Complete decoupling of systems of ordinary second-order differential equations

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Abstract

A theory is presented leading to a geometric characterization of separable systems of second-order ordinary differential equations. The idea is that such a characterization should provide necessary and sufficient conditions for the existence of coordinates, with respect to which a given system decouples. The specific problem of decoupling is merely chosen as a motivation to review differential geometric tools for the description of second-order dynamical systems. In particular, a survey is presented of the theory of derivations of scalar and vector-valued forms along the projection of the velocity-phase space onto the configuration manifold. It is explained how this theory relates to the more traditional calculus on tangent or first-jet bundles. In discussing this relation, the geometrical objects which are important for studying separability or other features of the evolution of second-order dynamics, come forward in a natural way. A few other applications of this theory are briefly touched upon.

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1. Introduction

Consider a system of, generally coupled and non-linear, second-order ordinary differential equations

$$\ddot{q}^i = f^i(t, q, \dot{q}) \qquad i = 1, \dots, n. \tag{1.1}$$

It is a truism to say that there is in general no hope for solving such a system analytically and that the best one can hope for is to obtain some reduction of the problem (for example via symmetries), or to get insight into qualitative aspects of the dynamics. Despite an enormous literature in this field of applied analysis, very few results are available when it concerns second-order equations with $n > 1$.

Suppose, however, that the given system (1.1) is actually a fairly simple problem in disguised form. This might be the case, for example, if it were possible to transform the equations into a system of (at least partially) decoupled equations or if it were possible to linearize them. How is one going to tell that this is the case?

The question we want to address here is the following: can one find new coordinates $Q^j = Q^j(t, q)$, such that the transformed system is completely decoupled, i.e. has the form

$$\ddot{Q}^j = F^j(t, Q^j, \dot{Q}^j), \tag{1.2}$$

where each F^j depends only on the corresponding Q^j and its derivative (and possibly also on time)? There are two aspects to this question. First of all, how can we tell if such Q^j exist, in a way which has practical relevance? In other words, we are certainly not interested in mere existence theorems, but want to discover necessary and sufficient conditions which can directly be applied to the given data, i.e. to the given functions f^i. Secondly, assuming the f^i pass all tests, we want to be able also to construct the good coordinates Q^j.

It is perhaps not so obvious to everyone that differential geometry has something to offer in that respect. Generally speaking, differential geometry is important in the study of dynamical systems when it comes to discussing global properties. In a somewhat less ambitious fashion, even in a local setting it may be very relevant to search for intrinsic properties, i.e. properties which do not depend on the choice of coordinates or, expressed differently, on a perhaps coincidental appearance of the system at hand.

Why then could differential geometry be relevant for the above raised question of separability? It is true of course that decoupling, if possible, will occur in a special set of coordinates. The existence of such coordinates, however, is coordinate independent, so that the tests which are to be discovered may well come from intrinsic tensorial objects and operators related to the given dynamical system.

Since the specific task for this workshop was to present a review lecture, the separability question will in fact be an alibi to give a review of differential geometric tools which have been developed to study systems of second-order differential equations, henceforth abbreviated to SODE's. The style of presentation of these elements of differential geometry is chosen to be very expository, in the hope that this will encourage students who have not specialised in this field to get at least a flavour of the general reasoning. Basically, the assumption will be that the reader has some notion about the concept of a differentiable manifold, vector fields and differential forms, and basic operations such as the exterior derivative and the Lie derivative. An easy access to these matters is offered by the book of Schutz (1980). To avoid extra technicalities, we will mostly restrict ourselves to an entirely autonomous framework! The natural environment for discussing SODE's then is a tangent bundle $\tau : TM \to M$.

We will enter into a short presentation of the following review topics. To begin with, we recall some basic elements of the geometry of a tangent bundle $\tau : TM \to M$ and discuss the additional structure coming from a SODE. By the end of that section, we will discover indications that it is of interest to introduce the concepts of differential forms (and in fact general tensor fields) along the projection τ. Faced with the problem of developing a calculus of differential forms along τ, Section 4 is about derivations. We recall the main ingredients and results of the standard work of Frölicher and Nijenhuis (1956) on the classification of derivations and sketch a survey of the corresponding theory for derivations of forms along τ. In Section 5, different ways are discussed for relating the newly developed calculus to the more traditional one on TM. In doing so, we come across the important covariant derivatives and the Jacobi endomorphism associated to a SODE.

After this review of general background information, we return to the separability problem in Section 6. For the concluding remarks in the final section, we will briefly mention some other applications of this theory.

Before starting our excursion now, we list a number of references for the various topics which will enter the discussion. The list is of course far from being exhaustive. For the geometry of a tangent bundle and its application to second-order dynamics in general and Lagrangian mechanics in particular, some key references are the work by Klein (see Klein (1962) for the early history and e.g. Klein (1992) for more recent developments), the contributions of Grifone (1972a,b) and a much cited paper by Crampin (1983a). Also of interest is an extensive review paper by Morandi *et al* (1990). The systematic study of forms along the projection τ was introduced in the thesis of Martínez (1991) and subsequently developed in a series of papers: see Martínez *et al* (1992, 1993a) and Sarlet *et al* (1995) for the corresponding theory in the time-dependent case. The main references for the application to complete separability are Martínez *et al* (1993b) and Cantrijn *et al* (1996). Some more references will be given underway.

2. Elements of the geometry of a tangent bundle and second-order dynamics

The tangent bundle TM of a manifold M is the union of all tangent spaces to M over all points $q \in M$. So each point of TM is a vector v_q attached to some point q of M and, if n is the dimension of M, it takes $2n$ coordinates (q^i, v^i) to specify v_q, namely the coordinates q^i which locate the point in M and the components v^i of the vector under consideration. The projection $\tau : TM \to M$ is the map $\tau : v_q \mapsto q$ and the inverse image $\tau^{-1}(q) = T_qM$ is called the fibre over q. TM itself, as a manifold, has its own tangent vectors of course, and those which are tangent to a fibre are said to be vertical. Natural objects on a manifold are things which can be defined in a canonical way, without needing any extra data or assumptions. On TM, there exist natural objects of the following kind. First, merely by the fact that TM is a vector bundle, we have a dilation vector field (sometimes called Liouville vector field) $\Delta = v^i\,\partial/\partial v^i$, characterizing indeed dilations in the fibres. More specifically for TM, there is a canonical type (1,1) tensor field (to be thought of as a linear map on vector fields, or dually on 1-forms)

$$S = dq^i \otimes \frac{\partial}{\partial v^i}\,, \tag{2.1}$$

called the *vertical endomorphism*:

$$X = \mu^i \frac{\partial}{\partial q^i} + \nu^i \frac{\partial}{\partial v^i} \quad \rightsquigarrow \quad S(X) = \mu^i \frac{\partial}{\partial v^i}.$$

We refer to the already cited literature and e.g. to Crampin (1983b), Crampin and Thompson (1985) and Yano and Davies (1975) for the properties of this tensor field and its role in characterizing the structure of a tangent bundle.

There are two ways of lifting vector fields on M canonically to vector fields on TM: the *vertical* and *complete* lift (or prolongation). For $X = X^i(q)\partial/\partial q^i \in \mathcal{X}(M)$, the vertical lift X^V is given by

$$X^V = X^i(q)\frac{\partial}{\partial v^i}\,. \tag{2.2}$$

This construction in fact lies at the heart of the definition of S. The complete lift on the other hand is obtained by prolonging the flow of X and reads:

$$X^c = X^i(q)\frac{\partial}{\partial q^i} + v^j \frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial v^i}. \tag{2.3}$$

More structure becomes available on TM when there are additional data. For example, a function $L \in C^\infty(TM)$ (satisfying some regularity condition) gives rise to a *symplectic form* $d\theta_L$, where

$$\theta_L = S(dL) = \frac{\partial L}{\partial v^i}\,dq^i \tag{2.4}$$

is the so-called Poincaré-Cartan 1-form associated to L. There is then a corresponding SODE field which represents the Euler-Lagrange equations coming from L. More generally, any SODE brings in itself some new canonical structure to TM. A SODE field Γ is determined by the intrinsic requirement $S(\Gamma) = \Delta$ and has the following coordinate representation:

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q,v) \frac{\partial}{\partial v^i}\,. \tag{2.5}$$

Its most important contribution to the structure on TM is that it comes along with a so-called non-linear connection. In general terms, a non-linear (Ehresmann) connection on $\tau : TM \to M$ is a smooth procedure for defining at each point v_q of TM a *horizontal subspace* of $T_{v_q}(TM)$, complementary to the space of vertical vectors. Every SODE Γ canonically defines such a connection in the following way. Observe for a start that, sitting on a manifold which carries a canonical structure (the tensor field S) and having been given a preferred direction to travel around on this manifold (the flow lines of the vector field Γ), it is quite natural to look at the way this canonical structure is being transported along the given flow. This is essentially the meaning of computing the Lie derivative of S with respect to Γ and it so happens that the square of the resulting tensor field is the identity. From the property $(\mathcal{L}_\Gamma S)^2 = I$, it follows that

$$P_{\scriptscriptstyle H} = \frac{1}{2}(I - \mathcal{L}_\Gamma S), \quad P_{\scriptscriptstyle V} = \frac{1}{2}(I + \mathcal{L}_\Gamma S) \tag{2.6}$$

are complementary projection operators, i.e. their sum is the identity and we have

$${P_{\scriptscriptstyle H}}^2 = P_{\scriptscriptstyle H}, \quad {P_{\scriptscriptstyle V}}^2 = P_{\scriptscriptstyle V}, \quad P_{\scriptscriptstyle H} \circ P_{\scriptscriptstyle V} = 0. \tag{2.7}$$

The horizontal subspace at each v_q then is, of course, the image of the projector $P_{\scriptscriptstyle H}$, restricted to $T_{v_q}(TM)$.

An alternative definition of the horizontal subspaces is worth to be mentioned here. With the aid of Γ, we have another lifting procedure, called *horizontal lift*, which can directly be defined as follows

$$X \in \mathcal{X}(M) \mapsto X^{\scriptscriptstyle H} \in \mathcal{X}(TM) = \frac{1}{2}(X^c + [X^{\scriptscriptstyle V}, \Gamma]). \tag{2.8}$$

For a coordinate explanation, it suffices to look at the lift of the coordinate fields on M, which are of the form

$$H_i = \left(\frac{\partial}{\partial q^i}\right)^{\scriptscriptstyle H} = \frac{\partial}{\partial q^i} - \Gamma^j_i \frac{\partial}{\partial v^j}\,, \tag{2.9}$$

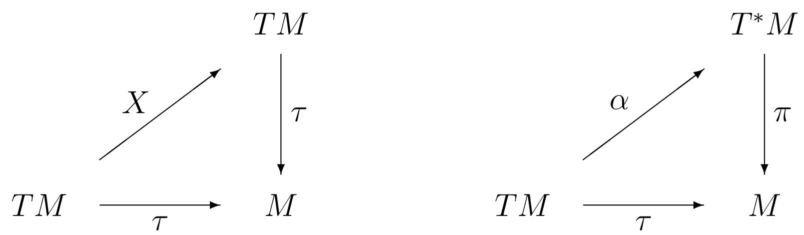
where the functions Γ^j_i are called connection coefficients and, for the case at hand, are given by

$$\Gamma^j_i = -\frac{1}{2}\frac{\partial f^j}{\partial v^i}. \tag{2.10}$$

It is of interest for our purposes to understand explicitly the link between these two equivalent ways of constructing horizontal subspaces. Clearly, every $Z \in \mathcal{X}(TM)$, through the use of the operators P_H and P_V, has a unique decomposition into a horizontal and a vertical part. Both of these parts can actually be regarded as lifts. But since their coefficients will generally be functions of q's and v's, we cannot be talking here about horizontal and vertical lifts of vector fields on M. What we are looking at are vector fields which in their derivation part look as if they live on M, but have their coefficients living on TM. They are vector fields along the projection $\tau : TM \to M$ and will be discussed in more detail in the next section. The link we were trying to explain then becomes clear if we observe that the above horizontal lift construction extends in a natural way to vector fields along τ by imposing linearity over $C^\infty(TM)$.

3. Vector fields and forms along the projection $\tau : TM \to M$

A vector field X along τ, and similarly a 1-form α along τ, are best visualized as maps defined by the following commutative schemes (where of course T^*M is the cotangent bundle of M, constructed in the same way as TM, with the dual space of covectors at each point replacing the vector space of tangent vectors, and with projection $\pi : T^*M \to M$).



As already indicated, the coordinate representation of these objects reads:

$$X = X^i(q,v)\frac{\partial}{\partial q^i}\,, \qquad\qquad \alpha = \alpha_i(q,v)\,dq^i\,. \tag{3.1}$$

More general tensor fields along τ can be constructed out of these building blocks by taking tensor products or wedge products in the usual way.

Let us introduce, as in Martínez *et al* (1992, 1993a), the following notations:

$\mathcal{X}(\tau)$: set of vector fields along τ,

$\bigwedge(\tau)$: set of scalar forms along τ,

$V(\tau)$: set of vector-valued forms along τ.

$V(\tau)$ is a graded module over $\bigwedge(\tau)$. Typically, an element $L \in V^\ell(\tau)$ is a tensor field of the form

$$L = \lambda^i \otimes \frac{\partial}{\partial q^i} \quad \text{with} \quad \lambda^i = \lambda^i_{j_1\cdots j_\ell}\, dq^{j_1} \wedge \cdots \wedge dq^{j_\ell} \in {\textstyle\bigwedge}^\ell(\tau). \tag{3.2}$$

Coming back to the remark at the end of the previous section, the vertical and horizontal lift operations from $\mathcal{X}(M)$ to $\mathcal{X}(TM)$ naturally extend to $\mathcal{X}(\tau)$. To be explicit, for

$$X = X^i(q,v)\frac{\partial}{\partial q^i} \in \mathcal{X}(\tau), \qquad \text{putting} \quad V_i = \frac{\partial}{\partial v^i}\,,$$

we have

$$X^V = X^i\,V_i, \qquad X^H = X^i\,H_i. \tag{3.3}$$

For the prolongation, however, the situation is slightly more complicated. Indeed, for $X \in \mathcal{X}(\tau)$, it is quite easy to define a prolongation $X^{(1)}$, but the result is, technically speaking, a vector field along the projection $\tau_{21} : T^2M \to TM$ (the $3n$-dimensional manifold T^2M has fibre coordinates, a^i say, to be thought of as modelling accelerations). With the aid of Γ, however, which can be regarded as a section γ of τ_{21} (think of γ as defining the submanifold $a^i = f^i(q,v)$ of T^2M), we obtain another lift operator $J_\Gamma : \mathcal{X}(\tau) \to \mathcal{X}(TM)$, defined by $J_\Gamma X = X^{(1)} \circ \gamma$ and determined, in coordinates, by the following prescription:

$$J_\Gamma X = X^i\frac{\partial}{\partial q^i} + \Gamma(X^i)\frac{\partial}{\partial v^i}\,. \tag{3.4}$$

A similar construction applies dually to 1-forms $\alpha = \alpha_i(q,v)dq^i \in \bigwedge^1(\tau)$ and results in a 1-form $I_\Gamma\alpha \in \bigwedge^1(TM)$, given by:

$$I_\Gamma\alpha = \alpha_i\, dv^i + \Gamma(\alpha_i)\, dq^i\,. \tag{3.5}$$

A couple of remarks are in order here. The image sets $J_\Gamma(\mathcal{X}(\tau))$ and $I_\Gamma(\bigwedge^1(\tau))$ were used at an earlier stage in a number of papers (see e.g. Sarlet *et al* (1984, 1987)) to develop calculational aspects adapted to given second-order dynamical systems. For example, $J_\Gamma(\mathcal{X}(\tau))$ is of interest because it contains all dynamical symmetries of Γ and, likewise, $I_\Gamma(\bigwedge^1(\tau))$ contains all so-called adjoint symmetries. The motivation which led later on to the idea of developing a "calculus along τ" came from the observation that many quantities of interest on TM appear to be generated by a corresponding object along τ. More precisely, they are such that the more familiar calculations on TM contain a certain degree of redundancy, because both the horizontal and the vertical part of the object in question come from the same corresponding object along τ. It is clear, for example, by looking at the coordinate expression of $J_\Gamma X$ that only the first term can carry relevant information, the second one then being determined automatically. Examples where this phenomenon occurs are the already mentioned symmetries and adjoint symmetries, but also recursion operators and the Poincaré-Cartan form in Lagrangian mechanics.

Before entering the review of the quite recent theory of derivations of forms along τ, this may be the place to say a few words about the picture for the more general case of time-dependent SODE's, where there should be room also to allow for time-dependent transformations.

For the time-dependent framework, the natural geometrical environment is the following. Second-order equations are governed by a vector field

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i\, \frac{\partial}{\partial q^i} + f^i(t,q,\dot{q})\frac{\partial}{\partial \dot{q}^i}\,, \tag{3.6}$$

on a first-jet bundle $J^1\pi$ of a bundle $\pi : E \to \mathbb{R}$. Here, (t, q^i) are local bundle coordinates on the $n+1$-dimensional space E and $(t, q^i, \dot{q}^i)$ then are the induced natural coordinates on $J^1\pi$, which is the set of equivalence classes of curves $t \mapsto (t, q^i(t))$ in E, whereby equivalence refers to first-order tangency. $J^1\pi$ also carries a canonical "vertical endomorphism", defined by the tensor field

$$S = \theta^i \otimes \frac{\partial}{\partial \dot{q}^i}, \quad \text{with} \quad \theta^i = dq^i - \dot{q}^i dt. \tag{3.7}$$

The local 1-forms θ^i are called contact forms. For a general reference on jet bundles, see Saunders (1989).

For any $X \in \mathcal{X}(E)$, say

$$X = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i}, \tag{3.8}$$

there is a well defined concept of prolongation again: $X^{(1)} \in \mathcal{X}(J^1\pi)$ is of the form

$$X^{(1)} = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + (\dot{X}^i - \dot{q}^i \dot{X}^0) \frac{\partial}{\partial \dot{q}^i}\,. \tag{3.9}$$

Obviously, there is also a vertical lift operation, in fact we have

$$X^V = S(X^{(1)}). \tag{3.10}$$

Finally, the SODE Γ this time defines a "non-linear connection" on the bundle $\pi_1^0 : J^1\pi \to E$; the subspaces of $T(J^1\pi)$, which are complementary to the vertical sub-bundle are determined by the following horizontal lift construction:

$$X^H = \frac{1}{2}\,(X^{(1)} + [X^V, \Gamma] + \langle X, dt\rangle \Gamma)\,. \tag{3.11}$$

Despite some clear analogies at the start of the description, the time-dependent theory is not a trivial copy of the autonomous one: there are many technical complications, but a corresponding calculus of forms along π_1^0 has been developed (see Sarlet *et al* (1995)).

4. Derivations – Classification

We have indicated several reasons why it is of interest to develop a calculus of forms along τ, but what does that really amount to? Apart from purely algebraic operations (such as the wedge product), the operations we know to be of interest from the standard calculus of forms (see e.g. Flanders (1963), Loomis and Sternberg (1968)) are things like the exterior derivative, the interior product with vector fields and the Lie derivative. Additional operations may become important when there is more structure available. For example, when there is a linear connection at our disposal, we will certainly be interested in covariant derivatives. What all these operations have in common is that they are derivations. A general theory of derivations of differential forms has been developed by Frölicher and Nijenhuis (1956). Since then, of course, many people have contributed to the field or have developed a similar machinery starting from different premises. One of the main contributors to the subject is Michor (see e.g. Michor (1987,1989) and also Kolář *et al* (1993)). It is reasonable to expect that, also for the case of forms along τ, most calculational aspects of interest will emerge from a general theory of derivations or at least will benefit from being interpreted within such a theory.

Introducing the concept of derivations of differential forms is a matter of defining operators with certain abstract properties and these are the same whether we are talking about ordinary differential forms on a manifold or forms along a map.

Definition: $D : \bigwedge(\tau) \to \bigwedge(\tau)$ is a derivation of degree r if

1. $D(\bigwedge^p(\tau)) \subset \bigwedge^{p+r}(\tau)$
2. $D(\alpha + \lambda\,\beta) = D\alpha + \lambda\,D\beta, \qquad \alpha, \beta \in \bigwedge(\tau),\ \lambda \in \mathbb{R}$
3. $D(\alpha \wedge \gamma) = D\alpha \wedge \gamma + (-1)^{pr}\alpha \wedge D\gamma, \qquad \alpha \in \bigwedge^p(\tau).$

To define derivations of vector-valued forms, we need somehow to impose in addition that the operator also satisfies the sort of graded Leibniz rule 3 with respect to multiplication of the vector field part of the tensor with scalar forms. Recall that $V(\tau)$ is a (graded) module over $\bigwedge(\tau)$, so that the wedge product of a scalar and a vector-valued form makes sense. Instead of giving a full definition then of a derivation of $V(\tau)$, it suffices to mention that a derivation D (of degree r) of $V(\tau)$ has an associated derivation of $\bigwedge(\tau)$, also denoted by D, and that the additional requirement to make everything consistent is: for $L \in V^\ell(\tau)$, $\omega \in \bigwedge^p(\tau)$,

$$D(\omega \wedge L) = D\omega \wedge L + (-1)^{pr}\omega \wedge DL. \tag{4.1}$$

One easily proves that every D of $\bigwedge(\tau)$ is completely determined by its action on functions on TM and on so-called *basic 1-forms* (i.e. elements of $\bigwedge^1(M)$). For an extension of a derivation of $\bigwedge(\tau)$ to $V(\tau)$, it then further suffices to specify the action on *basic vector fields* (elements of $\mathcal{X}(M)$) in a way which is consistent with multiplication by basic functions.

The set of derivations of $\bigwedge(\tau)$ (or $V(\tau)$) forms a graded Lie algebra. Indeed, if D_1 and D_2 are derivations of degree r_1 and r_2 respectively, then

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1 \tag{4.2}$$

is a derivation of degree $r_1 + r_2$. Considering in addition a D_3 of degree r_3, we have the graded Jacobi identity:

$$(-1)^{r_1 r_3}[D_1, [D_2, D_3]] + (-1)^{r_2 r_1}[D_2, [D_3, D_1]] + (-1)^{r_3 r_2}[D_3, [D_1, D_2]] = 0. \tag{4.3}$$

Having introduced derivations, the first main challenge is understanding their classification and for that purpose we will briefly review how the classification works in the standard theory of Frölicher and Nijenhuis (for derivations of scalar forms).

The algebra $\bigwedge(M)$ comes equipped with the exterior derivative d, a derivation of degree 1. Two types of derivations then can be distinguished: *derivations of type $i_\star$*, they are the ones which vanish on functions, and *derivations of type $d_\star$*, by definition the ones that commute with d. For $L \in V^r(M)$, one defines i_L, a derivation of degree $r-1$ by the following requirements: $i_L f = 0$ (on functions) and for $\alpha \in \bigwedge^1(M)$ we have,

$$i_L\alpha(X_1, \dots, X_r) = \alpha(L(X_1, \dots, X_r)). \tag{4.4}$$

The action on forms of degree higher than one then follows from the derivation property 3. The main classification results of Frölicher and Nijenhuis are summarized in the following theorem.

Theorem.

1. *Every type $i_\star$-derivation is of the form i_L for some L.*
2. *Every derivation of type $d_\star$ is of the form $d_L = [i_L, d]$ for some L.*
3. *Every derivation D has a unique decomposition of the form*

$$D = i_{L_1} + d_{L_2}, \quad \text{for some } L_1, L_2.$$

Coming back now to derivations of $\bigwedge(\tau)$, the situation turns out to be more involved: it is not possible to come to a complete classification scheme without extra data. One should, of course, first of all wonder whether there exists some canonical kind of exterior derivative to mimic the idea of $d_\star$-derivations. There is indeed such a derivation of degree one, called the *vertical exterior derivative*, denoted by d^V and determined by the following rules

$$d^V F = V_i(F)\, dq^i, \quad V_i = \frac{\partial}{\partial v^i}, \qquad \forall F \in C^\infty(TM) \tag{4.5}$$

$$d^V \alpha = 0 \qquad \text{for} \quad \alpha \in \textstyle\bigwedge^1(M). \tag{4.6}$$

Defining and constructing $i_\star$-derivations is no problem: derivations of the form i_L (with $L \in V(\tau)$) can simply be defined as before. One can, however, rightaway suspect that this is not enough to arrive at a full classification of derivations. Indeed, it suffices to think of functions to realize that information about $V_i(F)$ (coming from d^V) will not be sufficient to cover the full picture and that, so to speak, the horizontal information is missing. As we already learned above, 'horizontal information' is not canonically available and requires an extra tool. Assume therefore that we have a connection on the bundle $\tau : TM \to M$ at our disposal (not necessarily a Sode connection). Then, we know what horizontal vector fields on TM are and they are locally spanned by

$$H_i = \frac{\partial}{\partial q^i} - \Gamma^i_j \frac{\partial}{\partial v^i},$$

where the functions $\Gamma^i_j(q, v)$ are the connection coefficients. We define a degree 1 derivation d^H, the *horizontal exterior derivative* by the following properties,

$$d^H F = H_i(F)\, dq^i, \qquad F \in C^\infty(TM) \tag{4.7}$$

$$d^H \alpha = d\alpha \qquad \text{for} \quad \alpha \in \textstyle\bigwedge^1(M). \tag{4.8}$$

d^H, in some sense, is an extension to $\bigwedge(\tau)$ of the ordinary exterior derivative on basic forms. For completeness, let us mention that the action of d^V and d^H can easily be extended to $V(\tau)$. Remembering that it suffices to know what happens with basic vector fields for that purpose, the extension is determined by,

$$d^V\left(\frac{\partial}{\partial q^i}\right) = 0, \qquad d^H\left(\frac{\partial}{\partial q^i}\right) = V_i(\Gamma^k_j)\, dq^j \otimes \frac{\partial}{\partial q^k}. \tag{4.9}$$

Needless to say, these key derivations can be defined also in an intrinsic way, i.e. without using coordinate descriptions (see the literature quoted before).

We are sufficiently equipped now to arrive at a classification result. For the classification of derivations of $\bigwedge(\tau)$, one has to distinguish the following types of derivations:

- type $i_\star$: derivations vanishing on functions; they are determined as in the standard theory by some $L \in V(\tau)$ and of the form i_L;
- type $d^V_\star$: derivations of the form $d^V_L = [i_L, d^V]$;
- type $d^H_\star$: derivations of the form $d^H_L = [i_L, d^H]$.

For the extension to vector-valued forms along τ, one further ingredient is needed to characterize the non-zero derivations which vanish on the whole of $\bigwedge(\tau)$. These are called derivations of type $a_\star$. If D is of degree r and of type $a_\star$, it is determined by a tensor field $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$ and we write it as a_Q. To understand the meaning of such a derivation, it suffices to think of a Q which consists of only one term and hence is of the form $Q = \omega \otimes U$, with $\omega \in \bigwedge^r(\tau)$ and $U \in V^1(\tau)$, and to know what the action is on basic vector fields. In fact, for any $X \in \mathcal{X}(\tau)$ we have,

$$a_{\omega\otimes U}(X) = \omega \otimes U(X) \quad (\in V^r(\tau)). \tag{4.10}$$

Theorem. *Every derivation D of $V(\tau)$, of degree r, has a unique decomposition into the form:*

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3} + a_Q \tag{4.11}$$

for some $L_1 \in V^{r+1}(\tau)$, $L_2, L_3 \in V^r(\tau)$, $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$.

In further developing a calculus of such derivations, one will inevitably start computing commutators of the different types of derivations which entered the discussion so far and, of course, look for other derivations which come along when more specific data are being introduced. The computation of commutators of some of these derivations can be rather messy. We limit ourselves here to discussing just a few commutators which lead us directly to additional features of interest.

Derivations of type $d^V_\star$ constitute a subalgebra of the graded algebra of derivations. The interest of such a property is that it implies a bracket operation on the vector-valued forms themselves. The mechanism works as follows: starting from arbitrary elements L_1, L_2 of $V(\tau)$, there must be a third one which determines the commutator of the corresponding $d^V_\star$ derivations; this new vector-valued form is then said to be the bracket of L_1 and L_2. The (graded) Jacobi identity of the algebra of derivations automatically induces a Jacobi identity on the newly defined bracket. So, for the case at hand, we introduce a vertical bracket $[\,,\,]_V$ on $V(\tau)$, which is defined by the relation

$$[d^V_{L_1}, d^V_{L_2}] = d^V_{[L_1,L_2]_V}. \tag{4.12}$$

In particular, for $X, Y \in \mathcal{X}(\tau)$, the resulting operation explicitly reads:

$$[X,Y]_V = (X^k V_k(Y^i) - Y^k V_k(X^i))\frac{\partial}{\partial q^i}. \tag{4.13}$$

This result strongly suggests to introduce, by analogy, a horizontal bracket operation, which for the case of vector fields reads:

$$[X,Y]_H = (X^k H_k(Y^i) - Y^k H_k(X^i))\frac{\partial}{\partial q^i}\,. \tag{4.14}$$

This bracket, however, does generally not satisfy a Jacobi identity because the commutator of two $d^H_\star$ derivations need not be a derivation of the same type. We will see in a moment that the obstruction is related to the curvature of the connection.

One cannot seriously talk about the role of a connection in a theory without looking at properties of some important tensor fields related to this connection, namely its *torsion* and *curvature*. So how do torsion and curvature make their appearance within the present approach? If we restrict ourselves for a moment to derivations of the scalar forms $\bigwedge(\tau)$, it appears that the commutator of d^H and d^V is of type $d^V_\star$, while the commutator of d^H with itself has terms of type $i_\star$ and $d^V_\star$ only. Having proved these facts, one simply must look at the vector-valued forms which determine these terms. They appear to be related to the torsion $T \in V^2(\tau)$ and the curvature $R \in V^2(\tau)$ in the following way:

$$[d^H, d^V] = d^V_T, \tag{4.15}$$

$$\frac{1}{2}[d^H, d^H] = -i_{d^V R} + d^V_R. \tag{4.16}$$

Torsion and curvature of a connection on $\tau : TM \to M$ are of course well known concepts, but are in the traditional theory (see e.g. de León and Rodrigues (1989)) not regarded as elements of $V(\tau)$. To see that we are essentially talking about the same concepts here, it suffices to compare the coordinate expressions. The tensor fields T and R which make their appearance in our approach read:

$$T = \frac{1}{2}\left(\frac{\partial \Gamma^k_i}{\partial v^j} - \frac{\partial \Gamma^k_j}{\partial v^i}\right) dq^i \wedge dq^j \otimes \frac{\partial}{\partial q^k}\,, \tag{4.17}$$

$$R = \frac{1}{2}\left(\frac{\partial \Gamma^i_j}{\partial q^k} - \frac{\partial \Gamma^i_k}{\partial q^j} + \frac{\partial \Gamma^i_k}{\partial v^\ell}\Gamma^\ell_j - \frac{\partial \Gamma^i_j}{\partial v^\ell}\Gamma^\ell_k\right) dq^j \wedge dq^k \otimes \frac{\partial}{\partial q^i}\,. \tag{4.18}$$

Note in passing that the commutator of d^V with itself is zero or, in other words that $d^V \circ d^V = 0$. Moreover, the cohomology of d^V is trivial because of the linear vector space structure of the fibres where d^V is essentially acting.

To finish this section, it is important to bring a special class of derivations into the spotlight, namely the so-called *self-dual derivations of degree 0*. Such a derivation D is characterized by the property that for all $X \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$:

$$D\langle X, \alpha\rangle = \langle DX, \alpha\rangle + \langle X, D\alpha\rangle\,. \tag{4.19}$$

Via this different Leibniz-type property, the action of a self-dual derivation automatically extends to tensor fields of any type. As an example of a self-dual derivation of degree zero in the ordinary calculus of forms, think of the

Lie derivative. So far, we have not come across any such derivation in our discussion of derivations of $V(\tau)$. In fact, none of the degree zero derivations which would enter the decomposition of an arbitrary derivation of degree zero need be self-dual. But we can start from one of these fundamental degree zero derivations to construct new ones by a process of self-dualization (even in two different ways). We will not enter into the details of these constructions here, but instead give one simple example. According to the general classification results, a degree zero derivation of type $a_\star$ will be of the form a_U for some type (1,1) tensor field U along τ. It clearly is not self-dual, because it vanishes on 1-forms. But we can construct a new degree zero derivation, say μ_U by letting μ_U coincide with a_U on $\mathcal{X}(\tau)$, defining its dual action on 1-forms via the property (4.19) and extending this further to the whole of $\bigwedge(\tau)$ via the derivation property 3. It turns out that

$$\mu_U = a_U - i_U\,. \tag{4.20}$$

5. Interface with the calculus on TM and the special case of a SODE connection

Recall that, having a connection and its associated projectors P_H and P_V, every $Z \in \mathcal{X}(TM)$ has a unique decomposition:

$$Z = X^V + Y^H \quad \text{for some } X, Y \in \mathcal{X}(\tau). \tag{5.1}$$

Similar decompositions can be discovered for 1-forms, type (1,1) tensor fields, etcetera. Part of the motivation for introducing the calculus along τ came afterall from situations where different parts in such a decomposition essentially come from the same object along τ. It is nevertheless equally essential to keep in touch with the space where our dynamics lives and we have previously discussed various ways for doing that by certain lifting procedures. A nice additional aspect is that by doing calculations with these lifts on TM, one can learn new features about the calculus along τ. This is so because the result of such a calculation again has its decomposition into pieces which come from objects along τ and these in turn must inevitably originate from operations on the objects one started from. To illustrate this idea, think for example of the horizontal and vertical lift of two vector fields along τ and compute the Lie bracket of the resulting elements $X^H, Y^V \in \mathcal{X}(TM)$. Look subsequently at the decomposition of the new vector field and more specifically at the elements of $\mathcal{X}(\tau)$ which define both parts. What one discovers in this case are two degree zero derivations D^H_X and D^V_Y on the $C^\infty(TM)$-module $\mathcal{X}(\tau)$ (see the second of the formulas below). These derivations can subsequently be extended to the whole of $V(\tau)$ by imposing self-duality in the way explained before.

For completeness, let us now first write down the full set of bracket relations which are obtained this way.

$$[X^V, Y^V] = ([X,Y]_V)^V \tag{5.2}$$
$$[X^H, Y^V] = (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H \tag{5.3}$$
$$[X^H, Y^H] = ([X,Y]_H)^H + R(X,Y)^V. \tag{5.4}$$

As one can observe, if the curvature tensor field R and the horizontal and vertical brackets would not have been introduced before, we would be forced to think of them now.

In coordinates, D^V_X and D^H_X are determined by

$$\mathrm{D}^V_X F = X^i\, V_i(F), \quad \mathrm{D}^H_X F = X^i\, H_i(F) \tag{5.5}$$

on functions F, and by the following action on basic vector fields and basic 1-forms:

$$\mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0, \qquad \mathrm{D}^V_X dq^i = 0, \tag{5.6}$$

$$\mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^j V_i(\Gamma^k_j) \frac{\partial}{\partial q^k}, \qquad \mathrm{D}^H_X dq^i = -X^j V_k(\Gamma^i_j) dq^k. \tag{5.7}$$

Note that these derivations depend linearly on their vector argument. They are called, for that reason, the *vertical* and *horizontal covariant derivative*. An arbitrary self-dual derivation of degree zero D has a unique decomposition of the form

$$D = \mathrm{D}^V_X + \mathrm{D}^H_Y + \mu_Q, \qquad X, Y \in \mathcal{X}(\tau),\ Q \in V^1(\tau). \tag{5.8}$$

Everything that has been said so far about derivations of $V(\tau)$ is valid with respect to any chosen connection on $\tau : TM \to M$. Naturally, we are interested primarily in the case where this connection is the one coming from a SODE and will now look at the additional features which apply for this case.

To begin with, as one can easily see from the coordinate expression (4.17) of the torsion and the formula (2.10) for the connection coefficients, a SODE connection is torsion free, which implies, for example, that d^V and d^H will commute.

Using the mechanism explained in detail above, the additional lifting procedures $J_\Gamma : \mathcal{X}(\tau) \to \mathcal{X}(TM)$ and $I_\Gamma : \bigwedge^1(\tau) \to \bigwedge^1(TM)$ produce a new self dual derivation ∇. We have

$$J_\Gamma X = X^H + (\nabla X)^V, \quad I_\Gamma \alpha = (\nabla\alpha)^H + \alpha^V, \tag{5.9}$$

and the definition of ∇ is completed if we add that for functions F, $\nabla F = \Gamma(F)$. ∇ is called the *dynamical covariant derivative*, a name which reflects that it is the operator for controlling or describing the evolution of the system. The basic coordinate expressions read,

$$\nabla\left(\frac{\partial}{\partial q^i}\right) = \Gamma^k_i \frac{\partial}{\partial q^k}, \qquad \nabla(dq^i) = -\Gamma^i_j dq^j . \tag{5.10}$$

Equally important is a type (1,1) tensor field $\Phi \in V^1(\tau)$, called the *Jacobi endomorphism* and defined e.g. via the decomposition

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V. \tag{5.11}$$

In coordinates:

$$\Phi = \Phi^i_j \, dq^j \otimes \frac{\partial}{\partial q^i}, \quad \text{with} \quad \Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^k_j \Gamma^i_k - \Gamma(\Gamma^i_j). \tag{5.12}$$

The terminology for Φ is inspired by the fact that it reduces to the linear map with the same name which appears in the equation for geodesic deviation in the case of a spray (and its associated linear connection). For more general SODE's, the concepts of dynamical covariant derivative and

Jacobi endomorphism were first introduced by Foulon (1986), in the framework of a projectivized tangent bundle (and homogenized formulation of the second-order system).

To get a flavour of the importance of Φ, it suffices to mention that Φ completely determines the curvature of the connection and its dynamical evolution: we have

$$d^V\Phi = 3R, \qquad d^H\Phi = \nabla R. \tag{5.13}$$

For later use, let us mention here finally another interesting type (1,1) tensor field, usually called (in the traditional calculus on TM) the *tension* (see Grifone (1972a) or de León and Rodrigues (1989)). In our present context, the tension is defined by

$$\mathbf{t} = \, - d^H\mathbf{T} = \left(\Gamma^j_i - v^k \frac{\partial \Gamma^j_i}{\partial v^k}\right) dq^i \otimes \frac{\partial}{\partial q^j}, \tag{5.14}$$

where $\mathbf{T} = v^i\, \partial/\partial q^i$ is the canonical vector field along τ (the identity map in the commutative diagram of Section 3). Obviously, this tensor can provide useful information about homogeneity properties of the connection coefficients. It is, incidentally, not a tensor field which comes out of the blue here; it would be forced upon us, for example, if one were to look at the decomposition of the dynamical covariant derivative ∇.

6. The separability problem

The following preliminary considerations will help to understand the general context and justification for the methodology which will be sketched in this section. Generally speaking, it is clear that many of the properties of connections are governed by properties of its torsion and curvature. The idea to study dynamical systems via a connection also has a history: it probably arises first in work of Cartan on projective connections (Cartan (1937), see also Arnold (1983)). The Sode connection has no torsion and its curvature, as we observed in the previous section, is determined by Φ. It should therefore not come as a surprise that Φ is the key to many aspects related to second-order dynamics.

In Martínez *et al* (1993b) and Cantrijn *et al* (1996), we have studied first separability properties of general type (1,1) tensor fields along τ. Features of this study include: the characterization of tensor fields whose eigendistributions are basic, i.e. are generated by vector fields on the base manifold M, with the further property that these generators span an integrable distribution (in the sense of the Frobenius theorem); conditions under which algebraic diagonalizability of the coefficient matrix of such a tensor implies diagonalizability via a coordinate transformation on M; finally, the separability of such a tensor, meaning that its eigenfunctions depend on the coordinates of the corresponding eigendistributions only. We merely comment here on a couple of propositions which illustrate each of these features.

Let $U \in V^1(\tau)$ be a *diagonalizable* tensor field and D a self-dual derivation.

Proposition. *The eigendistributions of U are invariant under D if and only if $[DU, U] = 0$. DU is then also diagonalizable with eigenfunctions which follow from those of U by the action of D.*

The bracket operation involved here is of course the ordinary commutator of linear maps. The idea is to apply this result to the case where D is $\mathrm{D}^{\scriptscriptstyle V}_X$ or ∇. In the first case, we introduce the type (1,2) tensor field $C^{\scriptscriptstyle V}_U$ defined by

$$C^{\scriptscriptstyle V}_U(X, Y) = [\mathrm{D}^{\scriptscriptstyle V}_X U, U](Y)\,, \tag{6.1}$$

whose vanishing will imply that the eigendistributions of U are basic. If one further requires that $[\nabla U, U] = 0$, the implication will be that all eigendistributions are also $\mathrm{D}^{\scriptscriptstyle H}_X$-invariant (for all X). As a result, they will be simultaneously integrable and therefore U will be diagonalizable in coordinates. In fact, it can easily be seen from the coordinate expression (5.10) that the ∇-invariance will do something more in these new coordinates: it implies that connection coefficients with indices referring to different eigendistributions will be zero. This is clearly a step in the right direction, because it says that each of the right-hand sides f^i of the second-order system can only depend on a certain number of velocity variables (the ones coming from a single eigendistribution). In particular, if all eigendistributions were 1-dimensional, we would already have complete decoupling with respect to the velocity variables.

Proposition. *Let U be a diagonalizable tensor field, for which $C^{\scriptscriptstyle V}_U$ =*

0, $[\nabla U, U] = 0$, $d^{V}U = 0$ *and* $d^{H}U = 0$. *Then,* U *is separable and the degenerate eigenvalues are constant.*

The further gain in this statement is that we are now looking at situations were the eigenfunctions of U are either constant or depend on at most one position and one velocity variable. To relate such a property also to the f^i of our system, we clearly have to specify U now to be a tensor field associated to the given SODE and there can be little doubt that this tensor field must be the Jacobi endomorphism Φ. Besides, if the origin of the extra two conditions we have thrown in for the second proposition may look a bit mysterious, things should become clear when we have Φ in mind: indeed, via the results (5.13), these conditions refer directly to the curvature of the connection (note by the way that, in the case of Φ, the first extra condition implies the other).

Theorem. *Assume that* Φ *is diagonalizable and that:* $R = 0$, $[\nabla\Phi, \Phi] = 0$ *and* $C^{V}_{\Phi} = 0$. *Then the given system separates into single equations for each 1-dimensional eigenspace of* Φ *and into individual subsystems for each degenerate eigenvalue, which is then necessarily constant. Each such multi-dimensional subsystem further decouples iff the tension* $\mathbf{t}$ *is diagonalizable and* $C^{V}_{\mathbf{t}} = 0$.

The argumentation we have developed before the formulation of this theorem gives a fairly good idea already of the different steps in its proof. The only situation which needs some further comments concerns the case when some eigenvalues are degenerate. For each such eigenvalue, we are faced with a subsystem with $\Phi = \mu I$ (μ constant). Clearly, such a Φ contains no further information, hence we appeal to another tensor (the tension $\mathbf{t}$) to help us out, and we need less of the requirements for this tensor because some are identically satisfied in view of what preceeds. Are we, however, going to get stuck again if also the tension has degenerate eigenvalues? The answer is no, because by that time we are looking at an affine connection whose linear part has zero curvature, so there is a classical result (see e.g. Crampin and Pirani (1986)) which tells us that coordinates exist in which all connection coefficients are zero.

It may be worthwhile pointing out at this point that for the time-dependent version of this theory the situation is more complicated when Φ is a multiple of the identity. The reason is that an expression like (5.14) does not behave well under time-dependent coordinate transformations (even when we write, as we should, contact forms θ^i instead of the dq^i). In fact, there is no tension tensor in the time-dependent set-up: there is a canonical vector field **T** for sure, but when we take its horizontal exterior derivative we get zero. For general f^i, the subsequent analysis therefore can become quite complicated (see Cantrijn *et al* (1996)); fortunately, in most cases of interest the given f^i will polynomially depend on the velocity coordinates and there is an easy way out then as has been reported in Sarlet (1996).

Coming back to the autonomous situation, some comments are in order now about the practical applicability of this theory. There have been earlier results in the literature about complete or partial separability (see Ferrario *et al* (1985,1987) and Kossowski and Thompson (1991)). They rely on the existence of a tensor field or distribution with certain properties, but there are no conditions available to test the given f^i for the existence of such quantities. Our results, on the contrary, lead to algebraic conditions which can be tested directly on the given data. When a system passes all tests, finding the right coordinates is most of the time a matter of integrating integrable distributions which poses no great difficulties. The test computations, although algebraic in nature, may be very tedious however, so one will normally not try to do it all by hand and use computer algebra packages in support. Note also that the hardest computations come with the test of diagonalizability. So although this is the first requirement in the theory, it is the condition one will test at the very end, after all the vanishing tensor conditions (which in principle are very straightforward) have been matched.

One should realize also that complete decoupling is a very strong requirement, so we cannot expect many systems to pass the tests! If some model equations contain a number of as yet undetermined parameters or arbitrary functions, then there is a reasonable chance that a case of separability may be detected for specific values of the parameters or specific choices of the functions. We will not enter into such lengthy considerations here and merely give a few illustrative examples below.

As a first example, consider the following non-linearly coupled system:

$$\ddot{q}_1 = -a_1\,q_1 + b\,q_1 q_2 + (k_1 - \ell_1\,{q_1}^2)\,\dot{q}_2 \tag{6.2}$$

$$\ddot{q}_2 = -a_2\,q_2 + c\,{q_2}^3 + (k_2 - \ell_2\,{q_2}^2)\,\dot{q}_1 \tag{6.3}$$

Imposing all vanishing tensor requirements consecutively leads to the following restrictions: $\ell_1 = \ell_2 = 0$, $b = 0$, $c = 0$, $a_2 = a_1$. The diagonalizability of Φ in the end further requires: $k_1 k_2 > 0$. Decoupling then is achieved by the transformation

$$Q_1 = k_2\, q_1 + \sqrt{k_1 k_2}\, q_2 \tag{6.4}$$

$$Q_2 = -\sqrt{k_1 k_2}\, q_1 + k_1\, q_2. \tag{6.5}$$

This may look like a rather disappointing result, because decoupling apparently forces this system to be linear.

Consider next a system of the following type

$$\ddot{q}_1 = -c_1\, q_1 + b\, q_1^2 - a\, q_2^2\,, \tag{6.6}$$

$$\ddot{q}_2 = -c_2\, q_2 - 2m\, q_1 q_2\,. \tag{6.7}$$

The vanishing tensor conditions here imply (apart from the trivially decoupled case $a = m = 0$) that we must have: $c_1 = c_2$ and $m + b = 0$. Diagonalizability of Φ requires ab to be negative. Setting $s^2 = -ab$, the transformation $Q_1 = bq_1 + sq_2$, $Q_2 = bq_1 - sq_2$ decouples the equations. What we have recovered here is one of the integrable cases of the well-known Henon-Heiles system. Note, by the way, that another integrable case of this system is related to a form of separability, namely separability of the Hamilton-Jacobi equation. This, however, is an entirely different matter, for which of course it would also be interesting, if possible, to characterize it by conditions on the given data.

Finally, we give a simple example to illustrate the relevance of the generalization of the theory to a time-dependent framework:

$$\ddot{q}_1 = -\lambda\, \dot{q}_2\,, \qquad \lambda \text{ constant} \tag{6.8}$$

$$\ddot{q}_2 = \lambda\, \dot{q}_1\,. \tag{6.9}$$

No time-independent transformation will decouple this system. However, although the system itself is autonomous, it turns out that the following time-dependent transformation does the job:

$$Q_1 = (1 - \cos\lambda t) q_1 - (\sin\lambda t) q_2\,, \tag{6.10}$$

$$Q_2 = (\sin\lambda t) q_1 + (1 - \cos\lambda t) q_2\,. \tag{6.11}$$

7. Final comments

It is convenient at this stage to mention a few other fields of applications of the geometric tools which have been reviewed. To that end, we first make a digression about the quite remarkable fact that one can associate a linear connection to the non-linear SODE connection.

A (linear) connection on a bundle $\pi : E \to M$ is a map D associating to every $X \in \mathcal{X}(M)$ a derivation D_X of the $C^\infty(M)$-module of sections σ of π, so that for $f \in C^\infty(M)$:

$$D_X(f\sigma) = fD_X\sigma + X(f)\sigma\,, \tag{7.1}$$

$$D_{fX}\sigma = fD_X\sigma\,. \tag{7.2}$$

The bundle of interest for our purposes is the so-called pull-back bundle $\pi : \tau^*TM \to TM$. An element of the manifold τ^*TM can simply be regarded as a triplet (q, v, w), where (q, v) is the point of TM onto which it projects under π and w is a vector tangent to M at the point $q = \tau(q, v)$. It should be clear by comparison with the diagram of Section 3 that sections of this bundle are exactly vector fields along τ. For any $Z \in \mathcal{X}(TM)$, with decomposition

$$Z = X^H + Y^V, \qquad X, Y \in \mathcal{X}(\tau), \tag{7.3}$$

an associated derivation D_Z of $\mathcal{X}(\tau)$ is defined by

$$D_Z = \mathrm{D}^H_X + \mathrm{D}^V_Y, \tag{7.4}$$

and is easily seen to satisfy the requirements (7.1-2). This idea for a linear connection was developed in the thesis of Martínez (1991) and reported recently in Cariñena and Martínez (1995). A more extensive discussion of the same idea in the context of time-dependent SODE's is given in Crampin *et al* (1996). The main advantage of this approach is that it directly brings some of the most important derivations and tensor fields to the forefront. Some of the tensorial identities mentioned before can, for example, be understood in this picture as expressing Bianchi identities associated to the linear connection. Clearly, one will also be interested in its curvature and, in particular, in the meaning or interpretation of vanishing curvature.

The curvature of this linear connection is determined for a great deal by Φ again, but also by a tensor field of the form

$$\theta = \theta^k_{jml}\, dq^l \otimes dq^j \otimes \left(dq^m \otimes \frac{\partial}{\partial q^k} \right), \tag{7.5}$$

with

$$\theta^k_{jml} = -\frac{1}{2} \frac{\partial^3 f^k}{\partial v^j \partial v^m \partial v^l}. \tag{7.6}$$

It turns out that vanishing curvature for the linear connection roughly corresponds to *linearizability* of the SODE or reducability to free motion. One of the conditions then is $\theta = 0$, i.e. the equations can at most depend quadratically on the velocities. We refer to the above cited papers for details. Note, however, that there is no contradiction here with, for example, the work of Lie on linearizability (reported elsewhere in this volume): the reason why linearizable equations here are bound to be polynomials of degree 2 in the velocities (and not of degree 3) is that our framework does not allow for coordinate transformations which also transform the independent variable t.

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