The geometry of a class of mechanical systems with non-holonomic constraints

W. Sarlet
Department of Mathematical Physics and Astronomy
University of Ghent, Krijgslaan 281, B-9000 Ghent, Belgium

1 Introduction

Consider a dynamical system which is described by a mixed system of first and second-order ordinary differential equations, say of the general form,

$$\begin{aligned} \ddot{q}^\alpha &= f^\alpha(t, q^\beta, q^b, \dot{q}^\beta) \qquad & \alpha = 1, \ldots, k \\ \dot{q}^a &= g^a(t, q^\beta, q^b, \dot{q}^\beta) \qquad & a = 1, \ldots, m. \end{aligned}$$

In the context of mechanical systems with non-holonomic constraints, such equations arise, for example, when the (linear) constraints are solved for m of the velocities:

$$\dot{q}^a = B^a_\alpha(t, q)\dot{q}^\alpha + B^a(t, q), \qquad a = 1, \ldots, m,$$

(with summation over α from 1 to k). If the unconstrained physical system further is derivable from a Lagrangian L, the classical procedure for arriving at the equations of motion is to introduce Lagrange-multipliers λ_a and to consider, apart from the m constraint equations, the set of equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} &= -\lambda_a\, B^a_\alpha, \quad \alpha = 1, \ldots, k, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} &= \lambda_a, \qquad a = 1, \ldots, m. \end{aligned}$$

Eliminating the multipliers between these equations and making use of the constraints, it is clear that one will arrive at k second-order equations of the above type. Čaplygin systems (see [3]) are of this form, with the additional restriction that the functions B^a_α, B^a and the Lagrangian depend on the q^β only. Our model equations of course also encompass certain classes of Lagrangian systems with non-linear non-holonomic constraints.

In what follows, we briefly sketch the geometrical setting for such mixed equations, emphasize the existence of natural connections, and illustrate the role of their curvature in studying various properties of the dynamics.

2 Geometrical model

Modelling the occurrence of two different lots of coordinates is achieved by considering the configuration space E as fibred over another manifold M, where both E and M are fibred over $\mathbb{R}$. Local coordinates on M are denoted by (t, q^α), and on E by (t, q^α, q^a). The projections under consideration are denoted by $\pi : E \to M$ and $\tau_0 : M \to \mathbb{R}$. The first jet space $J^1\tau_0$ is a space with natural coordinates $(t, q^\alpha, \dot{q}^\alpha)$. Clearly, a vector field representing the kind of mixed systems under investigation will read in coordinates:

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + g^a \frac{\partial}{\partial q^a} + f^\alpha \frac{\partial}{\partial \dot{q}^\alpha},$$

and hence should live on a manifold with coordinates $(t, q^\alpha, q^a, \dot{q}^\alpha)$. Such a manifold is the so-called pull-back bundle $\pi^*J^1\tau_0$ over E. We thus get the following commutative scheme of spaces.

$$\begin{tikzcd} \pi^*J^1\tau_0 \arrow[r, "\pi_1"] \arrow[d, "\rho"] & J^1\tau_0 \arrow[d] \\ E \arrow[r, "\pi"] & M \arrow[d, "\tau_0"] \\ & \mathbb{R} \end{tikzcd}$$

An important observation to be made here is that there are intrinsic prescriptions which force a vector field on $\pi^*J^1\tau_0$ to have the form of Γ above ($\langle \Gamma, dt\rangle = 1$ and $S(\Gamma) = 0$, where S is the canonical vertical endomorphism which the space $\pi^*J^1\tau_0$ inherits from $J^1\tau_0$).

Each given vector field of the form Γ gives rise to two canonically defined (non-linear) connections, one on the fibration $\pi_1 : \pi^*J^1\tau_0 \to J^1\tau_0$ and one on $\rho : \pi^*J^1\tau_0 \to E$. We will now give a rudimentary description of these connections. With respect to the fibration ρ, the fibre coordinates on $\pi^*J^1\tau_0$ are the $\dot{q}^\alpha$ and vertical tangent vectors are those spanned by $\partial/\partial\dot{q}^\alpha$. A connection can be regarded as defining a complementary notion of horizontality, and this can be achieved, for example, by an intrinsic procedure for lifting the coordinate vector fields on the base manifold E to vector fields on $\pi^*J^1\tau_0$, which project onto the vector fields we start from and, together with the vertical fields, span the full tangent space to $\pi^*J^1\tau_0$. For the case at hand, these rules appear to be given by

$$\begin{aligned}\frac{\partial}{\partial t} &\rightsquigarrow \frac{\partial}{\partial t} + (f^\alpha + \dot q^\beta \Gamma^\alpha_\beta)\frac{\partial}{\partial \dot q^\alpha}\\ \frac{\partial}{\partial q^\alpha} &\rightsquigarrow \frac{\partial}{\partial q^\alpha} - \Gamma^\beta_\alpha \frac{\partial}{\partial \dot q^\beta}, \qquad \frac{\partial}{\partial q^a} \rightsquigarrow \frac{\partial}{\partial q^a}.\end{aligned}$$

Similarly, with respect to the fibration π_1, where the q^a play the role of fibre coordinates, a connection is defined by the following horizontal lift procedure:

$$\begin{aligned}\frac{\partial}{\partial t} &\rightsquigarrow \frac{\partial}{\partial t} + (g^a - \dot q^\beta B^a_\beta)\frac{\partial}{\partial q^a}\\ \frac{\partial}{\partial q^\alpha} &\rightsquigarrow \frac{\partial}{\partial q^\alpha} + B^a_\alpha \frac{\partial}{\partial q^a}, \qquad \frac{\partial}{\partial \dot q^\alpha} \rightsquigarrow \frac{\partial}{\partial \dot q^\alpha}.\end{aligned}$$

In these relations, the 'connection coefficients' Γ^α_β and B^a_β are defined by

$$\Gamma^\alpha_\beta = -\tfrac{1}{2}\frac{\partial f^\alpha}{\partial \dot q^\beta}, \quad B^a_\beta = \frac{\partial g^a}{\partial \dot q^\beta}.$$

Since these connections are automatically available whenever Γ is given, it should not be a surprise that features of, for example, their curvature play a distinctive role in various properties of the given dynamics.

3 Curvature aspects

Solution curves of the given dynamics on $\pi^*J^1\tau_0$ will be curves in E, so that it is natural to study maps from $\pi^*J^1\tau_0$ to tangent vectors on E. Such maps are called vector fields along ρ. In previous work on unconstrained second-order systems (see e.g. [7]), we have seen that an effective calculus of forms along a map can be developed for the coordinate free description of such systems. For the mixed systems under discussion here, elements of a calculus of forms along ρ are treated in [6]. We will limit ourselves here to just a few aspects which are sufficient to see curvature components of the connections make their appearance as vector-valued forms along ρ.

Vector fields along ρ (the set of which is denoted by $\mathcal{X}(\rho)$) formally look like vector fields on E, but with coefficients which depend on all coordinates of $\pi^*J^1\tau_0$. The most convenient local basis of such vector fields (which takes the information provided by the connections into account) is given by

$$\mathbf{T}_\Gamma, \quad X_\alpha = \frac{\partial}{\partial q^\alpha} + B^a_\alpha \frac{\partial}{\partial q^a}, \quad \frac{\partial}{\partial q^a},$$

where

$$\mathbf{T}_\Gamma = T\rho \circ \Gamma = \frac{\partial}{\partial t} + \dot q^\alpha \frac{\partial}{\partial q^\alpha} + g^a \frac{\partial}{\partial q^a}.$$

The span of these different parts have an intrinsic meaning as submodules of $\mathcal{X}(\rho)$. We write $\widetilde{\mathcal{X}}(\rho)$, $\overline{\mathcal{X}}(\rho)$ and $\widehat{\mathcal{X}}(\rho)$ for the modules spanned respectively by $\partial/\partial q^a$, X_α and $\{X_\alpha, \mathbf{T}_\Gamma\}$, and denote elements of each of these sets with the corresponding symbol. Each $X \in \mathcal{X}(\rho)$ has a unique decomposition in the form $X = \langle X, dt\rangle \mathbf{T}_\Gamma + \overline{X} + \widetilde{X}$.

Vector fields on the full space $\pi^*J^1\tau_0$ also have intrinsic decompositions into various distinct parts. In fact, as a result of the two fibrations with corresponding connections, there is a certain ambiguity there about what should be called 'horizontal'. It was observed in [6] that the connections on ρ and π_1 induce a third connection on $\pi \circ \rho : \pi^*J^1\tau_0 \to M$, and that the ambiguity is best resolved by keeping the vector fields which come from the lift along the sort of 'diagonal projection' $\pi \circ \rho$ in a separate class. In short, every $Z \in \mathcal{X}(\pi^*J^1\tau_0)$ has a unique decomposition in the form $Z = (Z_H)^H + (Z_D)^D + (Z_V)^V$, with $Z_H \in \widehat{\mathcal{X}}(\rho)$, $Z_D \in \widetilde{\mathcal{X}}(\rho)$ and $Z_V \in \overline{\mathcal{X}}(\rho)$. To explain what the 'horizontal', 'diagonal' and 'vertical' lifting operations are, it suffices to list their action on the local basis of $\mathcal{X}(\rho)$. Putting, for shorthand, $V_\beta = \partial/\partial \dot q^\beta$ and $V_a = \partial/\partial q^a$ (as vector fields on $\pi^*J^1\tau_0$), we have:

$$\begin{aligned}&\mathbf{T}_\Gamma{}^H = \Gamma, && X_\alpha{}^H = H_\alpha, && \left(\frac{\partial}{\partial q^a}\right)^H = 0\\ &\mathbf{T}_\Gamma{}^D = 0, && X_\alpha{}^D = 0, && \left(\frac{\partial}{\partial q^a}\right)^D = V_a\\ &\mathbf{T}_\Gamma{}^V = 0, && X_\alpha{}^V = V_\alpha, && \left(\frac{\partial}{\partial q^a}\right)^V = 0.\end{aligned}$$

The H_α can be regarded as being defined by this scheme. In coordinates, they read

$$H_\alpha = \frac{\partial}{\partial q^\alpha} + B^a_\alpha \frac{\partial}{\partial q^a} - \Gamma^\beta_\alpha \frac{\partial}{\partial \dot q^\beta}.$$

The set $\{\Gamma, H_\alpha, V_a, V_\alpha\}$ gives a local basis of $\mathcal{X}(\pi^*J^1\tau_0)$, which is perfectly adapted to the given dynamics and its associated connections.

One way to see a manifestation of curvature of a connection is to look at Lie brackets of vector fields on the full space which are not vertical. To be more specific, the vector field resulting from such a bracket operation will have its own decomposition; parts of this decomposition will contain derivations, but other parts will be completely algebraic and hence determine a certain tensor field; this tensor field can then be regarded as defining the curvature. In our situation, the brackets of interest will involve at least one diagonal or one horizontal lift, and we will encounter curvature tensors related to different connections. It turns out that some of the curvature components are zero. Indeed, the list of bracket relations which give rise to algebraic parts reads, if we leave the derivation part unspecified for simplicity,

$$[\widehat{X}^{H},\widehat{Y}^{H}] = (.)^{H} + \Big(R_1(\widehat{X},\widehat{Y})\Big)^{D} + \Big(R_2(\widehat{X},\widehat{Y})\Big)^{V}$$
$$[\widetilde{X}^{D},\widehat{Y}^{H}] = (.)^{H} - (.)^{D} + \Big(R_3(\widetilde{X},\widehat{Y})\Big)^{V}$$
$$[\overline{X}^{V},\widehat{Y}^{H}] = (.)^{H} - (.)^{V} + \Big(G(\overline{X},\widehat{Y})\Big)^{D}.$$

The tensor fields R_i are vector-valued 2-forms, as expected. G on the other hand turns out to be symmetric in its two arguments; this is due, apparently, to the fact that we wish to have a manifestation of curvature at the level of forms along ρ. In the more traditional approach, where curvature tensors are regarded as (vertical vector-valued) tensor fields on the full space, one would encounter two distinct vector-valued 2-forms for each of the two connections. One can show that R_2 and R_3 determine the curvature of the connection on the fibration ρ, whereas R_1 and G determine the curvature on π_1.

In the applications we are aware of so far, the tensor fields R_i seem to contain the most relevant features of curvature. An important property which to some extent underscores the relevance of the R_i is that they are completely determined by a tensorial object of lower covariant order, namely a type (1,1) tensor field along ρ. Explicitly, let $\Psi = i_{\mathbf{T}_\Gamma}R_1$, $\Phi = i_{\mathbf{T}_\Gamma}R_2$ and $\Lambda = i_{\mathbf{T}_\Gamma}R_3$. The coordinate expressions of these tensor fields read:

$$\begin{aligned}
\Psi &= (\Gamma(B^c_\beta) - X_\beta(g^c))\,\theta^\beta \otimes \frac{\partial}{\partial q^c}\\
\Phi &= -(\Gamma(\Gamma^\gamma_\beta) + X_\beta(f^\gamma) + \Gamma^\alpha_\beta\Gamma^\gamma_\alpha)\,\theta^\beta \otimes X_\gamma\\
\Lambda &= -\frac{\partial f^\gamma}{\partial q^b}\,\eta^b \otimes X_\gamma.
\end{aligned}$$

Here, the θ^β are the so-called contact forms $dq^\beta - \dot{q}^\beta dt$, whereas $\eta^b = dq^b - g^b dt - B^b_\alpha\theta^\alpha$. Together with dt, θ^β and η^b constitute a local basis of 1-forms along ρ, which is dual to the local basis for $\mathcal{X}(\rho)$ mentioned before. The relation between these tensors and the curvature fields R_i is as follows:

$$\begin{aligned}
R_1 &= \tfrac{1}{2}(d^V\Psi + dt\wedge\Psi)\\
R_2 &= \tfrac{1}{3}(d^V\Phi + 2dt\wedge\Phi)\\
R_3 &= \tfrac{1}{2}(d^V\Lambda + 2dt\wedge\Lambda)\,.
\end{aligned}$$

The 'vertical exterior derivative' d^V which figures in these expressions, is a canonically defined operation; for conciseness, however, we will not discuss it further here.

4 Applications

Working out full scale applications in a brief note is impossible, of course. We will limit ourselves, therefore, to discussing briefly a few fields of application in which the tensor fields introduced in the previous section clearly manifest themselves.

For a start, let us go back to the link with mechanics mentioned in the introduction. For Lagrangian systems of generalised Čaplygin type, consider the reduced Lagrangian function $\overline{L}(t, q^\alpha, q^b, \dot{q}^\alpha)$, obtained from the unconstrained L by the substitution imposed by the constraint equations. It turns out that one of the above mentioned type (1,1) tensor fields, namely Ψ, already makes its appearance in writing down simply the reduced second-order equations for the q^α. Indeed, these equations take the form:

$$\frac{d}{dt}\left(\frac{\partial\overline{L}}{\partial\dot{q}^\alpha}\right) = X_\alpha(\overline{L}) + \Psi^a_\alpha\,\frac{\partial L}{\partial\dot{q}^a}.$$

This observation has led us in [5] to a direct geometrical construction of these reduced equations, without the need of introducing first Lagrange multipliers which are in the end eliminated anyway. In [4], this construction was even extended to the case of non-linear constraints.

An interesting field of study concerning properties of dynamical systems is the search for symmetries or adjoint symmetries, which may lead, for example, to the identification of conservation laws or to a process of reduction of order of the system. Symmetries are transformations which map solution curves of the given dynamics Γ into solution curves. One-parameter groups of such transformations are generated by vector fields which are invariant under the flow of Γ. Adjoint symmetries, likewise, (leaving apart some details) are essentially invariant 1-forms. Obviously, since we are thinking here about the way certain objects evolve under the flow of Γ, there must be a kind of derivation operator which, at the level of tensor fields along ρ is the emanation of this evolution. This operator is what we call the *dynamical covariant derivative* ∇. It can be defined through the non-algebraic part of the decomposition of the Lie derivative with respect to Γ of the various kinds of lifts. Indeed, we have the relations:

$$\begin{aligned}
\mathcal{L}_\Gamma \overline{X}^V &= -\overline{X}^H + (\nabla\overline{X})^V \\
\mathcal{L}_\Gamma \widehat{X}^H &= (\nabla\widehat{X})^H + \Psi(\widehat{X})^D + \Phi(\widehat{X})^V \\
\mathcal{L}_\Gamma \widetilde{X}^D &= (\nabla\widetilde{X})^D + \Lambda(\widetilde{X})^V .
\end{aligned}$$

The action of ∇ on the local basis of $\mathcal{X}(\rho)$ is given by

$$\nabla \mathbf{T}_\Gamma = 0, \quad \nabla X_\beta = \Gamma^\alpha_\beta X_\alpha, \quad \nabla \frac{\partial}{\partial q^b} = -\frac{\partial g^c}{\partial q^b}\frac{\partial}{\partial q^c}.$$

If $Z \in \mathcal{X}(\pi^* J^1\tau_0)$ is going to be a symmetry vector field of Γ, its component along Γ is irrelevant and Z will therefore have a decomposition of the form

$$Z = \overline{X}^H + \widetilde{X}^D + \overline{Y}^V ,$$

for some $\overline{X}, \widetilde{X}, \overline{Y} \in \mathcal{X}(\rho)$. The point is that the symmetry requirement $[Z, \Gamma] = 0$ immediately fixes the vertical part: in fact we have $\overline{Y} = \nabla\overline{X}$. So, essentially, the analytical equations which must be solved to obtain symmetries (second- and first-order partial differential equations here) must be equations for the components of $\overline{X}$ and $\widetilde{X}$ only. A concise, coordinate free formulation of the symmetry conditions, which is the direct geometrical translation of these analytical conditions, is given by

$$\nabla^2\overline{X} + \nabla\widetilde{X} + (\Phi + \Lambda + \Psi)(X) = 0 .$$

There is a perfectly dual version of this, concerning the search for adjoint symmetries: the invariant 1-form at the level of $\pi^* J^1\tau_0$ will have a redundant part, so that the essential information is contained in two 1-forms $\overline{\alpha}$ and $\widetilde{\alpha}$ along ρ. We will not enter into details here about the definition of such 1-forms along ρ. Suffices it to say that the partial differential equations to be solved for constructing adjoint symmetries, are geometrically modelled by a condition of the form

$$\nabla^2\overline{\alpha} - \nabla\widetilde{\alpha} + (\Phi + \Lambda + \Psi)(\alpha) = 0,$$

which also in this description is seen to be the formal adjoint equation of the equation for symmetries.

The role of the tensor fields Φ, Ψ and Λ (together with the dynamical covariant derivative ∇) is of course very manifest here. It is worthwhile to observe that there is only one such tensor (Φ) in the geometrical study of unconstrained second-order equations. It is called the Jacobi endomorphism there, and its importance has already been recognised in quite different types of applications, such as the study of separability of differential equations (see [1]) and the so-called inverse problem of Lagrangian mechanics (see [2]). Needless to say, the three tensor fields which we encounter here (and the curvature tensors which they determine) are expected to be equally important in the study of mixed first- and second-order equations in general, and of Lagrangian systems with non-holonomic constraints in particular.

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