Adjoint symmetries, separability and volume forms

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Abstract. Two results of a preceding paper are generalized. The first is about characterizing to what extent preservation of the energy function of a Lagrangian of mechanical type turns dynamical symmetries into Noether symmetries. The generalization here is twofold: polynomial integrals of arbitrary degree are considered and the kinetic energy can have an arbitrary metric. The second result (here again for arbitrary metrics) is about the way separation variables for the Hamilton-Jacobi equation, when they are ensured to exist by Eisenhart's theorem, can be computed, in principle, from a factorization property of a certain volume form. The main novelty in the way the generalizations are discussed is that the emphasis is shifted from symmetries to the dual concept of adjoint symmetries.

1 Introduction

In a preceding paper [21], two aspects of the practical use of symmetries of Lagrangian systems were scrutinized: one was about conditions which will force a general dynamical symmetry to fall into the class of Noether symmetries; the other one was about the way separation variables for the Hamilton-Jacobi equation (if they exist) could be obtained from calculations involving the symmetry generators. Both of these investigations were prompted by certain potentially misleading statements in the work of others. To be more precise, for the first aspect, the idea was to show why in [9, 10, 11] calculations originating from the determining equations of general symmetries and thought not to be making use of Noether's theorem, turned out to give rise to Noether symmetries anyway. For the second aspect, there was need for an explanation why certain formal manipulations on the characteristic equations of the symmetry generators in the work of the same authors, can indeed produce separation variables under the right circumstances.

By the nature of the problems posed in [21], the emphasis was very much on computational aspects. For a start, therefore, attention was restricted to Lagrangians whose kinetic energy term has the standard Euclidean metric, and the whole analysis was about integrals of the motion which are polynomial functions of the velocities. Let us recall the two main conclusions, formulated as propositions in [21].

It is well known that Noether symmetries of autonomous Lagrangian systems preserve the energy function, so this is a condition which is the most likely candidate for having the effect of forcing symmetry generators towards matching the requirements of Noether symmetries. Symmetry generators were considered whose leading components are polynomials of odd degree and are at most of degree three in the velocities (so that corresponding Noether first integrals, if any, are bound to be even degree polynomials of at most degree four). Now, conditions for a vector field Y to be a symmetry of the given second-order system (SODE) Γ, or to be a Noether symmetry with respect to the given quadratic Lagrangian L, or to leave the energy function E invariant, all give rise to different determining equations (pde's) for the polynomial coefficients of the leading components ξ^i of Y and from a computational point of view it is by no means obvious how all these equations interrelate. It was shown that whenever the coefficients of the ξ^i are fully symmetric and energy preservation $Y(E) = 0$ is imposed, vanishing of the lowest-order terms in the symmetry requirement $[Y, \Gamma] = 0$ is enough to guarantee that all other terms will vanish as well and that in fact also all determining equations coming from the Noether requirement will be satisfied.

For the problem of explaining where the separation variables come from in [9, 10, 11], at least for systems with two degrees of freedom and an additional quadratic first integral, Eisenhart's theorem (see e.g. [7, 8, 1, 23]) was recalled in [21]. Always in the case that the kinetic energy term is the standard one, it was shown how this theorem implies that the volume form of the configuration manifold gives rise to a determinant, computed out of the symmetry generator corresponding to the additional integral, which factorizes into the product of linear functions in the velocities. These in turn, to within an integrating factor, are bound to be the derivatives of the separation variables. Adding an extra degree of freedom, such a mechanism of course produces a cubic expression which factorizes as the product of three linear functions (giving no support for the attempts in [11] to manipulate also quadratic expressions in that case).

The limitations which were built into the analysis of [21] gave enough freedom still to answer all the questions of computational nature which were posed. The conjecture was, moreover, that looking at more general situations (polynomial first integrals of degree higher than four and general metrics in the kinetic energy) would merely be a matter of more labour. In the present paper, we shall consider these generalizations anyway, because we feel that something substantial can be added to the discussion. Essentially, we shall look at a dual picture for proving more general results. The regular Lagrangian of the given SODE provides us with a symplectic form by which all statements concerning vector fields (symmetries in particular) can be translated, in principle, into equivalent statements on 1-forms ('adjoint symmetries' in particular). The point now is that by looking at this dual world, proving the more general results we have in mind turns out to become much more simple. In fact, for the first result, the proof for arbitrary degree polynomial first integrals becomes almost trivial and is carried out in the next section. With respect to the second result, the computation of a volume form with corresponding factorization property becomes more elegant and direct. This is presented in Section 3. In addition, we will arrive in Section 5 at a rather surprising new formulation of Eisenhart's theorem, which may inspire new developments in Hamilton-Jacobi theory in the future. Section 4 contains some illustrative examples for the computation of separation variables along the lines of the results of Section 3.

2 The dual picture of adjoint symmetries

Let the second-order vector field

$$\Gamma = \dot{x}^i \frac{\partial}{\partial x^i} + f^i(x,\dot{x}) \frac{\partial}{\partial \dot{x}^i}, \tag{1}$$

living on the tangent bundle TM of a manifold M, be derived from a regular Lagrangian function $L \in C^\infty(TM)$, i.e. we have

$$i_\Gamma d\theta_L = -dE, \tag{2}$$

where $E = \Delta(L) - L$ is the 'energy function' associated to L, and $\theta_L = S(dL)$ is the Poincaré-Cartan 1-form. These defining relations further refer to two canonically defined objects on TM, namely the Liouville vector field $\Delta = \dot{x}^i \partial/\partial \dot{x}^i$ and the type (1,1) tensor field

$$S = \frac{\partial}{\partial \dot{x}^i} \otimes dx^i, \tag{3}$$

usually called the vertical endomorphism (cf. [5]). The 2-form $d\theta_L$ is symplectic, so that the relation

$$i_Y d\theta_L = \beta \tag{4}$$

defines an isomorphism between the module of vector fields Y and the module of 1-forms β on TM. We will first discuss how a number of features of Y translate into corresponding features for β.

As was the case in [21], vector fields Y of interest will always be of the form

$$Y = \xi^i \frac{\partial}{\partial x^i} + \Gamma(\xi^i) \frac{\partial}{\partial \dot{x}^i}, \tag{5}$$

so that they are completely determined by their $\partial/\partial x^i$ components (referred to before as the leading components ξ^i). The intrinsic characterization of such vector fields is that they belong to the set $\mathcal{X}_\Gamma$ introduced in [17], determined by the condition $S(\mathcal{L}_\Gamma Y) = 0$. The corresponding 1-form β then will likewise be characterized by the property $S(\mathcal{L}_\Gamma \beta) = 0$. The set of such forms was denoted by $\mathcal{M}^*_\Gamma$ in [4] to distinguish it from the closely related set of 1-forms $\mathcal{X}^*_\Gamma$ considered in [17], which consists of those 1-forms β which have the property $\mathcal{L}_\Gamma(S(\beta)) = \beta$. The relation between these two sets is simply that $\mathcal{M}^*_\Gamma = \mathcal{L}_\Gamma S(\mathcal{X}^*_\Gamma)$, and this is an isomorphism in view of the property $(\mathcal{L}_\Gamma S)^2 = 1$. What elements of $\mathcal{M}^*_\Gamma$ and $\mathcal{X}^*_\Gamma$ have in common is that their $d\dot{x}^i$ components can be any functions, whereas, similar to the situation in (5), the dx^i components then are completely fixed. Elements of $\mathcal{X}_\Gamma^*$, for example, are of the form

$$\beta = b_i\, d\dot{x}^i + \Gamma(b_i)\, dx^i\,. \tag{6}$$

The relationship between the sets $\mathcal{X}_\Gamma$ (and $\mathcal{M}_\Gamma = \mathcal{L}_\Gamma S(\mathcal{X}_\Gamma)$) of vector fields on the one hand, and the sets $\mathcal{M}_\Gamma^*$ and $\mathcal{X}_\Gamma^*$ of 1-forms on the other hand, which are of course well defined for any SODE Γ (not necessarily coming from a Lagrangian), was also described in [15].

We are, in particular, interested in those elements of $\mathcal{X}_\Gamma$ which are symmetries of Γ, i.e. vector fields Y for which $[Y,\Gamma]=0$. In view of the property $\mathcal{L}_\Gamma d\theta_L = 0$, this is equivalent to saying that the corresponding β is invariant: $\mathcal{L}_\Gamma \beta = 0$. Originally in [18] (and [20] for time-dependent systems) the related 1-form $\alpha = \mathcal{L}_\Gamma S(\beta) \in \mathcal{X}_\Gamma^*$ was called an *adjoint symmetry*. To avoid too much terminology, however, we will also use the term adjoint symmetry for an invariant 1-form β. In fact, there is a deeper reason for that. Having recognised that for second-order systems Γ, objects of interest on TM are very frequently fully determined by only part of their components, Martínez *et al* [12, 13] developed a suitable calculus in which only this 'leading part' occurs, namely a calculus of derivations of forms along the projection $\tau : TM \to M$. In that approach, the adjoint symmetry would simply be the semi-basic 1-form $S(\alpha) = S(\beta) = b_i\, dx^i$, regarded as 1-form along τ (and satisfying of course a suitable condition, cf. [13]), and it becomes then a matter of taste or preference to choose whether one wants to think of this object as being associated to $\alpha \in \mathcal{X}_\Gamma^*$ or to $\beta \in \mathcal{M}_\Gamma^*$. We come back to this calculus along τ later. For the moment, and for the sake of generalizing the first result of [21], we stick to the more familiar calculus on the full space TM.

Let now L more specifically be a Lagrangian of the form

$$L = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j - V(x), \tag{7}$$

with $g_{ij}(x)$ symmetric and non-singular. The functions f^i in the expression for Γ thus are of the form

$$f^i = -\Gamma^i_{jk}\,\dot{x}^j\dot{x}^k - g^{il}\frac{\partial V}{\partial x^l}, \tag{8}$$

where the Γ^i_{jk} are the Christoffel symbols coming from the kinetic energy metric g. Since the f^i contain terms of even degree in the velocities only, whenever a polynomial function F is a first integral of Γ, its odd and even parts will be first integrals by themselves and we can discuss these two cases separately. Thinking of the even case first, let F be a polynomial (always to be understood as referring to the velocity variables) of degree $2r$. Then the condition that F be a first integral, $\Gamma(F)=0$, requires a polynomial of degree $2r+1$ to vanish identically. If F were a first integral indeed, its corresponding Noether symmetry would be a vector field Y of the form (5), whose leading components ξ^i would be given by

$$\xi^i = -g^{ij}\frac{\partial F}{\partial \dot{x}^j}, \tag{9}$$

and would accordingly be polynomials of degree $2r-1$, containing odd degree terms only. Suppose, on the other hand, that the construction of polynomial type symmetries of Γ

would be our first move (and that we would worry later about identifying which of these are of Noether type). Then, we would again be looking for vector fields of the form (5), with ξ^i purely odd polynomials of degree $2r-1$ say, satisfying the requirements (coming from $[Y,\Gamma]=0$):

$$\Gamma^2(\xi^i) = Y(f^i), \qquad i=1,\dots,n=\dim M. \tag{10}$$

Also this requires polynomials of degree $2r+1$ to vanish. Finally, the independent requirement that a vector field of type (5), with odd ξ^i of degree $2r-1$ preserves the energy function $E=T+V$, i.e. satisfies $Y(E)=0$, again gives rise to a polynomial condition of degree $2r+1$. In all three cases, moreover, the polynomials in question will contain terms of odd degree only, but the three conditions of course are drastically different in general, if only because in the second case there are n requirements, as opposed to only one in the first and third case. The result we want to generalize from [21], where g_{ij} was δ_{ij} and r was either 1 or 2, is the following: if the coefficients of the different powers of $\dot{x}$ in the ξ^i are symmetric in all their indices and $Y(E)=0$, then vanishing of the lowest order term in the polynomial expressions (10) is enough to ensure that Y is a Noether symmetry with respect to L. The first of these conditions is equivalent to saying that the ξ^i are of the form $\partial F/\partial\dot{x}^i$ for some function F. Obviously, it will have to be replaced here by a symmetry requirement with respect to the metric g_{ij}, which is the same as saying that the ξ^i are of the form (9) for some F (the sign is irrelevant for that matter). So we now state and prove the following result.

Proposition 1. *Consider the* SODE Γ *coming from a Lagrangian of type (7). Let Y be a vector field in $\mathcal{X}_\Gamma$, whose leading components ξ^i are polynomial functions of the velocities of degree $2r-1$ (and contain odd degree terms only). Then, if*

1. *the ξ^i are of the form (9) for some function F,*
2. *$Y(E)=0$, where $E=\Delta(L)-L$,*
3. *the lowest order terms in the expressions (10) cancel out,*

Y is a Noether symmetry and there exists a function $f\in C^\infty(M)$, such that $F+f$ is the corresponding first integral.

PROOF: Consider the 1-form β associated to Y via the relation (4). To say that Y belongs to $\mathcal{X}_\Gamma$ and satisfies the first condition, is exactly equivalent to saying that β is of the form

$$\beta = dF - S(d\Gamma(F)) \tag{11}$$

for some function $F\in C^\infty(TM)$. Indeed, $d\theta_L$ contains the term $g_{ij}d\dot{x}^j\wedge dx^i$ (and no other terms in $d\dot{x}^j$), so that the term dF in β will make sure that the ξ^i are of the form (9). The extra semi-basic part in β simply makes sure that β belongs to $\mathcal{M}^*_\Gamma$, as can be easily verified, remembering that when acting on 1-forms, we have the property $S\circ\mathcal{L}_\Gamma S = S$. Observe that there is a certain 'gauge freedom' in selecting functions F to construct a β of the form (11). Indeed, if f is any function on the base manifold M, putting $\tilde{F}=F+f$, we will have $d\tilde{F}-S(d\Gamma(\tilde{F})) = dF - S(d\Gamma(F))$.

The second condition on Y, namely $Y(E) = 0$, in view of (2) and (4) translates equivalently to the condition $i_\Gamma \beta = 0$. But with a β of the form (11) and remembering that $S(\Gamma) = \Delta$, we have $i_\Gamma\beta = \Gamma(F) - \Delta(\Gamma(F))$, so that the second condition immediately implies that $\Gamma(F)$ is homogeneous of degree 1 and (being a polynomial) therefore linear in the velocities. It further follows that

$$\begin{aligned} i_\Gamma d\beta &= -i_\Gamma d\Big(S(d\Gamma(F))\Big) \\ &= -\mathcal{L}_\Gamma\Big(S(d\Gamma(F))\Big) + d i_\Delta d\Gamma(F) \\ &= -\mathcal{L}_\Gamma\Big(S(d\Gamma(F))\Big) + d\Gamma(F). \end{aligned}$$

Turning now to the third condition, remember that the full symmetry requirements (10) translate to $\mathcal{L}_\Gamma\beta = 0$, which in view of $i_\Gamma\beta = 0$ reduces to $i_\Gamma d\beta = 0$. This in turn, from the computation just done, reduces to

$$\Gamma\left(\frac{\partial\Gamma(F)}{\partial\dot{x}^i}\right) - \frac{\partial\Gamma(F)}{\partial x^i} = 0, \qquad (12)$$

with $\Gamma(F)$ of the form $a_i(x)\dot{x}^i$ say. Hence, in this dual picture it is immediately clear that all terms but the lowest-order ones of the symmetry requirement, have already cancelled out, so that the third condition is going to make sure that Y is a symmetry. To see that it is actually going to be a Noether symmetry, it suffices to note that (12) expresses that $\Gamma(F)$ is the total time derivative of a function on M. In other words, there exists a function $f \in C^\infty(M)$ such that, with $\tilde{F} = F + f$, we will have $\Gamma(\tilde{F}) = 0$ and $\beta = d\tilde{F}$. □

Remark: For the case of polynomial functions F of even degree, it is clear that in defining the ξ^i via (9), information about the zeroth-order term in F is lost. Therefore, we know from the outset that $Y(E) = 0$ cannot be enough, in general, to guarantee that Y will become a Noether symmetry. The lowest order terms of the symmetry conditions (10), which will be second-order pde's for the potential V, then precisely provide the integrability conditions for existence of a function $f(x)$ which will complete the construction of a first integral.

The situation of course is different if F contains only terms of odd degree and is of degree, say, $2r-1$. Then, all conditions such as $\Gamma(F) = 0$, $Y(E) = 0$ or $[Y, \Gamma] = 0$ give rise to polynomials of degree $2r$ with even degree terms only. The coordinate free computations in the above proof remain perfectly valid, however, and still lead to the conclusion (from $\langle\Gamma, \beta\rangle = 0$) that $\Gamma(F)$ is linear in the velocities. This can now only be 'true', however, if $\Gamma(F) = 0$. Hence, we reach the following conclusion.

Proposition 2. *Consider the* SODE Γ *coming from a Lagrangian of type (7). Let* Y *be a vector field in* $\mathcal{X}_\Gamma$*, whose leading components* ξ^i *are polynomial functions of the velocities of degree* $2r-2$ *(and contain even degree terms only). Then, if*

1. *the* ξ^i *are of the form (9) for some function* F*,*
2. $Y(E) = 0$*, where* $E = \Delta(L) - L$*,*

F *is a first integral and* Y *is the corresponding Noether symmetry.* □

3 Adjoint symmetries and separability

Assume now, still for Lagrangians of type (7) with a general metric, that we are in the situation of Eisenhart's theorem, which gives necessary and sufficient conditions for the existence of a point transformation which will transform the kinetic energy part into Stäckel form. As in [21], for the sake of discussing the identification of separation variables, we can actually drop the potential energy term without loss of generality. So assume then that the SODE is a spray and that we know, apart from the energy function E, $n-1$ further (homogeneous) quadratic first integrals

$$F_\gamma = \frac{1}{2} a_{\gamma\, ij} \dot{x}^i \dot{x}^j, \qquad \gamma = 1, \ldots, n-1. \tag{13}$$

These are assumed to be linearly independent and all symmetric matrices involved are simultaneously diagonalizable in coordinates. More explicitly, the further assumptions in the contravariant version of Eisenhart's theorem (which is mentioned e.g. in [23, 1]) are that the roots of the $n-1$ eigenvalue problems $\det(a_\gamma^{ij} - \lambda_\gamma g^{ij}) = 0$ are simple and that there exist n common orthogonal closed eigenforms $\alpha^{(k)}$:

$$(a_\gamma^{ij} - \lambda_\gamma^{(k)} g^{ij})\, \alpha_j^{(k)} = 0, \qquad d\alpha^{(k)} = 0. \tag{14}$$

Separation variables y^k then follow from the local exactness of these eigenforms: $\alpha^{(k)} = dy^k$. We can rewrite these conditions equivalently with type (1,1) tensor fields while keeping the same 1-forms $\alpha^{(k)}$. That is to say, multiplying the above relations with g_{mi}, we obtain n equivalent conditions for each $\gamma = 1, \ldots, n-1$ and each $k = 1, \ldots, n$; but since all functions involved are basic, multiplying further by $\dot{x}^m$, these n conditions are still equivalent to the single condition, linear in the velocities:

$$\dot{x}^m a_{\gamma\, ml} g^{lj} \alpha_j^{(k)} = \lambda_\gamma^{(k)} \dot{x}^m \alpha_m^{(k)} \,. \tag{15}$$

As in [21] we recognise the symmetry generator in this expression. To be precise, in view of the symmetry of the matrices a_γ and g the left-hand side, up to a sign, contains the leading components ξ^j of the Noether symmetry Y corresponding to F_γ (cf. equation (9)). So we introduce (for each γ)

$$X_\gamma = -\xi_\gamma^j \frac{\partial}{\partial x^j}, \tag{16}$$

and recall that this is a well defined object, namely a vector field along the projection $\tau : TM \to M$, and that there are intrinsic operations by which the symmetry generator $Y \in \mathcal{X}(TM)$ can be constructed from this $X \in \mathcal{X}(\tau)$. We further recall that there is a canonical element in $\mathcal{X}(\tau)$, namely

$$\mathbf{T} = \dot{x}^i \frac{\partial}{\partial x^i}. \tag{17}$$

The 1-forms $\alpha^{(k)}$, being basic forms, can also be regarded as elements of $\bigwedge^1(\tau)$, i.e. as 1-forms along τ, and thus can be paired with elements of $\mathcal{X}(\tau)$. This way, the relations (15) acquire the simple form:

$$\langle X_\gamma, \alpha^{(k)} \rangle = \lambda_\gamma^{(k)} \langle \mathbf{T}, \alpha^{(k)} \rangle, \qquad k = 1, \ldots, n, \qquad \gamma = 1, \ldots, n-1. \tag{18}$$

It follows that

$$(\alpha^{(1)} \wedge \cdots \wedge \alpha^{(n)})(\mathbf{T}, X_1, \ldots, X_{n-1}) = \rho \, \langle \mathbf{T}, \alpha^{(1)} \rangle \cdots \langle \mathbf{T}, \alpha^{(n)} \rangle, \tag{19}$$

where ρ is the determinant with 1's in the first row and the eigenvalues $\lambda_\gamma^{(k)}$ in the rows 2 to n, and is non-zero in view of the linear independence of the integrals E, F_γ. We thus obtain, always as a corollary of Eisenhart's theorem, the following generalization (to arbitrary degrees of freedom n and arbitrary metrics g) of a procedure discussed in [21] by which, in principle, the separation variables y^k could be obtained from a computation on the symmetry generators: the left-hand side of (19) is a polynomial of degree n in the velocities, which is, up to a factor, the volume form $dx^1 \wedge \cdots \wedge dx^n$ acting on $(\mathbf{T}, X_1, \ldots, X_{n-1})$, and the right-hand side of (19) says that this polynomial can be factorized into the product of linear functions in the velocities which are total time derivatives of the separation variables.

What we wish to do now is to pass also for these considerations to the dual picture of adjoint symmetries and to show that one can express the result this way in an even more direct and transparant form.

In agreement with the discussion at the beginning of the previous section, the leading part of an adjoint symmetry β of a SODE Γ (whether regarded as element of $\mathcal{M}_\Gamma^*$ or as the corresponding α in $\mathcal{X}_\Gamma^*$) is the part $b_i d\dot{x}^i$. A way of singling out this part of a 1-form β on TM is in fact to act on it with the tensor field S, thus producing the semi-basic form (or 1-form along τ) $b_i dx^i$. It is in this more economical representation that adjoint symmetries can be discussed also within the calculus along τ (see [13]). If we are talking about an adjoint symmetry coming from a first integral F, then the b_i are of the form $b_i = \partial F / \partial \dot{x}^i$. The corresponding element of $\bigwedge^1(\tau)$ then is

$$\alpha = d^V F = \frac{\partial F}{\partial \dot{x}^i} \, dx^i. \tag{20}$$

We have hereby identified the canonically defined vertical exterior derivative d^V on $\bigwedge(\tau)$, at least for its action on functions on TM (its definition is completed by adding that d^V is a derivation of degree 1 and that $d^V dx^i = 0$). For the time being, however, there is even no need to use this notation, as $d^V F$ is also the Poincaré-Cartan 1-form associated to F and hence we can write θ_F instead. But we will continue to regard it now as a 1-form along τ and in that sense, it is an adjoint symmetry of Γ as soon as F is a first integral.

Consider now again property (18), and transfer in the left-hand side a g^{lj}-factor from one side of the pairing to the other, thereby defining the vector fields $\alpha^{(k)\sharp} \in \mathcal{X}(\tau)$ with components: $\alpha^{(k)\sharp^l} = g^{lj} \alpha_j^{(k)}$. Then we have

$$\langle X_\gamma, \alpha^{(k)} \rangle = \langle \alpha^{(k)\sharp}, \theta_{F_\gamma} \rangle, \tag{21}$$

and likewise (we write θ_E instead of θ_L although of course E and L are the same here)

$$\langle \mathbf{T}, \alpha^{(k)} \rangle = \langle \alpha^{(k)\sharp}, \theta_E \rangle. \tag{22}$$

As a result, the left-hand side of (19) can be rewritten as:

$$(\theta_E \wedge \theta_{F_1} \wedge \cdots \wedge \theta_{F_{n-1}})(\alpha^{(1)\sharp}, \ldots, \alpha^{(n)\sharp}).$$

All the velocity dependence this way is shifted to the volume form itself, so that it is the function appearing there which will have the factorization property.

Proposition 3. *Consider a system with Lagrangian (7) and assume that $n-1$ additional quadratic integrals have been found. Then, if we are in a situation where orthogonal separation variables exist, they can be found by taking only the homogeneous quadratic parts $E, F_1, \ldots, F_{n-1}$ of all integrals and factorizing the single component of the volume form $\theta_E \wedge \theta_{F_1} \wedge \cdots \wedge \theta_{F_{n-1}}$ into n factors which are linear in the velocities and integrable.* □

Remark: the single component of the volume form in question is of course the determinant of the matrix $\partial F_\gamma / \partial \dot{x}^i$, with $\gamma = 0, \ldots, n-1$ and $F_0 = E$.

4 Illustrative examples

We content ourselves in this section to giving a number of simple illustrations of the factorization ensured by Proposition 3. We leave the selection of suitable potentials for separability out of the discussion. For better legibility, we shall label coordinates here with lower indices.

Consider first a Lagrangian with the following kinetic energy term:

$$L = E = \frac{1}{2}\left(\dot{x}_1^2 + x_1^2 \dot{x}_2^2\right).$$

A second quadratic integral is given by

$$F_1 = x_1^2 (x_1 \dot{x}_2^2 \cos x_2 + \dot{x}_1 \dot{x}_2 \sin x_2).$$

We have

$$\theta_E \wedge \theta_{F_1} = x_1^2 \Big(2 x_1 \dot{x}_1 \dot{x}_2 \cos x_2 + (\dot{x}_1^2 - x_1^2 \dot{x}_2^2) \sin x_2 \Big)\, dx_1 \wedge dx_2 .$$

It is easy to see that, up to a factor, the component of this volume form is the product of the linear expressions

$$(1 - \cos x_2)\dot{x}_1 + x_1 \dot{x}_2 \sin x_2 \quad \text{and} \quad (1 + \cos x_2)\dot{x}_1 - x_1 \dot{x}_2 \sin x_2 ,$$

which are total time derivatives and thus provide the separation variables

$$y_1 = x_1 (1 - \cos x_2), \qquad y_2 = x_1 (1 + \cos x_2).$$

Another quadratic integral for the same metric Lagrangian could be taken to be

$$F_2 = x_1^2 (x_1 \dot{x}_2^2 \sin x_2 - \dot{x}_1 \dot{x}_2 \cos x_2).$$

It would lead by the same procedure to the separation variables

$$y_1 = x_1 (1 - \sin x_2), \qquad y_2 = x_1 (1 + \sin x_2).$$

These are of course well-known results: the Lagrangian we took can be thought of as the kinetic energy part of the Kepler problem in polar coordinates, and the two integrals F_1 and F_2 then are the quadratic parts of the Runge-Lenz vector.

As a second illustration, take L to have a constant (but non-Euclidian) metric:

$$L = \frac{1}{2}\left(\dot{x}_1^2 - \dot{x}_2^2\right) ,$$

and consider the additional quadratic integral

$$F = x_2\dot{x}_1\dot{x}_2 - x_1\dot{x}_2^2 .$$

We find

$$\theta_E \wedge \theta_F = \Big(x_2(\dot{x}_1^2 + \dot{x}_2^2) - 2x_1\dot{x}_1\dot{x}_2\Big) dx_1 \wedge dx_2 .$$

In domains where $x_1^2 - x_2^2 > 0$, putting $r = \sqrt{x_1^2 - x_2^2}$, a factorization is given by

$$\Big((r + x_1)\dot{x}_1 - x_2\dot{x}_2\Big)\Big(x_2\dot{x}_2 + (r - x_1)\dot{x}_1\Big),$$

and suitable integrating factors can be found which lead to the following separation variables

$$y_1 = \sqrt{x_1 + r}, \qquad y_2 = \sqrt{x_1 - r} .$$

Next, consider the Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \sin^2 x_1\, \dot{x}_2^2) ,$$

for which one can verify that the following functions are first integrals:

$$\begin{aligned} F_1 &= \sin^3 x_1\, \cos x_1\, \cos x_2\, \dot{x}_2^2 + \sin^2 x_1\, \sin x_2\, \dot{x}_1\dot{x}_2 \\ F_2 &= \sin^3 x_1\, \cos x_1\, \sin x_2\, \dot{x}_2^2 - \sin^2 x_1 \cos x_2\, \dot{x}_1\dot{x}_2 \ . \end{aligned}$$

For F_1 as additional integral, our volume form becomes

$$\begin{aligned} \theta_E \wedge \theta_{F_1} &= \sin^2 x_1(\sin x_2\, \dot{x}_1^2 + \cos x_2\, \sin 2x_1\, \dot{x}_1\dot{x}_2 \\ &\quad - \sin^2 x_1\, \sin x_2\, \dot{x}_2^2)\, dx_1 \wedge dx_2. \end{aligned}$$

Its component can be seen to factorize as the product of the linear functions:

$$\begin{aligned} &\left(\sqrt{1 - \cos^2 x_2\, \sin^2 x_1} + \cos x_1\, \cos x_2\right)\dot{x}_1 - \sin x_1\, \sin x_2\, \dot{x}_2 , \\ &\left(\sqrt{1 - \cos^2 x_2\, \sin^2 x_1} - \cos x_1\, \cos x_2\right)\dot{x}_1 + \sin x_1\, \sin x_2\, \dot{x}_2 . \end{aligned}$$

Both of these functions become total time derivatives if one divides by the square root they contain. One thus identifies the separation variables by the transformation formulas (in domains where they apply):

$$\begin{aligned} y_1 &= x_1 + \arcsin(\sin x_1\, \cos x_2) , \\ y_2 &= x_1 - \arcsin(\sin x_1\, \cos x_2) . \end{aligned}$$

For the case of F_2 as second integral, the calculations are completely similar.

Let us finally put the theory to a test on an example with n degrees of freedom for which we know what should come out from the start. Consider a so-called system of Liouville type, as described for example in [16]. We have (the summation convention cannot be used here),

$$L = E = \frac{1}{2} c \sum_{j=1}^{n} \frac{\dot{x}_j^2}{a_j(x_j)}, \qquad c(x) = \sum_{j=1}^{n} c_j(x_j)\,,$$

where the functions a_j and c_j depend on x_j only. Quadratic first integrals are

$$F_j = \frac{1}{2} \frac{c^2\, \dot{x}_j^2}{a_j} - c_j\, E\,, \qquad j = 1, \ldots, n.$$

We have $\sum_j F_j = 0$ but, for example, E and $F_2, \ldots, F_n$ can be chosen as linearly independent integrals. Writing $F_j = G_j - c_j\, E$, for shorthand, we have $\theta_{F_j} = \theta_{G_j} - c_j\, \theta_E$, so that the volume form reduces to

$$\theta_E \wedge \theta_{G_2} \wedge \cdots \wedge \theta_{G_n} = \theta_E \wedge \left(\frac{c^2 \dot{x}_2}{a_2} \right) dx_2 \wedge \cdots \wedge \left(\frac{c^2 \dot{x}_n}{a_n} \right) dx_n\,.$$

It is obvious then that only the term in dx_1 in θ_E survives, so that the result is

$$\frac{c^{2n-1}}{\prod_{j=1}^{n} a_j}\, \dot{x}_1 \cdots \dot{x}_n\, dx_1 \wedge \cdots \wedge dx_n\,.$$

This simple factorization of course does not come as a surprise: it tells us that the x_i are already separation variables, as expected. A completely similar result would be obtained if we took, somewhat more generally, Stäckel systems (see e.g. [16]) as the starting point.

5 Outlook for further study

Vector fields and 1-forms along τ have popped up in our analysis in a natural way, but we have avoided so far to refer too much to the related calculus developed in [12, 13]. We shall now embark into this area a bit more deeply. To some extent, showing how the same results could have been obtained by using that calculus at this stage, merely means rewriting the same formulas in another way. But it seems to us that the alternative formulation of Eisenhart's theorem we will arrive at, may open up an interesting avenue for future study.

The fundamental property following from Eisenhart's theorem, which leads to the factorization of a volume form, is (18) and was rewritten in the dual form using (21) and (22). Let us write X_k for the vector field $\alpha^{(k)\sharp}$ along τ (which for the time being can be seen as just a simplification of notations), and now use the notation $d^{\mathrm{V}}F$ for θ_F. Then this dual form of (18) reads

$$d^{\mathrm{V}}F_\gamma\,(X_k) = \lambda_\gamma^{(k)} d^{\mathrm{V}}E(X_k), \qquad k = 1, \ldots, n, \qquad \gamma = 1, \ldots, n-1.$$

or

$$\mathrm{D}^{V}_{X_k} F_\gamma = \lambda^{(k)}_\gamma \, \mathrm{D}^{V}_{X_k} E. \tag{23}$$

Here, D^V_X is a degree zero derivation, the *vertical covariant derivative* with respect to $X = X^i \partial/\partial x^i$, whose action on functions F is simply given by

$$\mathrm{D}^V_X F = X^i \frac{\partial F}{\partial \dot{x}^i}. \tag{24}$$

The volume form becomes

$$d^V E \wedge d^V F_1 \wedge \cdots \wedge d^V F_{n-1} = d^V (E \, d^V F_1 \wedge \cdots \wedge d^V F_{n-1}). \tag{25}$$

One can easily show that for any $X_i \in \mathcal{X}(\tau)$ (not necessarily basic vector fields):

$$(d^V E \wedge d^V F_1 \wedge \cdots \wedge d^V F_{n-1})(X_1, \ldots, X_n) = \begin{vmatrix} \mathrm{D}^V_{X_1} E & \cdots & \mathrm{D}^V_{X_n} E \\ \vdots & & \vdots \\ \mathrm{D}^V_{X_1} F_{n-1} & \cdots & \mathrm{D}^V_{X_n} F_{n-1} \end{vmatrix}. \tag{26}$$

If then the X_i are vector fields for which the relations (23) hold, we immediately find the factorization property again

$$(d^V E \wedge d^V F_1 \wedge \cdots \wedge d^V F_{n-1})(X_1, \ldots, X_n) = \rho \, \mathrm{D}^V_{X_1} E \cdots \mathrm{D}^V_{X_n} E, \tag{27}$$

where ρ is the same determinant as in (19).

Now, remember that relations like (23) contain a large part of the information in Eisenhart's theorem and note in fact that by dualizing the formulation (18) we used first via (21-22), we have returned from the contravariant tensors in (14) to covariant tensors. To be specific, the coordinate expressions for (23), with the F_γ given by (13) and writing the components of the vector X_k as X^j_k, read

$$\dot{x}^i a_{\gamma\, ij} X^j_k = \lambda^{(k)}_\gamma \dot{x}^i g_{ij} X^j_k \, . \tag{28}$$

Identifying the coefficients of each $\dot{x}^i$, they just say that the X_k are common eigenvectors of all matrices a_γ. A somewhat tricky remark is in order here. The (covariant) version of Eisenhart's theorem which can be found in [8] states that there exist such common eigenvectors which are in fact the coordinate vector fields $\partial/\partial y^k$ in the separation variables. We used the other version (14) first because we needed the closed 1-forms $\alpha^{(k)}$ in our arguments, and the point now is that the X_k we thus obtained as $\alpha^{(k)\sharp}$ then cannot be the coordinate vector fields, in general. This is not a contradiction because the theorem actually ensures that the system will have the Stäckel form in the new variables, with diagonal (g_{ij}). As a result, one can verify that the eigenvectors $\alpha^{(k)\sharp}$ will just be multiples of the coordinate eigenvectors X_k. This being said, it is obvious that in the present formulation (23) or (28), we rather work with X_k which will turn out to be coordinate vector fields in the good variables, because this can simply be expressed by requiring that they commute. We finally observe that the other assumptions of Eisenhart's theorem can also, just as is the case with (23), be written as conditions on the F_γ. Indeed, we want the F_γ to be homogeneous quadratic in the velocities, which by Euler's theorem can be expressed as

$\mathrm{D}^{V}_{\mathbf{T}} F_\gamma = 2F_\gamma$, and they have to be first integrals, meaning that $\Gamma(F) \equiv \nabla(F) = 0$. (The degree zero derivation ∇ is the *dynamical covariant derivative* of the calculus along τ, but coincides with the vector field Γ for its action on functions.)

We thus come to the conclusion that Eisenhart's theorem, as copied in [21] from [8], can be rewritten in the following form.

Proposition 4 (Eisenhart's theorem). *The necessary and sufficient conditions that a geodesic Lagrangian* $L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j = E$ *can be given the Stäckel form is that there exist* $n-1$ *functions* F_γ *such that:*

1. $\nabla F_\gamma = 0\,,$
2. $\mathrm{D}^{V}_{\mathbf{T}} F_\gamma = 2F_\gamma\,,$
3. *there exist* n *commuting basic vector fields* X_k*, such that for some functions* $\lambda^{(k)}_\gamma$ *which are all different for each fixed* γ*, we have* $\mathrm{D}^{V}_{X_k} F_\gamma = \lambda^{(k)}_\gamma \, \mathrm{D}^{V}_{X_k} E$*,*
4. $\begin{vmatrix} \mathrm{D}^{V}_{X_1} E & \cdots & \mathrm{D}^{V}_{X_n} E \\ \vdots & & \vdots \\ \mathrm{D}^{V}_{X_1} F_{n-1} & \cdots & \mathrm{D}^{V}_{X_n} F_{n-1} \end{vmatrix} \neq 0\,.$

PROOF: That the first three conditions are just transcripts of conditions in [8] was explained above. The last one, knowing already that the F_γ are quadratic, is just a way of saying that all quadratic integrals involved are linearly independent. □

Needless to say, we have made a point in this formulation of Eisenhart's theorem of expressing all conditions as differential conditions on the first integrals. The motivation for doing so is the following. Much of the old work on separability of the Hamilton-Jacobi equation is about conditions for checking whether a system is separable in the given coordinates. Examples in this respect are the Levi Civita conditions and Stäckel's theorem. Eisenhart's theorem was perhaps the first result which gives a sort of test for checking whether separation variables exist, although it is to a large extent an existence theorem, i.e. the test is not of the kind that could be applied directly on the given data. Important generalizations were obtained by Woodhouse [23] and, specifically for non-orthogonal separability by Benenti (see [2] and references therein). What these have in common is that a transition to separation variables, if they exist, will always be a point transformation. A couple of examples are known (see e.g. [3, 22]) of Hamiltonian systems with additional integrals of degree higher than two, for which a non-point canonical transformation exists such that the Hamilton-Jacobi equation of the transformed Hamiltonian is separable. To the best of our knowledge, no intrinsic characterization exists, for example of the kind of Eisenhart's theorem, of existence of such non-point transformations to separation variables. The idea is that our above reformulation of Eisenhart's theorem might lead to generalizations when not all additional integrals are quadratic (a generalization of condition 2). One might hope that an appropriate generalization of condition 3 could be found, which then presumably would involve vector fields $X_k \in \mathcal{X}(\tau)$ which are not basic (and also non-basic functions $\lambda^{(k)}_\gamma$). (Note that we have mentioned at least one result in what preceeds, which is valid for more than just basic vector fields.) The overall idea of the new approach would be to base the analysis on a study of the integrability conditions for formal integrability of the pde's on the F_γ. Admittedly, these ideas are rather speculative. But it seems to us that it would already be worthwhile to try to arrive at a new, independent proof of Eisenhart's theorem along such lines. An integrability analysis of the conditions in Proposition 4 might well lead to much more practical criteria for testing the existence of separation variables, i.e. conditions expressed directly in terms of the given data. The calculus for doing such an analysis is available and starts from commutator properties of the derivations involved. We may refer in this respect to a somewhat similar study on complete decoupling of systems of second-order equations which was carried out in [14] and did indeed give rise to fairly practical test criteria. Also, although this refers to an entirely different subject, an integrability study of equations expressed with the geometric derivations of the calculus along τ has successfully been carried out in the context of the inverse problem of the calculus of variations [6, 19].

Acknowledgements. W.S. thanks the Fund for Scientific Research, Flanders, Belgium, for continuing support. A.R. thanks the Spanish Ministerio de Educación y Cultura for a FPI grant (research project PB96-0717), which made a stay at the University of Ghent possible where most of this work was carried out.

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