the addition of an extra term to the Cartan 2-form Ω of the unconstrained system, which is dual to the traditional approach of adding a reaction force to the unconstrained dynamical vector field Γ. We show how this framework is closely related to the method of constructing a 2-form Ω_M when the constraints are given by a connection on an auxiliary bundle, as described in our earlier work.

1 Introduction

Consider a mechanical system with non-holonomic constraints. In an earlier work [10] we described a geometrical framework for such a system, where the bundle $\tau_1 : E \to \mathbb{R}$ represented the usual fibration of a configuration space, and where

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a connection $\tilde{\sigma}$ on an auxiliary bundle $\pi : E \to M$ was used to construct the constraint manifold J^1_σ as an affine sub-bundle of $J^1\tau_1 \to E$. Many of the objects familiar from the geometrical study of unconstrained systems had their analogues in this new framework: when the system was derived from a Lagrangian L we were able to use these objects to construct a fundamental 2-form Ω_M and, at the same time, give an intrinsic description of the dynamics of the system in terms of a second-order differential equation field (SODE) Γ defined on the constraint manifold.

The purpose of the present work is to look at two questions which arise naturally from the construction we have described. The first concerns the choice of the auxiliary bundle $\pi : E \to M$. If coordinates on E are (t, q^A) where $A = 1, \ldots, n$, and the constraints are given by the m equations

$$A_{aA}(t,q)\dot{q}^A + b_a(t,q) = 0$$

where the matrix A_{aA} has maximal rank, we may solve these equations to give m of the velocity coordinates (denoted $\dot{q}^a$) in terms of the other $k = n - m$ (denoted $\dot{q}^\alpha$):

$$\dot{q}^a = B^a_\alpha(t,q)\dot{q}^\alpha + B^a(t,q).$$

We would then denote the coordinates on E by (t, q^α, q^a) and select M to have coordinates (t, q^α). The point of our previous work was to consider the bundle structure $\pi : E \to M$ as part of the data, as is done also for example in the work of Bloch et al. [1]. Different choices of M, corresponding to different choices of the free and constrained velocity coordinates, will give a different fundamental 2-form Ω_M satisfying $\Gamma \lrcorner \Omega_M = 0$. It is therefore of interest to show that we may mimic some aspects of this construction to define a connection $\overline{\sigma}$ and a fundamental 2-form $\overline{\Omega}$ without making a particular choice of auxiliary bundle.

A second question arises when our approach is compared with the more traditional one of constructing the SODE for the unconstrained problem on $J^1\tau_1$, restricting it to the constraint manifold J^1_σ, and then adding an additional vector field (representing the reaction force) so that the result is tangent to J^1_σ; see, for example, [5, 6, 7] for descriptions of this approach, and [2, 11] for similar constructions in the autonomous case. When the additional reaction force is expressed in terms of a basis constructed from the constraints and the Lagrangian, its components are just the Lagrange multipliers used in the formulation of the Euler-Lagrange equations. Again, we shall see that the fundamental 2-form $\overline{\Omega}$ incorporates just these Lagrange multipliers in a natural way. In a significant sense, therefore, our approach is dual to the traditional one.

2 Results

Let $J^1(E, k+1)$ denote the manifold of $(k+1)$-dimensional contact elements over E. The elements of $J^1(E, k+1)$ are equivalence classes of immersions of $\mathbb{R}^{k+1}$ in E near a given point, with equivalence when the immersions are tangent to one another at that point. The bundle $J^1(E, k+1) \to E$ is therefore the $(k+1)$-dimensional Grassmannian bundle over E, and an element of $J^1(E, k+1)$ projecting to $a \in E$ may be considered as a $(k+1)$-dimensional subspace of $T_a E$. In fact, $J^1(E, k+1)$ is slightly too large for our purposes: if $\tau : E \to \mathbb{R}$ is the fibration of the configuration space E (this is the map denoted τ_1 in our earlier work [10]) we are interested in the open submanifold $J^1_\tau(E, k+1)$ constructed from immersions transversal to the fibres of τ.

Each section $\overline{\sigma} : E \to J^1_\tau(E, k+1)$ defines a $(k+1)$-dimensional distribution on E. Let J^1_σ be the set of jets $j^1_t\gamma \in J^1\tau$ where the tangent vector to the curve γ at $\gamma(t) \in E$ lies in the $(k+1)$-dimensional subspace $\overline{\sigma}(\gamma(t))$ of $T_{\gamma(t)}E$; this definition does not depend on the particular representative γ of the jet $j^1_t\gamma$, as it involves only first-order contact. With this definition, J^1_σ becomes a submanifold of $J^1\tau$, the *constraint submanifold*. Furthermore, for each point of J^1_σ the annihilator of the corresponding $(k+1)$-dimensional subspace of $T_a E$ is an m-dimensional subspace of $T^*_a E$; pulling this back to J^1_σ defines an m-dimensional co-distribution H on J^1_σ (this is called the Chetaev bundle in [6], where a somewhat different construction is used for constraints which are not necessarily linear). A 1-form on J^1_σ taking its values in H will be called a *constraint form*. Finally, a vector field $\overline{\Gamma}$ on J^1_σ will be called a SODE field if it satisfies the conditions $\langle \overline{\Gamma}, dt \rangle = 1$ and $\langle \overline{\Gamma}, i^*(\theta) \rangle = 0$, where θ is any contact form on $J^1\tau$ and $i : J^1_\sigma \to J^1\tau$ is the inclusion.

Our first result is given by the following theorem.

Theorem 1

Let L be a positive-definite Lagrangian on $J^1\tau$ with Cartan 1-form θ_L, and let $\overline{\sigma}$ be a $(k+1)$-dimensional distribution on E over τ. There is then a unique constraint form η on J^1_σ with the property that the 2-form $\overline{\Omega}$ defined by

$$\overline{\Omega} = i^*(d\theta_L) - dt \wedge \eta$$

contains exactly one SODE field $\overline{\Gamma}$ in its kernel.

(The proofs of this and the following two theorems will be given in Section 3.)

To see the relationship of this result to our previous work, let M be a $(k+1)$-dimensional manifold and $\pi : E \to M$ a bundle: then $J^1\pi$ is an open-dense submanifold of $J^1(E, k+1)$ (see, for example, [9] Theorem 3.28). The bundle $J^1(E, k+1) \to E$ may be thought of as the "projective completion" of the affine bundle $J^1\pi \to E$. If we also require M to be fibred over $\mathbb{R}$ in such a way that the composite map $E \to M \to \mathbb{R}$ is identical to τ, then we always have $J^1\pi \subset J^1_\tau(E, k+1)$.

A connection on π is a section $\tilde{\sigma} : E \to J^1\pi$ and is, *a fortiori*, also a section $\tilde{\sigma} : E \to J^1_\tau(E, k+1)$. There are, however, some sections $\overline{\sigma} : E \to J^1_\tau(E, k+1)$ which do not take their values entirely within $J^1\pi$, and so do not arise from (global) connections on the particular bundle $\pi : E \to M$: in coordinates adapted to this fibration, such sections yield "infinite derivatives" at certain points. To emphasize the link with connections on π, we could also call any section $\overline{\sigma} : E \to J^1_\tau(E, k+1)$ a $(k+1)$-*dimensional connection on* E *over* τ.

Let (t, q^α, q^a) be coordinates on E, chosen so that $\overline{\sigma}$ (restricted to this coordinate patch) takes its values in the coordinate patch on $J^1_\tau(E, k+1)$ represented by $(t, q^\alpha, q^a, q^a_t, q^a_\alpha)$: this situation arises automatically if we have a bundle $\pi : E \to M$ such that $\overline{\sigma}(E) \subset J^1\pi$, and if the corresponding coordinates on M are (t, q^α). The corresponding coordinates on $J^1\tau$ are $(t, q^\alpha, q^a, \dot{q}^\alpha, \dot{q}^a)$, and J^1_σ may be described as in [10] by

$$\dot{q}^a = B^a_\alpha \dot{q}^\alpha + B^a.$$

The 1-forms

$$\eta^a = dq^a - B^a_\alpha dq^\alpha - B^a dt$$

defined locally on J^1_σ span the co-distribution H. In these coordinates, the result stated in Theorem 1 is that there is a unique set of multiplier functions λ_a such that the 2-form $\overline{\Omega}$ defined locally by

$$\overline{\Omega} = i^*(d\theta_L) - dt \wedge (\lambda_a \eta^a)$$

contains exactly one SODE field $\overline{\Gamma}$ in its kernel.

In [10], we defined another 2-form Ω_M by

$$\Omega_M = d\theta_{\overline{L}} - dt \wedge (i^* \dot{\Psi}(dL) - N(d\overline{L}))$$

where $\overline{L} = i^* L$ and, in coordinates,

$$N = \frac{\partial}{\partial q^a} \otimes \eta^a$$

and

$$\dot{\Psi} = C^a_\alpha \theta^\alpha \otimes \frac{\partial}{\partial \dot{q}^a} \quad \text{with} \quad C^a_\alpha = \frac{dB^a_\alpha}{dt} - \left(\frac{\partial}{\partial q^\alpha} + B^b_\alpha \frac{\partial}{\partial q^b} \right) (B^a_\beta \dot{q}^\beta + B^a).$$

Theorem 2

If there exists a fibration $\pi : E \to M$ such that $\overline{\sigma}(E) \subset J^1\pi$, then the SODE field $\overline{\Gamma}$ (from Theorem 1) also satisfies $\overline{\Gamma} \lrcorner \Omega_M = 0$.

Corollary The SODE field Γ constructed in [10] is independent of the chosen fibration.

We shall now compare this approach with the more traditional one of finding the constrained SODE field by starting with the unconstrained SODE field and adding a "reaction" force. This uses the construction of a distinguished sub-bundle of the vertical bundle $VJ^1\tau$ to represent the possible reaction forces (see, for example, [5], Theorem 16; a similar approach is taken in [3] using almost-product structures). Each constraint form η^a then gives rise to a reaction vector field V^a according to the following rules.

First, the Lagrangian L is used to define a "fibre metric along $J^1\tau \to E$" in the usual way, so that if X, Y are vertical vector fields on E then

$$g(X,Y) = g_{\alpha\beta}X^\alpha Y^\beta + g_{a\beta}X^a Y^\beta + g_{\alpha b}X^\alpha Y^b + g_{ab}X^a Y^b$$

where we have written, for example, $g_{\alpha\beta}$ for the submatrix $\partial^2 L/\partial\dot{q}^\alpha\partial\dot{q}^\beta$ of the Hessian of L. In general $g(X,Y)$ is a function on $J^1\tau$, although when the Lagrangian is quadratic in the velocity coordinates we may also consider $g(X,Y)$ as a well-defined function on E. At each point $u \in J^1_\sigma$ projecting to $p \in E$, this "metric" defines a map $g^\#_u : V^*_pE \to V_pE$ such that, considering an element of V^*_pE as an equivalence class of cotangent vectors,

$$g^\#_u([\eta^a(u)]) = h^{ab}(u)\left.\frac{\partial}{\partial q^b}\right|_p + h^{a\beta}(u)\left.\frac{\partial}{\partial q^\beta}\right|_p .$$

In this formula we have written $h^{ab} = g^{ab} - B^a_\alpha g^{\alpha b}$ and $h^{a\beta} = g^{a\beta} - B^a_\alpha g^{\alpha\beta}$, where g^{AB} is the inverse of the Hessian matrix g_{AB}, and $g^{\alpha\beta}$, etc. are its various submatrices.

At each point $u \in J^1_\sigma$, there is also a canonical vertical lift operator v_u sending a vector in V_pE to a vector in $V_uJ^1\tau$, so that

$$\mathrm{v}_u\left(\left.\frac{\partial}{\partial q^b}\right|_p\right) = \left.\frac{\partial}{\partial \dot{q}^b}\right|_u , \qquad \mathrm{v}_u\left(\left.\frac{\partial}{\partial q^\beta}\right|_p\right) = \left.\frac{\partial}{\partial \dot{q}^\beta}\right|_u .$$

We then define

$$V^a(u) = \mathrm{v}_u(g^\#_u([\eta^a(u)])),$$

so that the vector field V^a along the inclusion $i : J^1_\sigma \to J^1\tau$ is represented in coordinates as

$$V^a = h^{ab}\frac{\partial}{\partial \dot{q}^b} + h^{a\beta}\frac{\partial}{\partial \dot{q}^\beta}.$$

We can now state our final result.

Theorem 3
Let Γ be the SODE field on $J^1\tau$ for the unconstrained Lagrangian system with Lagrangian L, so that $\Gamma \lrcorner d\theta_L = 0$. Then the vector field $\overline{\Gamma}$ along $i : J^1_\sigma \to J^1\tau$ defined locally by

$$\overline{\Gamma} = \Gamma|_{J^1_\sigma} + \lambda_a V^a$$

is tangent to J^1_σ, and is the unique SODE field on J^1_σ satisfying $\overline{\Gamma} \lrcorner \overline{\Omega} = 0$.

3 Proofs

To prove Theorem 1, we shall obtain the equations in local coordinates which must be satisfied by the components of $\overline{\Omega}$, and show that they have a unique solution. We start by writing $d\theta_L$ in coordinates as

$$d\theta_L = \frac{\partial^2 L}{\partial q^A \partial \dot{q}^B} \theta^A \wedge \theta^B + g_{AB} \omega^A \wedge \theta^B$$

where the "force forms" $\omega^A = d\dot{q}^A - F^A dt$ are chosen so that the above expression contains no separate terms in $dt \wedge \theta^B$. If $\overline{\Gamma}$ is a SODE field on J^1_σ then $\langle \overline{\Gamma}, i^* \theta^A \rangle = 0$, so that

$$\overline{\Gamma} \lrcorner i^* d\theta_L = i^*(g_{AB}) \langle \overline{\Gamma}, i^* \omega^A \rangle i^* \theta^B .$$

Letting θ^β also denote contact forms pulled back to J^1_σ as well as those forms on the whole of $J^1\tau$, we have

$$i^* \theta^\beta = \theta^\beta , \qquad i^* \theta^b = \eta^b + B^b_\beta \theta^\beta .$$

Put $\overline{\omega}^\alpha = d\dot{q}^\alpha - \overline{F}^\alpha dt$, where the new force functions $\overline{F}^\alpha$ are to be determined. Then

$$i^* \omega^a = \left(B^a_\alpha \overline{F}^\alpha - F^a + \dot{q}^\alpha \frac{dB^a_\alpha}{dt} + \frac{dB^a}{dt} \right) dt + \ldots ,$$

where "..." represents terms in $\overline{\omega}^\beta$, θ^β and η^b. In this expression, we have written d/dt for the restriction of the total time derivative to J^1_σ so that, for example,

$$\frac{dB^a}{dt} = \frac{\partial B^a}{\partial t} + \dot{q}^\beta \frac{\partial B^a}{\partial q^\beta} + (B^b_\beta \dot{q}^\beta + B^b) \frac{\partial B^a}{\partial q^b} ;$$

we have also omitted the pullback map i^* operating on functions where there will be no confusion. If $\overline{\Gamma}$ is such that $\langle \overline{\Gamma}, \overline{\omega}^\alpha \rangle = 0$, we find that

$$\begin{aligned} \overline{\Gamma} \lrcorner i^* d\theta_L \ = \ & \left(h_{a\beta} \left(B^a_\alpha \overline{F}^\alpha - F^a + \dot{q}^\alpha \frac{dB^a_\alpha}{dt} + \frac{dB^a}{dt} \right) + h_{\alpha\beta} (\overline{F}^\alpha - F^\alpha) \right) \theta^\beta \\ & + \left(g_{ab} \left(B^a_\alpha \overline{F}^\alpha - F^a + \dot{q}^\alpha \frac{dB^a_\alpha}{dt} + \frac{dB^a}{dt} \right) + g_{\alpha b} (\overline{F}^\alpha - F^\alpha) \right) \eta^b \end{aligned}$$

where $h_{a\beta} = g_{a\beta} + g_{ab}B^b_\beta$ and $h_{\alpha\beta} = g_{\alpha\beta} + g_{\alpha b}B^b_\beta$. We shall therefore be able to solve $\overline{\Gamma} \lrcorner (i^* d\theta_L - dt \wedge (\lambda_a \eta^a)) = 0$ if we can choose the functions $\overline{F}^\alpha$ so that the coefficients of θ^β vanish; the multiplier functions λ_a will then be determined automatically. But those coefficients may be written in the form

$$(B^a_\alpha h_{a\beta} + h_{\alpha\beta})\overline{F}^\alpha + \ldots$$

and the matrix $B^a_\alpha h_{a\beta} + h_{\alpha\beta}$ is non-singular by virtue of the positive-definiteness of g_{AB} (it is, essentially, just the restriction of g_{AB} to vectors whose vertical lifts are tangent to the constraint manifold). It follows that there is a unique local solution to the equations; uniqueness implies that the solutions may be glued together to give a unique global SODE field $\overline{\Gamma}$ and a unique constraint form η, completing the proof of Theorem 1.

A different way to arrive at the same conclusion, offering a further insight into the nature of the SODE field $\overline{\Gamma}$ and the multipliers λ_a, goes as follows. It is easy to verify that

$$i^*\theta_L = \overline{L}\, dt + \frac{\partial \overline{L}}{\partial \dot{q}^\alpha}\theta^\alpha + \left(i^* \frac{\partial L}{\partial \dot{q}^a}\right)\eta^a.$$

In computing the exterior derivative of this form, we make use of the following local basis of vector fields on J^1_σ (compare with [10]):

$$X_\alpha = \frac{\partial}{\partial q^\alpha} + B^a_\alpha \frac{\partial}{\partial q^a},\ \frac{\partial}{\partial q^a},\ \overline{\Gamma},\ \frac{\partial}{\partial \dot{q}^\alpha},$$

with dual basis of 1-forms

$$\theta^\alpha,\ \eta^a,\ dt,\ \varpi^\alpha = d\dot{q}^\alpha - \overline{F}^\alpha\, dt.$$

Here, of course, the $\overline{F}^\alpha$ and the corresponding expression for $\overline{\Gamma}$ are again as yet to be determined. For any $f \in C^\infty(J^1_\sigma)$ we therefore write

$$df = X_\alpha(f)\,\theta^\alpha + \frac{\partial f}{\partial q^a}\,\eta^a + \overline{\Gamma}(f)\, dt + \frac{\partial f}{\partial \dot{q}^\alpha}\,\varpi^\alpha.$$

Proceeding in this way we find, e.g.

$$\begin{aligned} d\eta^a &= -C^a_\alpha\, dt \wedge \theta^\alpha - \frac{\partial}{\partial q^b}(B^a_\alpha \dot{q}^\alpha + B^a)\,\eta^b \wedge dt \\ &\quad -X_\beta(B^a_\alpha)\,\theta^\beta \wedge \theta^\alpha - \frac{\partial B^a_\alpha}{\partial q^b}\,\eta^b \wedge \theta^\alpha. \end{aligned}$$

The calculation of $i^* d\theta_L$ is now a straightforward matter and yields

$$\begin{aligned} i^* d\theta_L &= \left[\overline{\Gamma}\left(\frac{\partial \overline{L}}{\partial \dot{q}^\alpha}\right) - X_\alpha(\overline{L}) - \left(i^* \frac{\partial L}{\partial \dot{q}^a}\right) C^a_\alpha\right] dt \wedge \theta^\alpha \\ &\quad + \left[\overline{\Gamma}\left(i^* \frac{\partial L}{\partial \dot{q}^a}\right) - i^* \frac{\partial L}{\partial q^a}\right] dt \wedge \eta^a + \ldots, \end{aligned}$$

where all the terms contained in the "…" part are made up of factors involving only θ^α, η^a and $\overline{\varpi}^\alpha$. Several things can be learned from this expression. First, we see another manifestation of Theorem 1: there is a unique way to eliminate the terms in $dt \wedge \eta^a$ and the resulting form $\overline{\Omega}$ then has a specific SODE field $\overline{\Gamma}$ in its kernel. What we see in addition here is that this SODE field $\overline{\Gamma}$ is determined by

$$\overline{\Gamma}\left(\frac{\partial \overline{L}}{\partial \dot{q}^\alpha}\right) = X_\alpha(\overline{L}) + \left(i^*\frac{\partial L}{\partial \dot{q}^a}\right) C^a_\alpha$$

and that the multipliers λ_a can in fact be expressed as

$$\lambda_a = \overline{\Gamma}\left(i^*\frac{\partial L}{\partial \dot{q}^a}\right) - i^*\frac{\partial L}{\partial q^a}.$$

The determining relation for $\overline{\Gamma}$ confirms that, in the case of a fibration $\pi : E \to M$, we are talking about the same reduced dynamics (on J^1_σ) as the one which was uniquely determined in [10] by the condition $\overline{\Gamma} \lrcorner \Omega_M = 0$, thus proving Theorem 2. Note further that the regularity of the matrix $B^a_\alpha h_{a\beta} + h_{\alpha\beta}$ in this alternative description corresponds to the regularity of the Hessian matrix $\partial^2 \overline{L}/\partial \dot{q}^\alpha \partial \dot{q}^\beta$, which again is consistent with [10].

To prove Theorem 3, we shall find the coordinate expression of the reaction force which must be added to an unconstrained SODE field Γ to give the constrained field $\overline{\Gamma}$, and show that the equations so found are identical to those obtained in the proof of Theorem 1. So let

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^A \frac{\partial}{\partial q^A} + F^A \frac{\partial}{\partial \dot{q}^A}$$

as usual, and require $\overline{\Gamma}$ to be a SODE field on J^1_σ: in terms of the coordinates on $J^1\tau$, we obtain

$$\begin{aligned}\overline{\Gamma} \;&=\; \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + (B^a_\alpha \dot{q}^\alpha + B^a)\frac{\partial}{\partial q^a} + \overline{F}^\alpha \frac{\partial}{\partial \dot{q}^\alpha} \\ &\quad + \left(B^a_\alpha \overline{F}^\alpha + \dot{q}^\alpha \frac{dB^a_\alpha}{dt} + \frac{dB^a}{dt}\right)\frac{\partial}{\partial \dot{q}^a}\end{aligned}$$

where the coefficient of $\partial/\partial \dot{q}^a$ comes from the requirement that $\overline{\Gamma}$ be tangent to J^1_σ. We want the difference between $\Gamma|_{J^1_\sigma}$ and $\overline{\Gamma}$ to be a reaction force, so put

$$\overline{\Gamma} = \Gamma|_{J^1_\sigma} + \lambda_a V^a.$$

Then the equations to be solved for $\overline{F}^\alpha$ are

$$\begin{aligned}\overline{F}^\alpha - F^\alpha \;&=\; \lambda_b h^{b\alpha} \\ \chi^a \;&=\; \lambda_b h^{ba}\end{aligned}$$

where we have written

$$\chi^a = B^a_\alpha \overline{F}^\alpha - F^a + \dot{q}^\alpha \frac{dB^a_\alpha}{dt} + \frac{dB^a}{dt}.$$

The theorem will be proved if we can show that these equations are the same as the equations we obtained in the proof of Theorem 1. But those earlier equations, rewritten in terms of χ^a, are

$$\begin{aligned} h_{\alpha\beta}(F^\alpha - \overline{F}^\alpha) &= h_{a\beta}\chi^a \\ g_{\alpha b}(F^\alpha - \overline{F}^\alpha) &= g_{ab}\chi^a - \lambda_b \end{aligned}$$

and it is straightforward to check, using the identities

$$h^{c\alpha} g_{\alpha b} = \delta^c_b - h^{ca} g_{ab}$$

and

$$h^{c\alpha} h_{\alpha\beta} = -h^{ca} h_{a\beta},$$

that the two sets of equations are, indeed, identical.

4 An example

Consider a sled which is constrained to move so that its velocity is always in the direction of its orientation (see, for example, [8] p.94). If the coordinates on the configuration manifold $E = \mathbb{R} \times (\mathbb{R}^2 \times S^1)$ are (t, x, y, ϕ), where x, y represent position and ϕ represents orientation, then the constraint may be written in the form

$$\dot{y} = \dot{x} \tan\phi$$

for most values of ϕ, and the Lagrangian (putting for simplicity the mass and the moment of inertia equal to 1) is

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\phi}^2).$$

To find the 2-form $\overline{\Omega}$ described above, we start with the Cartan form

$$\theta_L = \dot{x}\, dx + \dot{y}\, dy + \dot{\phi}\, d\phi - \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\phi}^2)\, dt$$

so that

$$\begin{aligned} i^* d\theta_L &= (\tan\phi\, d\dot{x} + \dot{x} \sec^2\phi\, d\phi) \wedge (\eta_y + \tan\phi\, \theta_x) \\ &\quad + d\dot{x} \wedge \theta_x + d\dot{\phi} \wedge \theta_\phi \end{aligned}$$

where θ_x, θ_ϕ are the contact forms and $\eta_y = dy - \tan\phi\, dx$ is the constraint form. Expressing $i^* d\theta_L$ in terms of force forms $\overline{\omega}_x$, $\overline{\omega}_\phi$, we find that the existence of a SODE field $\overline{\Gamma}$ satisfying

$$\overline{\Gamma} \lrcorner (i^* d\theta_L - \lambda\, dt \wedge \eta_y) = 0$$

gives the condition

$$(F_x + \dot{x}\dot{\phi}\tan\phi)\sec^2\phi\, \theta_x + F_\phi \theta_\phi + (F_x \tan\phi + \dot{x}\dot{\phi}\sec^2\phi - \lambda)\eta_y = 0$$

so that

$$\begin{aligned} F_x &= -\dot{x}\dot{\phi}\tan\phi \\ F_\phi &= 0 \\ \lambda &= \dot{x}\dot{\phi}. \end{aligned}$$

Hence

$$\begin{aligned} \overline{\Omega} &= \sec^2\phi\ \omega_x \wedge \theta_x + \tan\phi\ \omega_x \wedge \eta_y + \omega_\phi \wedge \theta_\phi \\ &\quad + \dot{x}\sec^2\phi\ \theta_\phi \wedge \eta_y + \dot{x}\sec^2\phi\tan\phi\ \theta_\phi \wedge \theta_x \end{aligned}$$

and

$$\overline{\Gamma} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{x}\tan\phi\frac{\partial}{\partial y} + \dot{\phi}\frac{\partial}{\partial \phi} - \dot{x}\dot{\phi}\tan\phi\frac{\partial}{\partial \dot{x}}.$$

Note in passing that the explicit formula we obtained for the multipliers would for this example read

$$\lambda = \overline{\Gamma}\left(i^*\frac{\partial L}{\partial \dot{y}}\right) = \overline{\Gamma}(\dot{x}\tan\phi)$$

and confirms that indeed $\lambda = \dot{x}\dot{\phi}$.

We may now take a manifold M with coordinates (t, x, ϕ) so that, in the notation of [10] with $\overline{L} = i^* L$,

$$\overline{L} = \tfrac{1}{2}(\dot{x}^2 \sec^2\phi + \dot{\phi}^2)$$

and

$$\begin{aligned} d\theta_{\overline{L}} &= \sec^2\phi\ d\dot{x} \wedge \theta_x + d\dot{\phi} \wedge \theta_\phi \\ &\quad + \dot{x}\tan\phi\sec^2\phi\,(d\phi \wedge \theta_x + d\phi \wedge dx). \end{aligned}$$

Then we use the formula

$$\Omega_M = d\theta_{\overline{L}} - dt \wedge (i^*\dot{\Psi}(dL) - N(d\overline{L}))$$

where

$$N = \frac{\partial}{\partial y} \otimes \eta_y$$

and

$$\dot{\Psi} = (\dot{\phi} \sec^2 \phi \, \theta_x - \dot{x} \sec^2 \phi \, \theta_\phi) \otimes \frac{\partial}{\partial y},$$

so that

$$\begin{aligned} \Omega_M &= \sec^2 \phi \; d\dot{x} \wedge \theta_x + d\dot{\phi} \wedge \theta_\phi \\ &\quad + \dot{x} \tan \phi \sec^2 \phi \; \theta_\phi \wedge \theta_x + \dot{x} \tan \phi \sec^2 \phi \; d\phi \wedge \theta_x. \end{aligned}$$

It is straightforward to check that $\overline{\Gamma} \lrcorner \Omega_M = 0$.

Now suppose we choose a different fibration $E \to M'$ where M' has coordinates (t, y, ϕ), so that

$$\begin{aligned} \dot{x} &= \dot{y} \cot \phi \\ \eta_x &= dx - \cot \phi \; dy \\ \overline{\Omega} &= -\dot{y} \; \mathrm{cosec}^2 \phi \; \theta_\phi \wedge \eta_x - \dot{y} \cot \phi \; \mathrm{cosec}^2 \phi \; \theta_\phi \wedge \theta_y \\ &\quad + \cot \phi \; \omega_y \wedge \eta_x + \mathrm{cosec}^2 \phi \; \omega_y \wedge \theta_y + \omega_\phi \wedge \theta_\phi \\ \Omega_{M'} &= \mathrm{cosec}^2 \phi \, \omega_y \wedge \theta_y - 2\dot{y} \; \mathrm{cosec}^2 \phi \; \cot \phi \, \theta_\phi \wedge \theta_y \\ &\quad + \omega_\phi \wedge \theta_\phi \\ \overline{\Gamma} &= \frac{\partial}{\partial t} + \dot{y} \cot \phi \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{\phi} \frac{\partial}{\partial \phi} + \dot{y} \dot{\phi} \cot \phi \frac{\partial}{\partial \dot{y}}. \end{aligned}$$

Observe that these expressions for $\overline{\Omega}$ and $\overline{\Gamma}$ are the ones which also follow from the coordinate transformation $\dot{x} = \dot{y} \cot \phi$ applied to the original expressions, whereas this is not the case for the relationship between Ω_M and $\Omega_{M'}$. This precisely reflects the fact that $\overline{\Omega}$ and $\overline{\Gamma}$ are independent of the choice of a fibration, whereas Ω_M is not.

Finally, we may consider the SODE for the unconstrained problem,

$$\Gamma = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{\phi} \frac{\partial}{\partial \phi}$$

and add to it a multiple of

$$V = \frac{\partial}{\partial \dot{y}} - \tan \phi \frac{\partial}{\partial \dot{x}}$$

so that the result is tangent to J^1_σ. We find that

$$\begin{aligned} \lambda \left(\frac{\partial}{\partial \dot{y}} - \tan \phi \frac{\partial}{\partial \dot{x}} \right) &= (\dot{x} \dot{\phi} \sec^2 \phi + F_x \tan \phi) \frac{\partial}{\partial \dot{y}} \\ &\quad + F_x \frac{\partial}{\partial \dot{x}} + F_\phi \frac{\partial}{\partial \dot{\phi}} \end{aligned}$$

so that, as before,

$$\begin{aligned} F_x &= -\dot{x}\dot{\phi}\tan\phi \\ F_\phi &= 0 \\ \lambda &= \dot{x}\dot{\phi}. \end{aligned}$$

5 Discussion

Given a system with a Lagrangian, a major objective is the construction of the associated SODE field, as this will describe the system's motion. For non-holonomic systems, the traditional method starts with the unconstrained SODE field and adds a vertical vector field to represent the reaction forces; at each point, the reaction vector must lie in a subspace described using the Lagrangian "metric". In contrast the construction of $\overline{\Omega} = \Omega - dt \wedge \eta$, as a 2-form with a suitable SODE field in its kernel, involves that metric only in the proof that there is a unique constraint form η with the requisite property. This latter approach, dual to the former one, is perhaps the more straightforward. (An approach described in [3] using almost-product structures apparently also leads to the 2-form $\overline{\Omega}$ [4].)

There are, nevertheless, significant questions which arise if this point of view is adopted. In holonomic dynamics, there is a clear prescription for constructing the 2-form Ω from which the motion of the system will be determined: that is, $\Omega = d\theta_L$. In non-holonomic dynamics, this is no longer the case. We may always construct the 2-form $\overline{\Omega}$, but given a suitable fibration $E \to M$ we may also construct the 2-form Ω_M: indeed, the latter is not unique, as a different fibration $E \to M'$ may give rise to a different 2-form $\Omega_{M'}$. Furthermore, the availability of such a fibration permits the further analysis of the system in terms of the dynamical covariant derivative and the Jacobi endomorphism, as we have described elsewhere. It is therefore of some interest to see how all these 2-forms are related.

Clearly, the difference between $\overline{\Omega}$ and Ω_M must arise from the difference between $i^*\theta_L$ and $\theta_{\overline{L}}$. From [10] the constrained Cartan form $\theta_{\overline{L}}$ is given by $\theta_{\overline{L}} = \overline{L}\,dt + \overline{S} \lrcorner d\overline{L}$, where the fibration $E \to M$ has been used to define a projection $\kappa : J^1\tau \to J^1_\sigma$ and the vertical endomorphism S on $J^1\tau$ then projects onto the vertical endomorphism $\overline{S}$ on J^1_σ. As we observed in the proof of Theorem 2, we have

$$i^*\theta_L = \theta_{\overline{L}} + i^* \frac{\partial L}{\partial \dot{q}^a}\eta^a,$$

from which it follows that

$$\begin{aligned} \overline{\Omega} &= \Omega_M + dt \wedge (i^*\dot{\Psi}(dL) - N(d\overline{L})) \\ &\quad + dt \wedge \lambda_a \eta^a + d\left(i^* \frac{\partial L}{\partial \dot{q}^a}\eta^a\right). \end{aligned}$$

Using the coordinate expressions for $\dot{\Psi}$ and N (see [10]) and the formulas for λ_a and $d\eta^a$ obtained earlier, it is thus easy to check that

$$\begin{aligned}\overline{\Omega} &= \Omega_M + \left(d\left(i^* \frac{\partial L}{\partial \dot{q}^a} \right) - \overline{\Gamma}\left(i^* \frac{\partial L}{\partial \dot{q}^a} \right) dt \right) \wedge \eta^a \\ &\quad - i^* \frac{\partial L}{\partial \dot{q}^a} \left(dB^a_\alpha - \overline{\Gamma}(B^a_\alpha)\, dt \right) \wedge \theta^\alpha.\end{aligned}$$

Essentially, $i^*\theta_L$ (and hence $\overline{\Omega}$) depend on the values of L close to J^1_σ, whereas $\theta_{\overline{L}}$ (and hence Ω_M) depend on the values of L on (rather than near) J^1_σ, spread out to a neighbourhood using the projection κ. It is therefore natural to ask whether the projection κ is the more fundamental object, and whether consideration of the fibration $E \to M$ is really necessary.

We may, however, be reassured by the fact that, if we are given an arbitrary affine projection $\kappa : J^1\tau \to J^1_\sigma$, we may define a distribution of vertical vectors on E: if two elements of $J^1\tau$ map to the same element of J^1_σ under κ, then the vertical vector representing their difference is deemed to be a member of the distribution (this is well-defined as κ is affine). In favourable circumstances this distribution will be integrable and the collection of integral manifolds will form a manifold M. The fibration $E \to M$ will then give rise to the projection κ which we started with. Indeed, there is always a local projection from $J^1\tau$ to J^1_σ which will give rise to a local fibration of E in just this way: we simply choose coordinates on E in such a way that the constraints can be expressed in solved form.

Acknowledgements. This research was partially supported by NATO Collaborative Research Grant (no. CRG 940195). W.S. and F.C. further thank the Belgian National Fund for Scientific Research for continuing support.

References

[1] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden and R.M. Murray, Nonholonomic Mechanical Systems with Symmetry, preprint (1995)

[2] J.F. Cariñena and M.F. Rañada, Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers, *J. Phys. A: Math. Gen.* **26** (1993) 1335–1351

[3] M. de León and D. Martín de Diego, Solving non-holonomic Lagrangian dynamics in terms of almost product structures, preprint (1995)

[4] M. de León and D. Martín de Diego, private communication

[5] G. Giachetta, Jet methods in nonholonomic mechanics, *J. Math. Phys.* **33** (1992) 1652–1665

[6] E. Massa and E. Pagani, Classical dynamics of non-holonomic systems: a geometric approach, *Ann. Inst. H. Poincaré* **55** (1991) 511–544

[7] E. Massa and E. Pagani, A new look at Classical Mechanics of constrained systems, preprint (1995)

[8] Ju.I. Neimark and N.A. Fufaev, Dynamics of nonholonomic systems, *Translations of Mathematical Monographs* **33** AMS, Providence (1972)

[9] P. Olver, Applications of Lie Groups to Differential Equations, *Graduate Texts in Math.* **107** Springer, Berlin (1986)

[10] W. Sarlet, F. Cantrijn and D.J. Saunders, A geometrical framework for the study of non-holonomic Lagrangian systems, *J. Phys. A. (Math. Gen.)* **28** (1995) 3253–3268

[11] A.M. Vershik, Classical and non-classical dynamics with constraints, in: Yu.G. Borisovich and Yu.E. Gliklikh, eds., *Global Analysis — Studies and Applications I*, Lecture Notes in Math. **1108**, (Springer-Verlag, Berlin, 1984) 278–301