Lagrangian equations on affine Lie algebroids

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Abstract. We recall the concept of a Lie algebroid on a vector bundle and the associated notion of Lagrange-type equations. A heuristic calculus of variations approach tells us what a time-dependent generalization of such equations should look like. In order to find a geometrical model for such a generalization, the idea of a Lie algebroid structure on a class of affine bundles is introduced. We develop a calculus of forms on sections of such a bundle by looking at its extended dual. It is sketched how the affine Lie algebroid axioms are equivalent to the coboundary property of the exterior derivative in such a calculus. The interest of the new formalism is further illustrated by the fact that one can define a notion of prolongation of the original algebroid. We briefly discuss how this prolongation will provide the key to various geometrical constructions which are the analogues of the well-known geometrical aspects of second-order ordinary differential equations in general, and Lagrangian dynamics in particular.

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1 Lie algebroid structure on a vector bundle and Lagrangian equations

Consider a vector bundle $\pi : V \to M$, which comes equipped with the following tools:

- a bracket operation $[\,\cdot\,,\,\cdot\,] : \; Sec\,\pi \times Sec\,\pi \to Sec\,\pi$ which satisfies the axioms of a real Lie algebra;

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- a linear bundle map $\rho : V \to TM$, called *anchor map*, which induces a map from $Sec\,\pi$ to $\mathcal{X}(M)$ (also denoted by ρ) and is such that the following *compatibility conditions* hold true $\forall \sigma, \eta \in Sec\,\pi,\ f \in C^\infty(M)$,

$$[\sigma, f\,\eta] = f[\sigma, \eta] + \rho(\sigma)(f)\,\eta, \tag{1}$$

$$[\rho(\sigma), \rho(\eta)] = \rho([\sigma, \eta]). \tag{2}$$

Under such circumstances, we say that the bundle π is equipped with a *Lie algebroid structure*, or we simply say that $\pi : V \to M$ is a Lie algebroid.

Note that the condition (2) can actually be derived from (1) and the Jacobi identity satisfied by the bracket on $Sec\,\pi$. Nevertheless, we will repeatedly refer to this property and actually take it as a point of reference for the comparison between the known Lie algebroid concept and the new one we will develop. For a general reference to the subject of Lie algebroids, see [3].

In coordinates, if x^i are coordinates on M and y^α fibre coordinates on V with respect to some local basis $\{e_\alpha\}$ for $Sec\,\pi$, we can put

$$\rho(e_\alpha) = \rho^i_\alpha(x)\frac{\partial}{\partial x^i}\,, \qquad [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}(x)\,e_\gamma\,,$$

and then the condition (2) requires that $[\rho(e_\alpha), \rho(e_\beta)] = \rho([e_\alpha, e_\beta])$, which in turn is equivalent to:

$$\rho^i_\alpha \frac{\partial \rho^j_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^j_\alpha}{\partial x^i} = \rho^j_\gamma C^\gamma_{\alpha\beta}. \tag{3}$$

The Jacobi identity for the bracket on $Sec\,\pi$, making use also of the compatibility condition (1), has the coordinate representation:

$$\sum_{\alpha,\beta,\gamma} \left(\rho^i_\alpha \frac{\partial C^\mu_{\beta\gamma}}{\partial x^i} + C^\mu_{\alpha\nu} C^\nu_{\beta\gamma} \right) = 0, \tag{4}$$

where the summation sign stands for cyclic sums over the indicated indices.

Now, Lagrangian equations on a Lie algebroid are differential equations of the form (cf. [5])

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha(x)\,y^\alpha \\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) &= \rho^i_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}\,, \end{aligned} \tag{5}$$

with $L \in C^\infty(V)$. The question we want to address in this paper is: what would be an appropriate *time-dependent version* of such equations? We shall first try to discover by analytical considerations how such generalized equations should look like and subsequently explore what the geometrical framework is for modelling them.

2 'Rudimentary' calculus of variations

As a preliminary remark, note that it is easy to convince oneself that for a time-dependent set-up, if one wants to arrive at equations which preserve their structure under time-dependent coordinate transformations, the 'constraint equations' should be of the form:

$$\dot{x}^i = \rho^i_\alpha(t, x) y^\alpha + \lambda^i(t, x). \tag{6}$$

So, let us consider the following calculus of variations problem for curves $t \mapsto (t, x^i(t), y^\alpha(t))$ in $\mathbb{R}^{n+k+1}$ say. Assume we have a given functional

$$\mathcal{J}(\gamma) = \int_{t_1}^{t_2} L(t, x(t), y(t))dt,$$

and want to find its extremals, within arbitrary one-parameter families of curves which satisfy the constraints (6). We will proceed in a very formal way, without worrying too much about the mathematical complications which come from constraints depending on velocities. Formally, taking variations of the constraint equations, we get:

$$\delta\dot{x}^i = \left(\frac{\partial\rho^i_\alpha}{\partial x^j}y^\alpha + \frac{\partial\lambda^i}{\partial x^j}\right)\delta x^j + \rho^i_\alpha \delta y^\alpha.$$

Multiplying these by Lagrange multipliers p_i and adding the result to the variation of the functional, one obtains (after an integration by parts on the term $p_i\delta\dot{x}^i$)

$$\int_{t_1}^{t_2}\left[\left(\frac{\partial L}{\partial x^j} - \dot{p}_j - p_i\Big(\frac{\partial\rho^i_\alpha}{\partial x^j}y^\alpha + \frac{\partial\lambda^i}{\partial x^j}\Big)\right)\delta x^j \left(\frac{\partial L}{\partial y^\alpha} - p_i\rho^i_\alpha\right)\delta y^\alpha\right]dt = 0$$

It is tacitly assumed that all variations δx^i and δy^α vanish at the endpoints, thereby skipping over the mathematical complications which come from the differential equations they have to satisfy. The traditional argument then is that one can choose the multipliers p_i in such a way that the coefficients of δx^j vanish, leaving only terms in δy^α, which are arbitrary, so that those coefficients must vanish in view of the fundamental lemma of the calculus of variations. We thus get the equations

$$\begin{aligned} \dot{p}_j &= \frac{\partial L}{\partial x^j} - p_i\Big(\frac{\partial\rho^i_\alpha}{\partial x^j}y^\alpha + \frac{\partial\lambda^i}{\partial x^j}\Big), \\ \frac{\partial L}{\partial y^\alpha} &= p_j\rho^j_\alpha. \end{aligned}$$

The next step one would like to take is to eliminate the p_i. Taking the total time derivative of the second equations and using the first to substitute for $\dot{p}_i$, one is left with a number of terms containing p_j, which will go away only if they combine in such a way that they pick up a factor ρ^j_α. Therefore, an interesting situation is the case that there exist functions $C^\gamma_{\alpha\beta}(t,x)$ and $C^\alpha_\beta(t,x)$ such that:

$$\begin{aligned} \rho^i_\alpha\frac{\partial\rho^j_\beta}{\partial x^i} - \rho^i_\beta\frac{\partial\rho^j_\alpha}{\partial x^i} &= \rho^j_\gamma C^\gamma_{\alpha\beta}, \\ \frac{\partial\rho^j_\beta}{\partial t} + \lambda^i\frac{\partial\rho^j_\beta}{\partial x^i} - \rho^i_\beta\frac{\partial\lambda^j}{\partial x^i} &= \rho^j_\alpha C^\alpha_\beta. \end{aligned} \tag{7}$$

These relations clearly generalize the conditions (3). The equations which result from the elimination then are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha\frac{\partial L}{\partial x^i} - (C^\gamma_{\alpha\beta}y^\beta - C^\gamma_\alpha)\frac{\partial L}{\partial y^\gamma}\,, \tag{8}$$

and they of course have to be supplemented by the constraints (6).

The point of this formal exercise is the following: if one carries out the same procedure in an autonomous framework, one arrives exactly at the equations (5), with functions $\rho^i_\alpha(x)$ and $C^\gamma_{\alpha\beta}(x)$ satisfying the relations (3); therefore, we can feel confident that the more general equations we derived in this section are indeed the ones we are looking for. Our programme thus becomes: identify now an appropriate geometrical framework for generalization of the classical notion of a Lie algebroid, which gives rise to compatibility conditions of the type we have just encountered, and within which time-dependent Lagrange equations of the form (8) can be accomodated.

Note in passing that we did not encounter Jacobi-type conditions in our formal analysis, which means that it may even make sense to relax the axioms of a Lie algebroid if the purpose merely would be to describe differential equations of the form (8), but this is a path we do not wish to explore at present. The solution to our programme will be to introduce the notion of Lie algebroid structure on affine bundles which are 'anchor mapped' into first jet bundles, rather than tangent bundles. For that purpose, of course, one cannot take just any affine bundle $E \to M$; we have to assume further that the base manifold M is fibred over $\mathbb{R}$.

3 Affine Lie algebroids

Consider thus an affine bundle $\pi : E \to M$, modelled on a vector bundle $\overline{\pi} : V \to M$. Assume further we have a fibration $\tau : M \to \mathbb{R}$ with associated jet bundle $\tau_1^0 : J^1M \to M$. In what follows, we shall distinguish sections of the affine bundle from sections of the underlying vector bundle by using boldface type for the latter.

The requirements for having a Lie algebroid structure on the bundle π are the following:

1. there exists a skew-symmetric and $\mathbb{R}$-bilinear bracket $[\,\cdot\,,\,\cdot\,]$ on $Sec\,\overline{\pi}$;
2. sections of π act on $Sec\,\overline{\pi}$ in such a way that, if we write $[\zeta, \boldsymbol{\sigma}] \in Sec\,\overline{\pi}$ for the action of $\zeta \in Sec\,\pi$ on $\boldsymbol{\sigma} \in Sec\,\overline{\pi}$, we have the properties
$$[\zeta, \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2] = [\zeta, \boldsymbol{\sigma}_1] + [\zeta, \boldsymbol{\sigma}_2], \qquad [\zeta + \boldsymbol{\sigma}, \boldsymbol{\eta}] = [\zeta, \boldsymbol{\eta}] + [\boldsymbol{\sigma}, \boldsymbol{\eta}], \tag{9}$$
$$[\zeta, [\boldsymbol{\sigma}, \boldsymbol{\eta}]] = [[\zeta, \boldsymbol{\sigma}], \boldsymbol{\eta}] + [\boldsymbol{\sigma}, [\zeta, \boldsymbol{\eta}]]; \tag{10}$$
3. there exists an affine bundle map $\lambda : E \to J^1M$, with corresponding vector bundle homomorphism $\rho : V \to VM$, such that $\forall f \in C^\infty(M)$
$$[\zeta, f\boldsymbol{\sigma}] = f\,[\zeta, \boldsymbol{\sigma}] + \lambda(\zeta)(f)\boldsymbol{\sigma}. \tag{11}$$

It follows from (9,10) that the bracket on $Sec\,\overline{\pi}$ satisfies the Jacobi identity, and it also follows from (10) and (11) that λ and ρ, which we both will call *anchor maps* have the compatibility property:
$$[\lambda(\zeta), \rho(\boldsymbol{\sigma})] = \rho([\zeta, \boldsymbol{\sigma}]). \tag{12}$$
One can now further extend the bracket operation to sections of π by putting $[\zeta_1, \zeta_2] = [\zeta_1, \zeta_2 - \zeta_1]$ and it then easily follows that we have
$$\sum_{i,j,k} [[\zeta_i, \zeta_j], \zeta_k] = \mathbf{0}, \tag{13}$$
$$[\lambda(\zeta_1), \lambda(\zeta_2)] = \rho([\zeta_1, \zeta_2]). \tag{14}$$

But observe that the bracket of two 'affine sections' is a 'vector section'! Needless to say, we regard J^1M and VM as subbundles of TM, so that the bracket on the left in both (12) and (14) is a bracket of vector fields on M.

Let us have a look at the coordinate representation of our basic conditions. Choosing a zero section $e_0 \in Sec\,\pi$ and a local basis $\{\mathbf{e}_\alpha\}$ of $Sec\,\overline{\pi}$, for $m \in M$, with coordinates (t, x^i), every $e \in E_m$ will be of the form $e = e_0(m) + y^\alpha\,\mathbf{e}_\alpha(m)$; (t, x^i, y^α) then are the coordinates of e. The anchor maps are determined by the functions λ^i and ρ^i_α, defined by

$$\lambda(e_0) = \frac{\partial}{\partial t} + \lambda^i(t,x)\frac{\partial}{\partial x^i} \qquad \rho(\mathbf{e}_\alpha) = \rho^i_\alpha(t,x)\frac{\partial}{\partial x^i}\,. \tag{15}$$

Putting

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = C^\gamma_{\alpha\beta}(t,x)\,\mathbf{e}_\gamma \qquad [e_0, \mathbf{e}_\alpha] = C^\beta_\alpha(t,x)\,\mathbf{e}_\beta, \tag{16}$$

the compatibility property (14) precisely translates into the conditions (7) we encountered in the previous section. The Jacobi-type properties further imply:

$$\frac{\partial C^\mu_{\alpha\beta}}{\partial t} + \lambda^i \frac{\partial C^\mu_{\alpha\beta}}{\partial x^i} + C^\gamma_{\alpha\beta}C^\mu_\gamma - C^\mu_{\alpha\gamma}C^\gamma_\beta + C^\mu_{\beta\gamma}C^\gamma_\alpha - \rho^i_\alpha\frac{\partial C^\mu_\beta}{\partial x^i} + \rho^i_\beta\frac{\partial C^\mu_\alpha}{\partial x^i} = 0\,, \tag{17}$$

and

$$\sum_{\alpha,\beta,\gamma}\left(\rho^i_\alpha\frac{\partial C^\mu_{\beta\gamma}}{\partial x^i} + C^\mu_{\alpha\nu}C^\nu_{\beta\gamma}\right) = 0\,, \tag{18}$$

which generalize (4).

We wish to provide now further evidence that this set-up of an affine Lie algebroid is of interest, by showing that it has enough of the properties which a standard Lie algebroid has, such as the availability of a coboundary operator and the existence of a Poisson structure on a suitable dual space.

4 Forms on $Sec\,\pi$

Consider $\pi^\dagger : E^\dagger \to M$, the *extended dual* of $\pi : E \to M$. Its total space is defined as the union of spaces $E^\dagger_m$ of affine functions on E_m and $\pi^\dagger$ is actually a vector bundle. For any $\theta \in Sec\,\pi^\dagger$ and $\zeta \in Sec\,\pi$, $\theta(\zeta)$ is a function on M and if $\zeta_0 \in Sec\,\pi$ is any reference section and $\zeta = \zeta_0 + \boldsymbol{\zeta}$, we have $\theta(\zeta) = \theta(\zeta_0) + \boldsymbol{\theta}(\boldsymbol{\zeta})$ for some $\boldsymbol{\theta} \in Sec\,\overline{\pi}^*$, where $\overline{\pi}^* : V^* \to M$ is the dual of $\overline{\pi}$. We wish to regard sections of $\pi^\dagger$ as 1-forms on $Sec\,\pi$. Observe, however, that there is no $C^\infty(M)$-linearity in the usual sense, but rather something like

$$\theta(\zeta_0 + f\boldsymbol{\zeta}) = \theta(\zeta_0) + f\,\boldsymbol{\theta}(\boldsymbol{\zeta}). \tag{19}$$

What then could k-forms be if we want to think of them as skew-symmetric maps on $Sec\,\pi$, which are 'multilinear' in some sense?

Definition. A k-form $\omega \in \bigwedge^k(\pi^\dagger)$ is a map $\omega : Sec\,\pi \times \cdots \times Sec\,\pi \to C^\infty(M)$ (k arguments) for which there exist associated maps $\omega_0, \boldsymbol{\omega}$, where $\omega_0 : Sec\,\pi \times Sec\,\overline{\pi} \times \cdots \times Sec\,\overline{\pi} \to C^\infty(M)$ is a skew-symmetric and $C^\infty(M)$-multilinear map in its $k-1$ vector arguments, and $\boldsymbol{\omega}$ is a (standard) k-form on $Sec\,\overline{\pi}$, such that the following properties hold:

$$\omega_0(\zeta + \boldsymbol{\sigma}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{k-1}) = \omega_0(\zeta, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{k-1}) + \boldsymbol{\omega}(\boldsymbol{\sigma}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{k-1})\,, \tag{20}$$

and with respect to any reference section ζ_0 of $Sec\,\pi$,

$$\omega(\zeta_1,\ldots,\zeta_k) = \sum_{i=1}^{k}(-1)^{i-1}\omega_0(\zeta_0,\boldsymbol{\zeta}_1,\ldots,\hat{\boldsymbol{\zeta}}_i,\ldots,\boldsymbol{\zeta}_k) + \boldsymbol{\omega}(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_k)\,. \tag{21}$$

Note that one needs a reference section to compute $\omega(\zeta_1,\ldots,\zeta_k)$, but the important point of course is that ω_0 and $\boldsymbol{\omega}$ do not depend on the choice of ζ_0! One can easily verify that this definition precisely covers the kind of construction one obtains by wedging 1-forms in the sense of (19).

For writing down coordinate expressions of forms, observe first that there exists a global section of $\pi^\dagger$, namely $e^0 : m \mapsto 1 \in E_m^\dagger$. Then, having chosen a zero section e_0 of π and a local basis $\{\mathbf{e}_\alpha\}$ for $Sec\,\overline{\pi}$, one can consider the dual basis $\{\mathbf{e}^\beta\}$ of $Sec\,\overline{\pi}^*$ and extend its action (keeping the notation unchanged) to $Sec\,\pi$ by $\mathbf{e}^\beta(\zeta) = \mathbf{e}^\beta(e_0 + \zeta^\alpha\,\mathbf{e}_\alpha) = \zeta^\beta$. It follows that every $\omega \in \bigwedge^k(\pi^\dagger)$ is of the form

$$\omega = \tfrac{1}{(k-1)!}\,\omega_{0\mu_1\cdots\mu_{k-1}}\,e^0 \wedge \mathbf{e}^{\mu_1} \wedge \cdots \wedge \mathbf{e}^{\mu_{k-1}} + \tfrac{1}{k!}\,\omega_{\mu_1\cdots\mu_k}\,\mathbf{e}^{\mu_1} \wedge \cdots \wedge \mathbf{e}^{\mu_k}, \tag{22}$$

with coefficients in $C^\infty(M)$ which are skew-symmetric in all indices.

The exterior derivative of forms on $Sec\,\pi$ can be defined by the kind of formula one expects, namely

$$\begin{aligned} d\omega(\zeta_1,\ldots,\zeta_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i-1}\lambda(\zeta_i)\Big(\omega(\zeta_1,\ldots,\hat{\zeta}_i,\ldots,\zeta_{k+1})\Big) \\ &+ \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([\zeta_i,\zeta_j],\zeta_1,\ldots,\hat{\zeta}_i,\ldots,\hat{\zeta}_j,\ldots,\zeta_{k+1})\,. \end{aligned} \tag{23}$$

There are, however, a number of remarks to be made. First of all, for (23) to make sense, we need to give a meaning also to the action of ω when the first argument is a vector section. This is done as follows

$$\omega(\boldsymbol{\sigma},\zeta_2,\ldots,\zeta_k) = \omega(\zeta_1+\boldsymbol{\sigma},\zeta_2,\ldots,\zeta_k) - \omega(\zeta_1,\zeta_2,\ldots,\zeta_k), \tag{24}$$

where ζ_1 is arbitrary and its choice does not affect the value of the left-hand side. Secondly, in view of our definition of forms, in order to show that $d\omega$ is a form, we have to identify a $(d\omega)_0$ and a $\mathbf{d}\boldsymbol{\omega}$ with all the right properties. This of course is a fairly technical matter, but everything works fine. In fact, roughly speaking, if we define an exterior derivative of ω_0 by sort of copying (23) with suitable changes to ensure that every term has a meaning (that is to say, replacing λ by ρ and ω by ω_0 or $\boldsymbol{\omega}$ where appropriate), one can prove that $(d\omega)_0 = d\omega_0$, whereas $\mathbf{d}\boldsymbol{\omega}$ is simply the familiar exterior derivative of the form $\boldsymbol{\omega}$ on $Sec\,\overline{\pi}$.

One further proves that d is a derivation of degree 1 on $\bigwedge(\pi^\dagger)$. As a result, for coordinate calculations, it is enough to know what d does on the basic ingredients in (22). For functions $f \in C^\infty(M)$, we have $df(\zeta) = \lambda(\zeta)(f)$, from which it follows in particular that

$$dt = e^0, \tag{25}$$

$$dx^i = \lambda^i e^0 + \rho^i_\alpha \mathbf{e}^\alpha. \tag{26}$$

One further finds that $de^0 = 0$ and

$$d\mathbf{e}^\alpha = -C^\alpha_\beta e^0 \wedge \mathbf{e}^\beta - \tfrac{1}{2} C^\alpha_{\beta\gamma} \mathbf{e}^\beta \wedge \mathbf{e}^\gamma. \tag{27}$$

The most important result, which we regard as further evidence that our set-up of a class of affine Lie algebroids is of interest, is the following.

Proposition: Assume that we have a bracket operation satisfying the three axioms of the definition of a Lie algebroid structure of the beginning of Section 3, with the exception, however, of condition (10). Then $d^2 = 0$ iff $\sum_{i,j,k} [\zeta_i, [\zeta_j, \zeta_k]] = \mathbf{0}$.

Let us repeat here that the property $[\lambda(\zeta_1), \lambda(\zeta_2)] = \rho([\zeta_1, \zeta_2])$ then is a further consequence of (11) and the Jacobi identity.

5 Admissible curves and pseudo-second-order dynamics

We now wish to bring some dynamics into the picture. A curve ψ in E, which is a section of $\tau \circ \pi$, is said to be λ-admissible if $\lambda \circ \psi = j^1(\pi \circ \psi)$.

$$\begin{tikzcd}
 & & J^1M \arrow[d] \\
E \arrow[urr, "\lambda"] \arrow[rr, "\pi"] \arrow[dr, shift left] & & M \arrow[dl, "\tau"] \\
 & \mathbb{R} \arrow[ul, shift left, "\psi"] &
\end{tikzcd}$$

In coordinates, $\psi : t \mapsto (t, x^i(t), y^\alpha(t))$ will be λ-admissible if for all t,

$$\dot{x}^i(t) = \rho^i_\alpha(t, x(t))\, y^\alpha(t) + \lambda^i(t, x(t)). \tag{28}$$

Note: if $\theta^i = dx^i - \dot{x}^i dt$ are the the contact forms on J^1M, and $\Theta^i = \lambda^*\theta^i$, we have that ψ is λ-admissible iff $\psi^*\Theta^i = 0$.

A vector field $\Gamma \in \mathcal{X}(E)$ is said to be a pseudo-second-order equation field (pseudo-SODE for short) if $T\pi \circ \Gamma = i \circ \lambda$, where $i : J^1M \hookrightarrow TM$ is the canonical injection.

$$\begin{tikzcd}
TE & TM \\
\cup & \cup \\
J^1E \arrow[r, "T\pi"] \arrow[d, shift left] & J^1M \arrow[d] \\
E \arrow[u, shift left, "\Gamma"] \arrow[ur, "\lambda"] \arrow[r, "\pi"] & M
\end{tikzcd}$$

In coordinates, such a Γ is of the form

$$\Gamma = \frac{\partial}{\partial t} + (\rho^i_\alpha(t,x) y^\alpha + \lambda^i(t,x)) \frac{\partial}{\partial x^i} + f^\alpha(t,x,y) \frac{\partial}{\partial y^\alpha} \tag{29}$$

and thus models the following type of differential equations:

$$\dot{x}^i = \rho^i_\alpha(t,x)\, y^\alpha + \lambda^i(t,x), \tag{30}$$
$$\dot{y}^\alpha = f^\alpha(t,x,y). \tag{31}$$

Another characterization of pseudo-SODEs is that all their integral curves are λ-admissible. A Lie algebra structure on $Sec\,\pi$ is of course not needed for these concepts.

Let us now look at pseudo-SODEs from a slightly different angle. The defining relation in fact states that Γ is a pseudo-SODE if and only if $\forall p \in E$, $(p, \Gamma(p))$ is a point of the pullback bundle $\lambda^* J^1 E$ which we henceforth denote as $J^1_\lambda E$, with projections as indicated in the diagram.

$$\begin{tikzcd}
J^1_\lambda E \arrow[r, "\lambda^1"] \arrow[d, "\pi_2"'] & J^1 E \arrow[d, "T\pi"] \\
E \arrow[r, "\lambda"'] & J^1 M
\end{tikzcd}$$

But if we project the point $\Gamma(p)$ in the image of λ^1 back to E via the projection $(\tau \circ \pi)^0_1$, we of course get back to the point p we started from. Putting $\pi_1 = (\tau \circ \pi)^0_1 \circ \lambda^1 : J^1_\lambda E \to E$, this gives rise to the following diagram in which we can regard the pseudo-SODE Γ just as well as a section of the bundle $\pi_1 : J^1_\lambda E \to E$, with the property that $\pi_2 \circ \Gamma = \pi_1 \circ \Gamma$.

$$\begin{tikzcd}
 & J^1 E \arrow[dr] \arrow[dd] & & \\
J^1_\lambda E \arrow[ur, "\lambda^1"] \arrow[rr, "\pi_1"] \arrow[dd, "\pi_2"] & & E \arrow[dr] \arrow[dd, "\pi"'] & \\
 & J^1 M \arrow[dr] & & \mathbb{R} \\
E \arrow[ur, "\lambda"] \arrow[rr, "\pi"'] & & M \arrow[ur, "\tau"'] &
\end{tikzcd}$$

The two triangles appearing in this diagram suggests the question: with more structure added by assuming that π carries an affine Lie algebroid structure, is it possible to prolong this to an affine Lie algebroid structure on π_1, with λ^1 in the role of anchor map? We briefly indicate how such a prolongation can indeed be constructed. A section Z of π_1 is completely determined by its projections $\pi_2 \circ Z : E \to E$ and $\lambda^1 \circ Z : E \to J^1 E$. If (t, x^i, y^α) are the coordinates of a point $e \in E$ (the copy of E on the right side in the diagram), and Z is a section of π_1, we will have:

$$\begin{aligned}
\pi_2 \circ Z : \qquad & (t, x, y) \longmapsto (t, x, z^\alpha(t, x, y)), \\
\lambda^1 \circ Z : \qquad & (t, x, y) \longmapsto \left(\frac{\partial}{\partial t} + (\lambda^i + \rho^i_\alpha z^\alpha) \frac{\partial}{\partial x^i} + Z^\alpha \frac{\partial}{\partial y^\alpha} \right) \Bigg|_e .
\end{aligned}$$

Z then can be locally represented as

$$Z = \mathcal{E}_0 + z^\alpha(t, x, y) \boldsymbol{\mathcal{X}}_\alpha + Z^\alpha(t, x, y) \boldsymbol{\mathcal{V}}_\alpha, \tag{32}$$

where $\mathcal{E}_0$ is a properly selected zero section of the affine bundle π_1 and $(\boldsymbol{\mathcal{X}}_\alpha, \boldsymbol{\mathcal{V}}_\alpha)$ is a local basis for the sections of the underlying vector bundle $\overline{\pi}_1$. To be precise, in the representation of sections by their two projections, we have:

$$\boldsymbol{\mathcal{X}}_\alpha(e) = \left(\mathbf{e}_\alpha(\pi(e)), \rho^i_\alpha(t,x)\frac{\partial}{\partial x^i}\bigg|_e\right) \qquad \boldsymbol{\mathcal{V}}_\alpha(e) = \left(\mathbf{0}(\pi(e)), \frac{\partial}{\partial y^\alpha}\bigg|_e\right), \tag{33}$$

and

$$\mathcal{E}_0(e) = \left(e_0(\pi(e)), \left(\frac{\partial}{\partial t} + \lambda^i(t,x)\frac{\partial}{\partial x^i}\right)\bigg|_e\right). \tag{34}$$

One can then consistently define a bracket on $Sec\,\overline{\pi}_1$ and an action of affine sections on vector sections, which satisfies all requirements of our definition of an affine Lie algebroid, and is locally determined by the following expressions:

$$\begin{array}{lll} [\mathcal{E}_0, \boldsymbol{\mathcal{X}}_\alpha]^1 = C^\beta_\alpha \boldsymbol{\mathcal{X}}_\beta, & [\mathcal{E}_0, \boldsymbol{\mathcal{V}}_\alpha]^1 = \mathbf{0}, & \\ [\boldsymbol{\mathcal{X}}_\alpha, \boldsymbol{\mathcal{X}}_\beta]^1 = C^\gamma_{\alpha\beta}\boldsymbol{\mathcal{X}}_\gamma, & [\boldsymbol{\mathcal{X}}_\alpha, \boldsymbol{\mathcal{V}}_\beta]^1 = \mathbf{0}, & [\boldsymbol{\mathcal{V}}_\alpha, \boldsymbol{\mathcal{V}}_\beta]^1 = \mathbf{0}. \end{array}$$

The fact that this all works, in perfect analogy with the situation on vector bundles, is a second extra evidence that we are looking at an interesting mathematical extension of Lie algebroids! To indicate why it is important to have this notion of prolongation, it suffices to mention the following results which have been derived by Martínez for the 'autonomous theory' (some of these results can be found in [4], others are still unpublished). The prolonged Lie algebroid is for the original quite similar to what the tangent bundle TM is for its base manifold M. By this we mean that among other things: (i) there exists a notion of *complete* and *vertical* lift from sections of the original bundle to sections of its prolongation; (ii) the prolonged bundle carries intrinsic objects similar to the Liouville vector field and the *vertical endomorphism* on a tangent bundle; (iii) (pseudo)-SODEs have intrinsic properties similar to those of SODEs on a tangent bundle, in particular they give rise to associated non-linear and linear connections; (iv) there exists an intrinsic geometrical construction of (pseudo)-SODEs of Lagrangian type, which makes use of these canonically defined concepts and of an analogue of Poincaré-Cartan type forms.

All of such properties will carry over to our affine generalization (with suitable adaptations)!

6 Discussion

We have put the emphasis on two features which we deem important for ensuring that the newly defined structure is the right one, but there are certainly more properties of interest which will emerge. For example, it is known that the dual of a vector bundle Lie algebroid carries a natural Poisson structure. The same is true for our affine Lie algebroid, where it is the extended dual $E^\dagger$ which carries a Poisson structure. As a matter of fact, every section σ of π can be regarded as a linear function on $E^\dagger$: if $\sigma = e_0 + \sigma^\alpha \mathbf{e}_\alpha$ and (t, x^i, p_0, p_α) are coordinates on $E^\dagger$, the corresponding function $\hat{\sigma}$ on $E^\dagger$ is given by $\hat{\sigma} = p_0 + p_\alpha\sigma^\alpha(t,x)$. We can define the bracket of two such functions as $\{\hat{\sigma}, \hat{\eta}\} = \widehat{[\sigma,\eta]}$, further put $\{\hat{\sigma}, f\} = \lambda(\sigma)(f)$ and impose the Leibniz rule for an extension to arbitrary functions.

Another remark which is worth mentioning here is that there are other ways of defining the affine Lie algebroid structure on $\pi : E \to M$ and developing a calculus of forms on sections of π. Roughly, a different approach can start from embedding the original affine bundle as an affine subspace in the dual of the extended dual $E^\dagger$, but a detailed exposition of these ideas will be discussed elsewhere.

Given the range of applications of Lie algebroid structures (see e.g. [1, 2]), there is no doubt that our affine generalization will be relevant for applications where an explicit time-dependence is required. We wish to finish here by returning to our starting point and give a sort of preview of a more geometrically justified 'calculus of variations'.

Recall that in a geometrical approach to calculus of variations (for autonomous second-order equations), vector fields along a curve take over the role of variations; they can be lifted to vector fields along the lifted curve in the tangent bundle.

For our present needs, a similar construction will work as follows. Starting from a section $\boldsymbol{\sigma}$ of $\overline{\pi}$ along a curve in M, and its image

$$X(t) = \rho^i_\alpha(t, x(t))\sigma^\alpha(t)\frac{\partial}{\partial x^i}$$

under the anchor map ρ, one can define a lift to a vector field along any λ-admissible curve in E, of the form

$$Y(t) = \rho^i_\alpha(t, x(t))\sigma^\alpha(t)\frac{\partial}{\partial x^i} + \Big(\dot{\sigma}^\alpha(t) - (C^\alpha_{\beta\gamma}(t, x(t))y^\gamma(t) - C^\alpha_\beta(t, x(t)))\sigma^\beta(t)\Big)\frac{\partial}{\partial y^\alpha}.$$

The calculus of variations problem can then be defined directly as the search for curves in M, with the property that

$$\int_{t_1}^{t_2} Y(L)(t)dt = 0,$$

for arbitrary $\boldsymbol{\sigma}$ along that curve with zero endpoints.

Working out the details of such geometrical construction will require the intrinsic features of the prolonged Lie algebroid on π_1.

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